26.01.2021

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Grupa 215

Lucrare scrisă la analiză matematică

sesiune iarnă 2021

① Studiați natura seriei cu termeni pozitivi în funcție de val. parametrului $a \in \mathbb{R}$.

$$\sum_{n=1}^{\infty} \frac{a^{2n^2}}{\sqrt{n}}$$

② Calculați integrala improprie:

$$\int_0^{\infty} x^3 \cdot e^{-x^2} \, dx$$

③ Se dă funcția $f(x,y) = \frac{x \cdot y}{x-y}$. Det. constanta $L \in \mathbb{R}$

a.î. $\frac{\partial^2 f}{\partial x^2}(x,y) + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{L}{x-y}$, $\forall\, x > y$

④ Fie $g : (0,+\infty)^2 \to \mathbb{R}$ o funcție de clasă C_1.

Exprimați relația $\frac{\partial g}{\partial u}(u,v) + \frac{v+1}{u} \cdot \frac{\partial g}{\partial v}(u,v) = \frac{u}{v+1}$,

$\forall (u,v) \in (0,\infty)^2$ în variabilele $(x,y) \in (0,\infty)^2$, efectuând transformarea $u = x+y$, $v = \frac{x}{y}$. Det. apoi o funcție g cu prop. de mai sus. Verificare.

① $\sum_{n=1}^{\infty} \frac{a^{2n^2}}{\sqrt{n}}$; $a \in \mathbb{R}$ $\Rightarrow x_n = \frac{a^{2n^2}}{\sqrt{n}}$

stp $\Rightarrow$

$$\Rightarrow D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{a^{2n^2}}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{a^{2(n+1)^2}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{a^{2n^2}}{a^{2n^2+4n+2}}$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}} \cdot \sqrt{1+\frac{1}{n}}}{\cancel{\sqrt{n}}} \cdot \frac{\cancel{a^{2n^2}}}{\cancel{a^{2n^2}} \cdot a^{4n} \cdot a^2} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{a^{4n+2}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{a^{2(2n+1)}}$$

$$= \lim_{n\to\infty} \frac{1}{a^{4n+2}} \Rightarrow$$

$\Rightarrow D < 1$ pt $\forall a \in (-\infty, -1) \cup (1, +\infty) \Rightarrow \sum_{n=1}^{\infty} \frac{a^{2n^2}}{\sqrt{n}}$ convergentă

$D > 1$ pt $\forall a \in (-1, 1) \setminus \{0\} \Rightarrow \sum_{n=1}^{\infty} \frac{a^{2n^2}}{\sqrt{n}}$ divergentă

pt $a = 0$, $\sum_{n=1}^{\infty} \frac{(0)^{2n^2}}{\sqrt{n}} = \sum_{n=1}^{\infty} 0 \Rightarrow$ serie convergentă pt $a = 0$

pt $a = \pm 1 \Rightarrow D$ - nu decide

$$R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \Big(\underbrace{\frac{1}{1^{4n+2}} - 1}_{0}\Big) \overset{\infty \cdot 0}{=}$$

pt $a = \pm 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ divergentă (seria armonică)

C.C.
$\Rightarrow \frac{1}{n} < \frac{1}{\sqrt{n}}$

② $\int_0^\infty x^3 \cdot e^{-x^2} dx$ $\qquad (e^{-x^2})' = (e^{-x^2}) \cdot (-2x)$

Aplicăm schimbare de variabilă

$u = x^2 \Rightarrow 2x\,dx = du \Rightarrow dx = \frac{1}{2x} du$

$x = 0 \Rightarrow u = 0$

$x = \infty \Rightarrow u = \infty$

$$\int_0^\infty u \cdot \cancel{x} \cdot e^{-u} \cdot \frac{1}{2\cancel{x}} du = \frac{1}{2}\int_0^\infty u \cdot e^{-u} du = \frac{1}{2}\int_0^\infty u \cdot (e^{-u})' \cdot (-1) du$$

$$= \frac{1}{2}\left(u \cdot (-1) \cdot e^{-u} \Big|_0^\infty - \int_0^\infty - e^{-u} du\right)$$

$$= \frac{1}{2}\left(\lim_{u \to \infty} \frac{-u}{e^u} - 0 - e^{-u} \Big|_0^\infty\right)$$

$$= \frac{1}{2}\left(0 - \lim_{u \to \infty} \frac{1}{e^u} + e^0\right)$$

$$= \frac{1}{2}(0 - 0 + 1)$$

$$= \frac{1}{2} \Rightarrow \int_0^\infty x^3 \cdot e^{-x^2} dx = \frac{1}{2}$$

③ $f(x,y) = \dfrac{x \cdot y}{x-y}$, $L \in \mathbb{R}$

$$\frac{\partial^2 f}{\partial x^2}(x,y) + 2\frac{\partial^2 f}{\partial x \partial y}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{L}{x-y}, \quad \forall\, x \neq y$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x-y) - x\cdot y \cdot 1}{(x-y)^2} = \frac{\cancel{xy} - y^2 - \cancel{xy}}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x-y) - x\cdot y(-1)}{(x-y)^2} = \frac{x^2 - \cancel{xy} + \cancel{xy}}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\nabla f(x,y) = \left(\frac{-y^2}{(x-y)^2} ; \frac{x^2}{(x-y)^2} \right)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(-y^2 \cdot (x-y)^{-2}\right)'_x = -y^2 \cdot (-2) \cdot (x-y)^{-3} = \frac{2y^2}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(-y^2 \cdot (x-y)^{-2}\right)'_y = -2y \cdot (x-y)^{-2} + (-y^2)\cdot(-2)\cdot(x-y)^{-3}\cdot(-1)$$

$$= -2y(x-y)^{-2} + (-2y^2)(x-y)^{-3} = \overset{(x-y)}{\frac{-2y}{(x-y)^2}} - \frac{2y^2}{(x-y)^3} = \frac{-2y(x-y) - 2y^2}{(x-y)^3} =$$

$$= \frac{-2xy + \cancel{2y^2} - \cancel{2y^2}}{(x-y)^3} = \frac{-2xy}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \left(x^2 \cdot (x-y)^{-2}\right)'_y = x^2 \cdot (-2) \cdot (x-y)^{-3} \cdot (-1) =$$

$$= \frac{2x^2}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) + 2\frac{\partial^2 f}{\partial x \partial y}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\alpha}{x-y}, \quad \forall\, x > y \iff$$

$$\iff \frac{2y^2}{(x-y)^3} + 2 \cdot \frac{-2xy}{(x-y)^3} + \frac{2x^2}{(x-y)^3} = \frac{\alpha}{x-y} \iff$$

$$\iff \frac{2y^2}{(x-y)^3} - \frac{4xy}{(x-y)^3} + \frac{2x^2}{(x-y)^3} = \frac{\alpha}{(x-y)}$$

$$\iff \frac{2(y^2 - 2xy + x^2)}{(x-y)^3} = \frac{\alpha}{(x-y)} \iff \frac{2(x^2 - 2xy + y^2)}{(x-y)^3} = \frac{\alpha}{(x-y)}$$

$$\iff \frac{2\cancel{(x-y)^2}}{(x-y)^{\cancel{3}}} = \frac{\alpha}{(x-y)} \iff \frac{2}{(x-y)} = \frac{\alpha}{(x-y)} \;\Big|\cdot (x-y) \Rightarrow$$

$$\Rightarrow \alpha = 2$$

(4) $g : (0,+\infty)^2 \to \mathbb{R}$ fct. de clasă C^1

$$\frac{\partial g}{\partial u}(u,v) + \frac{v+1}{u} \cdot \frac{\partial g}{\partial v}(u,v) = \frac{u}{v+1}, \quad \forall (u,v) \in (0,+\infty)^2$$

$(x,y) \in (0,+\infty)^2$; $u = x+y$, $v = \frac{x}{y}$

Considerăm o fc. $h(x,y) = g\left(x+y, \frac{x}{y}\right)$

$h = g \circ f$ unde $f(x,y) = (\underbrace{x+y}_{u(x,y)}, \underbrace{\tfrac{x}{y}}_{v(x,y)})$

$$\frac{\partial h}{\partial x}(x,y) = \frac{\partial g \circ f}{\partial x}(x,y) = \cdot \frac{\partial u}{\partial x}(x,y) \cdot \frac{\partial g}{\partial u}(f(x,y)) +$$

$$+ \frac{\partial g}{\partial v}(f(x,y)) \cdot \frac{\partial v}{\partial x}(x,y) =$$

$$= \frac{\partial g}{\partial u}(f(x,y)) + \frac{1}{y} \cdot \frac{\partial g}{\partial v}(f(x,y)) \quad (1)$$

$$\left.\begin{array}{l} u = x+y \Rightarrow y = u - x \\ v = \frac{x}{y} \Rightarrow x = vy \end{array}\right\} \Rightarrow y = u - vy \Rightarrow$$

$$\Rightarrow y + vy = u \Rightarrow y(1+v) = u \Rightarrow y = \frac{u}{v+1}$$

$$\frac{1}{y} = \frac{v+1}{u} \quad (2)$$

Din ① şi ② $\Rightarrow$

$$\Rightarrow \frac{\partial g}{\partial u}(u,v) + \frac{v+1}{u} \cdot \frac{\partial g}{\partial v}(u,v) = \frac{u}{v+1} =$$

$$= \frac{x+y}{\frac{x}{y}+1} = \frac{x+y}{\frac{x+y}{y}} = \frac{\cancel{x+y}}{1} \cdot \frac{y}{\cancel{x+y}} = y = \frac{\partial h}{\partial x}(x,y)$$

$$h(x,y) = \int y \, dx = xy + C(y)$$

Din $h = g \circ f \Rightarrow g = h \circ f^{-1}$

$$\left.\begin{array}{l} x = vy \\ y = \frac{u}{v+1} \end{array}\right\} \Rightarrow x = v \cdot \frac{u}{v+1} = \frac{vu}{v+1}$$

$$x = \frac{vu}{v+1}$$

$$y = \frac{u}{v+1}$$

$$f^{-1}(u,v) = \left(\frac{vu}{v+1} ; \frac{u}{v+1} \right)$$

$$g(u,v) = h(f^{-1}(u,v)) = \frac{vu}{v+1} \cdot \frac{u}{v+1} + C\left(\frac{u}{v+1}\right) =$$

$$= \frac{u^2 v}{(v+1)^2} + C\left(\frac{u}{v+1}\right)$$

$$h(x,y) = xy + C(y)$$