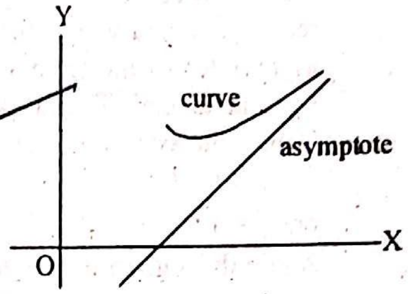
Unit 7

Asymptotes

Asymptote

A straight line is called an asymptote to a curve if the perpendicular distance of the straight line from a point on the curve tends to zero as the point moves to infinity along the curve.



Types of asymptotes

1. **Vertical asymptote (Asymptote parallel to Y-axis)**

   A straight line x = a is called vertical asymptote to the curve y = f(x) if $\lim_{x\to a} f(x) = \infty$.

2. **Horizontal asymptote (Asymptote parallel to x-axis).**

   A straight line y = b is called horizontal asymptote to the curve y = f(x) if $\lim_{x\to\infty} y = b$ or $\lim_{x\to\infty} f(x) = b$.

3. **Oblique asymptote:**

   A straight line y = mx + c is called oblique asymptote of the curve y = f(x) if $\lim_{x\to\infty}\left(\frac{y}{x}\right) = m$ and $\lim_{x\to\infty}(y - mx) = c$

Some important facts:

- The number of asymptotes of an algebraic curve does not exceed the degree of equation.
- If in an algebraic curve of degree n, y^n is absent then the asymptotes parallel to y-axis can be obtained by equating the coefficients of the highest degree terms of y to zero.
- If in an algebraic curve of degree n, x^n is absent then equation of asymptotes parallel to x-axis can be obtained by equating the coefficient of highest degree term of x to zero.
- If the terms involving both x^n and y^n are absent, then equate the coefficient of highest available powers of x, y to zero. Then we get the asymptotes parallel to x and y-axis.
- If the roots of $\phi_n(m) = 0$ are not repeated then use for each m from $\phi_n(m) = 0$ to $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$. Thus we get the asymptote y = m x + c.
- If the roots of $\phi_n(m) = 0$ are twice repeated, then we get a set of parallel asymptotes. Then in such a care, for corresponding to the repeated root m, c can be obtained from the equation $\frac{c^2}{2!}\phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$.
- If the term involving x_n and y_n are present, putting x = 1 and y = m on the homogeneous n^{th} degree term to get $\phi_n(m)$. Similarly we get $\phi_{n-1}(m)$ and so on.
- Finitely bounded curve has no asymptotes.

Exercise - 7

1. Find the asymptotes of the following curves:

(a) $y = \frac{x^2}{x^2+1}$

Soln: The equation can be written as

$$x^2y - x^2 + y = 0$$

The given curve is of degree three, so there are at the most three asymptotes real or imaginary. Absence of x^3 and y^3 in the curve shows that there are no asymptotes of the form $y = mx + c$. So, there are asymptotes parallel to the coordinate axes. The highest degree term in x is x^2 whose coefficient is $y - 1$.

Hence, the asymptote parallel to x axis is

$$y - 1 = 0$$

or, $y = 1$

Again the highest degree term in y is y whose coefficient is

$$x^2 + 1.$$

Hence, asymptote parallel to y-axis are

$$x^2 + 1 = 0$$

This does not give any real asymptote.

Hence, the required asymptote of the curve is $y = 1$

(b) $y = \frac{x^2+1}{1+x}$

Soln: The given equation can be written as

$$x^2 - xy - y + 1 = 0$$

The given curve is of degree two, so there are at the most two asymptotes real or imaginary. Due to the presence of x^2 in the curve shows that there are no asymptote parallel to x axis and due to the absence of y^2 shows that there are asymptotes parallel to y-axis.

The highest degree term in y is y and its coefficient is $-(x + 1)$.

Hence, the asymptote parallel to y-axis is

$$-(x + 1) = 0$$

or, $x + 1 = 0$

or, $x = -1$

To get the oblique asymptotes of the form $y = mx + c$ put $x = 1$, and $y = m$ in 2^{nd} and 1^{st} degree terms to get $\phi_2(m)$ and $\phi_1(m)$

$\therefore$ $\phi_2(m) = 1 - m$, $\phi_1(m) = -m$.

Now, $\phi_2(m) = 0$ gives $1 - m = 0$, i.e. $m = 1$

and $\phi_2'(m) = -1$

$$\text{Also, } c = \frac{-\phi_1(m)}{\phi_2'(m)} = \frac{-(-m)}{-1} = -m = -1$$

Now substituting values of m and c in $y = mx + c$, we get

$$y = 1x - 1$$

or, $y = x - 1$

$\therefore$ Required asymptotes of the curve are $x = -1$ and $y = x - 1$.

(c) $y = \dfrac{x}{(x-1)^2(x-2)}$

Soln: The given equation can be written as

$y(x^2 - 2x + 1)(x-2) - x = 0$

or. $y(x^3 - 2x^2 + x - 2x^2 + 4x - 2) - x = 0$

or, $x^3y - 4x^2y + 5xy - 2y - x = 0$

The equation of curve is of degree four, so there are four asymptotes real or imaginary. Due to the absence of x^4 and y^4 in the curve shows that there are no asymptotes of the form

$y = mx + c.$

To obtain the asymptote parallel to y–axis, we equate to zero the coefficient of highest degree term in y.

i.e. coefficient of y = 0

or, $x^3 - 4x^2 + 5x - 2 = 0$

or, $(x-1)^2(x-2) = 0$

$\therefore\ x = 1, x = 2$

Again, to obtain the asymptote parallel to x–axis, we equate to zero the coefficient of highest degree term in x.

i.e. coefficient of $x^3 = 0$

i.e. $y = 0$

$\therefore$ Required asymptotes are $x = 1$, $x = 2$ and $y = 0$

(d) $y = \dfrac{2x-3}{x^2 - 3x + 2}$

Soln: The given equation of curve is of degree 3, so there are at the most three asymptotes real or imaginary.

For the given curve

$y = \dfrac{2x-3}{(x-1)(x-2)}$

When $x \to 1$, $y \to \infty$ and when $x \to 2$, $y \to \infty$ and also when $x \to \infty$, $y \to 0$.

Therefore. asymptotes parallel to x-axis is $y = 0$ and parallel to y-axis are $x = 1$ and $x = 2$

$\therefore$ Required asymptotes are $x = 1$, $x = 2$ and $y = 0$.

(e) $y = \dfrac{(x+2)^2(x-3)}{(x-1)}$

Soln: The given equation of curve is,

$y = \dfrac{(x+2)^2(x-1)}{(x-1)}$

The given curve is of degree three, so there are at the most three asymptotes. When $x \to 1$, $y \to \infty$, therefore $x = 1$ is the asymptote parallel to y-axis. When $x \to \infty$, $y \to \infty$, so there are no asymptotes parallel to x-axis. The given equation can be written as

$(x^2 + 4x + 4)(x-3) = y(x-1)$

Or, $x^3 + x^2 - xy - 8x + y - 12 = 0$

To obtain the asymptote of the from $y = mx + c$, we substitute $x = 1$ in third, second and first degree term to get $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$ respectively.

$\phi_3(m) = 1.$

$\phi_2(m) = 1 - m$

and $\phi_1(m) = -8 + m$

Now, $\phi_3(m) = 0 \Rightarrow 1 = 0$ which is impossible, so there are no oblique asymptote.

Hence, the only asymptote of the curve is $x = 1$.

(f) $y = \frac{x}{x^2 - 1}$

Soln: the given curve is of degree three, so there are at the most three asymptotes real or imaginary for the given curve $y = \frac{x}{x^2 - 1}$.

When $x \to \pm 1$, $y \to \infty$.
Hence, asymptote parallel to y-axis are $x = \pm 1$.
Again, when $x \to \infty$, $y \to 0$.
Hence, asymptote parallel to x-axis is $y = 0$.
$\therefore$ Required asymptotes are $x = \pm 1$ and $y = 0$.

(g) $(x^2 + y^2)\, x - ay^2 = 0$

Soln: The equation of the curve can be written as $x^3 + xy^2 - ay^2 = 0$ which is of degree three, so there are at the most three asymptotes real or imaginary. x^3 is present, so there are no asymptotes parallel to x-axis.
Asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest degree term in y. The highest degree term in y is y^2 and its coefficient is $x - a$.
Now, coefficient of $y^2 = 0$
$\Rightarrow x - a = 0$
or $x = a$, which is the asymptotes parallel to y-axis.
Now, to find the asymptotes of the form $y = mx + c$, put $x = 1$ and $y = m$ in 3^{rd} and 2^{nd} degree term to obtain $\phi_3(m)$ and $\phi_2(m)$ respectively. $\phi_3(m) = 1 + m^2$ and $\phi_2(m) = -am^2$
We have $\phi_3(m) = 0 \Rightarrow m^2 + 1 = 0$, which gives no real solution of m. Hence there are no oblique asymptotes.
$\therefore$ Required asymptote of the curve is $x = a$.

(h) $x^3 - y^3 = 3ax^2$.

Soln: The given curve is
$x^3 - y^3 - 3ax^2 = 0$
This is of degree three, so there are at the most three asymptotes real or imaginary. Due to the presence of x^3 and y^3 in the curve shows that there are no asymptotes parallel to coordinate axes.
Now, to obtain the asymptote of the from $y = mx + c$, substitute $x = 1$ and $y = m$ in 3^{rd}, 2^{nd} degree terms to get $\phi_3(m)$, $\phi_2(m)$ and respectively.
Now, $\phi_3(m) = 1 - m^3$ and $\phi_2(m) = 3a$.
$\phi_3(m) = 0 \Rightarrow 1 - m^3 = 0$ gives $m = 1$ and other roots are not real we have,
$\phi_3'(m) = -3m^2$

$$\therefore\ c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-3a}{-3m^2} = \frac{a}{m^2} = a$$

Now, substituting $m = 1$ and $c = a$ in $y = mx + c$, we get $y = x + a$ only oblique asymptote of the given curve.

(i) $(y - a)^2 (x^2 - a^2) = x^4 + a^4$

Soln: The given equation of curve can be written as
$(y^2 - 2ay + a^2)(x^2 - a^2) = x^4 + a^4$
or, $x^2y^2 - a^2y^2 - 2ax^2y + 2a^3y + a^2x^2 - a^4 = x^4 + a^4$
or, $x^4 - x^2y^2 + 2ax^2y - a^2x^2 + a^2y^2 - 2a^3y + 2a^4 = 0$

This algebraic equation of curve is of fourth degree. So there are at most four asymptotes. Also x^4 is present and y^4 is absent. Hence there is no asymptotes parallel to x-axis and has asymptotes parallel to y-axis.

The equations of asymptotes parallel to y-axis are obtained by equating the coefficient of y^2 to zero.

i.e. $-x^2 + a^2 = 0, \quad \therefore\ x = \pm a$

Again to find the asymptotes of the form $y = mx + c$.

put $x = 1$, $y = m$ in fourth and third degree terms, we get

$\phi_4(m) = 1 - m^2, \phi_3(m) = 2am$

Now, $\phi_4'(m) = -2m$

Again, $\phi_4(m) = 0 \Rightarrow 1 - m^2 = 0 \Rightarrow m = \pm 1$

$$\text{Now, } c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{-2am}{-2m} = a$$

$\therefore$ Equation of asymptotes are $y = \pm 1x + a$ or, $y = \pm x + a$.

Hence, the asymptotes of the curve are,

$x = \pm a,\ y \pm x = a.$

(j) $x^3 + y^3 = 3axy$

Soln: The given equation of the curve is $x^3 + y^3 - 3axy = 0$.

This equation is of third degree and x^3 and y^3 are both present. So there are no asymptotes parallel to x-axis and y-axis.

Hence, to find the asymptotes of the form $y = mx + c$.

Putting $x = 1$ and $y = m$ in 3^{rd} and 2^{nd} degree terms, we get

$\phi_3(m) = 1 + m^3$

$\phi_2(m) = -3am$

Now, $\phi'_3(m) = 3m^2$

Also, $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$

$\therefore$ $m = -1$ and other values are imaginary.

$$\therefore\ c = \frac{-\phi_2(m)}{\phi'_3(m)} = -\frac{(-3\ am)}{3m^2} = \frac{a}{m} = -a$$

$\therefore$ Required asymptote of the curve is $y = -1.x + (-a)$

or, $y + x + a = 0$.

(k) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln: The given equation of curve can be writes as

$b^2x^2 - a^2y^2 - a^2b^2 = 0$

which is of degree two, so there are at the most two asymptotes real or imaginary. Due to the presence of x^2 and y^2 in the curve, this shows that there are no asymptotes parallel to x and y-axis.

Now, to obtain the asymptotes of the form

$y = mx + c$

we substitute $x = 1$ and $y = m$ in second and first degree terms to get $\phi_2(m)$ and $\phi_1(m)$ respectively.

$\therefore\ \phi_2(m) = b^2 - a^2m^2$. and $\phi_1(m) = 0$

We have $\phi_2(m) = 0 \Rightarrow b^2 - a^2m^2 = 0\ 1 \quad \therefore\ m = \pm\frac{b}{a}$

Now, $\phi'_2(m) = -2a^2m$

and $c = -\dfrac{\phi_1(m)}{\phi_2'(m)} = \dfrac{0}{-2a^2m} = 0$

Required asymptotes are

$y = \pm\dfrac{b}{a}x + 0$, *i.e.* $y = \pm\dfrac{b}{a}x$.

(l) $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$

Soln: The given equation of curve can be written as

$x^2y^2 - b^2x^2 - a^2y^2 = 0.$

The equation is of degree four, so there are at the most four asymptotes real or imaginary. Due to absence of x^4 and y^4 in the curve shows that there are only asymptotes parallel to coordinate axes.

Now, the asymptotes parallel to x-axis are obtained by equating to zero the coefficient of highest degree terms in x.

i.e. coefficient of $x^2 = 0$

$\Rightarrow y^2 - b^2 = 0$

$\therefore y = \pm x.$

Similarly, the equations of asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest degree terms in y.

i.e. coefficient of $y^2 = 0$

$\Rightarrow x^2 - a^2 = 0 \quad \therefore x = \pm a$

$\therefore$ Required asymptotes of the given curve are $x = \pm a, y = \pm b$.

(m) $y^2 = \dfrac{(x-a)^2}{(a^2+x^2)} \cdot x^2$

Soln: From the given equation of curve, we have

$(x-a)^2 x^2 = y^2(a^2+x^2)$

Or, $(x^2 - 2ax + a^2)x^2 = a^2y^2 + x^2y^2$

Or, $x^4 - x^2y^2 - 2ax^3 + a^2x^2 - a^2y^2 = 0$

This equation is of fourth degree and the term x^4 is present and y^4 is absent. Hence, there is no asymptote parallel to x–axis and there are asymptotes parallel to y-axis. Also, the number of asymptotes of the curve does not exceed 4.

The asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest degree term in y.

i. e. coefficient of $y^2 = 0$

$\Rightarrow -(x^2 + a^2) = 0$

$\therefore x = \pm i a$

Hence, there are no real asymptote parallel to y-axis.

To obtain the asymptote of the from $y = mx + c$, we substitute $x = 1$ and $y = m$ in 4^{th}, 3^{rd} and 2^{nd} terms in x and y to obtain $\phi_4(m)$, $\phi_3(m)$ and $\phi_2(m)$ respectively.

$\therefore \phi_4(m) = 1 - m^2$, $\phi_3(m) = -2a$, $\phi_2(m) = a^2 - a^2m^2$.

Now, $\phi_4(m) = 0 \Rightarrow 1 - m^2 = 0 \quad \therefore m = \pm 1.$

Again, $\phi_4'(m) = -2m$

Now, using $c = \dfrac{-\phi_3(m)}{\phi_4'(m)} = \dfrac{-(-2a)}{-2m} = \dfrac{-a}{m}$

When, $m = -1$, $c = a$ and when $m = 1$, $c = -a$

Required asymptotes are

$y = 1.x - a$ and $y = -1 . x + a$

or, $y = x - a$ and $y = -x + a$.

(n) $(a + x)^2 (b^2 + x^2) = x^2y^2$

Soln: From the given curve, we have

$$(a^2 + 2ax + x^2)(b^2 + x^2) = x^2y^2$$

or, $a^2b^2 + a^2x^2 + 2ab^2x + 2ax^3 + b^2x^2 + x^4 - x^2y^2 = 0$

or, $x^4 - x^2y^2 + 2ax^3 + (a^2 + b^2)x^2 + 2ab^2x + a^2b^2 = 0$

The equation of curve is of degree four. So, there are at the most four asymptotes real or imaginary. y^4 in the curve is absent and x^4 is present. So, there are no asymptotes parallel to x axis and there are asymptotes parallel to y axis.

The asymptotes parallel to y–axis are obtained by equating to zero the lowest degree term in y.

i.e. coefficient of $y^2 = 0$

$\Rightarrow x^2 = 0 \therefore x = 0$

Now, to obtain the asymptote of the form $y = mx + c$, we substitute $x = 1$ and $y = m$ in 4th and 3rd degree terms to get $\phi_4(m)$ and $\phi_3(m)$

$\phi_4(m) = 1 - m_2$ and $\phi_3(m) = 2a$

Now. $\phi_4(m) = 0 \Rightarrow 1 - m^2 = 0$, $\therefore m = \pm 1$ and $\phi_4'(m) = -2m$

$$\therefore c = -\frac{-\phi_3(m)}{\phi_4'(m)} = \frac{-2a}{-2m} = \frac{a}{m}$$

When $m = 1$, $c = a$ and when $m = -1$, $c = -a$

The oblique asymptotes are $y = 1.x + a$ and $y = 1.x - a$,

i.e. $y - x - a = 0$ and $y + x + a = 0$

$\therefore$ The required asymptotes are $x = 0$, $y - x - a = 0$ and $y + x + a = 0$.

2. **Find the asymptotes of the following curves:**

(a) $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$

Soln: The given equation of the curve is

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$$

This is of degree four, so there are four asymptotes real or imaginary. Absence of x^4 and y^4 in the curve shows that there are asymptotes parallel to the coordinate axis. The asymptotes parallel to x and y axis are obtained by equating the coefficient of the highest available power of x and y respectively to zero.

i.e. $y^2 - y = 0$ $\quad$ $[\because (y^2 - y)x^2 - xy^2 + x + y + 1 = 0]$

or, $y(y - 1) = 0$

$\therefore y = 0$ and $y = 1$

and $(x^2 - x) = 0$ $\quad$ $[\because (y^2 - y)y^2 - x^2y + x + y + 1 = 0]$

or, $x(x - 1) = 0$

$\therefore x = 0$ and $y = 1$

Hence, the required asymptotes of the given curve are

$x = 0$, $x - 1$, $y = 0$, $y = 1$

(b) $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ **[B.E. 2011]**

Soln: The given equation of curve can be written as

$$x^2(x^2 - 2xy + y^2) - a^2x^2 - a^2y^2 = 0$$

or, $x^4 - 2x^3y + x^2y^2 - a^2x^2 - a^2y^2 = 0$ .... (i) which is of degree four, so there are at the most four asymptotes real or imaginary. In the curve x^4 present and y^4 is absent. So, there is no asymptote parallel to x-axis and there are asymptotes parallel to y-axis.

To find the asymptote parallel to y-axis, we equate to the coefficient of highest degree term in y to zero.

i.e. $x^2 - a^2 = 0$ $\quad [\because x^4 - 2x^3y + (x^2 - a^2)y^2 - a^2x^2 = 0]$

$\therefore\ x = \pm a.$

To obtain the asymptote of the form $y = mx + c$, put $x = 1$ and $y = m$ in 4^{th}, 3^{rd} and 2^{nd} degree term, to get

$$\phi_4(m) = 1 - 2m + m^2,\ \phi_3(m) = 0 \text{ and } \phi_2(m) = -a^2 - a^2m^2$$

Now, $\phi_4(m) = 0$

$\Rightarrow\ 1 - 2m + m^2 = 0$

or, $(1 - m)^2 = 0$

or, $m = 1, 1$

Since the root $m = 1$ is repeated twice, so to find c, we shall use the formula

$$\frac{c^2}{2!}\phi_4''(m) + c\,\phi'(m) + \phi_2(m) = 0 \qquad \text{...... (ii)}$$

But $\quad \phi_4'(m) = -2 + 2m$

$\quad\quad \phi_4''(m) = 2$

and $\quad \phi_3(m) = 0$

Hence equation (i), given

$$\frac{c^2}{2!}.2 + c.0 + (-a^2 - a^2m^2) = 0$$

or, $c^2 - a^2 - a^2 = 0$ $\quad [\because m = 1]$

or, $c^2 = 2a^2$

$\therefore\ c = \pm\sqrt{2}\,a$

$\therefore$ The oblique asymptotes are $y = 1.x + (\pm\sqrt{2}\ a) =$ or

$$y = x \pm \sqrt{2}\,a$$

$\therefore$ Required asymptotes are $x = \pm a$ and $y = x \pm \sqrt{2}a$.

(c) $\quad x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0$

Soln: The given curve

$$x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0$$

which is of degree three, so there are at the most three asymptotes real or imaginary. x^3 and y^3 in the curve are present, so there are no asymptotes parallel with coordinate axes.

To find the oblique asymptotes put $x = 1$ and $y = m$ in 3^{rd}, 2^{nd} and 1^{st} degree terms, we get

$\phi_3(m) = 1 - 2m^3 + 2m - m^2$, $\phi_2(m) = m - m^2$ and $\phi_1(m) = 0$

Now, $\phi_3(m) = 0$

$\Rightarrow 1 - 2m^3 + 2m - m^2 = 0$

or, $(1 + 2m) - m^2(1 + 2m) = 0$

or, $(1 - m^2)(1 + 2m) = 0$

$\therefore\ m = \pm 1, \frac{-1}{2}$

12.13

Also, $\phi_3'(m) = -6m^2 + 2 - 2m$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(m-m^2)}{-6m^2+2-2m} = \frac{m^2-m}{-6m^2+2-2m}$$

If $m = 1$, $c = \dfrac{1-1}{-6.1+2-2.1} = 0$

If $m = -1$, $c = \dfrac{(-1)^2-(-1)}{-6(-1)^2+2-2(-1)} = \dfrac{1+1}{-6+2+2} = -1$

If $m = -\frac{1}{2}$, $c = \dfrac{\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)}{-6\left(-\frac{1}{2}\right)^2 + 2 - 2\left(-\frac{1}{2}\right)} = \dfrac{\frac{1}{4}+\frac{1}{2}}{-6.\frac{1}{4}+2+1} = \dfrac{1}{2}$

$\therefore$ Required asymptotes are

$y = 1.x + 0$, $y = -1.\,x + (-1)$ and $y = -\frac{1}{2}x + \frac{1}{2}$

or $y = x$, $y + x + 1 = 0$, and $2y + x = 1$.

(d) $\mathbf{(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0}$

Soln: The given equation of curve is

$(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$

or, $x^3 + 2x^2y + x^2 - xy^2 - 2y^3 - y^2 + x + y + 1 = 0$

or, $x^3 - 2y^3 + 2x^2y - xy^2 + x^2 - y^2 + x + y + 1 = 0$

This equation is of degree three, so there are at the most three asymptotes real or imaginary. x^3 and y^3 are present in the curve, so there are no asymptotes parallel to coordinate axes.

Now, to find asymptotes of the form $y = mx + c$, put $x = 1$ and $y = m$, we get

$\phi_3(m) = 1 - 2m^3 + 2m - m^2$, $\phi_2(m) = 1 - m^2$

Now, $\phi_3(m) = 0$

$\Rightarrow 1 - 2m^3 + 2m - m^2 = 0$

or, $(1 + 2m) - m^2(1 + 2m) = 0$

or, $(1 + 2m)(1 - m^2) = 0$

$\therefore \quad m = \pm 1, \frac{-1}{2}$

Also, $\phi_3'(m) = -6m^2 + 2 - 2m$

Now, $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = \dfrac{m^2-1}{2-2m-6m^2}$

If $m = 1$, $c = \dfrac{1-1}{2-2-6} = 0$

If $m = -1$, $c = \dfrac{(-1)^2-1}{2-2(-1)-6(-1)^2} = 0$

If $m = -\frac{1}{2}$, $c = \dfrac{\left(-\frac{1}{2}\right)^2 - 1}{2-2\left(-\frac{1}{2}\right)-6\left(-\frac{1}{2}\right)^2} = -\dfrac{1}{2}$

$\therefore$ Required asymptotes are

$y = 1.x + 0$, $y + x = 0$ and $2y + x + 1 = 0$

or, $y - x = 0$, $y + x = 0$ and $2y + x + 1 = 0$.

(e) $x(x-y)^2 - 3(x^2 - y^2) + 8y = 0$

Soln: The equation of curve is

$$x(x-y)^2 - 3(x^2-y^2) + 8y = 0$$

or, $x(x^2 - 2xy + y^2) - 3x^2 + 3y^2 + 8y = 0$

or, $x^3 - 2x^2y + xy^2 - 3x^2 + 3y^2 + 8y = 0$ .... (i)

which is of degree three, so there are at the most three asymptotes real or imaginary. x^3 in the curve is present and y^3 is absent, so there is no asymptote parallel to x–axis but there are asymptotes parallel to y-axis.

The asymptotes parallel to y–axis are obtained by equating the coefficient of highest degree term to zero.

i.e. $x + 3 = 0$ $[\because x^3 - 2x^3y + (x+3)y^2 - 3x^2 + 8y = 0]$

To get the asymptotes of the form $y = mx + c$, put $x = 1$ and $y = m$ in 3^{rd}, 2^{nd} and first degree terms, we get

$\phi_3(m) = 1 - 2m + m^2$, $\phi_2(m) = -3 + 3m^2$, $\phi_1(m) = 8m$

Now, $\phi_3(m) = 0$

$\Rightarrow$ $1 - 2m + m^2 = 0$

or, $(1-m)^2 = 0$

$\therefore$ $m = 1, 1$

Since $m = 1$ repeated twice, so we use the formula

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0 \quad \text{...... (ii)}$$

We have $\phi_3'(m) = -2 + 2m$

$\phi_3''(m) = 2$

And $\phi_2'(m) = 6m$

$\therefore$ from (ii), we have

$$\frac{c^2}{2}.2 + c\,.\,6m + 8m = 0$$

or, $c^2 + 6cm + 8m = 0$

or, $c^2 + 6c + 8 = 0$ $[\therefore m = 1)$

or, $(c+2)(c+4) = 0$

$\therefore$ $c = -2, -4$

$\therefore$ Oblique asymptotes are $y = 1.x - 2$ and $y = 1.x - 4$, i.e. $y = x - 2$, $y = x - 4$.

Required asymptotes are $x + 3 = 0$, $y = x - 2$, $y = x - 4$.

(f) $(x+y)^2(x-y) + 2y(x+y) - 3x + y = 0$

Soln: The given equation of curve is

$(x^2 + 2xy + y^2)(x-y) + 2y(x+y) - 3x + y = 0$

or, $x^3 + 2x^2y + xy^2 - x^2y - 2xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$

or, $x^3 - y^3 + x^2y - xy^2 + 2xy + 2y^2 - 3x + y = 0$ .... (i)

which is of degree three, so there are at the most three asymptotes real or imaginary. x^3 and y^3 both present in the curve shows that there are no. asymptotes parallel to the coordinate axes.

To obtain asymptote of the form $y = mx + c$, put $x = 1$, $y = m$, we get.

$\phi_3(m) = 1 - m^3 + m - m^2$, $\phi_2(m) - 2m + 2m^2$, $\phi_1(m) = -3 + m$

Now, $\phi_3(m) = 0$

$\Rightarrow$ $1 - m^3 + m - m^2 = 0$

or, $(1-m)(1 + m + m^2) + m(1-m) = 0$

or, $(1-m)(1 + m + m^2 + m) = 0$

or, $(1-m)(m^2 + 2m + 1) = 0$

or, $(1-m)(m+1)^2 = 0$

$\therefore$ m = 1 and m = −1, −1

For m = 1

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$$= \frac{-(2m + 2m^2)}{-3m^2 + 1 - 2m} \qquad [\because \ \phi_3'(m) = -3m^2 + 1 - 2m]$$

$$= \frac{-(2.1 + 2.1^2)}{-3.1^2 + 1 - 2.1} = 1$$

$\therefore$ $y = 1.x + 1$, i. e. $x - y + 1 = 0$ is asymptote

But m = − 1 is repeated root. So, to find corresponding value of c, we shall apply the formula

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0 \qquad \text{.......... (iii)}$$

Now, $\phi_3'(m) = -3m^2 + 1 - 2m$

$\phi_3''(m) = -6m - 2$

and $\phi_2'(m) = 2 + 4m$.

Hence, from (ii), we get

$$\frac{c^2}{2!}(-6m - 2) + c(2 + 4m) + (-3 + m) = 0$$

or, $\frac{c^2}{2}(-6 \times -1 - 2) + c(2 + 4 \times -1) + (-3 - 1) = 0$

or, $\frac{c^2}{2}.4 + c(-2) - 4 = 0$

or, $2c^2 - 2c - 4 = 0$

or, $c^2 - c - 2 = 0$

or, $(c + 1)(c - 2) = 0$

$c = -1, c = 2$

$\therefore$ $y = -1.x + 2$ and $y = -1.x - 1 = 0$ i.e. $x - y + 2 = 0$, $x + y + 1 = 0$

$\therefore$ Required asymptotes are

$x - y + 1 = 0, x - y + 2 = 0, x + y + 1 = 0$

(g) $x^2(x^2 + y^2 - 2xy) - 2x^2 - 2y^2 = 0$

Soln: The given equation of curve is

$x^4 + x^2y^2 - 2x^3y - 2x^2 - 2y^2 = 0$ .... (i)

which is of degree four, so there are at the most four asymptotes real or imaginary. x^4 present and y^4 is absent, so there is no asymptote parallel to x axis but there are asymptotes parallel to y-axis.

The asymptotes parallel to y axis are obtained by equating the highest degree term to zero.

i.e. $(x^2 - 2) = 0$ $\quad [\because \ x^4 + (x^2 - 2)y^2 - 2x^3y - 2x^2 - 2y^2 = 0]$

$\therefore$ $x = \pm\sqrt{2}$

To obtain the asymptotes of the form y = mx + c, put x = 1 and y = m in 4th, 3rd and 2nd degree term, we get.

$\phi_4(m) = 1 + m^2 - 2m$, $\phi_3(m) = 0$, $\phi_2(m) = -2 - 2m^2$.

Now, $\phi_4(m) = 0$

$\Rightarrow$ $(1 + m^2 - 2m) = 0$

or, $(1 - m)^2 = 0$

$\therefore$ m = 1,1

Since m = 1 is a repeated root, so we have to use the following formula to get the corresponding value of c.

$$\frac{c^2}{2!}\phi_4''(m) + c\,\phi_3'(m) + \phi_2(m) = 0 \qquad \text{.......... (ii)}$$

Now, $\phi_4'(m) = 2m - 2$

$\phi_4''(m) = 2$

And $\phi_3'(m) = 0$

$\therefore$ from equation (ii), we get

$$\frac{c^2}{2}\cdot 2 + c.\,0 + (-2 - 2m^2) = 0$$

or, $c^2 + (-2 - 2.1)$ $[\because m = 1]$

or, $c^2 - 4 = 0$

$\therefore$ $c = \pm 2$

$\therefore$ The oblique asymptotes are

$y = 1.\,x \pm 2$, i. e. $y = x \pm 2$

$\therefore$ Required asymptotes are

$x = \pm\sqrt{2}$, $y = x \pm 2$.

(h) $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2y + 2x + 1 = 0$

Soln: The given equation of curve is of degree three, so there are at the most three asymptotes real or imaginary.

Presence of x^3 and y^3 in the curve, this shows that there are no asymptotes parallel with x – and y-axis.

To find the asymptotes of the form y = mx + c, put x = 1 and y = m in 3rd, 2nd and 1st degree terms in x and y to get $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$ respectively.

$\phi_3(m) = m^3 - 2m^2 - m + 2$, $\phi_2(m) = 3m^2 - 7m + 2$,

$\phi_1(m) = 2m + 2$

Now, $\phi_3(m) = 0$

$\Rightarrow$ $m^3 - 2m^2 - m + 3 = 0$

or, $m^2(m - 2) - 1(m - 2) = 0$

or, $(m - 2)(m - 1)(m + 1)$

$\therefore$ $m = 2, 1, -1.$

Also, $\phi_3'(m) = 3m^2 - 4m - 1$

$$\therefore c = \frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(3m^2 - 7m + 2)}{3m^2 - 4m - 1}$$

When $m = 2$, $c = \dfrac{(3.2^2 - 7.2 + 2)}{3.2^2 - 4.2 - 1} = 0$

When $m = 1$, $c = \dfrac{(3.1^2 - 7.\,1 + 2)}{3.1^2 - 4.1 - 1} = -1$

When $m = -1$, $c = \dfrac{-[3(-1)^2 - 7(-1) + 2]}{3(-1)^2 - 4(-1) - 1} = -2$

$\therefore$ Required asymptotes are

$y = 2x + 0$, $y = 1.x - 1$ and $y = -x - 2$

i.e. $y = 2x$, $y = x - 1$ and $y = -x - 2$

(i) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy + y^2 - 1 = 0$

Soln: The given curve is of degree three, so there are at the most three asymptotes real or imaginary. The presence of x^3 and y^3 in the curve shows that there are no asymptotes parallel the coordinate axes.

To find the asymptotes of the form y = mx + c, put x = 1 and

$y = m$ in 3^{rd}, 2^{nd} and 1^{st} degree terms in x and y, we get

$$\phi_3(m) = 4 - 3m^2 - m^3$$
$$\phi_2(m) = 2 - m + m^2$$
$$\phi_1(m) = 0$$

So that $\phi_3'(m) = -6m - 3m^2$, $\phi_3''(m) = -6 - 6m$ and $\phi_2'(m) = -1 + 2m$

Now, $\phi_3(m) = 0$

$\Rightarrow 4 - 3m^2 - m^3 = 0$

or, $m^3 + 3m^2 - 4 = 0$

or, $m^3 - m^2 + 4m^2 - 4m + 4m - 4 = 0$

or, $m^2(m-1) + 4m(m-1) + 4(m-1) = 0$

or, $(m-1)(m^2 + 4m + 4) = 0$

or, $(m-1)(m+2)^2 = 0$

$\therefore$ $m = 1, m = -2, -2$

When $m = 1$

$$\therefore \; c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(2 - m + m^2)}{(-6m - 3m^2)} = \frac{-(2-1+1)}{-6.1 - 3.1^2} = \frac{2}{9}$$

$\therefore$ The asymptote is $y = 1.x + \frac{2}{9}$, i. e. $y = x + \frac{2}{9}$

Again,
$m = -2$ is a repeated root. So, we use the following formula to get the corresponding value of c,

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

or, $\frac{c^2}{2}(-6 - 6m) + c(-1 + 2m) + 0 = 0$

or, $\frac{c^2}{2}[-6 - 6(-2)] + c[-1 + 2(-2)] = 0$

or, $3c^2 - 5c = 0$

or, $c = 0, c = \frac{5}{3}$

$\therefore$ The asymptotes are $y = -2x + 0$ and $y = -2x + \frac{5}{3}$

i.e. $y + 2x = 0$ and $y + 2x - \frac{5}{3} = 0$

$\therefore$ Required asymptotes are

$$y = x + \frac{2}{9},\; y + 2x = 0 \text{ and } y + 2x - \frac{5}{3} = 0$$

(j) $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$

Soln: The given curve is of degree three, so there are at the most three asymptotes real or imaginary. x^3 and x^3 are present, this shows that there are no asymptotes parallel to both the coordinate axes.

To obtain the asymptote of the form $y = mx + c$, we substitute $x = 1$ and $y = m$ in 3^{rd}, 2^{nd} and 1^{st} degree terms in x, y, we get

$$\phi_3(m) = m^3 - m^2 - m + 1,\; \phi_2(m) = 1 - m^2,\; \phi_1(m) = 0$$

Now, $\phi_3(m) = 0$

$\Rightarrow m^3 - m^2 - m + 1 = 0$

or, $m^2(m-1) - 1(m-1) = 0$

or, $(m^2 - 1)(m-1) = 0$

or, $(m+1)(m-1)^2 = 0$

$\therefore$ m = –1 and m = 1,1

Also, $\phi_3'(m) = 3m^2 - 2m - 1$, $\phi_3''(m) = 6m - 2$, $\phi_2'(m) = -2m$.

For m = –1

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(1-m^2)}{3m^2 - 2m - 1} = \frac{-[1-(-1)^2]}{3(-1)^2 - 2(-1) - 1} = 0$$

$\therefore$ Asymptote is y = mx + c or, y = –1.x + 0 or, y + x = 0

For m = 1:

Since m = 1 is the repeated root, so we use the following formula to get the corresponding value of c.

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

or, $\frac{c^2}{2!}[6m - 2] + c(-2m) + 0 = 0$

or, $\frac{c^2}{2!}[6.1 - 2] + c(-2.1) = 0$ $\qquad [\because m = 1]$

or, $2c^2 - 2c = 0$

or, c = 0, c = 1

$\therefore$ The asymptotes are y = 1.x + 0 and y = 1, x + 1 i.e. y–x = 0, y = x + 1

$\therefore$ Required asymptote are y + x = 0, y – x = 0 and y = x + 1.

3. Find the asymptotes of the corves:

(a) $2r^2 = \tan 2\theta$.

Soln: The given curve can be written as

$$2r^2 = \frac{\sin 2q}{\cos 2q}$$

or, $2r^2 \cos 2\theta = \sin 2\theta$

or, $2r^2(\cos^2\theta - \sin^2\theta) = 2\sin\theta \cdot \cos\theta$

or, $2r^2(r^2\cos^2\theta - r^2\sin^2\theta) = 2r\sin\theta\, r\cos\theta$

or, $2(x^2 + y^2)(x^2 - y^2) = 2xy$ $\quad [\because x^2 + y^2 = r^2, x = r\cos\theta, y = r\sin\theta]$

or, $(x^4 - y^4) = xy$

or, $x^4 - y^4 - xy = 0$ .... (i)

The curve (i) is the algebraic curve of degree four, so there are at the most four asymptotes real as imaginary. Due to the presence of x^4 and y^4 in the curve shows that there are no asymptotes parallel to the coordinate areas.

Now, we find the asymptotes of the form,

y = mx + c .... (ii)

For this, putting x = 1 and y = m in fourth and third degree terms of x and y in (i) to obtain, $\phi_4(m)$ and $\phi_3(m)$, we have $\phi_4(m) = 1 - m^4$ and $\phi_3(m) = 0$

Now, $\phi_4(m) = 0$

$\Rightarrow 1 - m^4 = 0$

$\Rightarrow (1 + m^2)(1 - m^2) = 0$

i.e. m = $\pm$ 1 and other values of m are imaginary.

Also, $\phi_4'(m) = -4m^3$

We know that

$$c = \frac{-\phi_3(m)}{\phi_4'(m)} = \frac{-0}{-4m^3} = 0$$

Substituting the values of m and c in (ii), we get

$y = \pm 1\, x + 0$

or, $y = \pm\, x$

or, $\quad r \sin\theta = \pm\, r\cos\theta$

or, $\tan\theta = \pm 1$

or, $\tan\theta = \tan\left(\pm\frac{\pi}{4}\right)$

$\therefore\ \theta = \pm\frac{\pi}{4}$

Which are the required asymptotes of the curve

(b) $\quad r\theta\cos\theta = a\cos 2\theta$ **[B.E. 2016]**

Soln: The given equation can be written as,

$\frac{1}{r} = \frac{\theta\cos\theta}{a\cos 2\theta}$

or, $f(\theta) = \frac{\theta\cos\theta}{\theta\cos 2\theta} \qquad \left(\because\ \frac{1}{r} = f(\theta)\right)$

Differentiating, both with respect to θ, we get

$f'(\theta) = \frac{1}{a}\left[\frac{\cos 2\theta\,(-\theta\sin\theta + \cos\theta) - \theta\cos\theta\,(-2\sin 2\theta)}{(\cos 2\theta)^2}\right]$

Now, $\quad f(\theta) = 0$

$\Rightarrow \frac{\theta\cos\theta}{a\cos 2\theta} = 0$

or, $\theta\cos\theta = 0$

i.e. $\theta = 0$ or $\cos\theta = 0$

$\therefore\ \theta = 0$ or $\theta = \frac{\pi}{2}$

Let $\theta = 0 = \alpha$, and $\theta = \frac{\pi}{2} = \alpha_2$

Thus, $f'(\alpha_1) = \frac{1}{a}$

and the asymtote is

$r\sin(\theta - \alpha_1) = \frac{1}{f'(\alpha_1)}$

or, $\quad r\sin(\theta - 0) = \frac{1}{\frac{1}{a}}$

or $\quad r\sin\theta = a$

Similarly, $f'(\alpha_2) = \frac{1}{a}\left[\frac{(-1)(-\pi/2)}{(-1)^2}\right] = \frac{1}{a}\,(\pi/2)$

and the asymptote is

$r\sin(\theta - \alpha_2) = \frac{1}{f'(\alpha_2)}$

or, $r \sin(\theta - \pi/2) = \dfrac{1}{\dfrac{\pi}{2a}}$

or, $-r \cos\theta = \dfrac{2a}{\pi}$

or, $-\pi(r\cos\theta) = 2a$

or, $\pi(r\cos\theta) + 2a = 0$

Hence, the required asymptotes are $r\sin\theta = a$,

$\pi(r\cos\theta) + 2a = 0$

(c) $\mathbf{r \sin\theta = a\cos 2\theta}$

Soln: The given equation of the curve can be written an

$$\frac{1}{r} = \frac{\sin\theta}{a\cos 2\theta}$$

or, $f(\theta) = \dfrac{\sin\theta}{a\cos 2\theta}$ $\quad\left[\because \dfrac{1}{r} = f(\theta)\right]$

Differentiating both sides w. r. t. θ, we get

$$f'(\theta) = \frac{a\cos 2\theta.\cos\theta + 2a\sin 2\theta.\sin\theta}{(a\cos 2\theta)^2}$$

$$= \frac{1}{a}\left[\frac{\cos\theta\cos 2\theta + 2\sin\theta.\,\text{Sin}\,2\theta}{\cos^2 2\theta}\right]$$

Now, $f(\theta) = 0$

$\Rightarrow \dfrac{\sin\theta}{a\cos 2\theta} = 0$

or, $\sin\theta = 0$

$\therefore \theta = 0$

Let $\theta = 0 = \alpha$ (say)

Thus $f(\alpha) = \dfrac{1}{a}$

and the asymptote is

$r\sin(\theta - \alpha) = \dfrac{1}{f'(\alpha)}$

or, $r\sin(\theta - 0) = \dfrac{1}{\dfrac{1}{a}}$

or, $r\sin\theta = a$

$\therefore$ Required asymptote of the curve is $r\sin\theta = a$

(d) $\mathbf{r\sin\theta = a}$

Soln: The given equation of the curve is

$r\sin\theta = a$

Change it to cartesian form, we get

$y = a$ $\quad[\because r\sin\theta = y]$

which is the equation of straight line and it has no asymptote.

Some Additional Questions from Final Examination

1. Find the asymptotes of the curve $y^2(a^2 + x^2) = x^2(a^2 - y^2)$ [B.E. 2015]

Sol^n: The given curve is

$$y^2(a^2 + x^2) = x^2(a^2 - y^2)$$

This is of degree four, so there are four asymptotes real or imaginary. Absence of x^4 and y^4 in the curve shows that there are asymptotes parallel to X and Y-axis respectively. The asymptotes parallel to X-axis and Y-axis are obtained by equating the coefficients of the highest available power of x and y respectively to zero.

From the given curve

$$a^2y^2 + x^2y^2 - a^2x^2 + x^2y^2 = 0$$

or, $$a^2y^2 + 2x^2y^2 - a^2x^2 = 0 \quad \text{(i)}$$

or, $$(a^2 + 2x^2)\, y^2 - a^2x^2 = 0$$

The coefficient of highest power of y is $a^2 + 2x^2$ and equating it to zero, we have

$$a^2 + 2x^2 = 0$$

$\Rightarrow$ $x^2 = \dfrac{-a^2}{2}$, which gives no real asymptotes.

Again, from (i)

$$a^2y^2 + (2y^2 - a^2)\, x^2 = 0$$

The coefficient of highest power of x is $2x^2 - a^2$ and equating it to zero, we have

$$2y^2 - a^2 = 0$$

$$\Rightarrow \quad y^2 = \frac{a^2}{2}$$

$$\therefore \quad y = \pm\frac{a}{\sqrt{2}}$$

Hence the required asymptotes are $y = \pm\dfrac{a}{\sqrt{2}}$.

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