PREDICATE LOGIC

Section 1.4

PREDICATES AND QUANTIFIERS

Section 1.4

SECTION SUMMARY

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Equivalences in Predicate Logic
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic

PROPOSITIONAL LOGIC NOT ENOUGH

- If we have:

  "All men are mortal."

  "Socrates is a man."

- Does it follow that "Socrates is mortal" using propositional logic?
- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- It is difficult to draw inferences.

INTRODUCING PREDICATE LOGIC

- Predicate logic uses the following new features:
  - **Variables** （变量）： x, y, z
  - **Predicates** （谓词）： $P(x), M(x)$
  - **Quantifiers** （量词）: to be covered in later slides
- *Propositional functions* （命题函数） are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$.
  - P is also called an n-place predicate or a n-ary predicate (n元谓词).
  - Variables can be replaced by elements from their **domain**（论域）.

PROPOSITIONAL FUNCTIONS

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  P(-3)

  is false.

  P(3)

  is true.

- Often the domain is denoted by U. So in this example U is the integers.

QUANTIFIERS

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some students are from Hong Kong."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

UNIVERSAL QUANTIFIER

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall\, x\, P(x)$ is false.

EXISTENTIAL QUANTIFIER

- $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

UNIQUENESS QUANTIFIER

- $\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

QUANTIFIERS WITH RESTRICTED DOMAINS

- An abbreviated notation to restrict the domain of a quantifier.
- Examples: What do the statements $\forall x < 0\ (x^2 > 0)$, and $\exists z > 0\ (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?
  - $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$.
    $\forall x(x < 0 \rightarrow x^2 > 0)$.
  - $\exists z > 0\ (z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$.
    $\exists z\ (z > 0 \wedge z^2 = 2)$.

- The restriction of a universal quantification is the same as the universal quantification of a conditional statement.
- The restriction of an existential quantification is the same as the existential quantification of a conjunction.

PRECEDENCE OF QUANTIFIERS

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean
  $\forall x\, (P(x) \vee Q(x))$.

EQUIVALENCES IN PREDICATE LOGIC

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**:

  $\forall x \neg \neg S(x) \equiv \forall x\, S(x)$

THINKING ABOUT QUANTIFIERS AS CONJUNCTIONS AND DISJUNCTIONS

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

DE MORGAN'S LAWS FOR QUANTIFIERS

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**			
Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important.

TRANSLATING FROM ENGLISH TO LOGIC

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in C."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, define a propositional function $C(x)$ denoting "x has taken a course in C" and translate as $\forall x\, C(x)$.

Solution 2: But if U is all people, also define a propositional function $S(x)$ denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow C(x))$.

$\forall x\, (S(x) \wedge C(x))$?

TRANSLATING FROM ENGLISH TO LOGIC

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 1: But if *U* is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

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