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Chaos, randomness and multi-fractality in Bitcoin market

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ABSTRACT

Since its inception, the digital currency market is considerably growing, especially in the most recent years. The main purpose of this paper is to investigate, assess and detect chaos, randomness, and multi-scale temporal correlation structure in prices and returns of this specific virtual and speculative market throughout two distinct time periods; namely under a low-level regime period during which prices slowly increased, and during a high and turbulent regime time period whereby they exponentially increased. We found that chaos is only present in prices during both periods, whilst the level of uncertainty in returns has significantly increased during the high-price time period. Furthermore, both prices and returns exhibit long-range correlations and multi-fractality. The fat-tailed probability distributions are the main source of multi-fractality in the time series of prices and returns. Finally, short (long) fluctuations in returns are dominant during low (high) price-regime time period, respectively. Overall, the high-price regime phase has profoundly revealed consistent nonlinear dynamical patterns in the Bitcoin market.

1. Introduction

In econophysics, chaos, randomness and fractality are widely used alongside nonlinear statistics, to assess efficiency the stylized facts of time series. For instance, chaos theory has been applied to monetary [1], power [2], labor [3], international exchange rate [4–6], crude oil [7], stock exchange [8–10] and world major equity, currency and commodity markets [11,12]. Additionally, randomness was investigated in world major equity, currency and commodity markets [11] and their volatilities [12], in stock exchange [13], and currency markets [14], crude oil [15] and sovereign markets [16]. Lastly, fractality was investigated in stock [17–20], gold [21], electricity [22,23], crude oil [24] and shipping markets [25], yet at a smaller extent.

Although the aforementioned econophysics literature has explored the nonlinear statistical properties in numerous diverse equity and commodity markets [1–25], such rigorous investigations in Bitcoin market are missing to the best of our knowledge. Indeed, only a limited number of works have been conducted on Bitcoin market. For instance, long range dependence in Bitcoin market was examined by means of detrended fluctuation analysis (DFA) in [26] and by means of rescaled Hurst exponent (R/S) in [27]. It was found that DFA-based Hurst exponent changes significantly during the first years of existence of Bitcoin and tends to stabilize in recent times [26]. Besides, the R/S Hurst statistic indicates strong anti-persistence in returns of Bitcoin market [27].

As the market of digital (virtual) currencies was introduced in 2008, Bitcoin market is characterized by highly speculative features and by two distinct time periods as shown in Fig. 1 presented in Section 2. Indeed, during the first time period (18 July 2010 to 26 February 2013), Bitcoin prices followed a flat movement compared to the second time period (27 February 2013 to 23 October 2017), during which they followed a sharp upward trend. Therefore, the purpose of the current study is to examine the inherent nonlinear statistical properties of this crypto-currency market.

Indeed, our study makes three contributions to the econophysics literature: the first one stems from our methodological approach that examines chaos, randomness, and multi-fractality in prices and in returns of Bitcoin. Without a doubt, by examining these three issues, we can shed light on the complexity structure underlying Bitcoin market. Our second contribution relates to providing some stylized facts about the price and returns dynamics of Bitcoin disjointedly during time periods of low and high regime movements to achieve better understanding of its nonlinear dynamics. Finally, the third contribution is about examining not only multi-fractality but also its sources. Certainly, both long-range correlations and fat-tail distributions make important contributions to multi-fractality in a given time series. Those will be thoroughly investigated in our work.

Hence our study provides the first attempt to assess the existence of chaos, randomness, and multi-fractal statistics during low and high regime time periods in prices and returns in the Bitcoin

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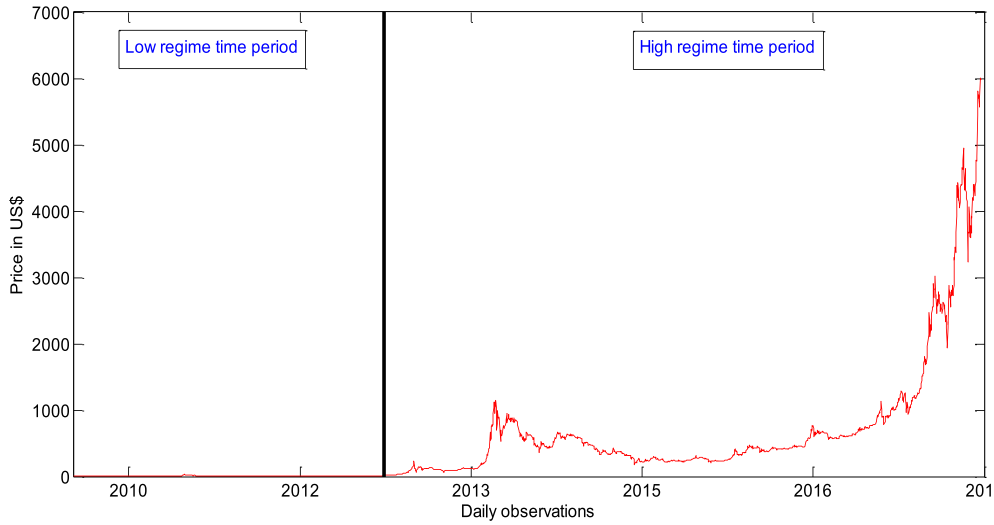

Fig. 1. Bitcoin price time series from 18 July 2010 to 23 October 2017. The low regime time period extends from 18 July 2010 to 26 February 2013. The high regime price period spans 27 February 2013 to 23 October 2017. Low and high regime periods are respectively characterized by low and high levels in prices.

market. To achieve this, the largest Lyapunov exponent (LLE) [28], Shannon entropy (*SE*) [29], and the multi-fractal detrended fluctuation analysis (MF-DFA) [30] are respectively employed to reveal chaos, randomness, and multi-fractality. Indeed, *LLE* is appropriate to quantify chaos in nonlinear dynamic systems, *SE* is suitable to capture randomness in a given time series, and *MF-DFA* is a valuable technique for the multi-fractal characterization of non-stationary signals.

The remainder of the paper is organized as follows: Section 2 describes the approaches utilized namely, largest Lyapunov exponent, Shannon entropy, and the multi-fractal detrended fluctuation analysis. Section 3 describes our dataset and presents the empirical results. Finally, Section 4 concludes.

2. Methodology

2.1. Largest Lyapunov exponent

The Lyapunov exponent determines whether a given dynamical system has divergent or convergent trajectories. For example, consider a noisy chaotic system represented by time series $\{x_t\}_{t=1}^{T}$ where:

$$x_t = f(x_{t-L}, x_{t-2L}, \ldots, x_{t-mL}) + \varepsilon_t \tag{1}$$

Herein, L is the time delay, m is the embedding dimension, ε noise term, f is an unknown function used to approximate a chaotic map, and t is time script. Then, the Lyapunov exponent λ of noisy chaotic system is written as [5]:

$$\lambda = \lim_{M\to\infty} \frac{1}{2M} \log(\upsilon_1) \tag{2}$$

where υ_1 is the largest eigenvalue of the matrix $T_M' T_M$ and T_M is given by [5]:

$$T_M = \prod_{t=1}^{M-1} J_{M-1} \tag{3}$$

where $M \leq T$ is the block-length of equally spaced evaluation points, and J is the Jacobian matrix of the chaotic map f. The Jacobian matrix J at a starting point x_0 is expressed as follows:

$$J^t(x_0) = \frac{d f^t(x)}{dx}\Big|_{x_0} \tag{4}$$

A multilayer feed-forward neural network trained with gradient descent algorithm [5] is able to approximate the chaotic map f as follows:

$$x_t \approx \alpha_0 + \sum_{j=1}^{q} \alpha_j A\left(\beta_{0,j} + \sum_{i=1}^{m} \beta_{i,j} x_{t-iL}\right) + \varepsilon_t \tag{5}$$

where q is the number of hidden layers, α_j are the layers connection weights, α_0 is the network bias, and A is a sigmoid function that processes data. The triplet (L,m,q) is set at high values; then varied so as the largest Lyapunov exponent is obtained [5]. In this regard, $\lambda \geq 0$ indicates that time series possess chaotic dynamics. In contrary, $\lambda < 0$ designates convergence between close trajectories which means existence of classic attractors.

2.2. Shannon entropy

In our study, Shannon entropy [29] denoted as *SE*, is used to gauge the degree of randomness in price and returns time series of the Bitcoin market. For instance, consider a time series $\{x_t\}_{t=1}^{n}$. Then, the Shannon entropy is given by:

$$SE(x) = -\sum_{y=1}^{n} p_i \log(p_i) \tag{6}$$

where p_i is a discrete probability such that $\sum_i p_i = 1$. For example, Shannon entropy (*SE*) reaches its maximum if all values of the underlying time series $\{x_t\}_{t=1}^{n}$ are equally probable. Hence, when *SE* approaches $\log(n)$, the time series is nearly random. Oppositely, *SE* reaches a minimum if a single x_i is assured to happen, i.e., with $Prob(x_i) = 1$.

2.3. Multi-Fractal detrended fluctuation analysis (MF-DFA)

As an extension to the original detrended fluctuation analysis (DFA) [31], the multi-fractal detrended fluctuation analysis (MF-DFA) [30] estimates the Hurst exponent of a time series at different scales. Let $\{x_k: k = 1,2,\ldots,N\}$ be a time series of length N. A brief description of MF-DFA follows. Firstly, the profile Y_i ($i = 1,2,\ldots,N$) is determined as $Y_i = \sum_{k=1}^{i} (x_k - \bar{x})$, where $\bar{x}$ is the average of the time series x_k. Secondly, the obtained profile is divided into $N_s = \mathrm{int}(\frac{N}{s})$ non-overlapping segments (windows) of equal length s. Thirdly, a least square fit is used to estimate the polynomial local trend for each of the $2N_S$ segments. As a result, the variance $F^2(s,v)$ is calculated by subtracting the local trend of each sub-interval v (for

$v = 1,2,\ldots,N_s$). Finally, the qth order fluctuation function $F_q(S)$ is calculated by averaging over all segments. It is expressed as follows:

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{v=1}^{2N_s} \left[F^2(s,v)\right]^{\frac{q}{2}} \right\}^{\frac{1}{q}}, \; q \neq 0 \tag{7}$$

$$F_q(s) = \exp\left\{ \frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln\left[F^2(s,v)\right] \right\}, \; q = 0 \tag{8}$$

Accordingly, a negative parameter q yields to an enhancement of small fluctuations, whilst a positive one yields to an enhancement of large fluctuations. In particular, the scaling behaviour $F_q(s)$ increases for large values of s when the series x_k are long range power law correlated. In this regard, the power law is expressed as follows:

$$F_q(s) \sim s^{h(q)} \tag{9}$$

where $h(q)$ represents the generalized Hurst exponent of MF-DFA for a given scale q. Usually, the generalized Hurst exponent $h(q)$ can be estimated by running the following linear regression:

$$\log(F_q(s)) = \log(A) + h(q)\log(s) \tag{10}$$

As a result, if $h(q)$ is independent of q the original series, x_k is mono-fractal and characterized by a single exponent over all time scales. On the contrary, if $h(q)$ is dependent on q, the original series x_k is multi-fractal as $h(q)$ varies with scale q.

Aside from $h(q)$, the multi-fractal spectrum distribution of the underlying time series can as well be described by the standard multi-fractal mass function $\tau(q)$ or by the singularity spectrum $f(\alpha)$. Indeed, the multi-fractal mass function $\tau(q)$ is given by:

$$\tau(q) = qh(q) - 1 \tag{11}$$

Afterwards, the singularity spectrum $f(\alpha)$ is obtained by applying the Legendre transform:

$$f(\alpha) = q\alpha - \tau(q) \tag{12}$$

with,

$$\alpha = \frac{d\tau}{dq} \tag{13}$$

where α is the is the singularity exponent (Hölder exponent) and $f(\alpha)$ represents the dimension of the subset of the series distinguished by α [32]. The range of exponents present in the underlying time series, is represented by the width of the spectrum such as the bigger is the width of α ($\Delta\alpha = \alpha_{max} - \alpha_{min}$) the more are violent data fluctuations. Clearly speaking, broad spectrum indicates high degree of multi-fractality.

Finally, to further examine any underlying nonlinear dynamics in prices and returns of the Bitcoin market, two other sources of multi-fractality are investigated. These two common sources of multi-fractality explicitly include a) the long-range temporal correlations in short and long fluctuations, and b) the fat-tailed probability distributions. In this regard, the fractal spectrum widths ($\Delta\alpha$) of shuffled and surrogate data are calculated. It is worth noting that, randomly shuffling original data allows destroying temporal correlations while preserving the distribution of events. Besides, constructing surrogate data by phase randomization of the original series, allows eliminating nonlinearities while preserving the original power spectrum as well as changing the distribution. In this study, we adopt the surrogate approach based on the popular amplitude adjusted Fourier transform as in [33], wherein the parameter q ranges between −20 and 20. Also, asymmetry in singularity spectrum $f(\alpha)$ is investigated. For instance, the widths of the left part ($\Delta\alpha(q>0)$) and the right part ($\Delta\alpha(q \langle\ 0)$) of $f(\alpha)$ spectrum are computed. Recall that $\Delta\alpha(q\ \rangle\ 0) = \alpha_0 - \alpha_{min}$ and $\Delta\alpha(q<0) = \alpha_{max} - \alpha_0$, where α_0 corresponds to the maximum of $f(\alpha)$ for the original series.

Table 1
Estimated LLE and SE statistics.

	Low regime time period	High regime time period
Prices		
LLE (λ)	0.7159	0.4037
SE	3.7039	3.4517
Returns		
LLE (λ)	-0.3571	−0.3343
SE	2.8504	3.1687

LLE: largest Lyapunov exponent. *SE*: Shannon entropy

3. Data and results

Bitcoin daily price time series are obtained from [34] in US dollars, as Bitcoin in such currency is the most traded one. The data sample ranges from 18 July 2010 to 23 October 2017, yielding to 2655 observations. Fig. 1 exhibits Bitcoin price time series from 18 July 2010 to 23 October 2017. Accordingly, two distinct regime time periods are identified. The first one is the *low regime* time period during which, price levels are significantly low. It spans from 18 July 2010 to 26 February 2013. The second one is the *high regime* period during which price levels are remarkably high, and spans from 27 February 2013 to 23 October 2017. Although, during the low regime time period, the Bitcoin price levels are low, one can observe that the price relatively evolves through time following an upward trend as shown in Fig. 2 wherein the low regime time period is solely exhibited.

Extensive nonlinear analysis is applied to both price and return time series. In this regard, return series are computed as the first difference of the logarithmic prices. For instance, if p_t is the price at time t, then the daily price return is $r_t = \log(p_t) - \log(p_{t-1})$. The estimated nonlinear statistics, namely the largest Lyapunov exponent (LLE) and Shannon entropy (SE) are all presented in Table 1, both for the low and high-price regime periods. As shown, the LLE is positive for prices during both low and high regime, whilst it is negative for return series during both time periods. Thus, the prices of Bitcoin exhibit chaotic dynamics during both time periods. Instead, the returns of Bitcoin are not chaotic during both periods. Moreover, according to Table 1, the level of uncertainty captured by *SE* is high for both prices and returns as the corresponding *SE* values are far away from zero during both low and high regime time periods. In particular, it is interesting to observe that uncertainty in returns increased during the high-price regime sample, in other words, less information was carried over during this sub-period. This issue can be further rationalized by the highly speculative behaviour of the Bitcoin market.

Figs. 3 and 4 display the log-log plot of fluctuation $F_q(s)$ versus s for negative and positive values of q, respectively for prices and returns throughout both regimes. For all series, the scaling behaviour of the $F_q(s)$ converges with an increasing pattern to s in the low as well as during the high regime time period. Therefore, the auto-correlated behaviour vis-à-vis both price and return time series, are scale dependent.

Next, Fig. 5 exhibits the generalized Hurst exponent $h(q)$ as a function of order q for prices and returns throughout low- and high-price time periods. As shown, for both prices and returns, $h(q)$ is high for negative values of q and is small for positive values of q. In other words, $h(q)$ decreases with order q. Therefore, short fluctuations (where $q<0$) exhibit high scaling exponents, and large fluctuations (where $q>0$) exhibit low scaling exponents in both series all through low and high regime periods. Hence, prices and returns of Bitcoin demonstrate a strong degree of multi-fractality during both phases. Particularly, the level of $h(q)$ is largely more well-pronounced in prices than in return series throughout both periods. In particular, the behaviour of the multi-fractal mass

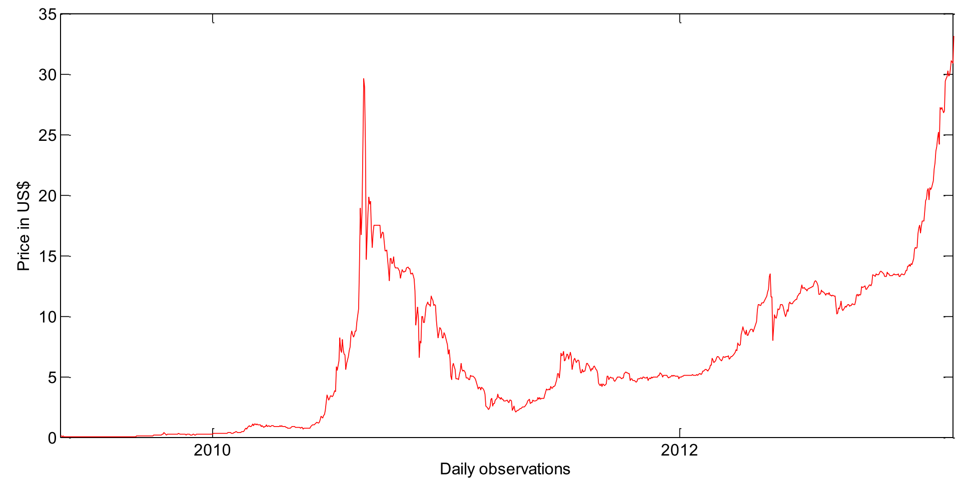

Fig. 2. Evolution of the price level during the low regime time period, i.e., 18 July 2010–26 February 2013.

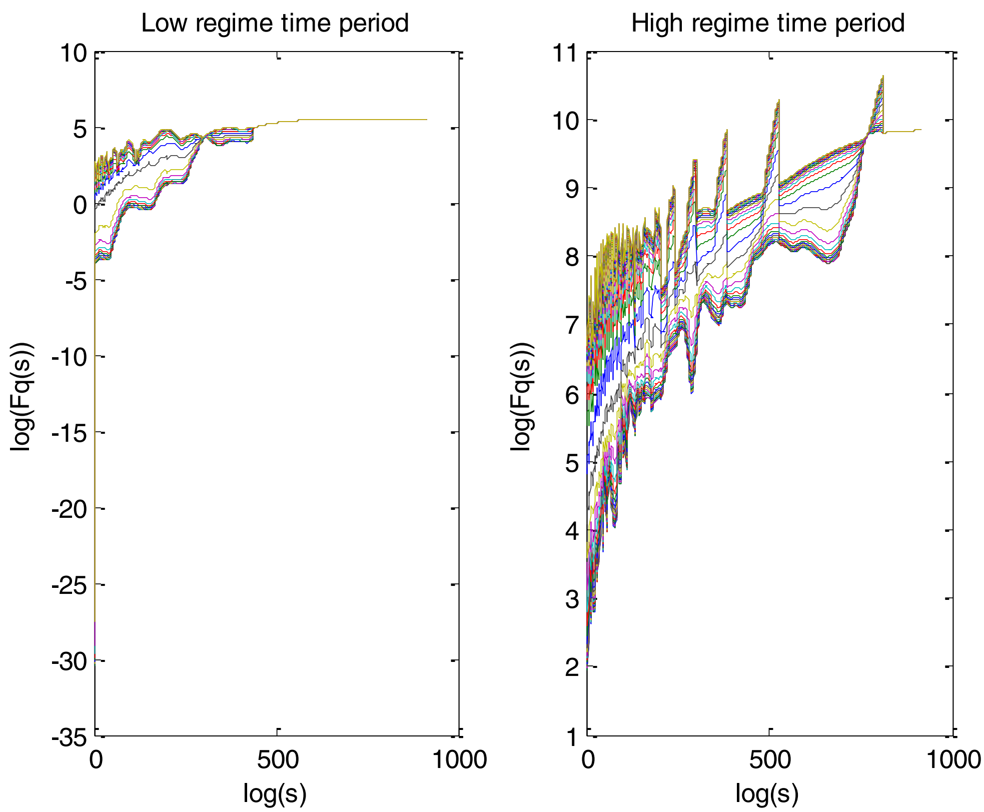

Fig. 3. Log-log plot of $F_q(s)$ versus s ($-20 \le q \le 20$) during low and high regime time periods: Price time series.

function $\tau(q)$ as a function of q is plotted in Fig. 6. As shown, there is strong evidence of the presence of multi-fractal patterns in prices and returns of the Bitcoin market during both time periods, as the curves of $\tau(q)$ seem to be straight lines when scale q is negative and also when it is positive. Importantly, for negative scales multi-fractality is stronger in the returns than in price series. On the contrary, for positive scales multi-fractality is stronger in prices than in returns. In other words, complexity in short fluctuations of returns is higher than that of prices, and complexity in long fluctuations of prices is higher than that of returns!

Finally, Table 2 presents the singularity spectra parameters obtained by MF-DFA for both low and high regime periods. The spread ($\Delta\alpha$) of singularity spectrum of prices is larger during the low regime. Similarly, the spread of singularity spectrum of returns is larger during the low regime time period than in the high one. Therefore, prices and returns exhibit stronger multi-fractality during low than high regime periods. Besides, it is evident that for both periods, the shuffled price time series demonstrate a lower $\Delta\alpha$ than the original price series. This surely indicates that serial correlation significantly influences the strength of the multi-fractal spectrum in prices. Similarly, serial correlation significantly influences the strength of the multi-fractal spectrum in returns as shuffled return time series present a lower $\Delta\alpha$ than the original ones. Above and beyond, surrogate time series of returns have higher $\Delta\alpha$ than the original return series during both time periods. This finding indicates that the presence of fat-tails in the original return series yields to narrow spectrum width. Also, it is found that the $\Delta\alpha$ of the surrogate prices is lower (higher) than that of the

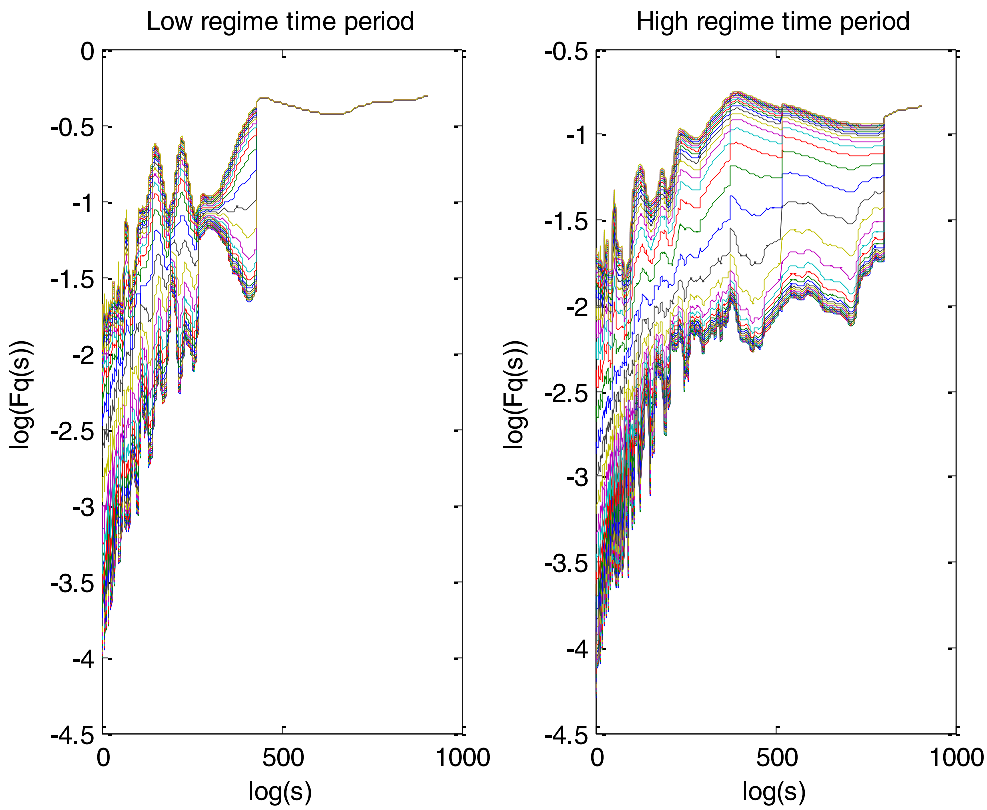

Fig. 4. Log-log plot of $F_q(s)$ versus s $(-20 \le q \le 20)$ during low and high regime time periods: Returns time series.

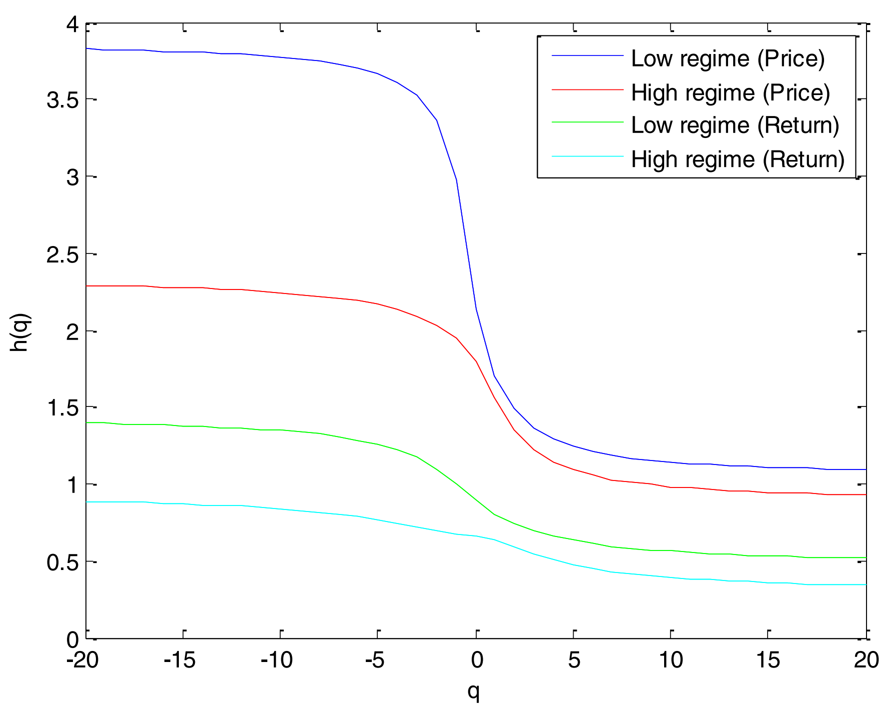

Fig. 5. Plot of generalized Hurst exponent $h(q)$ as a function of q during low and high regime time periods for price and return tie series.

original prices during the low (high) regime respectively. Hence, the presence of fat-tails in the original price data yields to narrow spectrum width of prices during the high-level regime.

Additionally, the asymmetry in spectrum $f(\alpha)$ is examined. For the original price time series, it was found that $\Delta\alpha(q>0)$ is lower than $\Delta\alpha(q<0)$ during both low and high regime time periods. In other words, the spectrum $f(\alpha)$ of the original prices is left-skewed. Therefore, short fluctuations in prices are dominant throughout both periods. Also for the original return series, it was found that during the low regime period $\Delta\alpha(q>0)$ is lower than $\Delta\alpha(q<0)$. This points towards the evidence of dominance of short fluctuations in returns during the low regime time sample. On

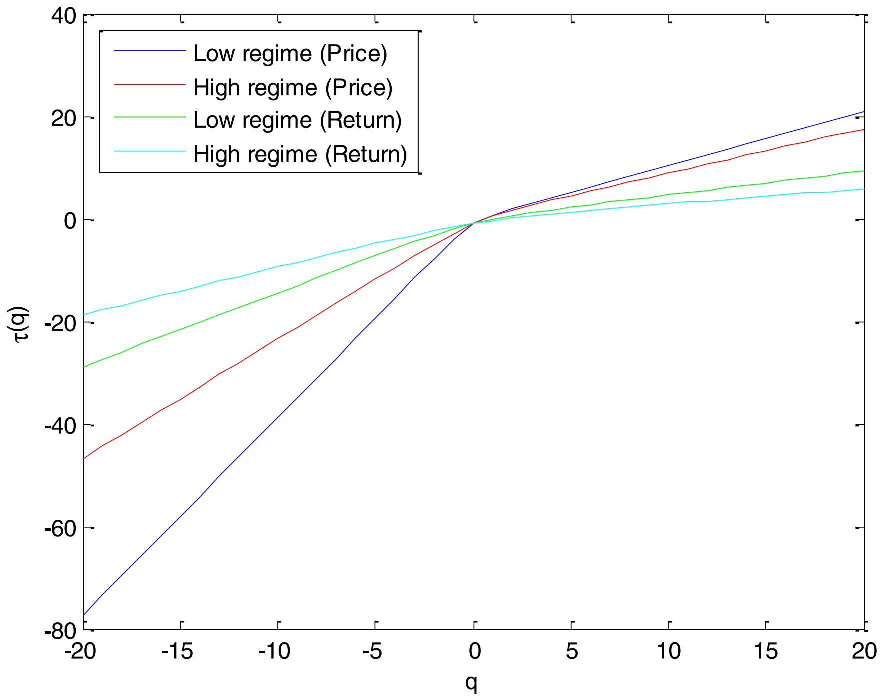

Fig. 6. Mass spectrum $\tau(q)$ as a function of q during low and high regime time periods for price and return series.

Table 2
Estimated parameters of singularity spectra.

	Low regime time period			High regime time period		
	$\Delta\alpha$	$\Delta\alpha$ $(q<0)$	$\Delta\alpha$ $(q>0)$	$\Delta\alpha$	$\Delta\alpha$ $(q<0)$	$\Delta\alpha$ $(q>0)$
Prices						
Original data	2.8435	2.1824	0.6611	1.4704	0.7797	0.6907
Shuffled data	0.2816	0.1506	0.1309	0.4208	0.2365	0.1843
Surrogate data	1.4447	1.0517	0.3930	1.7075	1.088	0.6195
Returns						
Original data	0.9810	0.6399	0.3411	0.6481	0.3028	0.3452
Shuffled data	0.4464	0.2340	0.2124	0.3786	0.2080	0.1706
Surrogate data	1.4587	1.0894	0.3692	1.5902	1.2115	0.3786

the contrary, during the high regime period $\Delta\alpha(q>0)$ is larger than $\Delta\alpha(q<0)$ which indicates that the associated spectrum $f(\alpha)$ is rightly-skewed. Consequently, long fluctuations in returns are dominant during the high regime period. Overall, for both prices and returns and throughout both time periods, $\Delta\alpha(q>0)$ is lower than $\Delta\alpha(q<0)$ for shuffled and surrogate data.

To sum-up, we examined chaos, randomness, and multi-fractal properties for the Bitcoin market throughout two different time periods: low and high regime time periods. The first one is mostly characterized by moderate increase in prices, whilst the second by an exponential increase in prices. Our findings include the following important information: firstly, Bitcoin prices reveal chaotic dynamics through both low and high regime time periods. Instead, their returns are not chaotic during those particular periods. Secondly, the level of uncertainty in prices and returns is high during both periods. In addition, uncertainty in returns increased during the high regime. The aforementioned results could be explained by the significant increase in speculative trading in Bitcoin market. Thirdly, the behaviour of $F_q(s)$ indicates that auto-correlated dynamics in both prices and returns are scale-dependent. In other words, prices and returns exhibit multi-fractality. Indeed, the generalized Hurst exponent and the characteristics of the mass function both confirm the presence of multi-fractality in returns and prices. In particular, we showed a strong evidence that for negative scales multi-fractality is stronger in return series than in price series, whilst for positive scales, multi-fractality is stronger in prices than in returns. Fourthly, fat-tails is the main source of multi-fractality in prices and returns during both low and high regime periods. Fifthly, there is strong evidence towards the dominance of short fluctuations in returns during the low-price (regime), as opposed to long fluctuations in returns which appear dominant during the high regime sample. We indicated that the high regime period has revealed profound shaped and consistent nonlinear patterns for the Bitcoin market.

4. Conclusion

We studied chaos, randomness and multi-fractal stylized properties of price and returns in Bitcoin market, via the utilization and estimation of the largest Lyapunov exponent, Shannon entropy, and the generalized Hurst exponent, also by derivation of singularity spectrums during low and high regime (price) time periods. Our empirical results showed that, as opposed to returns, prices incorporate and exhibit chaotic dynamics. Additionally, uncertainty level in returns significantly increased during the high-price regime period. Furthermore, there is strong evidence of multi-fractality in prices and returns throughout both investigated periods. It was shown that multi-fractality in prices and returns is mainly due to fat-tailed distributions. Finally, short (long) fluctuations in returns are dominant during the low (high) regime time period respectively. Overall, we concluded that the high-price level regime period has strongly revealed nonlinear dynamical patterns in the Bitcoin market.

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