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LOGIC PROGRAMMING
LEAST HERBRAND MODELS

Recall **duality** in Herbrand interpretation:

$$I_A \xleftrightarrow{1-1} A \subset B_L$$

$$I_A(atom) = \begin{cases} true; & atom \in A, \\ false; & atom \notin A. \end{cases}$$

Thus by **intersection** of 2 Herbrand interpretations we understand:

$$(I_{A_1} \cap I_{A_2})(atom) \stackrel{def.}{=} I_{A_1 \cap A_2}(atom).$$

$$\boxed{I_{A_1 \cap A_2}(atom) = \begin{cases} true; & atom \in A_1 \cap A_2 \\ false; & atom \notin A_1 \cap A_2 \end{cases}}$$

Important questions:

- is intersection of models a model?
- can we define the smallest model?

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Example 1:

$$\Delta = \begin{cases} p(a). \\ p(f(f(X))) \leftarrow p(X). \end{cases}$$

Then

$A_1 = \{p(a), p(f(a)), p(f(f(a))), \dots\} = B_\Delta$

$A_2 = B_\Delta \setminus \{p(f(a))\}$

$A_3 = \{p(a), p(f(f(a))), p(f(f(f(f(a))))), \dots\}$

Then all I_{A_1}, I_{A_2} & I_{A_3} as easily verifiable are Herbrand models.

$A_1 \cap A_2 = A_2$

$A_1 \cap A_3 = A_3$

$A_3 \cap A_2 = A_3$

So intersections are also models?

This will be proved in the next result.

$\square$.

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We show that intersection of all Herbrand models (the so-called least Herbrand model – LHM) __is a model__

Theorem 1:

Let P be __a definite logic program__

Then:

a) the set of Herbrand models __is non-empty__ (at least one model exists)

b) $\underset{\overset{\shortparallel}{M}}{LHM} = \bigcap_{k \in K} M_k$, where M_k are Herbrand models, is __a Herbrand model.__

* For non-definite clauses

a) fails for $\leftarrow A_1, \ldots, A_n$

b) fails for $\underline{\quad B, C \leftarrow A_1, A_2 \ldots A_n \quad}_3$

proof also follows for intersection of $\{I_1, \ldots, I_n\}$ some models

Proof:

a) Consider a Herbrand base $B_P \subseteq B_p$. Then I_{B_P} (under which all atoms in B_P are true) is a model for P.

Indeed for 2 types of definite logic program clauses we have to show that their ground instances are true

(*) $A^{G i_1} \leftarrow B_1^{G j_1}, B_2^{G j_2}, \dots, B_n^{G j_n}$ or $A^{G} \leftarrow$.

As all atoms $A^{G i_1}, B_1^{G j_1}, \dots, B_n^{G j_n}$ are true then (*) are true.

Thus I_{B_P} is a Herbrand model.

Note: the proof would break out for negative Horn clauses $\leftarrow A_1, A_2, \dots A_n$

but would work for non-Horn clauses

$$A_1, \ldots, A_k \leftarrow B_1, B_2, \ldots, B_n$$

This however would "spoil the proof of b). So

$$\left.\begin{array}{l} \text{— by a)} \quad k \geq 1 \\ \text{— by b)} \quad k \leq 1 \end{array}\right\} \Rightarrow k=1$$

That explains why exactly one head atom should be in the data (definite program clauses). clauses

(b) as we know now that (by a))

$$LHM = \bigcap_{k \in K} M_k = M$$

is correctly defined we show

that M is a Herbrand model.

Of course, by mere definition

it is the least model

We show that an arbitrary clause $\in P$ is true under M.

It suffices to show that arbitrary ground instance of such clause is true under M.

$\boxed{\text{a) } A^G \leftarrow}$ (empty body)

$I_K(A^G) = \text{true}$ for each H. model

Thus $\underset{\substack{\uparrow \\ \text{an atom}}}{A^G} \in M_K$ for each H. model

Thus $A^G \in \bigcap_{K \in \mathcal{K}} M_K \overset{M}{=} \Rightarrow \boxed{I_M(A^G) = \text{true}}$

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b) $A^{G\delta_1} \leftarrow B_1^{G\delta_1}, \dots, B_m^{G\delta_m}$ (*)

Case 1: at least one atom $B_k^{G\delta_k}$ is false under $I_M \Rightarrow I_M(\sim B_k^{G\delta_k}) = \text{true}$.

Thus (*) in DNF

$$(**)\ A^{G\delta_1} \vee \sim B_1^{G\delta_1} \vee \dots \vee \sim B_k^{G\delta_k} \vee \dots \vee \sim B_m^{G\delta_m}$$

true under I_M

$\Downarrow$

the whole disjunction (**) is true under I_M.

case 2: if all $B_i^{G\delta_i}$ are true under I_M then as they are atoms:

$$B_i^{G^{di}} \in M_K \quad \forall K \in \mathbb{K}$$

as $B_i^{G^{di}} \in M = \bigcap_{K \in \mathbb{K}} M_K$.

So $\mathcal{I}_{M_K}(B_i^{G^{di}}) = true$ for all B_i^{G} $\forall K \in \mathbb{K}$.

But $\mathcal{I}_{M_K}$ is a model, $\forall K \in \mathbb{K}$.

Therefore as:

$$\mathcal{I}_{M_K}(A^{G^{di}} \leftarrow B_1^{G^{d1}}, \ldots, B_m^{G^{dm}}) = true \quad \forall K \in \mathbb{K}$$

and

$$\mathcal{I}_{M_K}(B_i^{G^{di}}) = true$$

for all $B_i^{G^{di}}$ $\forall K \in \mathbb{K}$

$$\Downarrow$$

$$\boxed{\mathcal{I}_{M_K}(A^{G^{di}}) = true} \quad \forall K \in \mathbb{K}$$

But as $A^{G^{i_1}}$ is an atom (for non-Horn clauses a proof would fail here)

$$A^{G^{i_1}} \in M_k \quad \forall k \in K$$

$$\Downarrow$$

$$A^{G^{i_1}} \in \bigcap_{k \in K} M_k = M.$$

$$\Downarrow$$

$$\boxed{I_M(A^{G^{i_1}}) = true}$$

$$\Downarrow$$

$$I_M(A^{G^{i_1}} \leftarrow B_1^{G^{i_1}}, \ldots, B_m^{G^{i_n}}) = \underline{\underline{true}}$$

The proof is complete. $\square$

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Theorem 2:

Let P be a logic program.
Then $M_P = \{A \in B_P : A$ is a logical consequence of $P\}$.

<u>1st characterization of LHM.</u>

<u>**PROOF:**</u> Let A be a ground atom.

$P \models A$

$\Updownarrow$ iff

$P \cup \{\sim A\}$ is unsatisfiable

$\Updownarrow$

$P \cup \{\sim A\}$ has no Herbrand model (Th. previous lecture)

$\Updownarrow$

$\sim A$ is false with respect to each Herbrand model M

$\Updownarrow$ (as M_K are models for P)

A is true for each M_K.

$\Updownarrow$ $A \in M_K \; \forall K \in \mathcal{K} \Leftrightarrow A \in M_P$

$\square$

Example 2:

The last theorem gives the hint how to find M_P (LHM).

instead of finding all Herbrand models & then intersecting them we can:

find a set of all atoms s. that

$$\underline{P \models A} \quad \equiv M_P$$

$$P \equiv \begin{cases} p(a) & \text{(i)} \\ p(f(f(X))) \leftarrow p(X) & \text{(ii)} \end{cases}$$

By (i) $\underline{p(a) \in M}$

But as $P \models p(a)$ then by (ii)

$$\underline{p(f(f(a))) \in M}$$

& again by (ii) & the last one

$$p(f(f(f(f(a)))))\in M$$

etc

$\vdots$

$$M_P \geq \{p(a), p(f(f(a))), p(f(f(f(f(a))))),\dots\}$$

Clearly this is a minimal set.

$\square$

Example 3:

$$P \equiv \begin{cases} p(f(f(X))) \leftarrow p(X). & \text{(i)} \\ p(f(f(f(X)))) \leftarrow p(X). & \text{(ii)} \\ p(a). & \text{(iii)} \end{cases}$$

By (iii) & (i)

$$M_P \geqslant \{p(\overset{0}{a}), p(f(\overset{2}{f}(a))), p(f(f(\overset{4}{f}(f(a))))),\dots\}$$

By (ii) & (i)

$$M_P \geqslant \{p(\overset{0}{a}), p(f(\overset{3}{f}(f(a)))), p(f(f(f(f(\overset{6}{f}(f(a))))))),\dots\}$$

But if (3) → (5) → (7) ...

So
$$\underset{A}{\overset{\|}{M_P}} = B_P \setminus \{\underset{P}{\overset{\not\|}{p(f(a))}}\} \qquad \square$$

M_P does not capture all logical consequences – only those which are atoms!

<u>Recall PROLOG SCHEME</u>:

1. We wish to find whether there exists variables such that:

$$\Delta \models \underbrace{\exists x_1 \exists x_2 \ldots \exists x_k (A_1 \wedge \ldots \wedge A_m)}_{Q}$$

2. Q is negated to:

$$\forall x_1 \forall x_2 \ldots \forall x_k (\sim A_1 \vee \sim A_2 \vee \ldots \vee \sim A_m)$$

& added to Δ i.e. $\tilde{\Delta} = \Delta \cup \{\sim Q\}$

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In clausal form

$$\leftarrow A_1, A_2, \dots, A_m$$

3. Then PROLOG shows (by refutation) that $\tilde{\Delta}$ is unsatisfiable by showing that there are choices of variables that original goal is "true":

$$\exists x_1^1 \exists x_2^1 \dots \exists x_k^1((A_1 \wedge \dots \wedge A_m)\theta_1)$$

after;

$$\exists x_1^2 \exists x_2^2 \dots \exists x_k^2((A_1 \wedge \dots \wedge A_m)\theta_2)$$

$$\vdots$$

$$\exists x_1^l \exists x_2^l \dots \exists x_k^l(A_1 \wedge \dots \wedge A_m)\theta_l)$$

4. It may happen that once we bind some variables with θ we are still <u>left with vast</u>

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<u>degree of freedom</u> (free variables)

In lab 4 if there is no last fact:

"person who drinks tea lives to the next person who drinks milo."

then

h (yellow, norway, _||, fox, chess)

↑

free variable

Given binding θ for yellow, norway, fox & chess we have still

a free variable.

This means that <u>"correct answer"</u> has more than one answer even if we fix θ.

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Any choice of remaining free variables yields logical consequence

$\boxed{\text{DEFINITION 1:}}$

Let P be a logic programming & G be a goal:

$$\leftarrow A_1, A_2, \ldots, A_n . \equiv G$$

<u>A correct</u> <u>answer</u> for $P \cup \{G\}$ is a substitution θ for (some of) the variables of G such that

$$(*) \quad P \models \forall x_1' \forall x_2' \ldots \forall x_k' ((A_1 \wedge \ldots \wedge A_m)\theta).$$

free variables in

Note $(*) \equiv P \cup \{\sim G\theta\}$ is unsatisfiable

$P \cup \{\sim \forall x_1' \ldots \forall x_k' ((A_1 \wedge \ldots A_m)\theta)\}$ is unsatisfiable.

$\downarrow$

θ is a correct answer

Example 4:

$$\{P(a,X,Y)\} = \underline{P}$$

$$\text{Goal} \leftarrow P(X',Y',Z')$$

$$\Downarrow$$

$$\Theta = \{X'/a,\ Y'/X,\ Z'/Y\}$$ substitution!

Then Θ is a correct answer

$$\underline{P} \models \forall X \forall Y \big(P(X',Y',Z')\Theta\big)$$

$$\equiv$$

$$\underline{P} \models \forall X \forall Y\ P(a,\underset{\uparrow}{X},\underset{\uparrow}{Y}).$$

free variables

$\square$

Example 5:

PROLOG program for finding Min of 2 natural numbers N1 & N2

```
minimum (N1, N2, N1):- N1 ≤ N2.
minimum (N1, N2, N2):- N2 ≤ N1.
```

b)
```
writelist([H|T]):- write(H),
                    writelist(T).

writelist([ ]).

>> writelist([a,b,c,d])
   [a,b,c,d].
```

we use here built-in predicate write.

□