حل تمرینات تکمیلی مفصل روش های انتگرال گیری :

1) $\int \sin(\ln x)\,dx$; $I = \int \sin(\ln x)\,dx$

$$\begin{cases} u = \sin(\ln x) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\cos(\ln x)\,dx \\ v = x \end{cases}$$

$$I = x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$\begin{cases} u = \cos(\ln x) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{-1}{x}\sin(\ln x)\,dx \\ v = x \end{cases}$$

$$I = x\sin(\ln x) - x\cos(\ln x) - \underbrace{\int \sin(\ln x)\,dx}_{I}$$

بازگشتی

$$I = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C$$

2) $\int (\arcsin x)^2\,dx$

$$\begin{cases} u = (\arcsin x)^2 \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = 2\arcsin x \times \frac{1}{\sqrt{1-x^2}}\,dx \\ v = x \end{cases}$$

$$I = x(\arcsin x)^2 - \int \frac{2x}{\sqrt{1-x^2}}\arcsin x\,dx$$

$$\begin{cases} u = \arcsin x \\ dv = \frac{-2x}{\sqrt{1-x^2}}\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{dx}{\sqrt{1-x^2}} \\ v = 2\sqrt{1-x^2} \end{cases}$$

$$I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

3) $\int e^{3x}\cos 5x\,dx$

$$\begin{cases} u = e^{3x} \\ dv = \cos 5x\,dx \end{cases} \Rightarrow \begin{cases} du = 3e^{3x}\,dx \\ v = \frac{1}{5}\sin(5x) \end{cases}$$

$$I = \frac{1}{5}e^{3x}\sin(5x) - \frac{3}{5}\int e^{3x}\sin(5x)\,dx$$

$$\begin{cases} u = e^{3x} \\ dv = \sin(5x)\,dx \end{cases} \Rightarrow \begin{cases} du = 3e^{3x}\,dx \\ v = -\frac{1}{5}\cos(5x) \end{cases}$$

$$I = \frac{1}{5}e^{3x}\sin(5x) + \frac{3}{25}e^{3x}\cos(5x) - \frac{9}{25}\underbrace{\int e^{3x}\cos(3x)\,dx}_{I}$$ بازگشتی

$$I = \frac{1}{34}e^{3x}\left(5\sin(5x) + \cos(5x)\right) + C$$

4) $\int \cos(\sqrt{x})\,dx$

تغییر متغیر $t = \sqrt{x} \Rightarrow t^2 = x \Rightarrow 2t\,dt = dx$

$$\int \cos(\sqrt{x})\,dx = 2\int t\cos t\,dt$$

$$\begin{cases} u = t \\ dv = \cos t\,dt \end{cases} \Rightarrow \begin{cases} du = dt \\ v = \sin t \end{cases}$$

$$\int \cos(\sqrt{x})\,dx = 2t\sin t - 2\int \sin t\,dt = 2t\sin t + 2\cos t + C$$
$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

5) $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx$

$$I = \int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$$

$$\begin{cases} u = \sec^{n-2}x \\ dv = \sec^2 x\,dx \end{cases} \Rightarrow \begin{cases} du = (n-2)\sec^{n-3}x\,\sec x\tan x\,dx \\ v = \tan x \end{cases}$$

$$I = \tan x\sec^{n-2}x - (n-2)\int \sec^{n-2}x\,\underbrace{\tan^2 x}_{\sec^2 x - 1}\,dx$$

$$I = \tan x\sec^{n-2}x - (n-2)\underbrace{\int \sec^n x\,dx}_{I} + (n-2)\int \sec^{n-2}x\,dx$$

$$(n-1)I = \tan x\sec^{n-2}x + (n-2)\int \sec^{n-2}x\,dx$$

$$I = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{(n-2)}{(n-1)}\int \sec^{n-2}x\,dx.$$

6) $\int \frac{x^3+1}{x^3-8}dx = \int \frac{x^3-8+9}{x^3-8}dx = \int dx + \int \frac{9}{x^3-8}dx$

$$= x + 9\int \frac{dx}{(x-2)(x^2+2x+4)}$$

تجزیهٔ کسر

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$1 = A(x^2+2x+4) + (Bx+C)(x-2)$$

$$x=2 \Rightarrow 1 = 12A \Rightarrow A = \frac{1}{12}$$

$$x=0 \Rightarrow C = -\frac{1}{3}$$

$$x=1 \Rightarrow B = -\frac{1}{12}$$

$$\int \frac{dx}{(x-2)(x^2+2x+4)} = \frac{1}{12}\int \frac{dx}{x-2} - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx$$

$$= \frac{1}{12}\operatorname{Ln}|x-2| - \frac{1}{24}\int \frac{2x+2}{x^2+2x+4}dx - \frac{1}{4}\int \frac{dx}{(x+1)^2+3}$$

$$= \frac{1}{12}\operatorname{Ln}|x-2| - \frac{1}{24}\operatorname{Ln}|x^2+2x+4| - \frac{1}{4\sqrt{3}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$I = \int \frac{x^3+1}{x^3-8}dx = x + \frac{3}{4}\operatorname{Ln}|x-2| - \frac{3}{8}\operatorname{Ln}|x^2+2x+4| - \frac{3\sqrt{3}}{4}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

7) $\int \frac{x^4+x^2-x+1}{x^4-1}dx = \int \frac{x^4-1}{x^4-1}dx + \int \frac{x^2-x+2}{x^4-1}dx$

$$= \int dx + \int \frac{x^2-x+2}{(x-1)(x+1)(x^2+1)}dx$$

کسر دوم را تجزیه می‌کنیم :

$$\frac{x^2-x+2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x^2-x+2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$$x=1 \Rightarrow 2 = 4A \Rightarrow A = \frac{1}{2}$$

$$x=-1 \Rightarrow 4 = -4B \Rightarrow B = -1$$

$$x=0 \Rightarrow D = -\frac{1}{2}$$

$$x=2 \Rightarrow C = \frac{1}{2}$$

$$\int \frac{x^2-x+2}{(x-1)(x+1)(x^2+1)}dx = \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x+1} + \frac{1}{2}\int \frac{x-1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|x-1| - \ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\operatorname{arctg} x$$

$$I = \int \frac{x^4+x^2-x+1}{x^4-1}\,dx = x + \frac{1}{2}\ln|x-1| - \ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\operatorname{arctg} x + C$$

8) $$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\,dx$$

$$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$x=1 \Rightarrow B=-1$

$x=-1,\ x=2,\ x=0 \Rightarrow A=1,\ C=-1,\ D=1$

$$I = \int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{-x+1}{x^2+1}\,dx$$

$$I = \ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + \operatorname{arctg} x + C$$

9) $$\int \frac{dx}{4x^3+4x^2+x} = \int \frac{dx}{x(2x+1)^2}$$

$$\frac{1}{4x^3+4x^2+x} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$$

$$1 = A(2x+1)^2 + Bx(2x+1) + Cx$$

$x=0 \Rightarrow A=1 \qquad x=1 \Rightarrow B=-2$

$x=-\frac{1}{2} \Rightarrow C=-2$

$$I = \int \frac{dx}{4x^3+4x^2+x} = \int \frac{dx}{x} - 2\int \frac{dx}{2x+1} - 2\int \frac{dx}{(2x+1)^2}$$

$$I = \ln|x| - \ln|2x+1| + \frac{1}{2x+1} + C$$

10) $$\int \frac{x^3+x+1}{(x+1)(x^2+4)^2}\,dx \qquad \frac{x^3+x+1}{(x+1)(x^2+4)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$x^3+x+1 = A(x^2+4)^2 + (Bx+C)(x+1)(x^2+4) + (Dx+E)(x+1)$$

$x=-1 \Rightarrow A=-\frac{1}{25} \qquad x=0 \Rightarrow C=\frac{24}{25}$

$x=2i \Rightarrow D=-\frac{4}{5},\ E=-\frac{11}{5} \qquad x=1 \Rightarrow B=\frac{1}{25}$

$$I = -\frac{1}{25}\int \frac{dx}{x+1} + \frac{1}{25}\int \frac{x+24}{x^2+4}\,dx - \frac{1}{5}\int \frac{4x+11}{(x^2+4)^2}\,dx$$

$$I=-\frac{1}{25}\operatorname{Ln}|x+1|+\frac{1}{50}\int\frac{2x}{x^2+4}dx+\frac{24}{25}\int\frac{dx}{x^2+4}-\frac{2}{5}\int\frac{2x}{(x^2+4)^2}dx$$

$$\underbrace{-\frac{11}{5}\int\frac{dx}{(x^2+4)^2}}_{x=2tg\theta}=-\frac{1}{25}\operatorname{Ln}|x+1|+\frac{1}{50}\operatorname{Ln}|x^2+4|+\frac{12}{25}\operatorname{arctg}\left(\frac{x}{2}\right)+\frac{2}{5}\,\frac{1}{x^2+4}$$

$$-\frac{11}{5\times 16}\operatorname{arctg}\left(\frac{x}{2}\right)-\frac{11}{40}\,\frac{x}{x^2+4}+C\,.$$

نکته:

$$\int\frac{dx}{(x^2+4)^2}=\int\frac{2\sec^2\theta\, d\theta}{(4tg^2\theta+4)^2}=\frac{2}{16}\int\frac{\sec^2\theta}{\sec^4\theta}d\theta=\frac{1}{8}\int\cos^2\theta\, d\theta$$

$$x=2tg\theta \Rightarrow dx=2\sec^2\theta\, d\theta \qquad =\frac{1}{8}\int\frac{1+\cos 2\theta}{2}d\theta$$

$$=\frac{1}{16}\left(\theta+\frac{1}{2}\sin 2\theta\right)=\frac{1}{16}\left(\operatorname{arctg}\left(\frac{x}{2}\right)+\frac{1}{2}\,\frac{4x}{x^2+4}\right),\quad \sin 2\theta=\frac{2tg\theta}{1+tg^2\theta}$$

11) $\displaystyle\int\frac{dx}{(x^2+x)(x^2+9)}$ $\qquad \displaystyle\frac{1}{x(x+1)(x^2+9)}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2+9}$: تجزیه کسر

$$1=A(x+1)(x^2+9)+Bx(x^2+9)+(Cx+D)x(x+1)$$

$$x=0 \Rightarrow 1=9A \Rightarrow A=\frac{1}{9}$$

$$x=-1 \Rightarrow 1=-10B \Rightarrow B=-\frac{1}{10}$$

$$x=3i \Rightarrow C=\frac{-1}{90},\quad D=-\frac{1}{10}$$

$$I=\frac{1}{9}\int\frac{dx}{x}-\frac{1}{10}\int\frac{dx}{x+1}-\frac{1}{90}\int\frac{x+9}{x^2+9}dx$$

$$=\frac{1}{9}\operatorname{Ln}|x|-\frac{1}{10}\operatorname{Ln}|x+1|-\frac{1}{180}\operatorname{Ln}|x^2+9|-\frac{1}{10}\times\frac{1}{3}\operatorname{arctg}\left(\frac{x}{3}\right)+C$$

12) $\displaystyle\int\frac{x^3-3x^2+x+1}{(x^2+1)(x^2+4)}dx$; $\displaystyle\frac{x^3-3x^2+x+1}{(x^2+1)(x^2+4)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}$

$$x^3-3x^2+x+1=(Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

$$x=i \Rightarrow A=0,\quad B=\frac{4}{3}$$

$$x=2i \Rightarrow C=1,\quad D=\frac{-13}{3}$$

$$I=\frac{4}{3}\int\frac{dx}{x^2+1}+\int\frac{x-\frac{13}{3}}{x^2+4}dx=\frac{4}{3}\operatorname{arctg}x+\frac{1}{2}\operatorname{Ln}|x^2+4|$$

$$-\frac{13}{3}\times\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)+C\,.$$

13) $\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx$ $\qquad u^2 = x \quad \Leftarrow \quad u=\sqrt{x}$ تغییر متغیر

$2u\,du = dx$

$$\int \frac{\sin^3 u}{u}\times 2u\,du = 2\int \sin^3 u\,du = 2\int \sin^2 u \sin u\,du$$

$$= 2\int (1-\cos^2 u)\sin u\,du = 2\int \sin u\,du - 2\int \cos^2 u \sin u\,du$$

$t = \cos u$
$dt = -\sin u\,du$

$$= -2\cos u + \frac{2}{3}\cos^3 u + C$$

$$= -2\cos\sqrt{x} + \frac{2}{3}\cos^3\sqrt{x} + C.$$

14) $\int \frac{dx}{\sin x \cos^2 x} = \int \frac{\sin x\,dx}{\sin^2 x \cos^2 x} = \int \frac{\sin x\,dx}{(1-\cos^2 x)\cos^2 x}$

$u = \cos x$
$du = -\sin x\,dx$

$$= \int \frac{-du}{(1-u^2)u^2} = \int \frac{du}{(u-1)(u+1)u^2}$$

تجزیه کسر

$$\frac{1}{(u-1)(u+1)u^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} + \frac{D}{u+1}$$

$$1 = Au(u^2-1) + B(u^2-1) + Cu^2(u+1) + Du^2(u-1)$$

$u=0 \Rightarrow B=-1 \qquad u=-1 \Rightarrow D = -\frac{1}{2}$

$u=1 \Rightarrow C = \frac{1}{2} \qquad u=2 \Rightarrow A=0$

$$I = \int \frac{-du}{u^2} + \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u+1}$$

$$= \frac{1}{u} + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$= \sec x + \ln\sqrt{\frac{\cos x - 1}{\cos x + 1}} + C$$

15) $\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \tan x \sec x\,dx$

$$= \int (\sec^2 x - 1)\sec^2 x \tan x \sec x\,dx$$

$u = \sec x$
$du = \sec x \tan x\,dx$

$$= \int (u^2-1)u^2\,du = \int u^4 - u^2\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

16) $\int \frac{dx}{\sin x\,(۲+\cos x-۲\sin x)}$

تغییر متغیر

$u=\tan(\frac{x}{۲})$

$du=\frac{1}{۲}(1+u^۲)\,dx \Rightarrow dx=\frac{۲\,du}{1+u^۲}$

$\sin x=\frac{۲u}{1+u^۲}\quad,\quad \cos x=\frac{1-u^۲}{1+u^۲}$

$$I=\int \frac{\frac{۲\,du}{1+u^۲}}{\frac{۲u}{1+u^۲}\left(۲+\frac{1-u^۲}{1+u^۲}-\frac{۴u}{1+u^۲}\right)}=\int \frac{du}{\frac{u(u^۲-۴u+۳)}{1+u^۲}}=\int \frac{1+u^۲}{u(u-1)(u-۳)}$$

تجزیه کسر:

$$\frac{1+u^۲}{u(u-1)(u-۳)}=\frac{A}{u}+\frac{B}{u-1}+\frac{C}{u-۳}$$

$$u^۲+1=A(u-1)(u-۳)+B\,u(u-۳)+C\,u(u-1)$$

$u=0 \Rightarrow A=\frac{1}{۳}\quad,\quad u=1 \Rightarrow B=-1$

$u=۳ \Rightarrow C=\frac{۵}{۳}$

$$I=\frac{1}{۳}\int\frac{du}{u}-\int\frac{du}{u-1}+\frac{۵}{۳}\int\frac{du}{u-۳}=\frac{1}{۳}\ln|u|-\ln|u-1|+\frac{۵}{۳}\ln|u-۳|+C$$

$$=\frac{1}{۳}\ln\left|\tan(\tfrac{x}{۲})\right|-\ln\left|\tan(\tfrac{x}{۲})-1\right|+\frac{۵}{۳}\ln\left|\tan(\tfrac{x}{۲})-۳\right|+C$$

17) $\int \frac{dx}{\sin^۲x+۳\cos x\sin x-\cos^۲x}$

تغییر متغیر

$u=\tan x$

$du=\sec^۲x\,dx$

$$I=\int\frac{\frac{dx}{\cos^۲x}}{\tan^۲x+۳\tan x-1}=\int\frac{du}{u^۲+۳u-1}=\int\frac{du}{(u+\frac{۳}{۲})^۲-\frac{۱۳}{۴}}$$

از تجزیه کسر هم می توان استفاده کرد

$du=\frac{\sqrt{۱۳}}{۲}\sec\theta\,tg\theta\,d\theta \Leftarrow u+\frac{۳}{۲}=\frac{\sqrt{۱۳}}{۲}\sec\theta$ تغییر متغیر

$$I=\int\frac{\frac{\sqrt{۱۳}}{۲}\sec\theta\tan\theta\,d\theta}{\frac{۱۳}{۴}(\tan^۲\theta)}=\frac{۲}{\sqrt{۱۳}}\int\frac{\sec\theta}{\tan\theta}\,d\theta=\frac{۲}{\sqrt{۱۳}}\int\csc\theta\,d\theta$$

$$=\frac{۲}{\sqrt{۱۳}}\int\csc\theta\times\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\,d\theta=\frac{-۲}{\sqrt{۱۳}}\int-\frac{\csc^۲\theta+\csc\theta\cot\theta}{\cot\theta+\csc\theta}$$

$$=-\frac{۲}{\sqrt{۱۳}}\ln|\cot\theta+\csc\theta|=\frac{-۲}{\sqrt{۱۳}}\ln\left|\frac{1}{\sqrt{\sec^۲\theta-1}}+\frac{1}{\sqrt{1-\sec^{-۲}\theta}}\right|$$

$$=-\frac{۲}{\sqrt{۱۳}}\ln\left|\frac{\sqrt{۱۳}}{۲}\,\frac{1}{\sqrt{\tan^۲x+۳\tan x-1}}+\frac{1}{\sqrt{1-\frac{۱۳}{۴}(\tan x+\frac{۳}{۲})^{-۲}}}\right|+c=\dots$$

18) $\int \frac{\sqrt{x^2+1}}{x^3}dx \qquad x=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta$

$$I=\int\frac{\sqrt{x^2+1}}{x^3}=\int\frac{\sqrt{\tan^2\theta+1}}{\tan^3\theta}\sec^2\theta\, d\theta=\int\frac{\sec^3\theta}{\tan^3\theta}d\theta=\int\frac{1}{\sin^3\theta}d\theta$$

$$=\int \sin^{-3}\theta\, d\theta=\int(\sin^2\theta)^{-2}\sin\theta\, d\theta=\int(1-\cos^2\theta)^{-2}\sin\theta\, d\theta$$

از جزء به جزء نیز می توان استفاده کرد.

$$=-\int(1-u^2)^{-2}du=\int\frac{-du}{(1-u)^2(1+u)^2} \qquad u=\cos\theta \Rightarrow du=-\sin\theta\, d\theta$$

تجزیه کسر :

$$\frac{-1}{(1-u)^2(1+u)^2}=\frac{A}{1-u}+\frac{B}{(1-u)^2}+\frac{C}{1+u}+\frac{D}{(1+u)^2}$$

$$-1=A(1-u)(1+u)^2+B(1+u)^2+C(1-u)^2(1+u)+D(1-u)^2$$

$u=1 \Rightarrow -1=4B \Rightarrow B=\frac{-1}{4} \qquad \begin{matrix}u=0\\u=2\end{matrix} \Rightarrow A=C=-\frac{1}{4}$

$u=-1 \Rightarrow -1=4D \Rightarrow D=\frac{-1}{4}$

$$I=\frac{1}{4}\int\frac{du}{u-1}-\frac{1}{4}\int\frac{du}{(u-1)^2}-\frac{1}{4}\int\frac{du}{u+1}-\frac{1}{4}\int\frac{du}{(u+1)^2}$$

$$=\frac{1}{4}\ln|u-1|+\frac{1}{4}\frac{1}{u-1}-\frac{1}{4}\ln|u+1|+\frac{1}{4}\frac{1}{u+1}+C$$

$$=\frac{1}{4}\ln\left|\frac{u-1}{u+1}\right|+\frac{1}{2}\frac{u}{u^2-1}+C \qquad u=\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{1}{\sqrt{x^2+1}}$$

$$=\frac{1}{4}\ln\left|\frac{1-\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right|-\frac{\sqrt{1+x^2}}{2x^2}+C.$$

19) $\int\frac{x\, dx}{\sqrt{x-x^2}}$

عبارت زیر رادیکال را به صورت مربع کامل می نویسیم :

$$x-x^2=\frac{1}{4}-\left(x-\frac{1}{2}\right)^2$$

تغییر متغیر $\quad x-\frac{1}{2}=\frac{1}{2}\sin\theta \Rightarrow dx=\frac{1}{2}\cos\theta\, d\theta$

$$I=\int\frac{x\, dx}{\sqrt{x-x^2}}=\int\frac{\frac{1}{2}(\sin\theta+1)}{\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}}\times\frac{1}{2}\cos\theta\, d\theta=\int\frac{\frac{1}{2}(\sin\theta+1)}{\frac{1}{2}\cos\theta}\times\frac{1}{2}\cos\theta\, d\theta$$

$$=\frac{1}{2}\int(\sin\theta+1)d\theta=\frac{1}{2}\left[-\cos\theta+\theta\right]=\frac{-1}{2}\sqrt{1-(2x-1)^2}+\frac{1}{2}\arcsin(2x-1)+C$$

$$=-\sqrt{x-x^2}+\frac{1}{2}\arcsin(2x-1)+C$$

20) $\int \frac{\sqrt{4x^2+4x-3}}{2x+1}\,dx$

عبارت زیر رادیکال را به صورت مربع کامل می نویسیم :

$$4x^2+4x-3=(2x+1)^2-4$$

تغییر متغیر $2x+1=2\cosh t$ (یا $2x+1=2\sec\theta$)

$$2\,dx=2\sinh t\,dt \Rightarrow dx=\sinh t\,dt$$

$$I=\int \frac{\sqrt{4\cosh^2 t-4}}{2\cosh t}\times \sinh t\,dt=\int \frac{2\sinh t}{2\cosh t}\times \sinh t\,dt$$

$$=\int \sinh^2 t\,\cosh^{-1} t\,dt=\int \sinh^2 t\,(\cosh^2 t)^{-1}\cosh t\,dt$$

تغییر متغیر $u=\sinh t \Rightarrow du=\cosh t\,dt$

$$I=\int \sinh^2 t\,(1+\sinh^2 t)^{-1}\underbrace{\cosh t\,dt}_{du}=\int u^2(1+u^2)^{-1}\,du$$

$$=\int \frac{u^2}{1+u^2}\,du=\int du+\int \frac{-du}{1+u^2}=u-\operatorname{arctg}u+C$$

$$=\sinh t-\operatorname{arctg}(\sinh t)+C$$

$$=\frac{\sqrt{4x^2+4x-3}}{2}-\operatorname{arctg}\left(\frac{\sqrt{4x^2+4x-3}}{2}\right)+C$$

$$2x+1=2\cosh t$$

$$\Rightarrow \frac{2x+1}{2}=\cosh t$$

$$\sinh t=\sqrt{\cosh^2 t-1}=\sqrt{\left(\frac{2x+1}{2}\right)^2-1}=\frac{\sqrt{4x^2+4x-3}}{2}$$

21) $\int \frac{2x+4+\sqrt{x+2}}{\sqrt[3]{x+2}+\sqrt[4]{x+2}}\,dx$

تغییر متغیر $x+2=t^{12}$

$$dx=12t^{11}\,dt$$

$$I=\int \frac{2t^{12}+t^6}{t^4+t^3}\times 12t^{11}\,dt=12\int \frac{t^6(2t^6+1)}{t^3(t+1)}\times t^{11}\,dt=12\int \frac{t^{14}(2t^6+1)}{t+1}\,dt$$

$$=12\int t^{14}\left[\frac{(t+1)(2t^5-2t^4+2t^3-2t^2+2t-2)+3}{t+1}\right]dt$$

$$=24\int (t^{19}-t^{18}+t^{17}-t^{16}+t^{15}-t^{14})\,dt+36\int \frac{t^{14}}{t+1}\,dt$$

$$= 24\int (t^{19}-t^{18}+t^{17}-t^{16}+t^{15}-t^{14})\,dt + 36\int (t^{13}-t^{12}+t^{11}-\dots+t-1)\,dt$$

$$+36\int \frac{dt}{t+1} = 24\left[\frac{1}{20}t^{20}-\frac{1}{19}t^{19}+\frac{1}{18}t^{18}-\frac{t^{17}}{17}+\frac{1}{16}t^{16}-\frac{1}{15}t^{15}\right]+$$

$$36\left[\frac{1}{14}t^{14}-\frac{1}{13}t^{13}+\dots+\frac{1}{2}t^{2}-t\right]+36\,\mathrm{Ln}|1+t|+C$$

و درنهایت به جای t قرار دهیم $\sqrt[12]{x+2}$.

22) $$\int \frac{x+\sqrt{x}}{x\sqrt[4]{x}-\sqrt{x}}\,dx = \int \frac{x+x^{1/2}}{x^{5/4}-x^{1/2}}$$

تغییر متغیر: $x=t^4$

$dx = 4t^3\,dt$

$$= \int \frac{t^4+t^2}{t^5-t^2}\times 4t^3\,dt = \int \frac{t^2(t^2+1)}{t^2(t^3-1)}\,4t^3\,dt$$

$$\begin{array}{r|l} t^5+t^3 & t^3-1 \\ \hline t^5-t^2 & t^2+1 \\ \hline t^3+t^2 & \\ t^3-1 & \\ \hline t^2+1 & \end{array}$$

$$= 4\int \frac{t^5+t^3}{t^3-1}\,dt = 4\int \frac{(t^2+1)(t^3-1)+(t^2+1)}{t^3-1}\,dt$$

$$= 4\int (t^2+1)\,dt + 4\int \frac{t^2}{t^3-1}\,dt + 4\int \frac{dt}{t^3-1}$$

$$= \frac{4}{3}t^3 + 4t + \frac{4}{3}\mathrm{Ln}|t^3-1| + 4\underbrace{\int \frac{dt}{t^3-1}}$$

برای حل این انتگرال از تجزیه کسر استفاده می کنیم:

$$\frac{1}{t^3-1} = \frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1}+\frac{Bt+C}{t^2+t+1}$$

$$1 = A(t^2+t+1)+(Bt+C)(t-1)$$

$$t=1 \Rightarrow A=\frac{1}{3} \quad ; \quad t=0 \Rightarrow C=-\frac{2}{3} \quad ; \quad t=-1 \Rightarrow B=-\frac{1}{3}$$

$$\int \frac{dt}{t^3-1} = \frac{1}{3}\int \frac{dt}{t-1} - \frac{1}{3}\int \frac{t+2}{t^2+t+1} = \frac{1}{3}\mathrm{Ln}|t-1| - \frac{1}{6}\mathrm{Ln}|t^2+t+1| - \frac{1}{2}\int \frac{dt}{\underbrace{t^2+t+1}_{(t+\frac{1}{2})^2+\frac{3}{4}}}$$

$$= \frac{1}{3}\mathrm{Ln}|t-1| - \frac{1}{6}\mathrm{Ln}|t^2+t+1| - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

$$I = \frac{4}{3}t^3+4t+\frac{4}{3}\mathrm{Ln}|t^3-1| + \frac{4}{3}\mathrm{Ln}|t-1| - \frac{2}{3}\mathrm{Ln}|t^2+t+1| - \frac{4}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)+C.$$

و درنهایت به جای t مقدار $\sqrt[4]{x}$ را قرار می دهیم.

23) $\int \frac{dx}{1+e^x}$

$du = e^x dx \Leftarrow u = e^x$ تغییرمتغیر

$dx = \frac{du}{e^x} = \frac{du}{u}$

$\int \frac{dx}{1+e^x} = \int \frac{du}{u(1+u)}$ $\qquad \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ تجزیه کسر

$1 = A(u+1) + Bu$

$u = 0 \Rightarrow A = 1 \quad ; \quad u = -1 \Rightarrow B = -1$

$I = \int \frac{du}{u} - \int \frac{du}{u+1} = \ln|u| - \ln|u+1| + C$

$= \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{e^x}{e^x+1}\right| + C.$

24) $\int \frac{dx}{(x^2+x+1)^2}$ $\qquad x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$

$x + \frac{1}{2} = \frac{\sqrt{3}}{2}\,tg\theta$: تغییرمتغیر

$\Rightarrow dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$I = \int \frac{dx}{(x^2+x+1)^2} = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta}{(\frac{3}{4}tg^2\theta + \frac{3}{4})^2}$

$= \frac{\sqrt{3}}{2} \times \frac{16}{9} \int \frac{\sec^2\theta}{\sec^4\theta} d\theta = \frac{8}{3\sqrt{3}} \int \frac{d\theta}{\sec^2\theta} = \frac{8}{3\sqrt{3}} \int \cos^2\theta\, d\theta$

$= \frac{8}{3\sqrt{3}} \int \frac{1+\cos 2\theta}{2} d\theta = \frac{4}{3\sqrt{3}} \int 1+\cos 2\theta\, d\theta = \frac{4}{3\sqrt{3}} \left[\theta + \frac{1}{2}\sin 2\theta\right]$

$= \frac{4}{3\sqrt{3}} \left[\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{4}\, \frac{2x+1}{x^2+x+1}\right] + C$

$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\left(\frac{2x+1}{\sqrt{3}}\right)}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2} = \frac{\sqrt{3}}{2}\, \frac{2x+1}{x^2+x+1}$

25) $\int \frac{dx}{2+\sinh x}$ $\qquad \sinh x = \frac{e^x - e^{-x}}{2}$

$I = \int \frac{dx}{2+\sinh x} = \int \frac{dx}{2 + \frac{e^x - e^{-x}}{2}} = 2\int \frac{dx}{4 + e^x + e^{-x}}$

$dx = \frac{du}{u} \Leftarrow du = e^x dx \Leftarrow u = e^x$ تغییرمتغیر

$$I = 2\int \frac{\frac{du}{u}}{4+u+\frac{1}{u}} = 2\int \frac{du}{u^2+4u+1} = 2\int \frac{du}{(u+2)^2-3} = 2\int \frac{du}{(u+2-\sqrt{3})(u+2+\sqrt{3})}$$

تجزیه‌ی کسر :

$$\frac{1}{(u+2-\sqrt{3})(u+2+\sqrt{3})} = \frac{A}{u+2-\sqrt{3}} + \frac{B}{u+2+\sqrt{3}}$$

$$1 = A(u+2+\sqrt{3}) + B(u+2-\sqrt{3})$$

$$u = -2+\sqrt{3} \Rightarrow 1 = A(2\sqrt{3}) \Rightarrow A = \frac{1}{2\sqrt{3}}$$

$$u = -2-\sqrt{3} \Rightarrow 1 = -B(2\sqrt{3}) \Rightarrow B = \frac{-1}{2\sqrt{3}}$$

$$I = \frac{1}{\sqrt{3}}\int \frac{du}{u+2-\sqrt{3}} - \frac{1}{\sqrt{3}}\int \frac{du}{u+2+\sqrt{3}} = \frac{1}{\sqrt{3}} \ln\left|\frac{u+2-\sqrt{3}}{u+2+\sqrt{3}}\right| + C$$

$$= \frac{1}{\sqrt{3}} \ln\left|\frac{e^x+2-\sqrt{3}}{e^x+2+\sqrt{3}}\right| + C$$

26) $\int \arcsin(\sqrt{x})\,dx$

$\sqrt{x} = t$ تغییر متغیر

$x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int \arcsin t \times 2t\,dt$$

جزء به جزء $\begin{cases} u = \arcsin t \\ dv = 2t\,dt \end{cases} \Rightarrow \begin{cases} du = \frac{dt}{\sqrt{1-t^2}} \\ v = t^2 \end{cases}$

$$I = t^2 \arcsin t - \underbrace{\int \frac{t^2}{\sqrt{1-t^2}}dt}$$

$t = \sin\theta$ تغییر متغیر

$dt = \cos\theta\,d\theta$

$$\int \frac{-t^2}{\sqrt{1-t^2}}dt = \int \frac{-\sin^2}{\sqrt{1-\sin^2}} \times \cos\theta\,d\theta = \int \frac{-\sin^2\theta}{\cos\theta} \times \cos\theta\,d\theta$$

$$= -\int \sin^2\theta\,d\theta = -\frac{1}{2}\int(1-\cos 2\theta)\,d\theta = -\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= -\frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2} + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2t\sqrt{1-t^2}$

$$I = \left(x - \frac{1}{2}\right)\arcsin\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + C$$