CSC520 - Artificial Intelligence
Lecture 17

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Agenda

- Uncertainty
- Probability distributions
- Conditional distribution
- Probabilistic inference
- Product rule and chain rule
- Independence and conditional independence
- Bayes' rule

Uncertainty

- Real-world problems contain uncertainty
  - Partial observability
  - Non-determinitic actions
  - Adversaries
- Decision theory is about rational decision-making under uncertainty
  - *Decision theory* = *probability theory* + *utility theory*
  - Probability theory is about dealing with uncertainty
  - Utility theory is about dealing with preferences or utilities
- A rational agent chooses an action to maximize its expected utility
  - Principle of maximum expected utility (MEU)

Probability Distribution

- A random variable models some aspect of the world
  - W $\in$ {sun, rain, fog, ...}
  - T $\in$ {hot, cold}
- A probability distribution maps a variable's value to a real number

T	$P(T)$
hot	0.5
cold	0.5

W	$P(W)$
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

$P(T = \text{hot}) = 0.5$
$P(\text{hot}) = 0.5$

$P(W = \text{rain}) = 0.1$
$P(\text{rain}) = 0.1$

- Basic axioms of probability
  $\forall x \; P(x) \geq 0 \qquad \sum_x P(x) = 1$

Joint Distribution

- A probability distribution over a set of random variables $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$ OR $P(x_1, x_2, \ldots, x_n)$

T	W	$P(T, W)$
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- What can be a problem with representing probabilities in such joint distribution tables?
- Number of entries $= d^n$, for n variables with domain size d
- Can get very large!!

Marginal Distribution

- A distribution computed by summing out one or more variables

T	W	$P(T,W)$
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

T	$P(T)$
hot	0.5
cold	0.5

W	$P(W)$
sun	0.6
rain	0.4

- $P(T) = \sum_w P(T, W = w)$
- $P(W) = \sum_t P(T = t, W)$

Conditional Probability

- Conditional probability is the probability of a variable (or set of variables) given fixed values of other variables (or set of variables)

$$P(X|Y) = \frac{P(X, Y)}{P(Y)}$$

- What is the probability that it is sunny given it is cold, that is, $P(W = sun|T = cold)$?

T	W	$P(T, W)$
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Probability Distribution

- A conditional probability distribution is a distribution of conditional probabilities

T	W	$P(T, W)$
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

T	W	$P(W \mid T)$
hot	sun	0.8
hot	rain	0.2
cold	sun	0.4
cold	rain	0.6

Probabilistic Inference by Enumeration

- A joint probability distribution is given: $P(X_1, X_2, \ldots X_n)$
- Queries are of the form: $P(Q|e_1, \ldots, e_k)$
  - Evidence variables: $E_1, \ldots, E_k = e_1, \ldots, e_k$
  - Query variables: Q
  - Hidden variables: $H_1, \ldots, H_r$
- Steps in inference by enumeration
  1. Select rows in the joint distribution consistent with E
  2. Sum out the hidden variables H to get joint of query and evidence $P(Q, e_1, \ldots, e_k) = \sum_{h_1,\ldots,h_r} P(Q, h_1, \ldots, h_r, e_1, \ldots, e_k)$
  3. Normalize the result $Z = \sum_q P(Q, e_1, \ldots, e_k)$ $P(Q|e_1, \ldots, e_k) = \frac{1}{Z} P(Q, e_1, \ldots, e_k)$
- Space complexity $= O(d^n)$
  - E.g. $2^{30} \approx 1$Billion

Probabilistic Inference

$$P(W) = \sum_s \sum_t P(s,t,W)$$

$$\begin{aligned}
P(W = sun) = & P(s = summer, t = hot, W = sun) + \\
& P(s = summer, t = cold, W = sun) + \\
& P(s = winter, t = hot, W = sun) + \\
& P(s = winter, t = cold, W = sun) \\
= & 0.3 + 0.1 + 0.1 + 0.15 = 0.65 \\
P(W = rain) = & P(s = summer, t = hot, W = rain) + \\
& P(s = summer, t = cold, W = rain) + \\
& P(s = winter, t = hot, W = rain) + \\
& P(s = winter, t = cold, W = rain) \\
= & 0.05 + 0.05 + 0.05 + 0.20 = 0.35
\end{aligned}$$

S	T	W	$P(S,T,W)$
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

W	$P(W)$
sun	0.65
cold	0.35

Probabilistic Inference

$P(W|winter)$

W	$P(W\|winter)$
sun	0.5
rain	0.5

S	T	W	$P(S, T, W)$
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Probabilistic Inference

$P(W|winter, hot)$

W	$P(W\|winter, hot)$
sun	0.67
rain	0.33

S	T	W	$P(S, T, W)$
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Product Rule

- Compute joint distribution from a conditional and a marginal distribution (Product Rule)

$$P(X, Y) = P(X|Y)P(Y)$$

- Which is equivalent to:

$$P(X|Y) = \frac{P(X, Y)}{P(Y)}$$

Product Rule Example

$$P(X, Y) = P(X|Y)P(Y)$$

D	W	$P(D, W)$
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

D	W	$P(D\|W)$
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

W	$P(W)$
sun	0.8
rain	0.2

Chain Rule

- Generalization of product rule to more than two variables
- For four variables, X_1, X_2, X_3, X_4:

$$\begin{aligned}
\mathrm{P}(\mathrm{X}_1, X_2, X_3, X_4) &= \mathrm{P}(\mathrm{X}_1 | X_2, X_3, X_4) P(X_2, X_3, X_4) \\
&= \mathrm{P}(\mathrm{X}_1 | X_2, X_3, X_4) P(X_2 | X_3, X_4) P(X_3, X_4) \\
&= \mathrm{P}(\mathrm{X}_1 | X_2, X_3, X_4) P(X_2 | X_3, X_4) P(X_3 | X_4) P(X_4)
\end{aligned}$$

- For n variables:

$$P(X_1, X_2, \ldots, X_n) = \prod_i P(X_i | X_{i+1}, X_{i+2}, \ldots, X_n)$$

(Absolute) Independence

- Two variables X and Y are independent written as $X \perp\!\!\!\perp Y$ iff:

$$P(X|Y) = P(X) \qquad P(Y|X) = P(Y)$$

- From product rule:

$$\begin{aligned} P(X, Y) &= P(X|Y)P(Y) &&= P(X)P(X|Y) \\ P(X, Y) &= P(X)P(Y) &&= P(X)P(Y) \end{aligned}$$

- Which of these variables are independent: {Toothache, Catch, Cavity, Weather}?
- $P(Toothache, Catch, Cavity, Weather) = P(Toothache, Catch, Cavity)P(Weather)$

Conditional Independence

- Consider the joint distribution: $P(Toothache, Cavity, Catch)$
- The probability that the probe catches into a cavity does not depend on Toothache

$$P(Catch|Toothache, Cavity) = P(Catch|Cavity)$$

- $Catch$ is conditionally independent of $Toothache$ given $Cavity$
- We can derive following:

$$P(Toothache|Catch, Cavity) = P(Toothache|Cavity)$$
$$P(Toothache, Catch|Cavity) = P(Toothache|Cavity)P(Catch|Cavity)$$

Conditional Independence

- X is conditionally independent of Y given Z, written as: $X \perp\!\!\!\perp Y | Z$ iff:

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

  Or:

$$P(X|Z, Y) = P(X|Z)$$

- Conditional independence reduces space complexity of joint distribution

Bayes' Rule

- From the product rule, we get:

$$P(X,Y) = P(X|Y)P(Y)$$
$$P(X,Y) = P(Y|X)P(X)$$

- From the above equations, we get the Bayes' rule:

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

- $P(X|Y)$ = Posterior probability, $P(Y|X)$ = Likelihood, $P(X)$ = Prior probability, $P(Y)$ = Evidence
- Useful when a conditional distribution is available and we need to compute its reverse

Bayes' Rule Example

- Useful for accessing diagnostic probability from causal probability

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

- Suppose a doctor knows the following
  - Meningitis causes stiff neck 70% of the time: $P(s|m) = 0.7$
  - 1 out of 50,000 patients have meningitis: $P(m) = 1/50000 = 0.00002$
  - 1 out of 100 patients have stiff neck: $P(s) = 0.01$
- What is the probability that a patient with stiff neck has meningitis, that is, $P(m|s)$?

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times 1/50000}{0.01} = 0.0014$$

Bayes' Rule - Other Forms

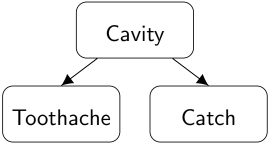
$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

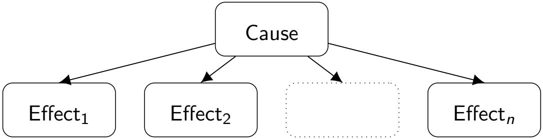
$$P(X|Y) = \frac{P(Y|X)P(X)}{\sum_x P(Y|x)P(x)}$$

$$P(X|Y) = \alpha P(Y|X)P(X)$$

Bayes' Rule and Conditional Independence

$$
\begin{aligned}
&P(Cavity \mid Toothache, Catch) \\
&= \alpha P(Toothache, Catch \mid Cavity) P(Cavity) && \text{(Bayes' Rule)} \\
&= \alpha P(Toothache \mid Cavity) P(Catch \mid Cavity) P(Cavity) && \text{(Cond. Ind.)}
\end{aligned}
$$





$$P(Cause \mid Effect_1, Effect_2, \ldots, Effect_n) = \alpha P(Cause) \prod_i P(Effect_i \mid Cause)$$

Probability Theory

- Probability Model: $\forall X_i : 0 \leq P(X_i) \leq 1 \quad \sum_i P(X_i) = 1$
- Conditional Probability: $P(X|Y) = \dfrac{P(X,Y)}{P(Y)}$
- Product Rule: $P(X,Y) = P(X|Y)P(Y)$
- Chain Rule: $P(X,Y,Z) = P(X|Y,Z)P(Y|Z)P(Z)$
- Chain Rule: $P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i|Parents(x_i))$
- Absolute Independence: $X \perp\!\!\!\perp Y \quad P(X,Y) = P(X)P(Y)$
- Conditional Independence: $X \perp\!\!\!\perp Y|Z \; P(X,Y|Z) = P(X|Z)P(Y|Z)$
- *Bayes' Rule:* $P(Cause|Effect) = \dfrac{P(Effect|Cause)P(Cause)}{P(Effect)}$

Class Exercise

Given:

D	W	$P(D\|W)$
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

W	$P(W)$
sun	0.8
rain	0.2

- What is $P(W|dry)$?