3.3.10) $f: \mathbb{R}^3 \to \mathbb{R}^2$

$f[x_1, x_2, x_3] = [x_2, -x_1]$

$v = [[-1,1,0], [0,1,1], [-1,0,1]]^t$

$w = [[1,1], [-1,-2]]^t$

a)

$f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

$f(x+y) = f(x) + f(y)$; $f(\alpha x) = \alpha f(x)$

$x = [x_1, x_2, x_3]$; $y = [y_1, y_2, y_3]$

$\Rightarrow f(x+y) = f([x_1+y_1, x_2+y_2, x_3+y_3])$

$= [x_2+y_2, -x_1-y_1]$ ①

$f(x) + f(y) = [x_2, -x_1] + [y_2, -y_1]$

$= [x_2+y_2, -x_1-y_1]$ ②

① ② $\Rightarrow f(x+y) = f(x) + f(y)$ ①

$f(\alpha x) = f(\alpha \cdot [x_1, x_2, x_3]) = f([\alpha x_1, \alpha x_2, \alpha x_3])$

$= [\alpha x_2, -\alpha x_1]$ ①

$\alpha \cdot f(x) = \alpha \cdot [x_2, -x_1] = [\alpha x_2, -\alpha x_1]$ ②

① ② $\Rightarrow f(\alpha x) = \alpha f(x)$ ②

① ② $\Rightarrow f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

b) $v = [[1,1,0],[0,1,1],[1,0,1]]^t$

$\Rightarrow [v] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow \det = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{matrix} = 1+1 = 2 \neq 0$

$\Rightarrow$ este lin. ind $\Rightarrow$ este bază în $\mathbb{R}^3$

$w = [[1,1],[1,-2]]^t$

$[w] = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow \det = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2-1 = -3 \neq 0$

$\Rightarrow$ este lin. ind $\Rightarrow$ este bază în $\mathbb{R}^2$

$[f]_{v,e} = f[v] = f[[1,1,0],[0,1,1],[1,0,1]]^t$

$= [[1,-1],[1,0],[0,-1]]$

$[f]_{v,w} = [[1,-1],[1,0],[0,-1]]_{e\ w}$ → trecem în bază w

$[fv]_w = [fv]_e \cdot [w]_e^{-1}$

$[w_e] = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{cases} a+c=1 \\ b+d=0 \\ a-2c=0 \\ b-2d=1 \end{cases} \Rightarrow [w]_e^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

$\Rightarrow [fv]_w = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

c) $f(x) = 0 \Rightarrow f([x_1, x_2 x_3]) = [x_2, -x_1] = [0,0]$

$\Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = 0 \end{matrix} \Rightarrow < [0,0,1] > = \ker f$

↳ este o bază

rank $\ker f = 1 \Rightarrow$ rank $\operatorname{Im} f = 2$

$f(<x>) = < f(x) > \rightarrow x \subseteq \mathbb{R}^3$

$\Rightarrow f(\mathbb{R}^3) = f(<[[1,0,0],[0,1,0],[0,0,1]]>) =$

$= < f([l_1, l_2, l_3]) > = <[f(l_1), f(l_2), f(l_3)]>$

$= <[[1,-1], [1,0], [0,-1]]>$

$v_1 \quad v_2 \quad v_3$

$\dim \operatorname{Im} f = \dim \mathbb{R}^3 - \dim \ker f \Rightarrow \dim \operatorname{Im} f = 2$

$\Rightarrow$ trebuie să eliminăm un vector

$v_1 = v_2 + v_3 \Rightarrow$ îl eliminăm

$\Rightarrow \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow [v_2, v_3]$ este lin. indep

$\Rightarrow$ este o bază pt $\operatorname{Im} f$