Chapter 9 Discrete Mathematics Notes

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Contents

9 Counting and Probability

9.1 Introduction to Probability

- A process is **random** when it leads to an outcome from some set of outcomes, and it is impossible to predict with certainty what it may be.
  - In a coin flip, the set of outcomes is heads and tails, but it cannot be predicted with certainty if it is heads or tails, even considering a calculated probability of 50%.

Definition:
A **sample space** is the set of all possible outcomes of a random process or experiment.
An **event** is a subset of a sample space.

- Formally, given S, a finite sample space where all outcomes are equally like, and E, an event in S, then **probability of** E, denoted $P(E)$ is:

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}$$

- Or, given $N(A)$, the number of elements in some set A:

$$P(E) = \frac{N(E)}{N(S)}$$

Counting the Elements of a List

- Sometimes, counting list elements can be easy, such as listing the number of elements from one integer through another.
- However, there is an exact formula for this, which is helpful for determining the number of iterations with variables.

Theorem 9.1.1:
If m and n are integers such that $m \le n$, then there are $n - m + 1$ integers from m to n, inclusive.

Example: Counting the Elements of a Sublist
How many three-digit integers exist are divisible by 5?
- For each three-digit integer divisible by 5, there is a corresponding integer from 20 to 199, inclusive.
- Hence, by **Theorem 9.1.1.**, there are $199 - 20 + 1 = 180$ three-digit integers divisible by 5.

What is the probability that a randomly chosen three-digit integer is divisible by 5?
- By **Theorem 9.1.1.**, there are $999 - 100 + 1 = 900$ total three-digit integers.
- Using the previous answer, we can now calculate the probability:

$$\frac{180}{900} = \frac{1}{5} \text{ by algebra}$$

9.2 Possibility Trees and the Multiplication Rule

- Tree structures are useful for keeping track of all the possibilities in situations where events happen in a particular order.

Example: Tournament Play Possibilities

Suppose there are two teams, A and B, who continuously play against each other until one team wins twice in a row or wins three times total. A series of wins may be denoted by the team names, like A,B,B.

How many ways can the tournament be played?

- Using a tree, we can outline all the possibilities up to 5 levels:

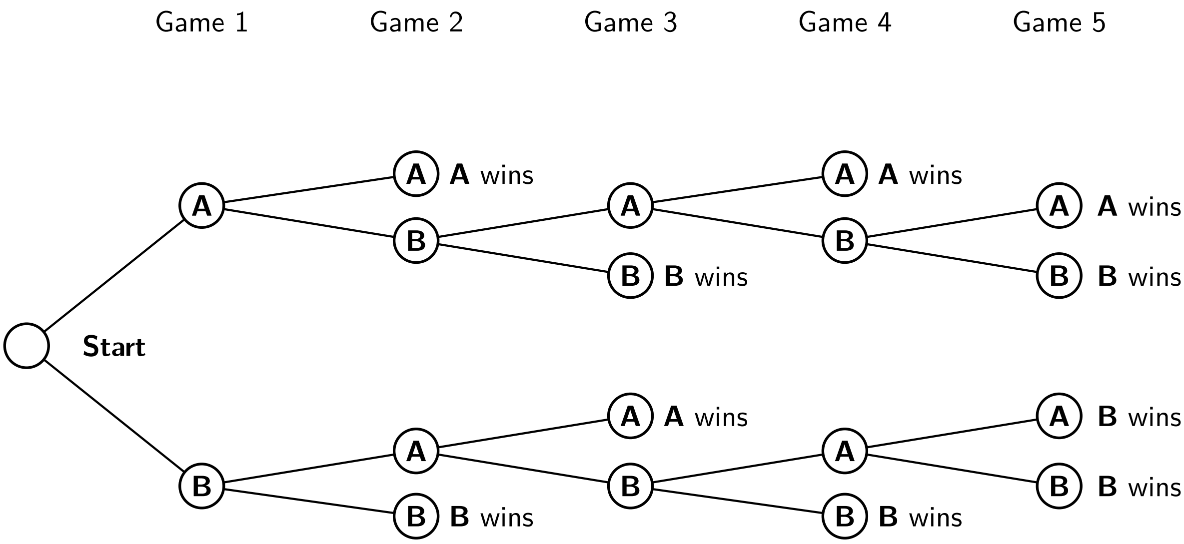

Diagram 9.2.1: The possibility tree for the tournament.

What are all the ways the tournament can be played?

- As outlined in the possibility tree, there are ten different ways the tournament can be played, with A and B each having five different events where they win.

If each outcome is assumed to be equally likely, what is the likelihood that a team wins after five games?

- There are 4 leaf nodes on the tree, each representing a win on the fifth game.
- Thus, the probability is as follows:

$$\frac{4}{10} = \frac{1}{5} \text{ by algebra}$$
$$= 40\% \text{ by converting to percent}$$

The Multiplication Rule

- Imagine a computer installation with four IO units A, B, C, and D, and three CPUs X, Y, and Z.
- There should be twelve ways to pair the IO units with the CPUs.
- Note that there are CPU options available for each IO unit, and there are four IO units.
- Thus, $4 \cdot 3 = 12$ pairs.

Theorem 9.2.1:
If an operation consists of k steps, and if every ith step for each integer $1 \le i \le k$ can be completed in n_i ways, then the entire operation may be completed in $n_1 \cdot n_2 \cdots n_k$ ways.

Example: Multiplication Rule with Nested Loops
How many times does the following nested loop run?

```
for i in range(4):
  for j in range(3):
    #Loop body
```

- The outer loop has four iterations.
- For each iteration in the outer loop, the inner loop has three iterations.
- By the multiplication rule, the nested loop has $4 \cdot 3 = 12$ iterations.

Permutations

- A **permutation** of a set of objects is a particular ordering of the objects in a row.
- For instance, the set $\{a, b, c\}$ has 6 permutations:
- Note that for each successive step, there is one less way to perform it.
- Thus, by the multiplication rule, there are $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ ways to perform the entire operation.
  - Note that this matches the recursive definition for factorial.

Theorem 9.2.6:
For any integer $n \ge 1$, a set with n elements has $n!$ permutations.

Permutations of Selected Elements

- Again, consider the set $\{a, b, c\}$.
- There are six different ways to select two different letters from the set and write them in a particular order:

$$ab, ac, ba, bc, ca, cb$$

- Altogether, these orderings are the 2-permutations of $\{a, b, c\}$.

Definition:
An r**-permutation** for a set of n elements is an ordered selection of r elements from the set. The number of r-permutations in a set of n elements is denoted $P(n, r)$.

Theorem 9.2.3:
If n and r are integers such that $1 \le r \le n$, the number of r-permutations of a set of n elements is given by the formula:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$
or
$$P(n,r) = \frac{n!}{(n-r)!}$$

- Logically, given a set of length n, the number of n-permutations in the set is equal to the number of permutations, or just $n!$.

9.3 Counting Elements of Disjoint Sets: The Addition Rule

- We may count the number elements in unions, intersections, and differences between sets.
- The number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the operands.

Theorem 9.3.1:
Suppose a finite set A equals the union of k distinct mutually disjoint sets $A_1, A_2, \ldots, A_k$. Thus,

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k)$$

- Logically, this only applies to mutually disjoint finite sets because overlapping elements are not counted separately.

The Difference Rule

- As a consequence of the addition rule, if the number of elements in set A and its subset B are known, then the number of elements in $A - B$ may be computed.

Theorem 9.3.2:
If A is a finite set and $B \subseteq A$, then:

$$N(A-B) = N(A) - N(B)$$

- Additionally, there is a related formula for the probability of an event's complement:

If S is a finite sample space, and A is an event in S, then

$$P\left(A^{C}\right) = 1 - P(A)$$

The Inclusion/Exclusion Rule

- Recall that the addition rule is for unions of sets that are *mutually disjoint*.
- Thus, to count the number of elements in the union of sets with overlapping elements, we must subtract the number of overlapping elements after applying the addition rule.
  - This is because overlapping elements will be counted more than once.

Theorem 9.3.3:
If A, b, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

9.4 The Pigeonhole Principle

- According to the **pigeonhole principle**, given $m < n$

9.5

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9.7