On the Stability of Operator Splitting Methods for Strongly Coupled Cardiac Electro-Mechanics

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ABSTRACT

We consider the electro-mechanical processes in cardiac tissue, which are governed by a mathematical model on the form

$$\frac{\partial s}{\partial t} = f(v, s, \lambda), \tag{1}$$

$$\frac{\partial v}{\partial t} + I_{ion}(v, s, \lambda) = \nabla \cdot (M_i \nabla v) + \nabla \cdot (M_i \nabla u_e), \tag{2}$$

$$\nabla \cdot ((M_i + M_e)\nabla u_e) = -\nabla \cdot (M_i \nabla v), \tag{3}$$

$$\nabla \cdot (FS) = 0, \tag{4}$$

$$S = S^p + S^a, \tag{5}$$

$$\Psi = \frac{1}{2} K(e^W - 1) \tag{6}$$

$$W = b_{ff} E_{11}^2 + b_{xx}(E_{22}^2 + E_{33}^2 + E_{23}^2) + b_{fx}(2E_{12}^2 + 2E_{13}^2). \tag{7}$$

$$S^p = \frac{\partial \Psi}{\partial E} + pC^{-1}, \tag{8}$$

$$S^a = JF^{-1}\sigma^a(s, \lambda, \dot{\lambda})F^{-T}, \tag{9}$$

Here (1) is a system of ordinary differential equations (ODEs) describing the electro-chemical state of the muscle cells, characterized by the state vector s. Furthermore v is the transmembrane potential, λ is the fiber stretch ratio. Eqs (2)-(3) constitute the bidomain model, which describes the electrical state of the tissue in terms of v and the extracellular potential u_e. The non-linear function I_{ion} describes the ionic current accross the cell membrane, while M_i, M_e are tissue conductivities. Mechanical equilibrium is described by (4), where we have neglected body forces and inertia terms, with constitive relations (5)-(9). Here F, J is the deformation gradient and its determinant, and S is the second Piola-Kirchoff stress tensor which is split into active and passive parts S^a and S^p. Furthermore W is the strain energy function which defines the transversely isotropic behavior of passive tissue, C is the right Cauchy-Green tensor, p is the hydrostatic pressure, E is the Green-Lagrange strain tensor, and σ^a is the active part of the Cauchy stress tensor. In local fiber coordinates we have $\sigma^a = \text{diag}(T_a, \eta T_a, \eta T_a)$, where η is a constant and T_a is the dynamic tension computed from the cell model. The model must be complemented with appropriate boundary conditions, see e.g. [2] for details.

The complexity of the system (1)-(9) motivates the use of numerical methods based on operator splitting. However, as reported by several authors [1, 3], splitting methods may suffer from severe stability problems when applied to realistic models of electro-mechanical coupling. We want to evaluate the stability of alternative splitting schemes compared with fully

implicit solution methods, and for this purpose we employ a simplified version of (1)-(9). We consider an unloaded tissue slab with homogeneous fiber distribution, which is subjected to a uniform electrical activation. Since the tissue is incompressible and transversely isotropic, this leads to a uniform deformation field with a diagonal deformation gradient with components $F_{11} = \lambda, F_{22} = F_{33} = \sqrt{1/\lambda}$. The right Cauchy-Green tensor C and the Green-Lagrange tensor E will also be diagonal, with components $C_{11} = \lambda^2, C_{22} = C_{33} = 1/\lambda, E_{11} = \frac{1}{2}(\lambda^2 - 1)$, and $E_{22} = E_{33} = \frac{1}{2}(\frac{1}{\lambda} - 1)$.

Inserting these expressions into the strain energy function above defines the passive stress in terms of $lambda, p$;

$$S_{11} = \frac{\partial \Psi}{\partial E_{11}} + p\{C^{-1}\}_{11} = \frac{1}{2} K b_{ff} (\lambda^2 - 1) e^W + p \frac{1}{\lambda^2}$$
$$S_{22} = S_{33} = \frac{\partial \Psi}{\partial E_{22}} + p\{C^{-1}\}_{22} = \frac{1}{2} K b_{xx} (1/\lambda - 1) e^W + p\lambda.$$

Since the slab is unloaded, and we have disregarded body forces and inertia effects, the active and passive stresses must balance in all points. Eq (4) may therefore be replaced by $S^a + S^p = 0$. Combining this with the cell model ODE system (1) leads to the differential-algebraic system

$$\frac{\partial s}{\partial t} = f(v, s, \lambda), \tag{10}$$
$$T^a(s, \lambda, d\lambda/dt) = S_{11}(\lambda, p), \tag{11}$$
$$\eta T^a(s, \lambda, d\lambda/dt) = S_{22}(\lambda, p). \tag{12}$$

The simplified model (10)-(12) retains the critical deformation-force feedback of (9), which is known to be the main source of stability problems [1]. The simple model is therefore well suited for studying alternative numerical schemes, and allows far more detailed analysis than what is possible with the full model. We intend to investigate the stability and accuracy of a number of alternative numerical methods using the (10)-(12), and then verify the results using (1)-(9) in a full-scale heart model with realistic geometry and boundary conditions.

References

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