# Chapter 8 Homework

# Problem 1

Let $\mathbb{R}$ be the relation defined on the set of all integers $\mathbb{Z}$ as follows:

*For all integers $m$ and $n$*

$$m\,R\,n \Leftrightarrow 5 \mid (m-n)$$

## Proof

Prove that $R$ is an equivalence relation.

*Showing that $R$ is reflexive:*

- Suppose that $m$ is any integer.

$$m - m = 0 \text{ by algebra}$$

- By definition of divisibility, $5$ divides $0$ because $0 = 5 \cdot 0$:

$$5 \mid 0$$

- Hence, by definition of $R$, $m\,R\,m$.
- $\boxed{R \text{ is reflexive.}}$

*Showing that $R$ is symmetric:*

- Suppose that $m$ and $n$ are any integers such that

$$m\,R\,n$$

- By definition of $R$, $5 \mid (m-n)$.
- By definition of divisibility, $m - n = 5k$ for some integer $k$.

$$\begin{aligned} m - n &= 5k \\ -m + n &= -5k \text{ by multiplying both sides by -1} \\ n - m &= -5k \text{ by commutative property} \\ n - m &= 5(-k) \text{ by associative property} \end{aligned}$$

- Because $\mathbb{Z}$ is closed under multiplication, $(-k)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (n-m)$.
- Hence, $n\,R\,m$.
- $\boxed{R \text{ is symmetric.}}$

*Showing that $R$ is transitive:*

- Suppose that $m$, $n$, and $p$ are any integers such that

$$m\,R\,n \text{ and } n\,R\,p$$

- By definition of $R$, $5 \mid (m-n)$ and $5 \mid (n-p)$.
- By definition of divisibility, $m - n = 5r$ and $n - p = 5s$ for some integers $r$ and $s$.

$$
\begin{aligned}
(m-n)+(n-p) &= 5r+5s \text{ by adding both equalities} \\
m-p &= 5r+5s \text{ by algebra} \\
m-p &= 5(r+s) \text{ by distributive property}
\end{aligned}
$$

- Because $\mathbb{Z}$ is closed under addition, $(r+s)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (m-p)$.
- Hence, $m\,R\,p$.
- $\boxed{R \text{ is transitive.}}$

**Problem 1 Answer:**

Because $R$ is reflexive, symmetric, and transitive, it is an equivalence relation.

# Problem 2

Let $S$ be the set of all strings of 0s and 1s of length 3. Define a relation $R$ on $S$ as follows:

*For all strings $s$ and $t \in S$,*

$$s\,R\,t \Leftrightarrow \text{the two left-most characters of } s \text{ are the same as the two left-most characters of } t$$

# Proof

Prove that $R$ is an equivalence relation on $S$.

*Showing that $R$ is reflexive:*

- Suppose $s$ is any string in $S$.
- Logically, it follows that the two left-most characters in $s$ are also the two left-most charaters in $s$.
- Hence, $s\,R\,s$.
- $\boxed{R \text{ is reflexive.}}$

*Showing that $R$ is symmetric:*

- Suppose that $s$ and $t$ are any strings in $S$ such that

$$s\,R\,t$$

- By definition of $R$, the two left-most characters in $s$ are the same as the two left-most characters in $t$.
- Logically, it follows that the two left-most characters in $t$ are the same as the two left-most characters in $s$.
- Hence, $t\,R\,s$.
- $\boxed{R \text{ is symmetric.}}$

*Showing that $R$ is transitive:*

- Suppose that $s$, $t$, and $u$ are any strings in $S$ such that

$$s\,R\,t \text{ and } t\,R\,u$$

- By definition of $R$, the two left-most characters in $s$ are the same as the two left-most characters in $t$, and the two left-most characters in $t$ are the same as the two left-most characters in $u$.
- Logically, it follows that the two left-most characters in $s$ are also the same as the two left-most characters in $u$.
- Hence, $s\,R\,u$.
- $\boxed{R \text{ is transitive.}}$

**Problem 2 Answer:**

Because $R$ is reflexive, symmetric, and transitive, it is an equivalence relation on $S$.