Chapter 9 Homework

Problem 1

Prove that for all integers n, k, and r for $r \leq k \leq n$ that

$$\binom{n}{k} \cdot \binom{k}{r} = \binom{n}{r} \cdot \binom{n-r}{k-r}$$

Proof

$$\begin{aligned}
\binom{n}{k} \cdot \binom{k}{r} &= \frac{n!}{k!(n-k)!} \cdot \frac{k!}{r!(k-r)!} \text{ by definition of combinations} \\
&= \frac{n!k!}{k!(n-k)!r!(k-r)!} \text{ by multiplication} \\
&= \frac{n!}{(n-k)!r!(k-r)!} \text{ by canceling out } k! \\
&= \frac{n!}{r!(n-k)!(k-r)!} \text{ by commutative property} \\
&= \frac{n!(n-r)!}{r!(n-r!)(n-k)!(k-r)!} \text{ by multiplying by } 1 = \frac{(n-r)!}{(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!(n-k)!} \text{ by commutative property} \\
&= \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!(n-r-k+r)!} \text{ by adding } 0 = r - r \text{ to } (n-k)! \\
&= \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!(n-r-(k-r))!} \text{ by distributive property} \\
&= \binom{n}{r} \cdot \binom{n-r}{k-r} \text{ by definition of combination}
\end{aligned}$$

Assuming that these integers are all positive, multiplying by $(n-r)!/(n-r)!$ *works for* $r \leq k \leq n$ *because the recursive definition for factorial states that* $0! = 1$.

Problem 1 Answer:

As shown in the proof above, by definition of equality

$$\binom{n}{k} \cdot \binom{k}{r} = \binom{n}{r} \cdot \binom{n-r}{k-r}$$

Problem 2

The binomial theorem states that for any real numbers a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \text{ for any integer } n \geq 0$$

Use this theorem to show that for any integer $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k = 1$$

Proof

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k &= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} 2^k (-1)^k \text{ by commutative property} \\
&= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (2 \cdot (-1))^k \text{ by distributive property} \\
&= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-2)^k \text{ by multiplication} \\
&= (3-2)^n \text{ by the binomial theorem} \\
&= 1^n \text{ by subtraction} \\
&= 1 \text{ because 1 to any power is 1}
\end{aligned}$$

Problem 2 Answer:

As show in the proof above, by definition of equality

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k = 1$$