FEniCS Course

Solving the cable equation in Fenics

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The cable equation

The standard cable equation is a reaction-diffusion equation given by

$$\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial x^2} + f(v, s),$$

where $f(v, s)$ is a reaction term describing ionic fluxes across the membrane.

- A linear $f(v)$ describes passive conductance through a leaky cable (dendrites).
- A cubic $f(v)$ gives the bistable equation with a propagating activation front.
- In general $f(v, s)$, where s is a vector describing the state of the cell membrane, typically governed by a system of ODEs.

The bistable equation

We want to solve the bistable equation on an interval $\Omega = [-L, L]$:

$$\begin{aligned} v_t &= \sigma v_{xx} + f(v) && \text{for } -L < x < L, \\ v_x &= 0 && \text{for } x = -L, x = L, \end{aligned} \tag{1}$$

with

- $f(v) = Av(1-v)(v-\alpha)$,
- $\sigma = 1.0$,
- $A = 1.0$,
- $\alpha = 0.1$,
- $L = 100$.

(Note that we have introduced the compact notation $v_t = \partial v/\partial t, v_{xx} = \partial^2 v/\partial x^2$.)

Weak formulation (1)

We multiply with a test function ϕ and integrate over the domain Ω:

$$\int_\Omega v_t \phi dx = \int_\Omega \sigma v_{xx} \phi dx + \int_\Omega f(v) \phi dx.$$

Integration by parts and using the boundary conditions gives

$$\int_\Omega v_t \phi dx = -\int_\Omega \sigma v_x \phi_x dx + \int_\Omega f(v) \phi dx.$$

Weak formulation (2)

A finite difference approximation of the time derivative gives the final weak form:

$$\int_\Omega \frac{(v^n - v^{n+1})}{\Delta t} \phi dx + \int_\Omega \sigma v_x \phi_x dx - \int_\Omega f(v) \phi dx = 0.$$

This is a non-linear weak form that can be implemented and solved directly in Fenics.

The FEniCS challenge!

Solve the bistable equations in FEniCS from $t = 0$ to $T = 250$ with a timestep of $dt = 2.5$. Use as initial condition

$$\frac{1}{2}(1 - \tanh(\sqrt{A/(8\sigma)})(x - 0.75L)) \tag{2}$$

Hint: You can set lower and upper limits on the plots with

Python code

```python
import pylab
p[0].axes.set_aspect('auto')
p[0].axes.set_ylim([0, 1])
pylab.show()
```

The Fitzhugh-Nagumo (FHN) model

A small modification of the bistable equation gives the Fitzhugh-Nagumo model:

$$
\begin{aligned}
v_t &= \sigma v_{xx} + f(v) - w && \text{for } -L < x < L, \\
w_t &= \epsilon(v - \gamma w) && \text{for } -L < x < L, \\
v_x &= 0 && \text{for } x = -L, x = L,
\end{aligned}
$$

Here we have introduced the *recovery variable* w, which is governed by a separate ODE. The additional parameters are set to

- $\epsilon = 0.005$
- $\gamma = 2.0$

Weak formulation of the FHN model (1)

We solve the two equations fully coupled. We multiply with test functions ϕ, η, and integrate over the domain Ω:

$$\int_\Omega v_t \phi dx = \int_\Omega \sigma v_{xx} dx + \int_\Omega f(v)\phi dx$$
$$\int_\Omega w_t \eta dx = \int_\Omega \epsilon(v - \gamma w)\eta dx$$

Integrating the first equation by parts gives the system

$$\int_\Omega v_t \phi dx = -\int_\Omega \sigma v_x \phi_x dx + \int_\Omega f(v)\phi dx,$$
$$\int_\Omega w_t \eta dx = \int_\Omega \epsilon(v - \gamma w)\eta dx$$

Weak formulation of the FHN model (2)

Again, a finite difference approximation of the time derivatives gives the final weak form:

$$
\begin{aligned}
\int_\Omega \frac{(v^n - v^{n+1})}{\Delta t}\phi dx + \int_\Omega \sigma v_x \phi_x dx - \int_\Omega f(v)\phi dx = 0, \\
\int_\Omega \frac{(w^n - w^{n+1})}{\Delta t}\eta dx - \int_\Omega \epsilon(v - \gamma w)\eta dx = 0 \qquad (3)
\end{aligned}
$$