PLP Project Paper

Team 2

Dmytro Daniuk, Lorenz Engelsberger

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Based on the assumption of the model of Czech mathematician Oldrich Vasicek from 2002 a loan defaults if the value of the borrower's assets at the loan maturity T falls below the contractual value B of its obligations payable. Vasicek lets A_i be the value of the i-th borrower's assets, described by the process

$$dA_i = \mu_i A_i dt + \sigma_i A_i dx_i$$

The asset value at T is than represented as

$$log A_i(T) = log A_i + \mu_i T - \frac{1}{2}{\sigma^2}_i T + \sigma_i \sqrt{T} X_i$$

where X_i is a standard normal variable. The variables X_i are jointly standard normal with equal pairwise correlations ρ, and are therefore represented as

$$X_i = Y\sqrt{\rho} + Z_i\sqrt{1-\rho}$$

where $Y, Z_1, Z_2, \ldots, Z_n$ are mutually independent standard normal variables (property of the equicorrelated normal distribution). The variable Y can be interpreted as a portfolio common factor, such as an economic index, over the interval (0,T). Then the term $Y\sqrt{\rho}$ is the company's exposure to the common factor and the term $Z_i\sqrt{1-\rho}$ represents the company specific risk.

Based on this formula, we can prove that $X_i \sim N(0,1)$. This can be done by proving that $E[X_i] = 0$, $Var(X_i) = 1$ and $corr(X_i, X_j) = \rho$, using fact that $Y, Z_1, Z_2, \ldots, Z_n$ are mutually independent standard normal variables:

$$E[X_i] = E[Y\sqrt{\rho} + Z_i\sqrt{1-\rho}] = \sqrt{\rho}E[Y] + \sqrt{1-\rho}E[Z_i] = \sqrt{\rho} \times 0 + \sqrt{1-\rho} \times 0 = 0$$

$$\begin{aligned}
Var(X_i) &= E[{X_i}^2] - (E[X_i])^2 = E[{X_i}^2] - 0 = E[{X_i}^2] = E[(Y\sqrt{\rho} + Z_i\sqrt{1-\rho})^2] \\
&= E[Y^2\rho + 2(Y\sqrt{\rho}Z_i\sqrt{1-\rho}) + {Z_i}^2(1-\rho)] = E[Y^2\rho + 2(YZ_i\sqrt{\rho(1-\rho)}) + {Z_i}^2(1-\rho)] \\
&= \rho E[Y^2] + 2\sqrt{\rho(1-\rho)}E[YZ_i] + (1-\rho)E[{Z_i}^2] = \rho E[Y^2] + 2\sqrt{\rho(1-\rho)}E[Y]E[Z_i] + (1-\rho)E[{Z_i}^2] \\
&= \rho \times 1 + 2\sqrt{\rho(1-\rho)} \times 0 \times 0 + (1-\rho) \times 1 = \rho + 0 + 1 - \rho = 1
\end{aligned}$$

$$\begin{aligned}
corr(X_i, X_j) =& \frac{Cov(X_i, X_j)}{\sigma(X_i) \times \sigma(X_j)} = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)} \times \sqrt{Var(X_j)}} = \frac{Cov(X_i, X_j)}{\sqrt{1} \times \sqrt{1}} = Cov(X_i, X_j) \\
&= E[X_iX_j] + E[X_i]E[X_j] = E[X_iX_j] + 0 \times 0 = E[X_iX_j] = E[(Y\sqrt{\rho} + Z_i\sqrt{1-\rho})(Y\sqrt{\rho} + Z_j\sqrt{1-\rho})] \\
&= E[Y^2\rho + YZ_j\sqrt{\rho(1-\rho)} + YZ_i\sqrt{\rho(1-\rho)} + Z_iZ_j(1-\rho)] \\
&= \rho E[Y^2] + \sqrt{\rho(1-\rho)}E[Y]E[Z_j] + \sqrt{\rho(1-\rho)}E[Y]E[Z_i] + (1-\rho)E[Z_iZ_j] \\
=& \rho E[Y^2] + \sqrt{\rho(1-\rho)}E[Y]E[Z_j] + \sqrt{\rho(1-\rho)}E[Y]E[Z_i] + (1-\rho)E[Z_i]E[Z_j] \\
& \rho \times 1 + 2 \times (\sqrt{\rho(1-\rho)} \times 0 \times 0) + (1-\rho) \times 0 \times 0) = \rho + 0 + 0 = \rho
\end{aligned}$$