



# FEniCS Course

## Lecture 5: Solving the cable equation in FEniCS

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*Contributors*

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# Outline

- The cable equation
- The bistable equation
  - Weak formulation
- The Fitzhugh-Nagumo model
  - Weak formulation

# The cable equation

The standard cable equation is a reaction-diffusion equation given by

$$\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial x^2} + f(v, s),$$

where  $f(v, s)$  is a reaction term describing ionic fluxes across the membrane.

- A linear  $f(v)$  describes passive conductance through a leaky cable (dendrites).
- A cubic  $f(v)$  gives the bistable equation with a propagating activation front.
- In general  $f(v, s)$ , where  $s$  is a vector describing the state of the cell membrane, typically governed by a system of ODEs.

# The bistable equation

We want to solve the bistable equation on an interval  $\Omega = [-L, L]$ :

$$\begin{aligned}v_t &= \sigma v_{xx} + f(v) \text{ for } -L < x < L, \\v_x &= 0 \qquad \qquad \text{for } x = -L, x = L,\end{aligned}\tag{1}$$

with

- $f(v) = Av(1-v)(v-\alpha)$ ,
- $\sigma = 1.0$ ,
- $A = 1.0$ ,
- $\alpha = 0.1$ ,
- $L = 100$ .

(Note that we have introduced the compact notation  $v_t = \partial v / \partial t$ ,  $v_{xx} = \partial^2 v / \partial x^2$ .)

## Weak formulation (1)

We multiply with a test function  $\phi$  and integrate over the domain  $\Omega$ :

$$\int_{\Omega} v_t \phi dx = \int_{\Omega} \sigma v_{xx} \phi dx + \int_{\Omega} f(v) \phi dx.$$

Integration by parts and using the boundary conditions gives

$$\int_{\Omega} v_t \phi dx = - \int_{\Omega} \sigma v_x \phi_x dx + \int_{\Omega} f(v) \phi dx.$$

## Weak formulation (2)

A finite difference approximation of the time derivative gives the final weak form:

$$\int_{\Omega} \frac{(v^{n+1} - v^n)}{\Delta t} \phi dx + \int_{\Omega} \sigma v_x^{n+1} \phi_x dx - \int_{\Omega} f(v^{n+1}) \phi dx = 0.$$

This is a non-linear weak form that can be implemented and solved directly in Fenics.

# Solving non-linear problems in FEniCS

For non-linear problems, use a “Function” instead of a “TrialFunction” in the variational form.

Once a variational problem has been defined, it may be solved by calling the `solve` function (as for linear problems):

*Python code*

```
solve(F == 0, v1, bcs)
```

Or more verbosely

*Python code*

```
dF = derivative(F, v1)
pde = NonlinearVariationalProblem(F, v1, bcs, dF)
solver = NonlinearVariationalSolver(pde)
solver.solve()
```

## *The FEniCS challenge!*

Solve the bistable equations in FEniCS from  $t = 0$  to  $T = 250$  with a timestep of  $dt = 2.5$ . Use as initial condition

$$\frac{1}{2}(1 - \tanh(\sqrt{A/(8\sigma)})(x - 0.75L)) \quad (2)$$

Hint: You can set lower and upper limits on the plots with

*Python code*

```
import pylab
[...]  
p = plot(v1)  
p[0].axes.set_aspect('auto')  
p[0].axes.set_ylim([0, 1])  
pylab.show()
```



# The Fitzhugh-Nagumo (FHN) model

A small modification of the bistable equation gives the Fitzhugh-Nagumo model:

$$\begin{aligned}v_t &= \sigma v_{xx} + f(v) - w \quad \text{for } -L < x < L, \\w_t &= \epsilon(v - \gamma w) \quad \text{for } -L < x < L, \\v_x &= 0 \quad \text{for } x = -L, x = L,\end{aligned}$$

Here we have introduced the *recovery variable*  $w$ , which is governed by a separate ODE. The additional parameters are set to

- $\epsilon = 0.005$
- $\gamma = 2.0$

# Weak formulation of the FHN model (1)

We solve the two equations fully coupled. We multiply with test functions  $\phi, \eta$ , and integrate over the domain  $\Omega$ :

$$\begin{aligned}\int_{\Omega} v_t \phi dx &= \int_{\Omega} \sigma v_{xx} dx + \int_{\Omega} f(v) \phi dx \\ \int_{\Omega} w_t \eta dx &= \int_{\Omega} \epsilon(v - \gamma w) \eta dx\end{aligned}$$

Integrating the first equation by parts gives the system

$$\begin{aligned}\int_{\Omega} v_t \phi dx &= - \int_{\Omega} \sigma v_x \phi_x dx + \int_{\Omega} f(v) \phi dx, \\ \int_{\Omega} w_t \eta dx &= \int_{\Omega} \epsilon(v - \gamma w) \eta dx\end{aligned}$$

## Weak formulation of the FHN model (2)

Again, a finite difference approximation of the time derivatives gives the final weak form:

$$\begin{aligned} \int_{\Omega} \frac{(v^n - v^{n+1})}{\Delta t} \phi dx + \int_{\Omega} \sigma v_x \phi_x dx - \int_{\Omega} f(v) \phi dx &= 0, \\ \int_{\Omega} \frac{(w^n - w^{n+1})}{\Delta t} \eta dx - \int_{\Omega} \epsilon(v - \gamma w) \eta dx &= 0 \end{aligned} \tag{3}$$

## *The FEniCS challenge (homework)!*

Adapt the previous solution so that it solves the FHN model.