Discrete Mathematics Exam 3 Practice

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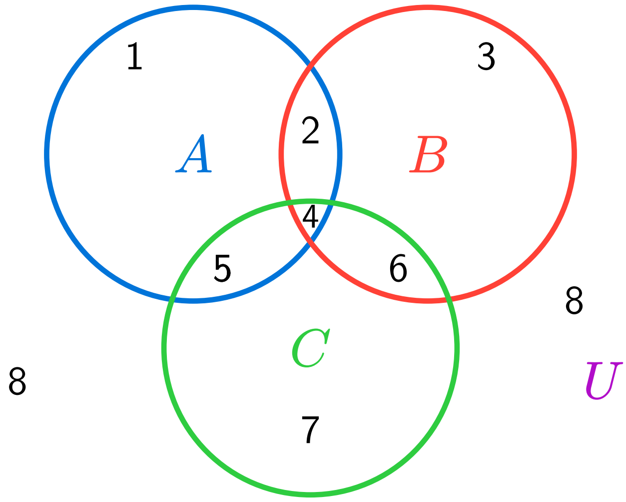
Problem 1

Considering the Venn diagram below:



$$A = \{1, 2, 4, 5\}$$
$$B = \{2, 3, 4, 6\}$$
$$C = \{4, 5, 6, 7\}$$
$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

Prove that $(A \cap C)^{\mathsf{C}} = \{1, 2, 3, 6, 7, 8\}$ and that $(B \cup C) - (A \cap C) = \{2, 3, 6, 7\}$

Answer

Proving part 1:

$$\begin{aligned} A \cap C &= \{4, 5\} \\ (A \cap C)^{\mathsf{C}} &= U - \{4, 5\} \\ &= \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{6}, \mathbf{7}, \mathbf{8}\} \end{aligned}$$

Proving part 2:

$$\begin{aligned} B \cup C &= \{2, 3, 4, 5, 6, 7\} \\ A \cap C &= \{4, 5\} \\ (B \cup C) - \{A \cap C\} &= \{\mathbf{2}, \mathbf{3}, \mathbf{6}, \mathbf{7}\} \end{aligned}$$

Problem 2

C is the circle relation on $\mathbb{R}$ such that:

For every $x, y \in \mathbb{R}$,

$$x\,C\,y \Leftrightarrow x^2 + y^2 = 1$$

Determine whether C is reflexive, symmetric, and/or transitive.

Answer

Is C reflexive?

- Suppose that x is any element of $\mathbb{R}$.

$$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \end{aligned}$$

- Plug in $x = 1$.

$$1^2 = \frac{1}{2}$$

$$1 \neq \frac{1}{2}$$

- ***C* is not reflexive.**

Is C symmetric?

- Suppose that x and y are any elements of $\mathbb{R}$ such that

$$x^2 + y^2 = 1$$

- By definition of C, $x\,C\,y$.
- By the commutative property of addition:

$$y^2 + x^2 = 1$$

- Hence, by definition of C, $y\,C\,x$.
- ***C* is symmetric.**

Is C transitive?

- Suppose that x, y, and z are any elements of $\mathbb{R}$ such that

$$x^2 + y^2 = 1 \text{ and } y^2 + z^2 = 1$$

- By definition of C, $x\,C\,y$ and $y\,C\,z$.
- Now:

$$\begin{aligned} y^2 + z^2 &= 1 \\ y^2 &= 1 - z^2 \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + 1 - z^2 &= 1 \\ x^2 &= z^2 \end{aligned}$$

- Plug in $x = 1$ and $z = 2$.

$$1^2 = 2^2$$
$$1 \neq 4$$

- **C is not transitive.**

Problem 3

O is the relation on $\mathbb{Z}$ as follows:

For every $m, n \in \mathbb{Z}$,

$$m\,O\,n \Leftrightarrow m - n \text{ is odd}$$

Determine whether O is reflexive, symmetric, and/or transitive.

Answer

Is O reflexive?

- Suppose that m is any integer.

$$m - m = 0$$

- By definition of even, 0 is even, and thus is not odd.
- By definition of O, m is not related to m by O.
- ***O* is not reflexive.**

Is O symmetric?

- Suppose that m and n are any integers such that

$$m\,O\,n$$

- By definition of O, $m - n$ equals an odd integer.
- By definition of odd, $m - n = 2k + 1$ for some integer k.
- Now,

$$\begin{aligned}
m - n &= 2k + 1 \\
-m + n &= -2k - 1 \text{ by multiplying by} -1 \\
n - m &= -2k - 1 \text{ by commutative property} \\
n - m &= -2k - 2 + 1 \text{ by subtracting 1 and adding 1} \\
n - m &= 2(-k - 1) + 1 \text{ by distributive property}
\end{aligned}$$

- Because $\mathbb{Z}$ is closed under subtraction and multiplication, $(-k - 1)$ is an integer.
- By definition of odd, $2(-k - 1) + 1$ is odd.
- By definition of O, $n\,O\,m$.
- ***O* is symmetric.**

Is O transitive?

- Suppose that m, n, and p are any integers such that

$$m\,O\,n \text{ and } n\,O\,p$$

- By definition of O, $m - n$ and $n - p$ both equal odd integers.
- By definition of odd, $m - n = 2r + 1$ and $n - p = 2s + 1$ for some integers r and s.
- Now,

$$\begin{aligned}
m - n &= 2r + 1 \\
m - n + n - p &= 2r + 1 + 2s + 1 \text{ by adding } n\,O\,p \\
m - p &= 4s + 2 \\
m - p &= 2(2s + 1)
\end{aligned}$$

- Because $\mathbb{Z}$ is closued under multiplication and addition, $(2s + 1)$ is an integer.
- By definition of even, $2(2s + 1)$ is even, and by extension, not odd.
- By definition of O, m is not related to p by O.
- **O is not transitive**.

Problem 4

Find inverse function G for

$$F(x) = \frac{x-1}{4}$$

Answer

$$\begin{aligned}
F(x) &= \frac{x-1}{4} \\
y &= \frac{x-1}{4} \text{ by definition of } F \\
4y &= x - 1 \\
4y + 1 &= x \\
x &= 4y + 1 \text{ because equality is symmetric} \\
\boldsymbol{G(y)} &= \boldsymbol{4y+1} \text{ by definition of inverse}
\end{aligned}$$

Problem 5

Define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{Z}$ as follows:

$$\textit{For all } x \in \mathbb{R},$$

$$F(x) = \frac{x^2}{4}$$

$$G(x) = \lfloor x \rfloor$$

What is $(G \circ F)(7)$?

Answer

$$(G \circ F)(x) = \left\lfloor \frac{x^2}{4} \right\rfloor$$

$$\begin{aligned}(G \circ F)(7) &= \left\lfloor \frac{7^2}{4} \right\rfloor \\ &= \left\lfloor \frac{49}{4} \right\rfloor \\ &= \lfloor 12.25 \rfloor \\ &= \mathbf{12}\end{aligned}$$

Problem 6

If 5 integers are chosen from $\{1, 2, 3, 4, 5, 6, 7, 8\}$, must there be at least two integers with the property that the larger of the two minus the smaller of the two equals 2?

Answer

- We can start by partitioning the set into pairs of integers that satisfy the property:

$$\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}$$

- Now, these sets are all disjoint, and altogether make up the partition of the original set.
  - This means that every integer appears exactly once across all these subsets.
- Thus, if we think about the different integers as pigeons, and the different pair subsets they may be part of as the pigeonholes, then there are 5 pigeons as opposed to 4 pigeonholes.
- Thus, by the pigeonhole principle, after choosing 5 different integers from the original set, at least one of the subset will have both of their integers chosen.
- **Now, because each of the subsets formed is a pair of integers that satisfy the aforementioned property, this means that when 5 integers are chosen, it is guaranteed that there will be at least two integers that satisfy the property.**

Problem 7

Prove that for each even integer n for $6 \leq n \leq 18$, n can be written as the sum of exactly two prime numbers.

Answer

$$
\begin{aligned}
6 &= 3 + 3 \\
8 &= 3 + 5 \\
10 &= 5 + 5 \\
12 &= 5 + 7 \\
14 &= 7 + 7 \\
16 &= 5 + 11 \\
18 &= 7 + 11
\end{aligned}
$$

Problem 8

A club has seven members. Three are chosen to go as a group to a national meeting.

1. How many distinct groups of three can be chosen?
2. If the club contains four men and three women, how many distinct groups of three contain two men and one woman?

Answer

Question 1:

$$\begin{aligned}\binom{7}{3} &= \frac{7!}{3!(7-3)!} \\ &= \frac{7!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5}{3!} \\ &= 7 \cdot 5 \\ &= \textbf{35 distinct groups of three}\end{aligned}$$

Question 2:

$$\begin{aligned}\binom{4}{2} \cdot \binom{3}{1} &= \frac{4!}{2!(4-2)!} \cdot 3 \\ &= \frac{4!}{2!2!} \cdot 3 \\ &= \frac{24}{4} \cdot 3 \\ &= 6 \cdot 3 \\ &= \textbf{18 distinct groups with two men and one woman}\end{aligned}$$

Problem 9

Let S be the set of all strings of 0s and 1s of length 3. Define a relation R on S as follows:

For all strings s and $t \in S$,

$$s\,R\,t \Leftrightarrow \text{The two leftmost characters in } s \text{ are the same as the two leftmost characters in } t$$

Prove that R is an equivalence relation on S.

Answer

Proving that R is reflexive:

- Suppose that s is any string in S.
- Logically, it follows that the two leftmost characters of s are the same as the two leftmost characters of s.
- Hence, by definition of R, $s\,R\,s$.
- **Thus, R is reflexive.**

Proving that R is symmetric:

- Suppose that s and t are any strings in S such that

$$s\,R\,t$$

- By definition of R, the two leftmost characters in s are the same as the two leftmost characters in t.
- Logically, it follows that the two leftmost characters in t are also the same as the two leftmost characters in s.
- Hence, by definition of R, $t\,R\,s$.
- **Thus, R is symmetric.**

Proving that R is transitive:

- Suppose that s, t, and u are any strings in S such that

$$s\,R\,t \text{ and } t\,R\,u$$

- By definition of R, the two leftmost characters in s are the same as the two leftmost characters in t and the two leftmost characters in t are the same as the two leftmost characters in u.
- Logically, it follows that the two leftmost characters in s are the same as the two leftmost characters in u.
- Hence, by definition of R, $s\,R\,u$.
- **Thus, R is transitive.**

Because R is reflexive, symmetric, and transitive, it is an equivalence relation.

Problem 10

Let R be the relation defined on $\mathbb{Z}$ as follows:

For integers m and $n \in R$,

$$m\,R\,n \Leftrightarrow 5 \mid (m-n)$$

Prove that R is an equivalence relation.

Answer

Proving that R is reflexive:

- Suppose m is any integer.

$$m - m = 0$$

- By definition of divisibility, 5 divides 0 because $0 = 5 \cdot 0$.
- By definition of R, $5 \mid (m-m)$.
- **Thus, R is reflexive.**

Proving that R is symmetric:

- Suppose that m and n are any integers such that

$$m\,R\,n$$

- By definition of R, $5 \mid (m-n)$
- By definition of divisibility, $m - n = 5k$ for some integer k.

$$\begin{aligned} m - n &= 5k \\ -m + n &= -5k \text{ by multiplying by} -1 \\ n - m &= -5k \text{ by commutative property} \\ n - m &= 5(-k) \text{ by associative property} \end{aligned}$$

- Because $\mathbb{Z}$ is closed under multiplication, $(-k)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (n-m)$.
- By definition of R, $n\,R\,m$.
- **Thus, R is symmetric.**

Proving that R is transitive:

- Suppose that m, n, and p are any integers such that

$$m\,R\,n \text{ and } n\,R\,p$$

- By definition of R, $5 \mid (m-n)$ and $5 \mid (n-p)$.
- By definition of divisibility, $m - n = 5r$ and $n - p = 5s$ for some integers r and s.

$$\begin{aligned} m - n &= 5r \\ m - n + n - p &= 5r + 5s \text{ by adding them together} \\ m - p &= 5r + 5s \\ m - p &= 5(r+s) \text{ by distributive property} \end{aligned}$$

- Because $\mathbb{Z}$ is closed under addition, $(r+s)$ is an integer.
- Thus, by definition of divisibility, $5 \mid (m-p)$.
- By definition of R, $m\,R\,p$.
- **Thus, R is transitive.** *(Final answer on next page).*

Because R is reflexive, symmetric, and transitive, it is an equivalence relation.

Problem 11

In a class with 30 students, three quizzes were given.

- 15 scored 12 or higher on quiz $\#1$.
- 12 scored 12 or higher on quiz $\#2$.
- 18 scored 12 or higher on quiz $\#3$.
- 7 scored 12 or higher on quiz $\#1$ and quiz $\#2$.
- 11 scored 12 or higher on quiz $\#1$ and quiz $\#3$.
- 8 scored 12 or higher on quiz $\#2$ and quiz $\#3$.
- 4 scored 12 or higher on quiz $\#1$, quiz $\#2$, and quiz $\#3$.

How many students scored 12 or higher on quiz $\#1$ and $\#2$, but not quiz $\#3$?

Answer

- Out of the 7 students who scored 12 or higher on quiz $\#1$ and $\#2$, 4 of them also scored 12 or higher on quiz $\#3$.
- Thus $7 - 4 = \mathbf{3}$ **students scored 12 or higher on quiz #1 and #2, but not on quiz #3.**

Problem 12

Let R be a congruence modulo 5 relation. Which of the following equivalence classes are equal?

$$[35], [3], [12], [0], [-2], [17], [492]$$

Answer

$$\begin{aligned} [0] &= [35] \\ [-2] &= [3] \\ [12] &= [17] = [492] \end{aligned}$$

Problem 13

A function, $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined as follows:

$$\textit{For all } (x, y) \in \mathbb{R} \times \mathbb{R},$$

$$F(x, y) = (3y - 1, 1 - x)$$

Prove that F is a one-to-one correspondence.

Answer

Proving that F is one-to-one:

- Suppose x_1, y_1, x_2, and y_2 are real numbers such that $F(x_1, y_1) = F(x_2, y_2)$.

$$\begin{aligned} F(x_1, y_1) &= F(x_2, y_2) \\ (3y_1 - 1, 1 - x_1) &= (3y_2 - 1, 1 - x_2) \end{aligned}$$

- Equating the left sides of both tuples:

$$\begin{aligned} 3y_1 - 1 &= 3y_2 - 1 \\ 3y_1 &= 3y_2 \\ y_1 &= y_2 \end{aligned}$$

- Equating the right sides of both tuples:

$$\begin{aligned} 1 - x_1 &= 1 - x_2 \\ -x_1 &= -x_2 \\ x_1 &= x_2 \end{aligned}$$

- Thus, by definition of equality for tuples, $(x_1, y_1) = (x_2, y_2)$.
- **Therefore, F is one-to-one.**

Proving that F is onto:

- Suppose (u, v) is any tuple in $\mathbb{R} \times \mathbb{R}$.
- Let $x = 1 - v$ and $y = \frac{1}{3}u + \frac{1}{3}$.

$$\begin{aligned} F(x, y) &= F\left(1 - v, \frac{1}{3}u + \frac{1}{3}\right) \\ &= \left(3\left(\frac{1}{3}u + \frac{1}{3}\right) - 1, 1 - (1 - v)\right) \\ &= (u + 1 - 1, 1 - 1 + v) \\ &= (u, v) \end{aligned}$$

- **Therefore, F is onto.**

Because F is one-to-one and onto, it is a one-to-one correspondence.

Problem 14

Consider the Boolean function f defined as follows:

For every (x_1, x_2, x_3) *of 0s and 1s,*

$$f(x_1, x_2, x_3) = (7x_1 + 2x_2 + 6x_3) \bmod 2$$

Compute $f(0,0,0)$, $f(0,0,1)$, $f(1,1,0)$, $f(0,1,1)$, and $f(0,1,0)$.

Answer

$$\begin{aligned} f(0,0,0) &= (7 \cdot 0 + 2 \cdot 0 + 6 \cdot 0) \bmod 2 \\ &= 0 \bmod 2 \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} f(0,0,1) &= (7 \cdot 0 + 2 \cdot 0 + 6 \cdot 1) \bmod 2 \\ &= 6 \bmod 2 \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} f(1,1,0) &= (7 \cdot 1 + 2 \cdot 1 + 6 \cdot 0) \bmod 2 \\ &= (7 + 2) \bmod 2 \\ &= 9 \bmod 2 \\ &= \mathbf{1} \end{aligned}$$

$$\begin{aligned} f(0,1,1) &= (7 \cdot 0 + 2 \cdot 1 + 6 \cdot 1) \bmod 2 \\ &= (6 + 2) \bmod 2 \\ &= 8 \bmod 2 \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} f(0,1,0) &= (7 \cdot 0 + 2 \cdot 1 + 6 \cdot 0) \bmod 2 \\ &= 2 \bmod 2 \\ &= \mathbf{0} \end{aligned}$$

Problem 15

Let $A = \{a, b, c, d\}$ and let R be defined on A as follows:

$$R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$$

Additionally, R is an equivalence relation. How many distinct equivalence classes does R have?

Answer

$$\begin{aligned}
[a] &= \{a\} \\
[b] &= \{b, d\} \\
[c] &= \{c\} \\
[d] &= \{b, d\} = [b]
\end{aligned}$$

- Thus, R has **3 distinct equivalence classes: $\{a\}$, $\{b, d\}$ and $\{c\}$.**

Problem 16

There are 50 people at a party whose ages range from 18 to 25 years old. What is the largest amount of people you can be certain are the same age at the party?

Answer

- Because the possible ages at the party are between 18 and 25, inclusive, there are 8 possible ages people could be.
- Imagining the people at the party as pigeons and the possible ages as pigeonholes, there must be at least one age that $k + 1$ people share by the generalized pigeonhole principle.
- We can maximizing k by finding the integer quotient of the number of people divided by the number of possible ages.

$$\begin{aligned} k &= \left\lfloor \frac{50}{8} \right\rfloor \\ &= \lfloor 6.25 \rfloor \\ &= 6 \end{aligned}$$

- Thus, there are at least $6 + 1 =$ **7 people who have the same age at the party.**

Problem 17

Find a positive inverse for $5 \bmod 49$.

Answer

- Let n be the positive inverse for $5 \bmod 49$

$$5n \bmod 49$$

$$1 = 5n - 49k \text{ for some integer } k$$

$$1 + 49k = 5n$$

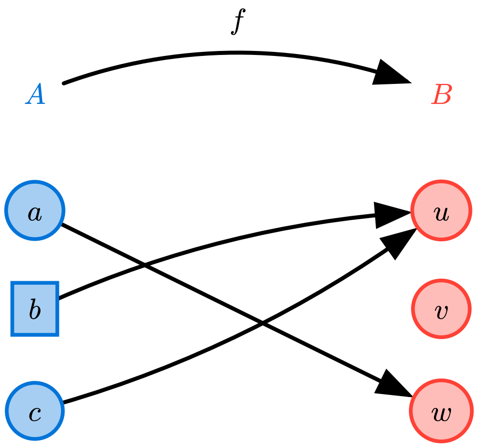
- Now, k may be any integer. Thus, we can choose $k = 1$.

$$\begin{aligned} 1 + 49 &= 5n \\ 50 &= 5n \\ 5n &= 50 \\ n &= 10 \end{aligned}$$

- **10 is a positive inverse for $\mathbf{5 \bmod 49}$.**

Problem 18

Consider the following diagram representing a function f:



1. What is the domain of f?
2. What is the co-domain of f?
3. State the range of f.
4. State the image of a under f.
5. Represent f as a set of ordered pairs.

Answer

Question 1:

- **The domain of f is $\{a, b, c\}$.**

Question 2:

- **The co-domain of f is $\{u, v, w\}$.**

Question 3:

- **The range of f is $\{u, w\}$.**

Question 4:

- **The image of a under f is w.**

Question 5:

$$\boldsymbol{f = \{(a, w), (b, u), (c, u)\}}$$

Problem 19

Define a function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ as follows:

For all nonzero real numbers x,

$$f(x) = \frac{x+3}{x}$$

1. Prove that f is one-to-one.
2. Find the inverse function for f, f^{-1}.

Answer

Question 1:

- Suppose that x_1 and x_2 are any real numbers such that

$$f(x_1) = f(x_2)$$

- Now,

$$\begin{aligned}
\frac{x_1+3}{x_1} &= \frac{x_2+3}{x_2} \text{ by definition of } f \\
(x_1+3) &= x_1\frac{x_2+3}{x_2} \\
x_2(x_1+3) &= x_1(x_2+3) \\
x_1x_2 + 3x_2 &= x_1x_2 + 3x_1 \\
3x_2 &= 3x_1 \\
x_2 &= x_1
\end{aligned}$$

- **Thus, f is one-to-one.**

Question 2:

$$\begin{aligned}
y &= f(x) \\
y &= \frac{x+3}{x} \text{ by definition of } f \\
xy &= x+3 \\
xy - x &= 3 \\
x(y-1) &= 3 \\
x &= \frac{3}{y-1} \\
\boldsymbol{f^{-1}(y)} &= \boldsymbol{\frac{3}{y-1}} \text{ by definition of inverse}
\end{aligned}$$

Problem 20

Prove that the set of all integers and the set of all odd integers have the same cardinality.

Answer

- If $\mathbb{Z}$ denotes the set of all integers, then $2\mathbb{Z}+1$ denotes the set of all odd integers by definition of odd.
- Thus, a function $f : \mathbb{Z} \to 2\mathbb{Z}+1$ may be defined as follows:

For all integers x

$$f(x) = 2x + 1$$

Showing that f is one-to-one:

- Suppose x_1 and x_2 are any integers such that

$$f(x_1) = f(x_2)$$

- Now,

$$\begin{aligned} 2x_1 + 1 &= 2x_2 + 1 \text{ by definition of } f \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

- Thus, f is one-to-one.

Showing that f is onto:

- Suppose y is any odd integer.
- Let $x = \frac{1}{2}y - \frac{1}{2}$

$$\begin{aligned} f(x) &= f(y) \\ &= 2\left(\frac{1}{2}y - \frac{1}{2}\right) + 1 \\ &= y - 1 + 1 \\ &= y \end{aligned}$$

- Thus, f is onto.
- Now, because f is one-to-one and onto, it is a one-to-one correspondence.
- **Because there exists a one-to-one correspondence between $\mathbb{Z}$ and $2\mathbb{Z}+1$, they have the same cardinality.**

Problem 21

Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as follows:

$$R = \{(0,2),(0,3),(2,0),(2,1)\}$$

1. State the inverse of R, R^{-1}.
2. Is R reflexive?
3. Is R symmetric?
4. Is R transitive?
5. Compute the transitive closure of R.

Answer

Question 1:

$$\boldsymbol{R^{-1}} = \{(\mathbf{2},\mathbf{0}),(\mathbf{3},\mathbf{0}),(\mathbf{0},\mathbf{2}),(\mathbf{1},\mathbf{2})\}$$

Question 2:

- $(2,2)$ is not an element of R.
- **Thus, $\boldsymbol{R}$ is not reflexive.**

Question 3:

- While $(3,0)$ is an element of R, $(0,3)$ is not an element of R.
- **Thus, $\boldsymbol{R}$ is not symmetric.**

Question 4:

- While $(0,2)$ and $(2,1)$ are elements of R, $(0,1)$ is not an element of R.
- **Thus, $\boldsymbol{R}$ is not transitive.**

Question 5:

$$\boldsymbol{R^t} = \{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1}),(\mathbf{0},\mathbf{2}),(\mathbf{0},\mathbf{3}),(\mathbf{2},\mathbf{0}),(\mathbf{2},\mathbf{2}),(\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{1})\}$$

NOTE: The loops in the relation are from elements' transitive connections to themselves through parallel edges.

Problem 22

How many elements are in the one-dimensional array shown below?

$$A[7], A[8], ..., A\left[\left\lfloor \frac{145}{2} \right\rfloor\right]$$

Answer

$$\begin{aligned}\left\lfloor \frac{145}{2} \right\rfloor &= \lfloor 72.5 \rfloor \\ &= 72\end{aligned}$$

- Thus, the index of the last element in the provided array is 72.
- Thus, there are $72 - 7 + 1 = \mathbf{66}$ **elements in the one-dimensional array.**

Problem 23

Three officers—president, treasurer, and secretary—are chosen from among four people: Ann, Bob, Cyd, and Dan.

Bob is not qualified to be a treasurer.

Cyd's other commitments make it impossible for her to be a secretary.

Dan is irresponsible with money (cannot be treasurer) and might be corrupted by power (cannot be president).

How many ways can the officers be chosen?

Answer

- We can start off by recalling the restrictions.
- Dan cannot be president (3 remaining).
- Bob and Dan cannot be treasurer (2 remaining).
- Cyd cannot be secretary (3 remaining).
- Based on this the only the following lists of officers can be created:

$$\{\text{Ann}, \text{Cyd}, \text{Bob}\}$$
$$\{\text{Ann}, \text{Cyd}, \text{Dan}\}$$
$$\{\text{Bob}, \text{Ann}, \text{Dan}\}$$
$$\{\text{Bob}, \text{Cyd}, \text{Ann}\}$$
$$\{\text{Bob}, \text{Cyd}, \text{Dan}\}$$
$$\{\text{Cyd}, \text{Ann}, \text{Bob}\}$$
$$\{\text{Cyd}, \text{Ann}, \text{Dan}\}$$
$$\textit{The format is } \{\textit{president}, \textit{treasurer}, \textit{secretary}\}$$

- There are **7 ways the officers can be chosen.**

Problem 24

A club has seven members. Three are chosen to go as a group to a national meeting.

1. If the club contains four men and three women, how many distinct groups of three contain at least one woman?

$$\begin{aligned}
\binom{7}{3} - \binom{4}{3} &= \frac{7!}{3!(7-3)!} - \frac{4!}{3!(4-1)!} \\
&= \frac{7!}{3!4!} - \frac{4!}{3!} \\
&= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!4!} - \frac{4 \cdot 3!}{3!} \\
&= \frac{7 \cdot 6 \cdot 5}{3!} - 4 \\
&= \frac{7 \cdot 6 \cdot 5}{6} - 4 \\
&= 7 \cdot 6 - 4 \\
&= 35 - 4 \\
&= \mathbf{31\ distinct\ teams}
\end{aligned}$$

Problem 25

In a class with 30 students, three quizzes were given.

- 15 scored 12 or higher on quiz $\#1$.
- 12 scored 12 or higher on quiz $\#2$.
- 18 scored 12 or higher on quiz $\#3$.
- 7 scored 12 or higher on quiz $\#1$ and quiz $\#2$.
- 11 scored 12 or higher on quiz $\#1$ and quiz $\#3$.
- 8 scored 12 or higher on quiz $\#2$ and quiz $\#3$.
- 4 scored 12 or higher on quiz $\#1$, quiz $\#2$, and quiz $\#3$.

How many students scored 12 or higher on quiz $\#1$ or $\#2$, but not quiz $\#3$?

Answer

- The total number of students who scored 12 or higher on quiz $\#1$ or quiz $\#2$ is $15 + 12 - 7 = 20$ students.
  - *The minus 7 is to account for the double counting of the subset of people who scored 12 or higher on both.*
- Thus, $\mathbf{20 - 11 - 8 + 4}$ **students scored 12 or higher on quiz #1 or #2, but not on quiz #3.**
  - *Similarly, the plus 4 is to account for the double subtraction of the subset of people who scored 12 or higher on all quizzes.*

Problem 26

In a certain state, license plates each consist of 2 letters followed by 3 digits.

1. How many different license plates are there?
2. How many different license plates are there that have no repeated letters or digits?
3. How many different license plates are there that begin with the letters "MC."

Answer

Question 1:

- There are $26^2 \cdot 10^3 = 676 \cdot 1000 = \mathbf{676000}$ **different license plates.**

Question 2:

- There are $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = \mathbf{468000}$ **different license plates with no repeated letters or digits.**

Question 3:

- There are $10^3 = \mathbf{1000}$ **different license plates beginning with the letters "MC."**

Problem 27

How many three-letter arrangements are there of the word, “FINAL?”

- Because we are looking for arrangements, each letter is distinct and there are no repeats.
- Thus, there are

$$
\begin{aligned}
P(5,3) &= \frac{5!}{(5-3)!} \\
&= \frac{5!}{2!} \\
&= \frac{120}{2} \\
&= \textbf{60 three-letter arrangements}
\end{aligned}
$$