$$= 2\pi c \int_0^a \cos h\frac{x}{c} \cdot \cos h\frac{x}{c}\, dx$$

$$= 2\pi c \int_0^a \left(\frac{1+\cos h\frac{2x}{c}}{2}\right) dx$$

$$= \pi c \left[x + \sin h\frac{2x}{c} \cdot \frac{c}{2}\right]_0^a$$

$$= \pi c \left[a + \frac{c}{2}\sin h\frac{2a}{c} - 0\right]$$

$$= \pi c \left[a + \sin h\frac{a}{c} \cdot \cos h\frac{a}{c}\right] \text{ Ans.}$$

(c) The given equation of parabola is

$y^2 = 4ax$ .... (i)

The latus rectum AB of the parabola cuts x-axis at (a, 0).

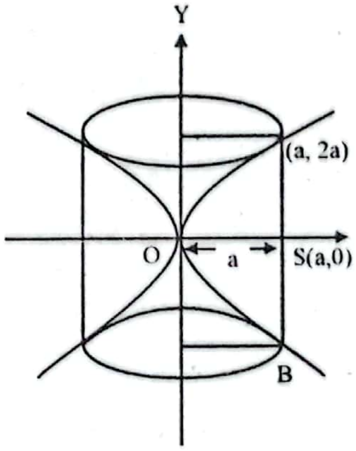
Let V be the volume of the solid formed by the revolution of region bounded by the latus rectum of parabola about x-axis, then

$$V = 2\int_0^{2a} \pi x^2 dy$$

$$= 2\pi \int_0^{2a} \frac{y^4}{16a^2}\, dy$$

$$= \frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{\pi}{8a^2}\left[\frac{32a^5}{5} - 0\right] = \frac{4\pi a^3}{5} \text{ Ans.}$$



Again, if S be the required surface area, then

$$S = 2\int_0^{2a} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 4\pi \int_0^{2a} \frac{y^2}{4a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Differentiating $y^2 = 4ax$ both sides w.r.t. y, we have

$$2y = 4a\frac{dx}{dy}$$

or, $\dfrac{dx}{dy} = \dfrac{y}{2a}$

$$\therefore\ S = \frac{\pi}{a}\int_0^{2a} y^2\sqrt{1+\frac{y^2}{4a^2}}\ du$$

$$= \frac{\pi}{2a^2}\int_0^{2a} y^2\sqrt{4a^2+y^2}\ dy$$

Put $y = 2a\tan\theta$ $\quad\therefore\ dy = 2a\sec^2\theta\ d\theta$

When $y = 0, \theta = 0$ and when $y = 2a, \theta = \frac{\pi}{4}$

$$\therefore\ S = \frac{\pi}{2a^2}\int_0^{\pi/4} 4a^2\tan^2\theta\ \sqrt{4a^2(1+\tan^2\theta)}\ \ 2a\sec^2\theta\ d\theta$$

$$= \frac{\pi}{2a^2}\ 16a^4\int_0^{\pi/4}\tan^2\theta\sec\theta\ \sec^2\theta\ d\theta$$

$$= 8\pi a^2\int_0^{\pi/4}(\sec^2\theta - 1)\sec^3\theta\ \ d\theta$$

$$= 8\pi a^2\left[\int_0^{\pi/4}\sec^5\theta\ d\theta - \int_0^{\pi/4}\sec^3\theta\ d\theta\right] \ \ldots\ldots\ldots\ldots \text{(i)}$$

Now, $\displaystyle\int_0^{\pi/4}\sec^5 d\theta = \int_0^{\pi/4}\sec^3\theta\sec^2\theta\ d\theta$

$$= [\sec^3\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 3\sec^3\theta\tan\theta\tan\theta\, d\theta$$

$$= (\sqrt{2})^3 - \int_0^{\pi/4} 3\sec^3\theta(\sec^2\theta - 1)\,d\theta$$

$$= 2\sqrt{2} - 3\int_0^{\pi/4}\sec^5\theta\ d\theta + 3\int_0^{\pi/4}\sec^3\theta\ d\theta$$

$$\therefore\ 4\int_0^{\pi/4}\sec^5\theta\ d\theta = 2\sqrt{2} + 3\int_0^{\pi/4}\sec^3\theta\ d\theta$$

$$= \frac{1}{\sqrt{2}} + \frac{3}{4}\int_0^{\pi/4}\sec^3\theta\ \ d\theta \ \ldots\ldots\ldots\ldots \text{(ii)}$$

Again, $\int_0^{\pi/4} \sec^3\theta \, d\theta = \int_0^{\pi/4} \sec^2\theta \sec\theta \, d\theta$

$$= [\sec\theta \tan\theta]_0^{\pi/4} - \int_0^{\pi/4} \sec\theta \,.\, \tan\theta \tan\theta \, d\theta$$

$$= \sqrt{2} - \int_0^{\pi/4} \sec\theta \,(\sec^2\theta - 1)\, d\theta$$

$$= \sqrt{2} - \int_0^{\pi/4} \sec^3\theta \, d\theta + \int_0^{\pi/4} \sec\theta \, d\theta$$

$$2\int_0^{\pi/4} \sec^3\theta \, d\theta = \sqrt{2} + [\log(\sec\theta + \tan\theta)]_0^{\pi/4}$$

$$= \sqrt{2} + \log(\sqrt{2} + 1)$$

$$\therefore \int_0^{\pi/4} \sec^3\theta \, d\theta = \frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2} + 1) \quad \text{....... (iii)}$$

$\therefore$ From (i), (ii) and (iii), we have

$$S = 8\pi a^2\left[\frac{1}{\sqrt{2}} + \frac{3}{4}\int_0^{\pi/4} \sec^3\theta \, d\theta - \int_0^{\pi/4} \sec^3\theta \, d\theta\right]$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4}\int_0^{\pi/4} \sec^3\theta \, d\theta\right]$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4}\left\{\frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2} + 1)\right\}\right]$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}} - \frac{1}{8}\log(\sqrt{2} + 1)\right]$$

$$= 8\pi a^2\left[\frac{3}{4\sqrt{2}} - \frac{1}{8}\log(\sqrt{2} + 1)\right]$$

$$= \pi a^2\left[\frac{2 \times 3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} - \log(\sqrt{2} + 1)\right]$$

$$= \pi a^2\left[3\sqrt{2} - \log(\sqrt{2} + 1)\right] \text{ Ans.}$$

Find the volume and surface area of solid generated by revolving

(a) the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

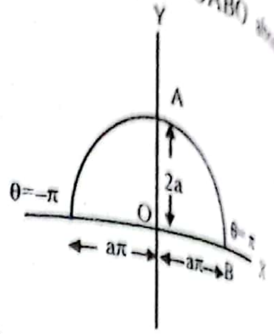
(b) the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.

Soln: (a) The given equation of cycloid is $x = a(\theta + \sin\theta)$,

$y = a(1 + \cos\theta)$

Let V be the required volume, then

V = 2 (Volume of the solid generated by the area OABO about x-axis.)



$$= 2\int_0^{a\pi} \pi y^2\, dx$$

$$= 2\pi \int_0^{a\pi} y^2\, dx$$

But $y = a(1 + \cos\theta)$ and $x = a(\theta + \sin\theta)$

$\therefore\ dx = a(1 + \cos\theta)$

When $y = 0 \ \Rightarrow a(1 + \cos\theta) = 0$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pi$

$\therefore\ x = a(\pi + \sin\pi) = a\pi$

Also, if $x = 0 \Rightarrow \theta = 0$ and if $x = a\pi \Rightarrow \theta = \pi$

$$\therefore\ V = 2\pi \int_0^{\pi} a^2(1 + \cos\theta)^2\ a(1 + \cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (1 + \cos\theta)^3 d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= 16\,\pi a^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t,\ d\theta = 2dt$

If $\theta = 0,\ t = 0$ and if $\theta = \pi,\ t = \frac{\pi}{2}$

$$\therefore\ V = 16\pi a^3 \int_0^{\pi/2} \cos^6 t \,.\, 2dt$$

$$= 32\pi a^3 \int_0^{\pi/2} \cos^6 t\, dt$$

$$= 32\pi a^3 \frac{\Gamma\left(\frac{6+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{6+2}{2}\right)}$$

$$= 32\pi a^3 \frac{\Gamma\left(\frac{7}{2}\right)\sqrt{\pi}}{2\,\Gamma(4)}$$

$$= 16\pi a^3 \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,.\sqrt{\pi}}{3\,.\,2.\,1}$$

$$= 5\pi^2 a^3 \text{ Ans.}$$

Again, if S be the required surfaced area, then

$$S = 2\int_0^{\pi} 2\pi y\ \frac{ds}{dq}\ d\theta$$

$$\text{But } \frac{ds}{dq} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2(-\sin\theta)^2}$$

$$= a\sqrt{2(1+\cos\theta)}$$

$$= 2a\cos\frac{\theta}{2}$$

$$\therefore\ S = 4\pi\int_0^{\pi} a(1+\cos\theta)\ 2a\cos\frac{\theta}{2}\ d\theta$$

$$= 8\pi a^2\int_0^{\pi} 2\cos^3\frac{\theta}{2}\ d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^3\frac{\theta}{2}\ d\theta$$

Put $\frac{\theta}{2} = t$, then $d\theta = 2dt$

When $\theta = 0$, $t = 0$ and $\theta = \pi$, $t = \frac{\pi}{2}$

$$\therefore\ S = 16\pi a^2\int_0^{\pi/2} 2\cos^3 t\ dt$$

$$= 32\pi a^2 \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3+2}{2}\right)} = 32\pi a^2 \frac{\Gamma(2)\,.\sqrt{\pi}}{2\Gamma\left(\frac{5}{2}\right)}$$

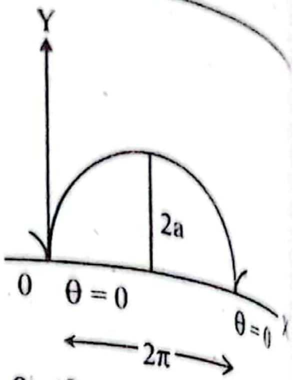
$$= 16\pi a^2 \frac{1\,.\sqrt{\pi}}{1\,.\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{64}{3}\ \pi a^2 \text{ Ans.}$$

(b) The given equation of curve is

$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

If V be the volume formed by the revolution of give cycloid about its base, then



$$V = \int_0^{2\pi a} \pi y^2 dx$$

Now, $x = a(\theta - \sin\theta) \Rightarrow dx = a(1 - \cos\theta)\, d\theta$

If $x = 0$, $t = 0$, if $x = 2\pi a$, $t = 2\pi$

$$\therefore V = \pi\int_0^{2\pi} a^2(1-\cos\theta)^2 \cdot a(1-\cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1-\cos\theta)^3\, d\theta$$

$$= 2\pi a^3 \int_0^{2\pi} (1-\cos\theta)^3\, d\theta$$

$$\left[\because f(2a-x) = f(x) \Rightarrow \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx\right]$$

$$\therefore V = 2\pi a^3 \int_0^{\pi} \left[2\cos^2\frac{\theta}{2}\right]^3 d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$

When $\theta = 0$, $t = 0$ and when $\theta = \pi$, $t = \frac{\pi}{2}$

$$\therefore V = 16\pi a^3 \int_0^{\pi/2} \cos^6 t \cdot 2dt$$

$$= 32\pi a^3 \int_0^{\pi/2} \cos^6 t\, dt$$

$$= 32\pi a^3 \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{6+2}{2}\right)}$$

$$= 16\pi a^3 \frac{\Gamma\left(\frac{7}{2}\right) \cdot \sqrt{\pi}}{\Gamma(4)}$$

$$= \frac{16\pi a^3 \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}\sqrt{\pi}}{3 \,.\, 2 \,.\, 1} = 5\pi^2 a^3 \text{ Ans.}$$

Again, the surface area S is given by

$$S = \int_0^{2\pi a} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

But $y = a(1 - \cos\theta) \Rightarrow \frac{dy}{d\theta} = a \sin\theta$

and $x = a(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2 \sin^2\frac{\theta}{2}} = \cot\frac{t}{2}$$

Also, if $x = 0$, $\theta = 0$ and if $x = 2\pi a$, $\theta = 2\pi$

$$\therefore S = 2\pi \int_0^{2\pi} (a - \cos\theta) \sqrt{1 + \cot^2\frac{\theta}{2}}\, (1 - \cos\theta)\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} (1 - \cos\theta)^2 \sqrt{\operatorname{cosec}^2\frac{\theta}{2}}\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} (1 - \cos\theta)^2 \operatorname{cosec}\frac{\theta}{2}\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} \left(2 \sin^2\frac{\theta}{2}\right)^2 \operatorname{cosec}\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2} \,.\, \operatorname{cosec}\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^3\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$ when $\theta = 0$, $t = 0$ and when $\theta = \pi$, $t = \frac{\pi}{2}$

$$S = 16\pi a^2 \int_0^{\pi/2} \sin^3 t \, . \, 2dt$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin^3 t \, dt$$

$$= 32\pi a^2 \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3+2}{2}\right)} = 16\pi a^2 \frac{\Gamma(2)\sqrt{\pi}}{\Gamma\left(\frac{5}{2}\right)}$$

$$= 16\pi a^2 \frac{1 . \sqrt{\pi}}{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} = \frac{64\ \pi a^2}{3} \text{ Ans.}$$

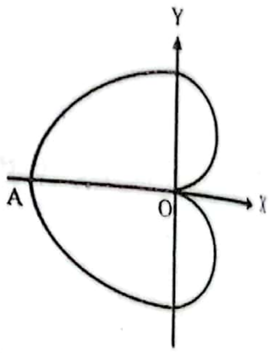
6. **Find the volume and surface area of solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.**

Soln: The given curve is

$r = a(1 - \cos\theta)$ .... (i)

Here, $\theta = \pi$, $r = 2a$, so that the coordinate of A is $(-2a, 0)$

Required volume V is given by



$$V = \int_{-2a}^{0} \pi y^2 \, dx$$

Now, $x = r\cos\theta$

$= a(1 - \cos\theta)\cos\theta = a\cos\theta - a\cos^2\theta$

and $y = r\sin\theta$

$= a(1 - \cos\theta)\sin\theta$

$\therefore$ $dx = (-a\sin\theta + 2a\sin\theta\cos\theta)\,d\theta$

Also, when $x = 0$, $\theta = 0$ and $x = -2a$, $\theta = \pi$

$$\therefore V = \int_{\pi}^{0} a^2(1 - \cos\theta)^2 \sin^2\theta\,(-a\sin\theta + 2a\sin\theta\cos\theta)\,d\theta$$

$$= -\pi a^3 \int_{\pi}^{0} (1 - \cos\theta)^2 \sin^3\theta\,(1 - 2\cos\theta)\,d\theta$$

$$= \pi a^3 \int_0^{\pi} 4\sin^4\frac{\theta}{2}\ 8 . \sin^3\frac{\theta}{2}\cos^3\frac{\theta}{2}\left(3 - 4\cos^2\frac{\theta}{2}\right) d\theta$$

$$= 96\pi a^3 \int_0^{\pi} \sin^7\frac{\theta}{2}\cos^3\frac{\theta}{2}\,d\theta - 128\ \pi a^3 \int_0^{\pi} \sin^7\frac{\theta}{2}\cos^5\frac{\theta}{2}\,d\theta$$

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt.$

When $\theta = 0, t = 0$ and when $\theta = \pi, t = \frac{\pi}{2}$.

$$\therefore V = 96\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^3 t \;.\; 2dt - 128\pi a^3 \int_0^{\pi/2} \sin^7 t \,.\, \cos^5 t \; 2dt$$

$$= 192\pi a^3 \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{7+3+2}{2}\right)} - 256\pi a^3 \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{7+5+2}{2}\right)}$$

$$= 96\pi a^3 \frac{\Gamma(4)\,\Gamma(2)}{\Gamma(6)} - 128\pi a^3 \frac{\Gamma(4)\,\Gamma(3)}{\Gamma(7)}$$

$$= 96\pi a^3 \,.\, \frac{3.2.1.1}{5.4.3.2.1} - 128\,\pi a^3 \frac{3.2.1.2.1}{6.5.4.3.2.1}$$

$$= \frac{24\pi a^3}{5} - \frac{32}{15}\pi a^3 = \left(\frac{24}{5} - \frac{32}{15}\right)\pi a^3 = \frac{8}{3}\pi a^3 \text{Ans.}$$

Let S be the required surface area, then

$$S = \int_0^{\pi} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta$$

We have, $y = r \sin\theta$

$= a(1 - \cos\theta)\sin\theta$

$r = a(1 - \cos\theta) \Rightarrow \frac{dr}{dq} = a \sin\theta$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{a\sqrt{1 - 2\cos\theta + 1}}$$

$$= a\sqrt{2(1-\cos\theta)}$$

$$= a\sqrt{2.2.\,\sin^2\frac{\theta}{2}} = 2a\sin\frac{\theta}{2}$$

$$\therefore S = 2\pi \int_0^{\pi} a(1-\cos\theta)\sin\theta \;\; 2a\sin\frac{\theta}{2}\; d\theta$$

$$= 4a^2\pi \int_0^{\pi} 2\sin^2\frac{\theta}{2}\;\; 2\sin\frac{\theta}{2}\;\cos\frac{\theta}{2}\sin\frac{\theta}{2}\; d\theta$$

$$= 16a^2\pi \int_0^{\pi} \sin^4\frac{\theta}{2}\;\cos\frac{\theta}{2}\; d\theta$$

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$ When $\theta = 0, t = 0$ and $\theta = \pi, t = \frac{\pi}{2}$

$$\therefore\ S = 32a^2\pi \int_0^{\pi/2} \sin^4 t\ \cos t\, dt$$

$$= 32a^2\pi \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\Gamma\left(\frac{4+1+2}{2}\right)}$$

$$= 32a^2\pi \frac{\Gamma\left(\frac{5}{2}\right).\Gamma(1)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= 16a^2\pi \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{32\pi a^2}{5} \text{ Ans.}$$

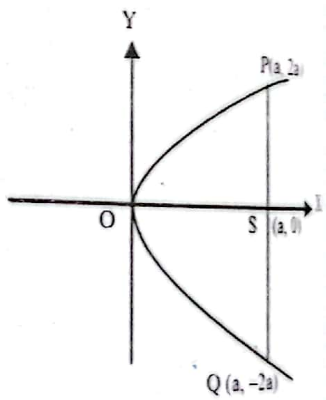
7. **An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume generated when the arc is rotated about the latus rectum.**

Soln: We know that the parabola with latus rectum 4a is $y^2 = 4ax$. The coordinates of end points of latus return PQ are P (a, 2a) and Q (a, –2a)
Since the axis of revolution is the latus rectum, so the required volume V is given by

$V = 2 \times$ volume generated by the area OPQO about PQ

$$= 2\int_0^{2a} \pi (a-x)^2\, dy$$

$$= 2\pi \int_0^{2a} \left(a - \frac{y^2}{4a}\right)^2 dy$$



$$= \frac{2\pi}{16a^2} \int_0^{2a} (4a^2 - y^2)^2\, dy$$

$$= \frac{\pi}{8a^2} \int_0^{2a} (16a^4 - 8a^2y^2 + y^4)\, dy$$

$$= \frac{\pi}{8a^2} \left[16a^4 y - \frac{8a^2y^3}{3} + \frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{\pi}{8a^2} \left[16a^4 . 2a - \frac{8a^2}{3}(2a)^3 + \frac{(2a)^5}{5}\right]$$

$$= \frac{\pi}{8a^2} \left[32a^5 - \frac{64a^5}{3} + \frac{32}{5}a^5\right]$$

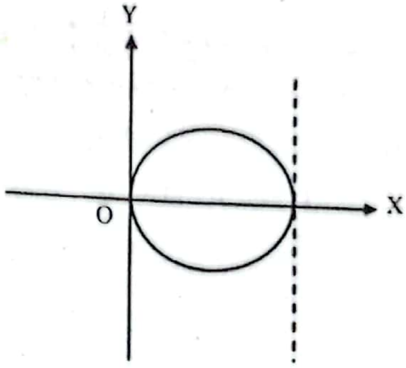
$$= \frac{32\pi a^5}{8a^2} \left[1 - \frac{2}{3} + \frac{1}{5}\right] = 4\pi a^3 \times \frac{8}{15} = \frac{32}{15}\pi a^3 \text{ Ans.}$$

Find the volumes of the solids formed by the revolution of following curves about the x-axis :

(a) $y^2 = x^2 (a - x)$

(b) $y^2 (a + x) = x^2 (a - x)$

Soln: (a) The given curve is $y^2 = x^2 (a - x)$



$$\text{Required volume } V = \int_0^a \pi y^2 \, dx$$

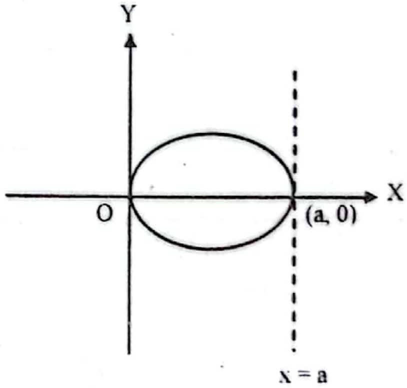
$$= \pi \int_0^a x^2 (a - x) \, dx$$

$$= \pi \int_0^a (ax^2 - x^3) \, dx$$

$$= \pi \left[a\frac{x^3}{3} - \frac{x^4}{4} \right]_0^a = \pi \left[\frac{a^4}{3} - \frac{a^4}{4} \right] = \frac{\pi a^4}{12} \text{ Ans.}$$

(b) The given curve is $y^2 (a + x) = x^2 (a - x)$

The required volume V is given by



$$V = \int_0^a \pi y^2 dx$$

$$= \pi \int_0^a \frac{x^2 (a - x)}{a + x} \, dx$$

Put $a + x = t \Rightarrow dx = dt$

$x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 2a$

$$\therefore V = \pi \int_a^{2a} \frac{(t - a)^2 (a - t + a)}{t} \, dt = \pi \int_a^{2a} \frac{(t^2 - 2t + a^2)(2a - t)}{t} \, dt$$

$$= \pi \int_a^{2a} \frac{(4at^2 - 5a^2t + 2a^3 - t^3)}{t} \, dt = \pi \int_a^{2a} \left(4at - 5a^2 + \frac{2a^3}{t} - t^2 \right) dt$$

$$= \pi \left[\frac{4at^2}{2} - 5a^2t + 2a^3 \log t - \frac{t^3}{3} \right]_a^{2a}$$

$$= \pi \left[2a.4a^2 - 5a^2 . 2a + 2a^3 . \log 2a - \frac{8a^3}{3} - 2a.a^2 + 5a^2. a - 2a^3 \log a + \frac{a^3}{3} \right]$$

$$= \pi \left[2a^3 \log 2 + a^3 - \frac{7a^3}{3} \right]$$

$$= 2\pi a^3 \left(\log 2 - \frac{2}{3} \right) \text{ Ans.}$$

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Unit 18

Trapezoidal and Simpson's Rules

Trapezoidal and Simpson's rule

1. **Trapezoidal Rule**

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_n)],$$

where $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$ and so on.

2. **Simpson's rule**

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

when n is even and $h = \dfrac{b-a}{n}$

While applying this rule it is to be noted that n should be even, i.e. the interval of integration must be divided into an even number of equal subinterval.

Exercise - 18

1. **Estimate $\int_{1.00}^{1.30} \sqrt{x}\,dx$ using the trapezoidal rule and the data:**

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$f(x) = \sqrt{x}$	1.000	1.02470	1.04881	1.07238	1.09545	1.11803	1.14018

Soln: Here, $x_0 = 1.00$ $\quad$ $y_0 = 1.000$

$x_1 = 1.05$ $\quad$ $y_1 = 1.02470$

$x_2 = 1.10$ $\quad$ $y_z = 1.04881$

$x_3 = 1.15$ $\quad$ $y_3 = 1.07238$

$x_4 = 1.20$ $\quad$ $y_4 = 1.09545$

$x_5 = 1.25$ $\quad$ $y_5 = 1.11803$

$x_6 = 1.30$ $\quad$ $y_6 = 1.14018$

By trapezoidal rule,

$$\int_{1.00}^{1.30} \sqrt{x}\,dx = \frac{1.30-1.00}{2\times 6}[(1.000 + 1.14018) + 2(1.02470 + 1.04881 + 1.07238 + 1.09545 + 1.11803)]$$

$$= \frac{0.3}{12}[2.14018 + 2 \times 5.35937]$$

$$= 0.025 \times 12.85892 = 0.321473 \text{ Ans.}$$

Estimate $\int_0^1 \frac{1}{1+x}$ dx using trapezoidal rule with h = 0.25 upto 5 decimal place.

ln: Here, $x_0 = 0$ $\quad y_0 = 1.00000$

$x_1 = 0.25$ $\quad y_1 = 0.80000$

$x_2 = 0.50$ $\quad y_2 = 0.66667$

$x_3 = 0.75$ $\quad y_3 = 0.57143$

$x_4 = 1.00$ $\quad y_4 = 0.50000$

By trapezoidal rule, we have,

$$\int_0^1 \frac{1}{1+x}\, dx = \frac{1-0}{2\times 4}\,[(1+0.5)+2\,(0.80000 + 0.66667 + 0.57143)]$$

$$= \frac{1}{8}\,[1.5 + 2 \times 2.03810]$$

$$= \frac{1}{8} \times 5.5762 = 0.69703 \text{ Ans.}$$

Estimate the integral in No. 2 by Simpson's rule with h = 0.25.

ln: Here, we have to find $\int_0^1 \frac{1}{1+x}\, dx$

We have, a = 0, b = 1, h = 0.25, so n = 4

$\therefore$ $x_0 = 0$ $\quad y_0 = 1.00000$

$x_1 = 0.25$ $\quad y_1 = 0.80000$

$x_2 = 0.50$ $\quad y_2 = 0.66667$

$x_3 = 0.75$ $\quad y_3 = 0.57143$

$x_4 = 1.00$ $\quad y_4 = 0.50000$

By Simpson's rule, we have

$$\int_0^1 \frac{1}{1+x}\, dx = \frac{h}{3}\,[y_0 + y_n + 4\,(y_1 + y_3) + 2y_2]$$

$$= \frac{0.25}{3}\,[(1.0000 + 0.50000) + 4\,(0.80000 + 0.57143) + 2 \times 0.66667]$$

$$= \frac{0.25}{3}\,[1.50000 + 4 \times 1.37143 + 1.33334]$$

$$= \frac{0.25}{3} \times 8.31906 = 6.9325 \text{ Ans.}$$

Estimate the following integrals with n = 4

(i) **using Trapezoidal rule.**

(ii) **using Simpson's rule and also estimated values with their exact values.**

(a) $\int_0^2 x\, dx$

Soln: (i) Trapezoidal's rule :

Here, $x = 4$, $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$, so that

$x_0 = 0$ $y_0 = 0$
$x_1 = 0.5$ $y_1 = 0.5$
$x_2 = 1.0$ $y_2 = 1.0$
$x_3 = 1.5$ $y_3 = 1.5$
$x_4 = 2.0$ $y_4 = 2.0$

∴ Using trapezoidal rule, we have

$$\int_0^2 x\,dx = \frac{2-0}{2\times 4}[(0+2.0)+2(0.5+1.0+1.5)]$$

$$= \frac{2}{8}[2.0+6.0] = \frac{2}{8}\times 8.0 = 2 \text{ Ans.}$$

The exact value of integral is

$$\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - 0 = 2$$

Error $= 2 - 2 = 0$ Ans.,

(ii) Simpson's rule :

$$\int_0^2 x\,dx = \frac{2-0}{4\times 3}[(0+2.0)+4(0.5+1.5)+2\times 1.0]$$

$$= \frac{2}{4\times 3}[2+8+2] = 2.0 \text{ Ans.}$$

Error $=$ Exact value $-$ Estimated value
$= 2 - 2 = 0$ Ans.

(b) $\int_0^2 x^2\,dx$

Soln: (i) Trapezoidal's rule

Here, $a = 0$, $b = 2$, $n = 4$, $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$, so that

$x_0 = 0$ $y_0 = 1$
$x_1 = 0.5$ $y_1 = 0.25$
$x_2 = 1.0$ $y_2 = 1.00$
$x_3 = 1.5$ $y_3 = 2.25$
$x_4 = 2.0$ $y_4 = 4$

∴ Using Trapezoidal's rule, we have

$$\int_0^2 x^2\,dx = \frac{b-a}{2n}[(y_0+y_4+2(y_1+y_2+y_3)]$$

$$= \frac{2-0}{2\times 4}[(0+4)+2(0.25+1.00+2.25)]$$

$$= \frac{1}{4}(4+7) = 2.75 \text{ Ans.}$$

The exact value of integral is $\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \left(\frac{2^3}{3} - 0\right) = \frac{8}{3}$

$$\text{Error} = \frac{2.75 - \frac{8}{3}}{\frac{8}{3}} \times 100\%$$

$$= \frac{0.25}{8} \times 100\% = 3.125 \text{ greater than exact value.}$$

(ii) Sampson's rule :

$$\int_0^2 x^2\,dx = \frac{b-a}{3 \times n} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{2-0}{3 \times 4} [(0.00 + 4.00) + 4(0.25 + 2.25) + 2 \times 1.00]$$

$$= \frac{2}{12} [4 + 10 + 2] = \frac{2 \times 16}{12} = 2.6667 \text{ Ans.}$$

$$\text{Error} = \frac{2.6667 - \frac{8}{3}}{\frac{8}{3}} \times 100\% = 0 \text{ Ans.}$$

(c) $\int_0^2 x^3\,dx$

Soln: (i) Trapezoidal's rule

Here, $a = 0$, $b = 2$, $n = 4$, $h = \frac{b-a}{4} = \frac{2-0}{4} = 0.5$, so that

$x_0 = 0$ $\quad y_0 = 0$
$x_1 = 0.5$ $\quad y_1 = 0.125$
$x_2 = 1.0$ $\quad y_2 = 1.000$
$x_3 = 1.5$ $\quad y_3 = 3.375$
$x_4 = 2.0$ $\quad y_4 = 8.000$

Using Trapezoidal rule, we have

$$\int_0^2 x^3\,dx = \frac{b-a}{2 \times n} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{2-0}{2 \times 4} [(0 + 8) + 2(0.125 + 1.000 + 3.375)]$$

$$= \frac{1}{4} [8 + 9]$$

$$= \frac{1}{4} \times 17 = 4.25 \text{ Ans.}$$

Exact value of integral is

$$\int_0^2 x^3 dx = \left[\frac{x^4}{4}\right]_0^2 = \frac{2^4}{4} = 4$$

$\therefore$ Error $= \dfrac{4.25 - 4}{4} \times 100\%$

$= 6.25\%$ greater than the exact value.

(ii) Simpson's Rule :

$$\int_0^2 x^3\,dx = \frac{b-a}{3.n}\left[(y_0 + y_n) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{2-0}{3 \times 4}\left[(0 + 8.000) + 4(0.125 + 3.375) + 2 \times 1.000\right]$$

$$= \frac{1}{6}[8 + 14 + 2]$$

$$= \frac{1}{6} \times 24 = 4 \text{ Ans.}$$

Error $= \dfrac{4-4}{4} \times 100\% = 0$ Ans.

(d) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

Soln: Here, $a = 1$, $b = 2$ $n = 4$, $h = \dfrac{b-a}{4} = \dfrac{2-1}{4} = 0.25$, so that

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.25$	$y_1 = 0.64$
$x_2 = 1.50$	$y_2 = 0.4444$
$x_3 = 1.75$	$y_3 = 0.3265$
$x_4 = 2.00$	$y_4 = 0.25$

Using Trapezoidal rule, we have

$$\int_1^2 \frac{1}{x^2}\,dx = \frac{b-a}{2 \times n}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{2-1}{2 \times 4}\left[(1 + 0.25) + 2(0.64 + 0.4444 + 0.3265)\right]$$

$$= \frac{1}{8}[1.25 + 2.8218]$$

$$= \frac{1}{8} \times 4.718 = 0.5090 \text{ Ans.}$$

Exact value of integral is

$$\int_0^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2$$

$$= -\frac{1}{2} + 1 = 0.5$$

Error $= \dfrac{0.5090 - 0.5}{0.5} \times 100\%$

$= 1.8\%$ greater than the exact value.

(ii) Simpson's rule :

$$\int_0^1 \frac{1}{x^2}\,dx = \frac{b-a}{3\times n}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

$$= \frac{2-1}{3\times 4}[(1+0.25)+4(0.64+0.3265)+2\times 0.4444]$$

$$= \frac{1}{12}[1.25+3.866+0.8888]$$

$$= \frac{1}{12}\times 6.0048 = 0.5004 \text{ Ans.}$$

$$\text{Error} = \frac{0.5004-0.5}{0.5}\times 100\%$$

$= 0.08\%$ greater than exact value

(e) $\int_1^4 \sqrt{x}\,dx$

Soln: Trapezoidal's rule

Here, $a = 1$, $b = 4$, $n = 4$, $h = \frac{b-a}{n} = \frac{4-1}{4} = \frac{3}{4} = 0.75$;

so that

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.75$	$y_1 = 1.3229$
$x_2 = 2.5$	$y_2 = 1.5811$
$x_3 = 3.25$	$y_3 = 1.8028$
$x_4 = 4$	$y_4 = 2$

Now, using Trapezoidal's rule, we have

$$\int_1^4 \sqrt{x}\,dx = \frac{b-a}{2n}[(y_0+y_4)+2(y_1+y_2+y_3)]$$

$$= \frac{4-1}{2\times 4}[(1+2)+2(1.3229+1.5811+1.8028)]$$

$$= \frac{3}{8}[3+4.7069]$$

$$= \frac{3\times 12.4136}{8} = 4.6551 \text{ Ans.}$$

Exact value of integral is

$$\int_1^4 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4$$

$$= \frac{2}{3}[4^{3/2}-1]$$

$$= \frac{2}{3}\times 7 = 4.6667$$

$$\therefore \text{ Error } = \frac{4.6667 - 4.6551}{4.6667} \times 100\%$$

$= 0.25\%$ less than exact value.

(ii) Simpson's rule

$$\int_1^4 \sqrt{x}\, dx = \frac{b-a}{3n}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{4-1}{3\times 4}[(1+2) + 4(1.3229 + 1.8028) + 2 \times 1.5811]$$

$$= \frac{1}{4}[3 + 12.5028 + 3.1622]$$

$= 4.66625$ Ans.

$$\text{Error} = \frac{4.6667 - 4.66625}{4.6667} \times 100\%$$

$= 0.009\%$ less than exact value

(f) $\int_0^\pi \sin x\, dx$

Soln: (i) Trapezoidal rule

Here, $a = 0$, $b = \pi$, $n = 4$, $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$,

so that

$x_0 = a = 0$	$y_0 = 0$
$x_1 = a + \frac{\pi}{4} = \frac{\pi}{4}$	$y_1 = \frac{1}{\sqrt{2}}$
$x_2 = a + \frac{2\pi}{4} = \frac{\pi}{2}$	$y_2 = 1$
$x_3 = a + \frac{3\pi}{4} = \frac{3\pi}{4}$	$y_3 = \frac{1}{\sqrt{2}}$
$x_4 = a + \frac{4\pi}{4} = \pi$	$y_4 = 0$

Using trapezoidal rule, we have

$$\int_0^\pi \sin x\, dx = \frac{b-a}{2\times n}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\pi - 0}{2\times 4}\left[(0+0) + 2\left(\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{8}\left[0 + \left(\frac{2}{\sqrt{2}} + 1\right)\right]$$

$$= \frac{\pi}{4} \times 2.4142 = 1.8961 \text{ Ans.}$$

Exact value of integral is

$$\int_0^\pi \sin x\, dx = [-\cos x]_0^\pi$$

$$= -\cos\pi + \cos 0 = 1 + 1 = 2$$

$$\text{Error} = \frac{2 - 1.8961}{2} \times 100\%$$

$= 5.2\%$ less than exact value

Simpson's rule

$$\int_0^{\pi} \sin x \, dx = \frac{b-a}{3n} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{\pi - 0}{3 \times 4}\left[(0+0) + 4\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2 \times 1\right]$$

$$= \frac{\pi}{2}\left[0 + \frac{8}{\sqrt{2}} + 2\right]$$

$$= \frac{\pi}{12} \times 7.6569 = 2.00464 \text{ Ans.}$$

$$\text{Error} = \frac{2.0464 - 2}{2} \times 100\%$$

$= 2.32\%$ more than exact value.

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