# LECTURE 14.
## LOGIC PROGRAMMING
## HERBRAND'S THEOREM

S <u>is unsatisfiable</u> iff there is <u>no model</u> for S.

For PREDICATE LOGIC

- there is plenty of different interpretations (so possible models)
- thus it is very inconvenient to consider all of them

It would be nice to have <u>a special domain</u> of discourse D such that

if there is no model for S for D $\Rightarrow$ there is no model for arbitrary domain.

<u>Satisfiability test</u> would be reduced to smaller classes I.

Fortunately there exists such domain called <u>**HERBRAND UNIVERSE OF $S$**</u>.

Also to derive such purely symbolic interpretation helps in building a computer system which performs on mechanical manipulation of symbols.

**DEFINITION 1:** Let L be a predicate logic language*.
The <u>**Herbrand Universe**</u> $U_L$ of the language is the set of all ground terms that can be formed in language L.

* the so-called first-order language.

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**Example 1:**

(i) $L_1 = \{a; f/1; p/1\}$ predicate

$U_{L_1} = \{a, f(a), f(f(a)), f(f(f(a))), \dots\}$ $\boxed{\text{infinite set}}$

(ii) $L_2 = \{a; g/2; q/2\}$ predicate

$U_{L_2} = \{a, g(a,a), g(a, g(a,a)), g(g(a,a),a), g(g(a,a), g(a,a)), \dots\}$ $\boxed{\text{infinite set}}$

(iii) $L_3 = \{a; b; q/1; p/2\}$ predicates

$U_{L_3} = \{a, b\}$ $\boxed{\text{finite set}}$

(iv) $L_4 = \{a; =\}$

here there are no constants.
By default we add some constant "a" to form ground terms.
The same for predicate we add "="  $\square$

**DEFINITION 2:** Let $\mathcal{L}$ be a predicate logic language.

The <u>**HERBRAND BASE**</u> is the set of all atoms that can be formed from the predicate symbols with arguments from $U_L$.

**Example 2:** Consider $\mathcal{L}$ as in Example 1.

(i) $B_{L_1} = \{ p(a), p(f(a)), p(f(f(a))), \dots \}$ $\boxed{\text{infinite}}$

(ii) $B_{L_2} = \{ q(a,a), q(a, g(a,a)), q(g(a,a),a), \dots \}$ $\boxed{\text{infinite}}$

(iii) $B_{L_3} = \{ q(a), q(b), p(a,a), p(a,b), p(b,a), p(b,b) \}$

$\boxed{\text{finite}}$ □

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## DEFINITION 3:

**Herbrand Interpretation:**

Let $\mathcal{L}$ be a first order language. A Herbrand pre-interpretation of $\mathcal{L}$ is the following pre-interpretation:

(a) the domain of discourse

$$D \longrightarrow U_L \leftarrow \text{Herbrand}$$

Universe

(b) constants in $\mathcal{L}$ are assigned to themselves

$$a_i \longrightarrow a_i$$

(c) a function symbol $f/n$ is assigned to $f/_n \Rightarrow f'/_n$

$$f' : U_L \times \ldots \times U_L \longrightarrow U_L$$

such that:

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$$f'(t'_1, t'_2, \ldots, t'_n) = f(t_1, t_2, \ldots, t_n).$$

(d) $p/n \xrightarrow[\text{predicate}]{} p'/n$ ?

As (d) is still not done that is why we call pre-interpretation.

$\boxed{\text{How to assign the meaning ?}}$

to finish the definition of Herbrand Interpretation

**Hint:**

the clue comes from PROPOSITIONAL LOGIC.

Recall: $\Delta = \{f_1, f_2, \ldots, f_n\}$

we find an interpretation as

(i) $\Delta_a = \{p_1, p_2, \ldots, p_m\}$

where $\underline{p_i}$ are propositional symbols building compound formulas in $\Delta$ i.e $f_1, \dots, f_n$.

(ii) then a given interpretation for $\Delta$ is uniquely determined by assigning {true, false} to each $p_i$:

$$\{p_1, p_2, \dots, p_m\} \xrightarrow{I} \{\text{true}, \text{false}\} \quad (\text{or } \{0,1\})$$

?

And can be viewed as

$$A \subset \{p_1, p_2, \dots, p_m\} \underset{1-1}{\Longleftrightarrow} I_A$$

$$I_A(p_i) = \begin{cases} \text{true} & \text{if } p_i \in A \\ \text{false} & \text{if } p_i \notin A. \end{cases}$$

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There is one-to-one identification between <u>Interpretation</u> (in Propositional logic) & <u>subsets of $\Delta_a$</u>.

We follow this clue to define(d) in DEFINITION 3.

Here for PREDICATE LOGIC & HERBRAND PRE-INTERPRETATION:

$$\Delta_a \equiv B_L \qquad \text{(Herbrand Base)}$$

this time this set can be infinite – if there is at least one function $f \in \mathcal{L}$.

We can define now

$$I_H^A \xleftrightarrow{1-1} A \subseteq B_L.$$

Here we use the fact that as formulas are closed it is sufficient to $p \rightarrow p'$ for ground atoms.

$$(*) \quad I_H^A(\text{ground\_atom}) = \begin{cases} \text{true}, & \text{if ground\_atom} \in A \\ \text{false}, & \text{if ground\_atom} \notin A \end{cases}$$

So we have not reduced the vast number of all possible interpretations to a single one.
But by introducing Herbrand interpretation notion this <u>degree of freedom</u> is reduced only

to all subsets of Herbrand Base.

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From now on:
Herbrand Interpretation
will be viewed by using:
**DUAL IDENTITY**

as either:

1) a mapping $I_H^A$
$I_H^A : B_L \to \{true, false\}$

2) or equivalently as
a subset $A \subset B_L$.

The cementing entity between both representations is (*).

Otherwise to test satisfiability we would need to analyze all possible interpretations with various a) b), c) & d).

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**DEFINITION 4:**

Let $\Delta$ be a set of closed formulas for 1-st order language $\mathcal{L}$. Then <u>**a Herbrand Model**</u> for $\Delta$ is a Herbrand Interpretation that is a model for $\Delta$.

**Example 1:**

Consider the following program:

$$\Delta \begin{cases} (i) & p(a). \\ (ii) & p(f(f(X))) \leftarrow p(X). \end{cases}$$

a) <u>**Non-Herbrand Model**</u>

let I: $D = \mathbb{Z}$ (integers)

$a \rightarrow a' = 0 \in \mathbb{Z}$

$f/_1 \rightarrow f'/_1$ such that

$f'(x) = x + 1 \qquad f' : \mathbb{Z} \rightarrow \mathbb{Z}$

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$p/1 \longrightarrow p'/1$ $\quad p'(x)$ is true iff $x$ is even.

It is obvious that I is not a Herbrand Interpretation as $D = \mathbb{Z} \neq B_L$ (Herbrand Base)

But as both (i) & (ii) are true under I as:

(i) $p'(0) \equiv$ true $\quad$ 0 is even

(ii) if $x$ is even $\Rightarrow$ $x+2$ is even true.

if $x$ is odd $\Rightarrow$ $x+2$ is odd - true

<u>So I is a non-Herbrand model</u>.

## b) <u>Herbrand models</u>

$U_L = D = \{a, f(a), f(f(a)), \ldots\}$ infinite set

$a \longrightarrow a' = a$ &

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$$B_L = \{p(a), p(f(a)), p(f(f(a))), \ldots\}$$

(i)

Consider now $\boxed{\mathcal{I}_H^{A_1}:}$

$$\underset{B_L}{\underset{\cap}{A_1}} = \{p(a), p(f(f(a))), p(f(f(f(f(a))))), \ldots\}$$

Under $\mathcal{I}_H^{A_1}$ we have

$$\overset{(*)}{p'(a')} = p'(f'(f'(a'))) = p'(f'(f'(f'(f'(a'))))) = \ldots = \text{true}$$

and for other atoms $\in B_L$ we set false.

1st clause program (i) <u>is true under $\mathcal{I}_H^{A_1}$</u> as (*)

For 2nd clause we need to consider possible 2 scenarios:

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**if** $\tilde{X} = \underbrace{f(f\ldots(f(a))}_{}\ldots)$

even times f composed with itself

$\Downarrow$

$\tilde{Y} = f(f(\tilde{x}))$ is also with even number of compositions of f.

so $\tilde{X}$ & $\tilde{Y} \in A_1$ & thus are both true. Hence (ii) <u>is true</u>.

**if** $\tilde{\tilde{X}} = \underbrace{f(\cdots f(a)}_{odd}\cdots) \Rightarrow \tilde{\tilde{Y}} = \underbrace{f(f(\tilde{\tilde{X}}))}_{odd}$

so both $\tilde{\tilde{X}}$ & $\tilde{\tilde{Y}} \notin A_1$ & are false

Hence (ii) <u>is true</u> under $I_H^{A_1}$.

Thus $I_H^{A_1}$ Is <u>a Herbrand Model for $\Delta$</u>.

(ii) let $\boxed{I_H^{A_2}:}$

$B_L \supseteq A_2 = \{p(a), p(f(f(a))), p(f(f(f(a))))\}$.

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Again here $A_2 \subseteq B_L$ so $I_H^{A_2}$ is a Herbrand Interpretation under which:

$$p'(a') \overset{(**)}{=} p'(f'(f'(a'))) \overset{(***)}{=} p'(f'(f'(f'(a')))) = true$$

& elsewhere we set false.

Now 1st clause <u>under</u> $I_H^{A_2}$ <u>is true</u> (see (i) & (**)).

The <u>second is not</u>!

Take $x = f(f(a))$ then for (ii)

$$\underbrace{\underbrace{p(f(f(f(f(a)))))}_{\text{false } \notin A_2} \leftarrow \underbrace{p'(f(f(a)))}_{\text{true as } (***)}}_{}$$

hence ground instance of (ii) is false – but it was to be true $\forall x\ p(f(f(x))) \leftarrow p(x)$.