CSC520 - Artificial Intelligence

Lecture 20

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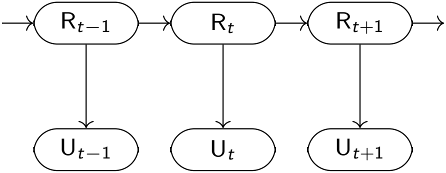
Agenda

- Hidden Markov model (HMM)
- HMM filtering
- HMM prediction
- HMM most likely explanation
- Axioms of utility theory
- Decision networks
- Value of perfect information

Hidden Markov Model

Consider this scenario

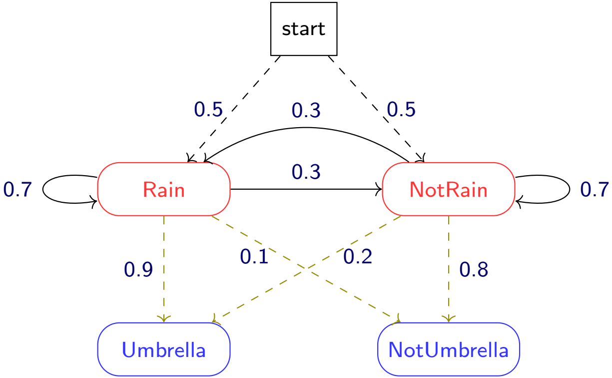
- You are a graduate student with an office that has no windows
- You never leave your office
- You meet your advisor daily who comes with or without an umbrella
- You want to know whether it is raining



R_{t-1}	$P(R_t \mid R_{t-1})$
T	0.7
F	0.3

R_t	$P(U_t \mid R_t)$
T	0.9
F	0.2

Hidden Markov Model: Example

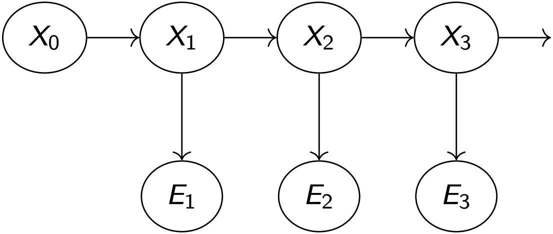


Hidden Markov Model

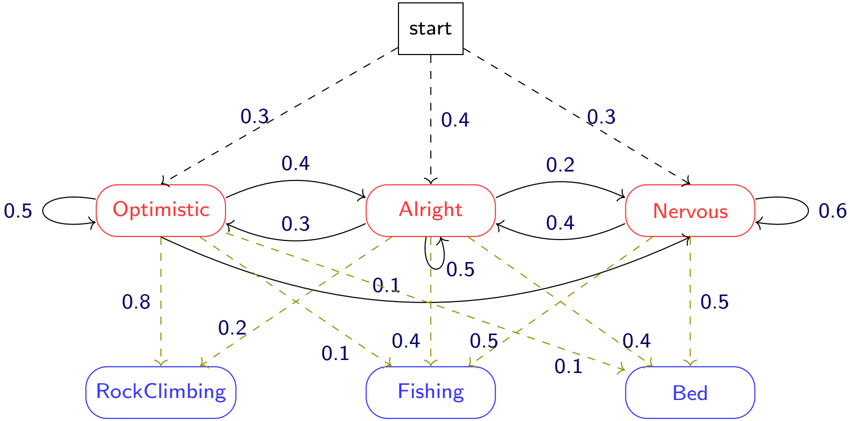
- Temporal probabilistic model
- Hidden state variables: X_t at time t
- Evidence variables: E_t at time t
- Initial distribution: $P(X_0)$
- First-order Markov assumptions
  - Transition Markov assumption
    $P(X_t|X_{0:t-1}) = P(X_t|X_{t-1})$
  - Sensor Markov assumption
    $P(E_t|X_{0:t-1}, E_{0:t-1}) = P(E_t|X_t)$
- Joint distribution

$$p(X_{0:t}, E_{1:t}) = P(X_0) \prod_{i=1}^{t} P(E_i|X_i)P(X_i|X_{i-1})$$

Hidden Markov Model Example



Hidden Markov Model: Example



Hidden Markov Model: Typical Uses

- Filtering: Estimate the current state of the hidden variable from the sequence of all evidence
- Prediction: Estimate a future state from the current evidence
- Smoothing: Improve our estimate of past states from all evidence
- Most likely explanation: Given a sequence of evidence, estimate the sequence of states that generated the evidence

Filtering

Forward Message

- Filtering: Estimate the current state of the hidden variable from the sequence of all evidence

$$
\begin{aligned}
P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}|e_{1:t}, e_{t+1}) && \text{Split evidence} \\
&= \alpha P(e_{t+1}|X_{t+1}, e_{1:t}) P(X_{t+1}|e_{1:t}) && \text{Bayes' rule} \\
&= \alpha P(e_{t+1}|X_{t+1}) P(X_{t+1}|e_{1:t}) && \text{Markov Sensors} \\
&= \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t}) && \text{Condition on } X_t \\
&= \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t}) && \text{Markov Transitions} \\
f_{1:t+1} &= \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t) f_{1:t} \\
f_{1:t+1} &= \alpha [\text{sensor model}] \sum_{x_t} [\text{transition model}][\text{recursion}]
\end{aligned}
$$

- Where $f_{1:t+1} i = P(X_i|e_{1:i}) = FORWARD(f_{1:t}, e_{t+1})$ is the *forward message*
- This is a linear computation

Filtering

Forward Message

$$
\begin{aligned}
P(X_0|e_{1:0}) =& P(X_0) \\
P(X_1|e_{1:1}) =& P(e_1|X_1) \sum_{x_1} P(X_1|x_0) P(x_0|e_{1:0}) \\
P(X_2|e_{1:2}) =& P(e_2|X_2) \sum_{x_2} P(X_2|x_1) P(x_1|e_{1:1}) \\
P(X_3|e_{1:3}) =& P(e_3|X_3) \sum_{x_3} P(X_3|x_2) P(x_2|e_{1:2}) \\
P(X_4|e_{1:4}) =& P(e_4|X_4) \sum_{x_4} P(X_4|x_3) P(x_3|e_{1:3})
\end{aligned}
$$

Filtering

- Suppose we want to calculate $P(H_2|F,B)$
- First we calculate $P(H_1|F)$

$$P(H_1|F) = \alpha * P(F|H_1) * P(H_1)$$
$$P(H_1) = \sum_{h_0} P(H_1|h_0)P(h_0)$$

H_1	$P(H_1)$
O	$0.15 + 0.12 + 0 = 0.27$
A	$0.12 + 0.20 + 0.12 = 0.44$
N	$0.03 + 0.08 + 0.18 = 0.29$

H_0	H_1	$P(H_0, H_1)$
O	O	0.15
O	A	0.12
O	N	0.03
A	O	0.12
A	A	0.20
A	N	0.08
N	O	0
N	A	0.12
N	N	0.18

H_0	H_1	$P(H_1|H_0)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_0	$P(H_0)$
O	0.3
A	0.4
N	0.3

Filtering

- Next we calculate $P(H_1|F)$, that is, prediction given the observation

$$P(H_1|F) = \alpha * P(F|H_1) * P(H_1)$$

H_1	$P(H_1\|F)$
O	2.87*0.027=0.078
A	2.87*0.176=0.505
N	2.87*0.145=0.417

H_1	$\alpha P(H_1\|F)$
O	$0.1*0.27 = 0.027$
A	$0.4*0.44 = 0.176$
N	$0.5*0.29 = 0.145$

H_1	$P(F\|H_1)$
O	0.1
A	0.4
N	0.5

H_1	$P(H_1)$
O	0.27
A	0.44
N	0.29

$$\alpha = \frac{1}{.027 + .176 + .145}$$

$$\alpha \approx 2.87$$

Filtering

- Now we can continue the process to calculate $P(H_2|F, B)$

$$P(H_2|F,B) = \alpha * P(B|H_2) * P(H_2|F)$$
$$P(H_2|F) = \sum_{H_1} P(H_2|H_1) * P(H_1|F)$$

H_2	$P(H_2\|F)$
O	.039+.151+0=.190
A	.031+.252+.167=.450
N	.007+.101+.250=.358

H_1	H_2	$P(H_2\|H_1) * P(H_1\|F)$
O	O	0.5*.078=.039
O	A	0.4*.078=.031
O	N	0.1*.078=.007
A	O	0.3*.505=.151
A	A	0.5*.505=.252
A	N	0.2*.505=.101
N	O	0*.417=0
N	A	0.4*.417=.167
N	N	0.6*.417=.250

H_1	H_2	$P(H_2\|H_1)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_1	$P(H_1\|F)$
O	0.078
A	0.505
N	0.417

Filtering

- Now we can continue the process to calculate $P(H_2|F, B)$

$$P(H_2|F,B) = \alpha * P(B|H_2) * P(H_2|F)$$

H_2	$P(H_2\|F,B)$
O	2.65*.019=0.05
A	2.65*.180=0.48
N	2.65*.179=0.47

H_2	$\alpha P(H_2\|F,B)$
O	.1*.190=.019
A	.4*.450=.18
N	.5*.358=.179

$\alpha \approx 2.65$

H_2	$P(B\|H_2)$
O	0.1
A	0.4
N	0.5

H_2	$P(H_2\|F)$
O	.190
A	.450
N	.358

- Transition Model

$$\mathbf{T}_{i,j} = P(X_t = j | X_{t-1} = i) = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

- Sensor Model

$$\mathbf{O}_1 = P(e_1 | X_t = i) = \begin{pmatrix} 0.9 & 0.0 \\ 0.0 & 0.1 \end{pmatrix}; \mathbf{O}_2 = \begin{pmatrix} 0.2 & 0.0 \\ 0.0 & 0.8 \end{pmatrix}$$

- Forward Message
$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^T \mathbf{f}_{1:t}$

Prediction

- Estimate a future state from the current evidence
- Filtering without new evidence
- $P(X_{t+k+1}|e_{1:t}) = \sum_{x_{t+k}} P(X_{t+k+1}|x_{t+k})P(X_{t+k}|e_{1:t})$
- For a large enough k, the prediction converges to a *stationary distribution*

Prediction

- Suppose we want to predict $P(H_4|F, B)$

$$P(H_4|F, B) = \sum_{H_3} P(H_4|H_3)P(H_3|F, B)$$

$$P(H_3|F, B) = \sum_{H_2} P(H_3|H_2)P(H_2|F, B)$$

H_3	$P(H_3\|F, B)$
O	0.025+.144=.169
A	0.02+.24+.188=.448
N	.005+.096+.282=.383

H_2	H_3	$P(H_3\|H_2)P(H_2\|F, B)$
O	O	.5*.05=.025
O	A	.4*.05=.020
O	N	.1*.05=.005
A	O	.3*.48=.144
A	A	.5*.48=.240
A	N	.2*.48=.096
N	O	0*.47=0
N	A	.4*.47=.188
N	N	.6*.47=.282

H_2	H_3	$P(H_3\|H_2)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_2	$P(H_2\|F, B)$
O	0.05
A	0.48
N	0.47

Prediction

- Suppose we want to predict $P(H_4|F,B)$

$$P(H_4|F,B) = \sum_{H_3} P(H_4|H_3)P(H_3|F,B)$$

$$P(H_3|F,B) = \sum_{H_2} P(H_3|H_2)P(H_2|F,B)$$

H_4	$P(H_4\|F,B)$
O	.085+.134=.219
A	.068+.224+.153=.445
N	.017+.090+.230=.337

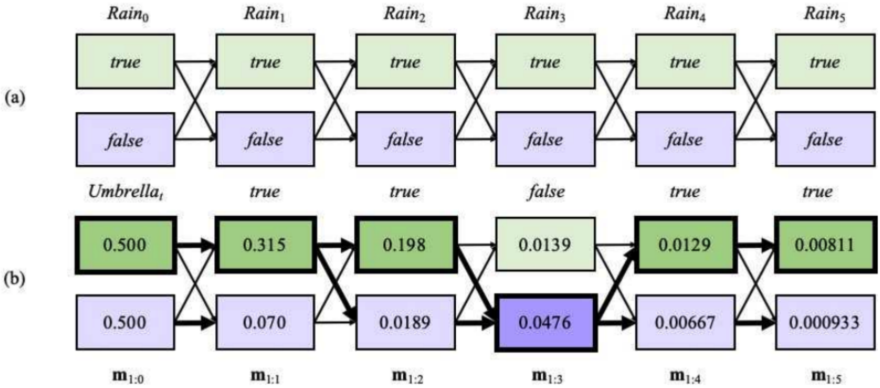
H_3	H_4	$P(H_4\|H_3)P(H_3\|F,B)$
O	O	.5*.169=.085
O	A	.4*.169=.068
O	N	.1*.169=.017
A	O	.3*.448=.134
A	A	.5*.448=.224
A	N	.2*.448=.090
N	O	0*.383=0
N	A	.4*.383=.153
N	N	.6*.383=.230

H_3	H_4	$P(H_4\|H_3)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_3	$P(H_3\|F,B)$
O	.169
A	.448
N	.383

Most Likely Explanation

- Estimate the sequence of states that generated a given sequence of evidence



Most Likely Explanation

- Estimate the sequence of states that generated a given sequence of evidence
- Compute recursively since most likely path to X_{t+1} is recursively linked to the path to X_t

$$
\begin{aligned}
m_{1:t} &= \max_{x_{1:t-1}} P(x_{1:t-1}, X_t, e_{1:t}) \\
m_{1:t+1} &= \max_{x_{1:t}} P(x_{1:t}, X_{t+1}, e_{1:t+1}) \\
&= \max_{x_{1:t}} P(x_{1:t}, X_{t+1}, e_{1:t}, e_{t+1}) \\
&= \max_{x_{1:t}} P(e_{t+1}|x_{1:t}, X_{t+1}, e_{1:t}) P(x_{1:t}, X_{t+1}, e_{1:t}) \\
&= P(e_{t+1}|X_{t+1}) \max_{x_{1:t}} P(X_{t+1}|x_t) P(x_{1:t}, e_{1:t}) \\
&= P(e_{t+1}|X_{t+1}) \max_{x_t} P(X_{t+1}|x_t) \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})
\end{aligned}
$$

Most Likely Explanation

- Suppose we want to find the most likely sequence of states that generated the sequence F, B

$$m_{1:t+1} = P(e_{t+1}|H_{t+1}) \max_{h_t} P(H_{t+1}|h_t) m_{1:t}$$

$$m_{1:1} = P(F|H_1) \max_{h_0} P(H_1|h_0) m_{1:0}$$

H_1	$m_{1:1}$
O	.15*.1=.015
A	.20*.4=.080
N	.18*.5=.090

H_1	$P(F\|H_1)$
O	0.1
A	0.4
N	0.5

H_0	H_1	$P(H_1\|H_0)m_{1:0}$
O	O	0.15
O	A	0.12
O	N	0.03
A	O	0.12
A	A	0.20
A	N	0.08
N	O	0
N	A	0.12
N	N	0.18

H_0	H_1	$P(H_1\|H_0)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_0	$m_{1:0}$
O	0.3
A	0.4
N	0.3

Most Likely Explanation

- Now we continue the process to calculate $m_{1:2}$

$$m_{1:2} = P(B|H_2) \max_{h_1} P(H_2|h_1) m_{1:1}$$

H_2	$m_{1:2}$
O	.024*.1=.0024
A	.04*.4=.016
N	.054*.5=.027

H_2	$P(B\|H_2)$
O	0.1
A	0.4
N	0.5

H_1	H_2	$P(H_2\|H_1)m_{1:1}$
O	O	0.0075
O	A	0.0060
O	N	0.0015
A	O	0.0240
A	A	0.0400
A	N	0.0160
N	O	0
N	A	0.0360
N	N	0.0540

H_1	H_2	$P(H_2\|H_1)$
O	O	0.5
O	A	0.4
O	N	0.1
A	O	0.3
A	A	0.5
A	N	0.2
N	O	0
N	A	0.4
N	N	0.6

H_1	$m_{1:1}$
O	0.015
A	0.080
N	0.090

Decision-making Under Uncertainty

- *Decision theory* = *Probability theory* + *Utility theory*

Decision-making Under Uncertainty

- *Decision theory* = *Probability theory* + *Utility theory*
- Consider an agent in a partially observable environment
  - $P(s)$ is the probability distribution over the current state s
- Suppose the action outcomes are also non-deterministic
  - $P(s'|s, a)$ is the transition model

Axioms of Utility Theory

- Lottery L is possible outcomes $S_1, \ldots, S_n$ with probabilities $p_1, \ldots, p_n$ of an action
  - S_i can be a state lottery or another lottery
- Orderability: $A \succ B$, $B \succ A$, or $A \sim B$
- Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
- Continuity: $A \succ B \succ C$, then $\exists p$ s.t. $L_1 \sim L_2$

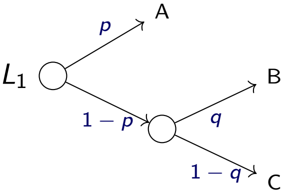
Axioms of Utility Theory

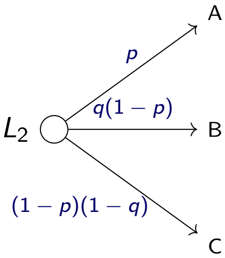
- Substitutability: $(A \sim B)$ then $(L_1 \sim L_2)$

- Monotonicity: $A \succ B$, then $(p > q) \Leftrightarrow (L_1 \succ L_2)$

Axioms of Utility Theory

- Decomposability: $L_1 \sim L_2$





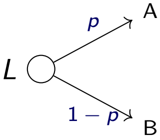
Existence of Utility Function

- If an agent's preferences satisfy the axioms, then a utility function exists such that:

$$U(A) > U(B) \Leftrightarrow (A \succ B)$$
$$U(A) = U(B) \Leftrightarrow (A \sim B)$$

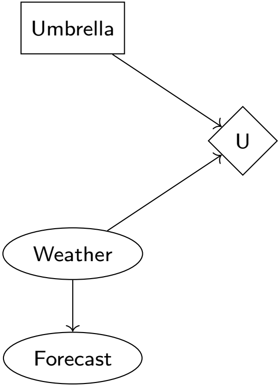
- Utility of a lottery is the expected utility of its outcomes

$$U(L) = pU(A) + (1 - p)U(B)$$

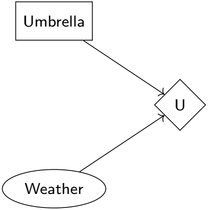


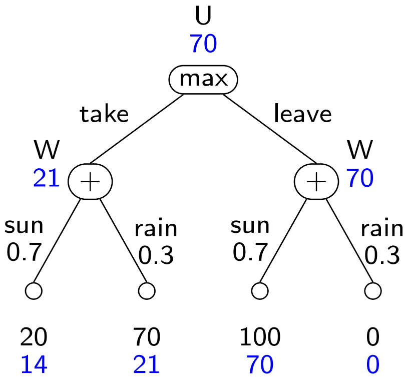
Decision Network

- Three type of nodes
  - Chance (oval) nodes like in Bayes Net
  - Action (rectangle) nodes
  - Utility (diamond) nodes
- Links like in Bayes Net
- Links connecting action and chance nodes to utility node



Decision Network



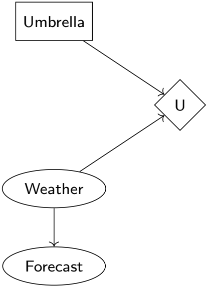


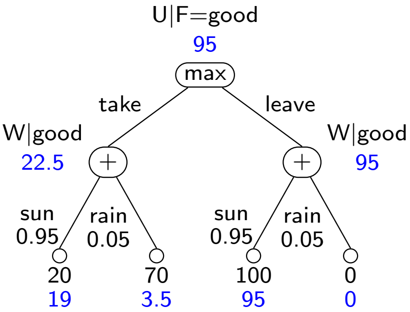
A	W	$U(A,W)$
take	sun	20
take	rain	70
leave	sun	100
leave	rain	0

W	$P(W)$
sun	0.7
rain	0.3

- $MEU(\phi) = \max((.7 * 20 + .3 * 70), (.7 * 100 + .3 * 0)) = \max(35, 70) = 70$

Decision Network





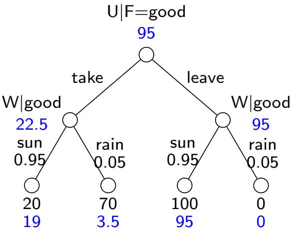
A	W	$U(A,W)$
take	sun	20
take	rain	70
leave	sun	100
leave	rain	0

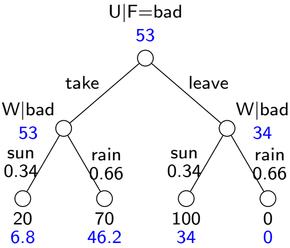
F	W	$P(W\|F)$
bad	sun	0.34
bad	rain	0.66
good	sun	0.95
good	rain	0.05

W	$P(W)$
sun	0.7
rain	0.3

- $MEU(\phi) = \max((.95 * 20 + .05 * 70), (.95 * 100 + .05 * 0)) = \max(22.5, 95) = 95$

Value of Perfect Information (VPI)





F	W	$P(W\|F)$
bad	sun	0.34
bad	rain	0.66
good	sun	0.95
good	rain	0.05

- $MEU(U|F = good) = \max((.95 * 20 + .05 * 70), (.95 * 100 + .05 * 0)) = \max(22.5, 95) = 95$
- $MEU(U|F = bad) = \max((.34 * 20 + .66 * 70), (.34 * 100 + .66 * 0)) = \max(53, 34) = 53$
- Suppose the forecast probability is: $P(F = good) = 0.59, P(F = bad) = 0.41$
- Then expected MEU if forecast is given is: $MEU(F) = .59 * 95 + .41 * 53 = 77.8$
- The value of knowing F is: $VPI(F) = MEU(F) - MEU(\phi) = 77.8 - 70 = 7.8$