CSC520 - Artificial Intelligence

Lecture 13

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Feb 20, 2025

Agenda

- First Order Logic (FOL)
- Quantifiers
- Reduction to propositional inference
- Forward chaining
- Proof by resolution

First-order Logic

- Similar to a natural language, FOL assumes that world contains objects, relations, and functions
- Objects
  - E.g., people, houses, numbers, Ronald McDonald, colors, ...
- Relations
  - Unary relations, properties, e.g., red, round, bogus, ...
  - n-ary relations, e.g., brother of, bigger than, has color, ...
- Functions
  - Relations with one value for a given input
  - E.g., father of, best friend, one more than, ...

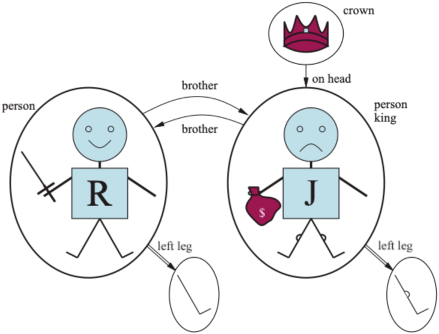
FOL Syntax in Backus-Naur Form (BNF)

$$
\begin{array}{rcl}
\textit{Sentence} & \rightarrow & \textit{AtomicSentence} \mid \textit{ComplexSentence} \\
\textit{AtomicSentence} & \rightarrow & \textit{Predicate} \mid \textit{Predicate}(\textit{Term}, \ldots) \mid \textit{Term} = \textit{Term} \\
\textit{ComplexSentence} & \rightarrow & (\ \textit{Sentence}\ ) \\
& \mid & \neg\ \textit{Sentence} \\
& \mid & \textit{Sentence} \land \textit{Sentence} \\
& \mid & \textit{Sentence} \lor \textit{Sentence} \\
& \mid & \textit{Sentence} \ \Rightarrow\ \textit{Sentence} \\
& \mid & \textit{Sentence} \ \Leftrightarrow\ \textit{Sentence} \\
& \mid & \textit{Quantifier}\ \textit{Variable}, \ldots\ \textit{Sentence} \\
\\
\textit{Term} & \rightarrow & \textit{Function}(\textit{Term}, \ldots) \\
& \mid & \textit{Constant} \\
& \mid & \textit{Variable} \\
\\
\textit{Quantifier} & \rightarrow & \forall \mid \exists \\
\textit{Constant} & \rightarrow & A \mid X_1 \mid \textit{John} \mid \cdots \\
\textit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\textit{Predicate} & \rightarrow & \textit{True} \mid \textit{False} \mid \textit{After} \mid \textit{Loves} \mid \textit{Raining} \mid \cdots \\
\textit{Function} & \rightarrow & \textit{Mother} \mid \textit{LeftLeg} \mid \cdots \\
\textsc{Operator Precedence} & : & \neg, =, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

Truth in FOL

- Sentences are true with respect to a *model*
  - A model contains a set of objects, relations and functions
  - Additionally model contains an *interpretation*
- Interpretation specifies the referents for the symbols
  - Constant symbols to objects
  - Predicate symbols to relations
  - Function symbols to functional relations
- Atomic sentence $predicate(term_1, \ldots, term_n)$ is *true* iff the objects referred by $term_1, \ldots, term_n$ are in the *relation* referred by the *predicate*

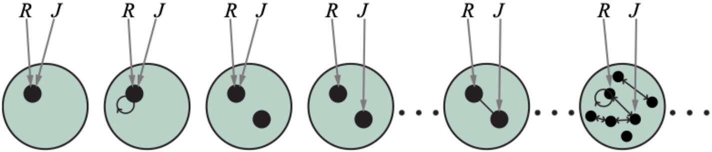
Interpretation Example for FOL



Consider the below interpretation

- Object symbols
  Richard $\rightarrow$ Richard the Lionheart
  John $\rightarrow$ the evil King John
- Predicate symbols
  Brother $\rightarrow$ the brotherhood relation
- Atomic sentence
  Is $Brother(Richard, John)$ true?
- Complex sentence
  Are these sentences true?
  $\neg Brother(LeftLeg(Richard), John)$
  $\neg King(Richard) \Rightarrow King(John)$

Number of Models in FOL

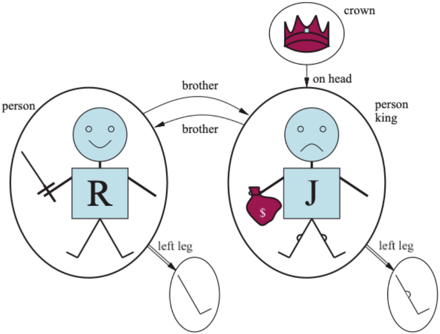


- Number of FOL models is unbounded since it can have infinite number of objects, relations, and functions
- Not possible to prove entailment by model enumeration
- Even if the number of objects, relations, and functions is restricted, the number of combinations can get very large making model enumeration intractable

Universal Quantification

- Universal quantifier $\forall$; read as, "For all . . ."
- Syntax: $\forall$ (*variables*)(*sentence*)
  - For e.g., $\forall x\ P(x)$
- Semantics: $\forall x\ P(x)$ is *true* in a model iff $P(x)$ is true for every object x in the model

Universal Quantification

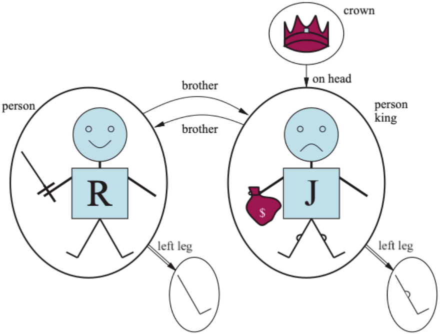


- Is $\forall x\ King(x)$ true?
- Is $\forall x\ King(x) \Rightarrow Person(x)$ true?
- Richard the Lionheart is a king $\Rightarrow$ Richard the Lionheart is a person
- King John is a king $\Rightarrow$ King John is a person
- Richard's left leg is a king $\Rightarrow$ Richard's left leg is a person
- John's left leg is a king $\Rightarrow$ John's left leg is a person
- The crown is a king $\Rightarrow$ The crown is a person

Existential Quantification

- Existential quantifier $\exists$; read as, "There exists ..."
- Syntax: $\exists$ (*variables*)(*sentence*)
  - For e.g., $\exists x\ P(x)$
- Semantics: $\exists x\ P(x)$ is *true* in a model iff $P(x)$ is true for at least one object x in the model

Existential Quantification



- Is $\exists x\ King(x)$ true?
- Is $\exists x\ Crown(x) \wedge OnHead(x, John)$ true?

Common Mistake with Quantification

- Which is an incorrect formulation of "All kings are persons" in FOL?
  - $\forall x\ King(x) \wedge Person(x)$
  - $\forall x\ King(x) \Rightarrow Person(x)$
- Typically $\Rightarrow$ is the connective used with $\forall$

- Which is an incorrect formulation of "King John has a crown on his head" in FOL?
  - $\exists x\ Crown(x) \wedge OnHead(x, John)$
  - $\exists x\ Crown(x) \Rightarrow OnHead(x, John)$
- Typically $\wedge$ is the connective used with $\exists$

Properties of Nested Quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$
- But $\exists x\ \forall y$ is NOT the same as $\forall x\ \exists y$
  - "Everybody loves somebody": $\forall x\ \exists y\ Loves(x, y)$
  - "There is someone who is loved by everyone": $\exists y\ \forall x\ Loves(x, y)$
- $\forall$ and $\exists$ are related to each other
  - Everyone likes ice cream $\equiv$ There is no one who dislikes ice cream
    $\forall x\ Likes(x, IceCream) \equiv \neg\exists x\ \neg Likes(x, IceCream)$
  - Someone likes broccoli $\equiv$ Everyone does not dislike broccoli
    $\exists x\ Likes(x, Broccoli) \equiv \neg\forall x\ \neg Likes(x, Broccoli)$
- De Morgan's rules for quantified sentences
  - $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$
  - $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$
  - $\forall x\ P(x) \equiv \neg\exists x\ \neg P(x)$
  - $\exists x\ P(x) \equiv \neg\forall x\ \neg P(x)$

Equality

- $term_1 = term_2$ is *true* under a given interpretation iff $term_1$ and $term_2$ refer to the same object
- E.g., $Father(John) = Henry$ is true if the object referred by $Father(John)$ is the same as the object referred by $Henry$ in a given interpretation
- E.g., Suppose we want to say that Richard has two brothers
  - Is this correct: $\exists x\ \exists y\ Brother(x, Richard) \wedge Brother(y, Richard)$?
  - We need to say: $\exists x\ \exists y\ Brother(x, Richard) \wedge Brother(y, Richard) \wedge \neg(x = y)$

FOL Sentence Examples

- Sentences from Wumpus world
  - Squares adjacent to the wumpus are smelly
    $\forall x\ Wumpus(x) \Leftrightarrow \exists y\ Smelly(y) \land Adjacent(x, y)$
- Sentences from the family relationship domain
  - Predicates: $Parent, Sibling, Brother, Sister, Child, \ldots$
  - Functions: $Mother, Father$
  - *Brothers are siblings*
  - *Siblinghood is symmetric*
  - *One's mother is one's female parent*
  - *A first cousin is a child of a parent's sibling*

Universal Instantiation

- Every instantiation of a universally quantified sentence is entailed by it
- $\dfrac{\forall v \; \alpha}{Subst(\{v/g\}, \alpha)}$, for any variable v and ground term g
- E.g., $\forall x \; King(x) \land Greedy(x) \Rightarrow Evil(x)$,
  John and *Richard* are the objects and *Father* is a function
  - $King(John) \land Greedy(John) \Rightarrow Evil(John)$
  - $King(Richard) \land Greedy(Richard) \Rightarrow Evil(Richard)$
  - $King(Father(John)) \land Greedy(Father(John)) \Rightarrow Evil(Father(John))$
  - $King(Father(Father(John))) \land Greedy(Father(Father(John))) \Rightarrow Evil(Father(Father(John)))$
  - $\ldots$

Existential Instantiation

- Replace an existentially quantified variable with a single *new* constant symbol called as a Skolem constant

- $\dfrac{\exists v\ \alpha}{Subst(\{v/k\}, \alpha)}$, k is a Skolem constant

- E.g., $\exists x\ Crown(x) \wedge OnHead(x, John)$. Then, we can infer:
  - $Crown(C_1) \wedge OnHead(C_1, John)$

Reduction to Propositional Inference

- Apply universal and existential instantiation to all sentences in the KB
- Replace ground atomic sentences with proposition symbols
- Apply propositional inference algorithms to prove entailment

Original KB	After universal instantiation
$\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$	$King(John) \wedge Greedy(John) \Rightarrow Evil(John)$
$King(John)$	$King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$
$Greedy(John)$	$King(John)$
$Brother(Richard, John)$	$Greedy(John)$
	$Brother(Richard, John)$

- After substituting ground atomic sentences with propositions, we get:

$$JohnIsKing \wedge JohnIsGreedy \Rightarrow JohnIsEvil$$
$$RichardIsKing \wedge RichardIsGreedy \Rightarrow RichardIsEvil$$
$$\ldots$$

Reduction to Propositional Inference

- With function symbols, there are infinitely many ground terms
  - E.g., $Father(Father(John)), Father(Father(Father(John))), \ldots$
- Herbrand's Theorem: If a sentence is entailed by FOL KB, it is entailed by a finite subset of of the propositionalized KB
- To prove entailment:
  for $n = 0$ to ∞ **do**:
  Create a propositional KB by instantiating with depth-n terms
  Check if α is entailed by the KB
- But this works only if α is entailed by KB! Otherwise, it loops forever
- Turing and Church proved that entailment in FOL is *semidecidable*
  - Algorithms exist that say yes to entailed sentence
  - No algorithm exists that says no to every nonentailed sentence

Unification

- Consider the KB

$$\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$$
$$\forall y\ Greedy(y)$$
$$King(John)$$

- How can we prove $Evil(John)$?

Unification

- Unification involves finding a substitution that makes different sentences look identical
- $\text{UNIFY}(\alpha, \beta) = \theta$ where $\alpha/\theta = \beta/\theta$

α	β	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, Bill)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, Elizabeth)$	

- Standardizing apart eliminates overlap of variables

Generalized Modus Ponens

- Generalized Modus Ponens is a lifted version of Modus Ponens
- All variables are assumed to be universally quantified

$$\frac{p_1', p_2', \ldots, p_n' \qquad p_1 \wedge p_2 \ldots \wedge p_n \Rightarrow q}{q/\theta}$$

where, $p_i'/\theta = p_i/\theta$ for all i

- Example

$\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
$\forall y\ Greedy(y)$
$King(John)$

p_1' is $King(John)$, p_1 is $King(x)$
p_2' is $Greedy(y)$, p_2 is $Greedy(x)$

$\theta = \{x/John, y/John\}$

q/θ is $Evil(John)$

Proof by Resolution

- FOL resolution rule is a lifted version of the propositional resolution rule

$$\frac{l_1 \vee \ldots l_k, \qquad m_1 \vee \ldots m_n}{\textsc{subst}(\theta, l_1 \vee \ldots l_{i-1}, l_{i+1} \vee \ldots \vee l_k \vee m_1 \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n)}$$

  where $\textsc{unify}(l_i, \neg m_j) = \theta$.
- FOL literals are complementary if one unifies with the negation of the other
- E.g., $\frac{\neg Rich(x) \vee Unhappy(x) \qquad Rich(Ken)}{Unhappy(Ken)}$ where $\theta = \{x/Ken\}$
- Apply resolution steps to $CNF(KB \wedge \neg\alpha)$

Conversion to CNF

- Everyone who loves all animals is loved by someone

$$\forall x\ [\forall y\ Animal(y) \Rightarrow Loves(x, y)] \Rightarrow [\exists y\ Loves(y, x)]$$

- Eliminate implications

$$\forall x\ \neg[\forall y\ Animal(y) \Rightarrow Loves(x, y)] \lor [\exists y\ Loves(y, x)]$$
$$\forall x\ \neg[\forall y\ \neg Animal(y) \lor Loves(x, y)] \lor [\exists y\ Loves(y, x)]$$

- Move $\neg$ inwards using $\neg\exists x\ P \equiv \forall x\ \neg P$ and $\neg\forall x\ P \equiv \exists x\ \neg P$

$$\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x, y))] \lor [\exists y\ Loves(y, x)]$$
$$\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x, y)] \lor [\exists y\ Loves(y, x)]$$
$$\forall x\ [\exists y\ Animal(y) \land \neg Loves(x, y)] \lor [\exists y\ Loves(y, x)]$$

Conversion to CNF

- Standardize variables such that each quantifier uses a different one

$$\forall x\ [\exists y\ Animal(y) \land \neg Loves(x, y)] \lor [\exists y\ Loves(y, x)]$$
$$\forall x\ [\exists y\ Animal(y) \land \neg Loves(x, y)] \lor [\exists z\ Loves(z, x)]$$

- Skolemization or existential instantiation: each existential variable is replaced by a Skolem function of the enclosing universally quantified variables

$$\forall x\ [Animal(F(x)) \land \neg Loves(x, F(x))] \lor [Loves(G(x), x)]$$

- Drop universal quantifiers

$$[Animal(F(x)) \land \neg Loves(x, F(x))] \lor [Loves(G(x), x)]$$

Conversion to CNF

- Distribute $\vee$ over $\wedge$

$$[Animal(F(x)) \wedge \neg Loves(x, y)] \vee [Loves(G(x), x)]$$
$$[Animal(F(x)) \vee Loves(G(x), x)] \wedge [\neg Loves(x, y) \vee Loves(G(x), x)]$$

Conversion to CNF

$\forall x\ [\forall y\ Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y\ Loves(y,x)]$	
$\forall x\ \neg[\forall y\ Animal(y) \Rightarrow Loves(x,y)] \vee [\exists y\ Loves(y,x)]$	Eliminate implication
$\forall x\ \neg[\forall y\ \neg Animal(y) \vee Loves(x,y)] \vee [\exists y\ Loves(y,x)]$	Eliminate implication
$\forall x\ [\exists y\ \neg(\neg Animal(y) \vee Loves(x,y))] \vee [\exists y\ Loves(y,x)]$	Negate universal
$\forall x\ [\exists y\ \neg\neg Animal(y) \wedge \neg Loves(x,y)] \vee [\exists y\ Loves(y,x)]$	DeMorgan
$\forall x\ [\exists y\ Animal(y) \wedge \neg Loves(x,y)] \vee [\exists y\ Loves(y,x)]$	Double negative
$\forall x\ [\exists y\ Animal(y) \wedge \neg Loves(x,y)] \vee [\exists z\ Loves(z,x)]$	Standardize variables
$\forall x\ [Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee [Loves(G(x), x)]$	Skolem constants
$[\mathrm{Animal(F(x))}\ \wedge \neg Loves(x, F(x))] \vee [Loves(G(x), x)]$	Drop universals
$[\mathrm{Animal(F(x))}\ \vee Loves(G(x), x)] \wedge [\neg Loves(x, y) \vee Loves(G(x), x)]$	Distribute $\vee$ over $\wedge$

Previous Example in CNF

- Original KB

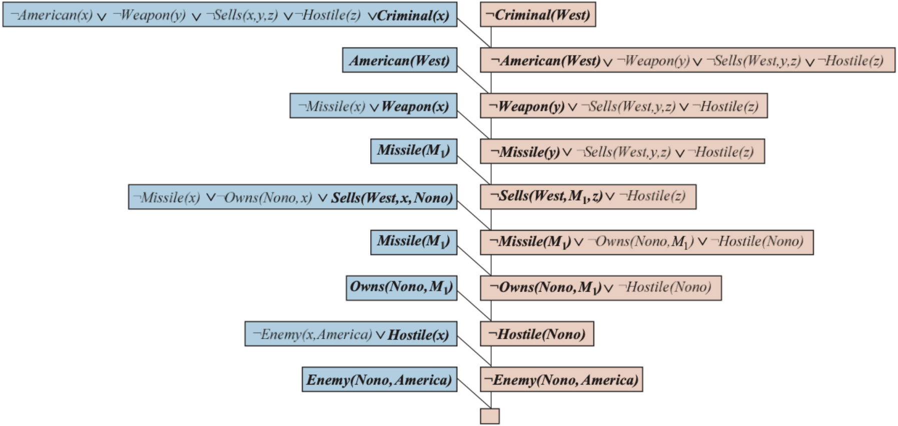
$$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$$
$$Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$$
$$Missile(x) \Rightarrow Weapon(x)$$
$$Enemy(x, America) \Rightarrow Hostile(x)$$
$$Owns(Nono, M_1) \quad Missile(M_1)$$
$$American(West) \quad Enemy(Nono, America)$$

- KB converted to CNF

$$\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z) \vee Criminal(x)$$
$$\neg Missile(x) \vee \neg Owns(Nono, x) \vee Sells(West, x, Nono)$$
$$\neg Missile(x) \vee Weapon(x)$$
$$\neg Enemy(x, America) \vee Hostile(x)$$
$$Owns(Nono, M_1) \quad Missile(M_1)$$
$$American(West) \quad Enemy(Nono, America)$$

- Negation of sentence we are trying to prove: $\neg Criminal(West)$

FOL Resolution Proof



Class Exercise

Write the following sentences in FOL.

- A grandchild is a child of a child.
- A brother is a male sibling.
- A daughter is a female child.
- One's uncle is a brother of one's father or mother.