Chapter 8 Homework

Problem 1

Let $\mathbb{R}$ be the relation defined on the set of all integers $\mathbb{Z}$ as follows:

$$\textit{For all integers } m \textit{ and } n$$

$$m\,R\,n \Leftrightarrow 5 \mid (m - n)$$

Proof

Prove that R is an equivalence relation.

Showing that R is reflexive:

- Suppose that m is any integer.

$$m - m = 0 \text{ by algebra}$$

- By definition of divisibility, 5 divides 0 because $0 = 5 \cdot 0$:

$$5 \mid 0$$

- Hence, by definition of R, $m\,R\,m$.
- $\boxed{R \text{ is reflexive.}}$

Showing that R is symmetric:

- Suppose that m and n are any integers such that

$$m\,R\,n$$

- By definition of R, $5 \mid (m - n)$.
- By definition of divisibility, $m - n = 5k$ for some integer k.

$$\begin{aligned} m - n &= 5k \\ -m + n &= -5k \text{ by multiplying both sides by -1} \\ n - m &= -5k \text{ by commutative property} \\ n - m &= 5(-k) \text{ by associative property} \end{aligned}$$

- Because $\mathbb{Z}$ is closed under multiplication, $(-k)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (n - m)$.
- Hence, $n\,R\,m$.
- $\boxed{R \text{ is symmetric.}}$

Showing that R is transitive:

- Suppose that m, n, and p are any integers such that

$$m\,R\,n \text{ and } n\,R\,p$$

- By definition of R, $5 \mid (m - n)$ and $5 \mid (n - p)$.
- By definition of divisibility, $m - n = 5r$ and $n - p = 5s$ for some integers r and s.

$$(m-n)+(n-p)=5r+5s \text{ by adding both equalities}$$
$$m-p=5r+5s \text{ by algebra}$$
$$m-p=5(r+s) \text{ by distributive property}$$

- Because $\mathbb{Z}$ is closed under addition, $(r+s)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (m-p)$.
- Hence, $m\,R\,p$.
- $\boxed{R \text{ is transitive.}}$

Problem 1 Answer:
Because R is reflexive, symmetric, and transitive, it is an equivalence relation.

Problem 2

Let S be the set of all strings of 0s and 1s of length 3. Define a relation R on S as follows:

For all strings s and $t \in S$,

$s \, R \, t \Leftrightarrow$ the two left-most characters of s are the same as the two left-most characters of t

Proof

Prove that R is an equivalence relation on S.

Showing that R is reflexive:

- Suppose s is any string in S.
- Logically, it follows that the two left-most characters in s are also the two left-most charaters in s.
- Hence, $s \, R \, s$.
- $\boxed{R \text{ is reflexive.}}$

Showing that R is symmetric:

- Suppose that s and t are any strings in S such that

$$s \, R \, t$$

- By definition of R, the two left-most characters in s are the same as the two left-most characters in t.
- Logically, it follows that the two left-most characters in t are the same as the two left-most characters in s.
- Hence, $t \, R \, s$.
- $\boxed{R \text{ is symmetric.}}$

Showing that R is transitive:

- Suppose that s, t, and u are any strings in S such that

$$s \, R \, t \text{ and } t \, R \, u$$

- By definition of R, the two left-most characters in s are the same as the two left-most characters in t, and the two left-most characters in t are the same as the two left-most characters in u.
- Logically, it follows that the two left-most characters in s are also the same as the two left-most characters in u.
- Hence, $s \, R \, u$.
- $\boxed{R \text{ is transitive.}}$

Problem 2 Answer:

Because R is reflexive, symmetric, and transitive, it is an equivalence relation on S.