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Thus I_H is <u>not a Herbrand</u> <u>model</u> for Δ.

(iii) Let $\boxed{I_H^{A_3}:}$

$B_L \supseteq A_3 = \{p(a)\}$

Of course $I_H^{A_3}$ <u>is not a model</u> for Δ as although (i) from Δ is true under $I_H^{A_3}$ but for (ii) from Δ & $\tilde{x} = a$ & $\tilde{\tilde{x}} = f(f(a))$

$$\underbrace{\underbrace{p'(f'(f'(a')))}_{\text{false}} \leftarrow \underbrace{p'(a')}_{\text{true}}}_{\text{false.}}$$

So $\forall x(p(f(f(x))) \leftarrow p(x))$ is obviously false.

(iv) Let $\boxed{I_H^{A_4}:}$

$$\emptyset \subseteq B_L$$

"all atoms from B_L are false".

Clearly, $I_H^{\emptyset}$ is <u>not a model</u> for Δ as (i) from Δ is false under $I_H^{\emptyset}$.

(v) Let $\boxed{I_H^{A_5}:}$

$$A = B_L \subseteq B_L$$

"all atoms in B_L are true".

Then of course by repeating an argument from above (i) is true under $I_H^{B_L}$ & (ii) also is true as: $\underbrace{\text{true} \leftarrow \text{true}}_{\text{true}}$

So $I_H^{B_L}$ is another Herbrand model for Δ. $\square$

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(iv) $\boxed{I_H^{A_4}:}$ $A_4 = \emptyset \subseteq B_L$

"all atoms are false"

Clearly $I_H^{\emptyset}$ is <u>not a Herbrand</u> model for Δ as (i) from Δ is false under $I_H^{\emptyset}$ (note (ii) in Δ is true under $I_H^{\emptyset}$) $\leftarrow - \rightarrow \underbrace{\text{false} \leftarrow \text{false}}_{\text{true}}$.

(v) Let $\boxed{I_H^{A_5}:}$ $A_5 = B_L \subseteq B_L$

"all atoms are true".

Then $I_H^{B_L}$ is <u>a Herbrand model</u> for Δ as (i) is true under $I_H^{B_L}$ (since $p(a) \in B_L$) & (ii) is true under $I_H^{B_L}$ (since whatever X is we have $\underbrace{\text{true} \leftarrow \text{true}}_{\text{true}}$).

$\square$

So we may have more than one Herbrand model or may not have at all

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RECALL (for PROPOSITIONAL LOGIC)

$$\boxed{\Delta \models Q \iff \Delta \cup \{Q\} \text{ is unsatisfiable}}$$

This result with new extended meanings (already defined) i.e.

$$\Delta \models Q$$

&

$$\Delta \cup \{\sim Q\} \text{ is unsatisfiable}$$

can be immediately proved for PREDICATE LOGIC (the same proof).

Now the issue of satisfiability

of $\Delta \cup \{\sim Q\}$

shall be reduced to the satisfiability

of $\Delta \cup \{\sim Q\}$

but only with all possible Herbrand Interpretations.

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A ground instance of a clause C is a clause obtained by replacing variables in C by members of Herbrand Universe.

Clearly:

if $\tilde{\Delta}$ has a Herbrand model

$\Rightarrow$ it has a model & thus $\tilde{\Delta}$ is satisfiable.

More important question:

If $\tilde{\Delta}$ has a model

$\Rightarrow$ has it got a Herbrand model ?

$\boxed{?}$

If the answer is positive then:

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$\boxed{\text{Herbrand Theorem:}}$

Let $\tilde{\Delta}$ be a set of clauses & suppose that $\tilde{\Delta}$ has a model Then $\tilde{\Delta}$ has a Herbrand model.

Note: this result holds for arbitrary clauses – not neccessarily Horn clauses.

__PROOF__:

To shorten notation assume that all predicates are of arity 1

We define now (from arbitrary non-Herbrand Interpretation I being a model) a new corresponding Herbrand interpretation $\tilde{I}_H^A$:

Shown later to be a Herbrand model!

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$$\mathcal{I} \longrightarrow \tilde{\mathcal{I}}_H^A \overset{1-1}{\longleftrightarrow} A \subset B_L$$

(▲) $$A = \{ \underbrace{p_i}_{\text{predicate}}(\underbrace{t}_{\text{ground term}}) \in B_L : \text{ such that } p'(t') \text{ is true under } \mathcal{I} \}.$$

So all atoms $\in B_L$ which are true under model $\mathcal{I}$ they are also to be true under $\tilde{\mathcal{I}}_H^A$.

We show $\forall \text{clause} \in \tilde{\Delta}$

$$\boxed{\tilde{\mathcal{I}}_H^A(\text{clause}) \overset{?}{=} \text{true}} \quad (*)$$

(*) would imply that $\tilde{\mathcal{I}}_H^A$ is a Herbrand Model for $\tilde{\Delta}$.

To prove (*) it suffices to show

$$\boxed{\tilde{\mathcal{I}}_H^A(\text{ground\_instance\_clause}) = \text{true}}$$

for arbitrary ground_instance_clause.

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Take arbitrary clause in Δ:

$$q_1(X_{j_1}), q_2(X_{j_2}), \ldots, q_m(X_{j_m}) \qquad (*)$$

$$\leftarrow p_1(X_{i_1}), p_2(X_{i_2}), \ldots, p_n(X_{i_n}).$$

Take now arbitrary ground terms

$t_{i_1}, t_{i_2}, \ldots, t_{i_n}, t_{j_1}, t_{j_2}, \ldots, t_{j_m}$.

Then an arbitrary ground instance of $(*)$ reads:

$$(**) \quad \boxed{q_1(t_{j_1}), \ldots, q_m(t_{j_m}) \leftarrow p_1(t_{i_1}), \ldots, p_n(t_{i_n}).}$$

We represent $(**)$ in DNF:

$$\boxed{q_1(t_{j_1}) \vee \ldots \vee q_m(t_{j_m}) \vee \sim p_1(t_{i_1}) \vee \ldots \vee \sim p_n(t_{i_n}).}$$

<u>Assume</u> $n > 0$ (non-empty body)

if any of $p'_k(t_{i'_k})$ is false under I then by $(\blacktriangle)$ (as it is an atom)

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$$\tilde{\mathcal{I}}_H^A(p_k(t_{i_k})) = \text{false}$$

$$\Downarrow$$

$$\tilde{\mathcal{I}}_H^A(\sim p_k(t_{i_k})) = \text{true}$$

$$\Downarrow$$

(***)

$$q_1(t_{j_1}) \vee \ldots \vee q_{j_m}(t_{j_m}) \vee \sim p_1(t_{i_1}) \vee \ldots \vee \sim p_k(t_{i_k}) \vee \ldots \vee \sim p_n(t_{i_n}).$$

is true under $\tilde{\mathcal{I}}_H^A$.

(ii) if all $p_1'(t_{i_1}'), \ldots, p_n'(t_{i_n}')$ are true under $\mathcal{I}$ then as $\mathcal{I}$ is a model, clause (**) must be true ~~$\mathcal{I}$~~ under $\mathcal{I}$ therefore as all body atoms are true under $\mathcal{I}$ $\Rightarrow$

to może być tylko możliwe dla klauzul postaci nie w fon ← P1...Pn

$$\mathcal{I}(q_1'(t_{j_1}') \vee \ldots \vee q'(t_{j_m}')) = \text{true}$$

Thus at least one $q_k'(t_{j_k})$ is

true under $\mathcal{I}$. As it is an atom by (▲)

$$\widetilde{\mathcal{I}}_H^A(q_k(t_{j_k})) = \text{true}$$

$$\Downarrow$$

$$\widetilde{\mathcal{I}}_H^A(q_1(t_{i_1}) \vee \ldots \vee q_k(t_{j_k}) \vee \ldots \vee q_m(t_{j_m})) = \text{true}$$

$$\Downarrow$$

the whole clause (***) is true under $\widetilde{\mathcal{I}}_H^A$.

b) Assume $n = 0$ (empty body) i.e. (**) reads

$$q_1(t_{j_1}) \vee \ldots \vee q_m(t_{j_m})$$

Again since $\mathcal{I}$ is a model

$$\mathcal{I}(q'_1(t'_{j_1}) \vee \ldots \vee q'_m(t'_{j_m})) = \text{true}$$

$$\Downarrow$$

$$\exists_K \quad \mathcal{I}(q'_K(t'_{j_K})) = \text{true}$$

as this is an atom by (▲)

$$\Downarrow$$

$$\tilde{\mathcal{I}}^{\blacktriangle}_H(q_K(t_{j_K})) = \text{true}$$

$$\Downarrow$$

$$\tilde{\mathcal{I}}^{\blacktriangle}_H(q_1(t_{j_1}) \vee \ldots \vee q_K(t_{j_K}) \vee \ldots q_m(t_{j_m})) = \text{true}.$$

So the whole (***) is true under $\tilde{\mathcal{I}}^{\blacktriangle}_H$.

The proof is complete ■

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Lemma 1:

Let S be a set of clauses.
Then

S is unsatisfiable
$\Updownarrow$ iff
S has no Herbrand model.

This is a simple conclusion from previous theorem.

Lemma 1 (& Herbrand Th.) is not true for S not being a set of clauses.

Example 2: a) $\Delta \begin{cases} p(a). \\ p(f(f(x))) \leftarrow p(x). \end{cases}$

$I_H^A \leftrightarrow A = \{p(a), p(f(f(a))), p(f(f(f(f(a))))), \dots\}$
is a Herbrand model $\Rightarrow$ <u>Δ is satisfiable.</u>

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b) $\boxed{S' = \{ p(x), \leftarrow p(a) \}}$.

$U_L = \{a\}$ and $B_L = \{p(a)\}$.

There are only 2 subsets of B_L (and thus 2 Herbrand interpretations):

(i) $\phi \subseteq B_L \overset{1-1}{\Longleftrightarrow} \underset{\sim}{I}_H^{\phi}(p(a)) = false$

$$\Rightarrow \underset{\sim}{I}_H^{\phi}(\sim p(a)) = true$$

(ii) $B_L \subseteq B_L \overset{1-1}{\Longleftrightarrow} \underset{\sim}{I}_H^{B_L}(p(a)) = true$

$$\Rightarrow \underset{\sim}{I}_H^{B_L}(\sim p(a)) = false$$

Under (i) & (ii) one of the clauses from S' is false.

Thus <u>no Herbrand model for S</u>.

Thus by Lemma 1 no model at all $\Rightarrow$ S is unsatisfiable. $\square$

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Remark:

1. If $\mathcal{L}$ has no function symbols $\Rightarrow$ U_L is finite $\Rightarrow$ B_L is finite as for a given language $\mathcal{L}$ we have finite number of constants & predicates.

Hence if B_L is finite (say $\overline{\overline{B_L}} = n$) $\Rightarrow$ 2^n different subsets (so different Herbrand interpretations) of B_L.

$\Rightarrow$ the upper bound for the number of Herbrand models is $\boxed{2^n}$.

2. If there is at least one function

symbol in $\mathcal{L} \Rightarrow \overline{\overline{U_L}} = \infty$

$$\Downarrow$$

$$\overline{\overline{B_L}} = \infty$$

(infinite)

Then number of all Herbrand Interpretations (the same as number of all subsets of B_L) <u>is infinite</u>.

Clearly number of Herbrand models can be now

- either infinite
- or finite

<u>REMARK</u>:

a) If $\mathcal{L}$ is defined <u>a priori</u> then even if Program is not using all constant & function & predicate symbols from $\mathcal{L} \Rightarrow$

U_L & B_L are built based on all symbols from $\mathcal{L}$.

b) If $\mathcal{L}$ is not defined a priori the $\mathcal{L}$ is inherited from program P (i.e. we use all constant, function & predicate symbols from P). So U_L & B_L are built from P.