NOTES ON SET THEORY

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1. Preliminaries

Definition 1.1 (Syntax of one-order logic)**.** A **signature** σ is a set of relation symbols, function symbols and constant symbols, each with a specified arity. A **first-order language** $\mathcal{L}$ over a signature σ consists of:

1. a countable set of variables $x_0, x_1, x_2, \ldots$
2. the symbols in σ
3. the logical symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, =$
4. the punctuation symbols (,), and ,.

The terms and formulas of $\mathcal{L}$ are defined inductively as follows:

1. every variable and constant symbol is a term
2. if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term
3. if R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is an atomic formula
4. if φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$
5. if φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

A variable x is free in a formula φ if it is not bound by a quantifier in φ. A formula with no free variables is called a sentence.

2. Axioms

Axiom 2.1 (Axiom of Extensionality)**.** Two sets are equal if and only if they have the same elements:

$$\forall A \forall B (\forall x (x \in A \iff x \in B) \iff A = B).$$

Axiom 2.2 (Axiom of Empty Set)**.** There exists a set with no elements, denoted by $\emptyset$.

$$\exists A \forall x (x \notin A).$$

Axiom 2.3 (Axiom of Pairing)**.** For any sets A and B, there exists a set $\{A, B\}$ whose elements are exactly A and B.

$$\forall A \forall B \exists C \forall x (x \in C \iff (x = A \vee x = B)).$$

Axiom 2.4 (Axiom of Union)**.** For any sets A and B, there exists a set $A \cup B$ whose elements are exactly the elements of A or B.

$$\forall A \forall B \exists C \forall x (x \in C \iff (x \in A \vee x \in B)).$$

Axiom 2.5 (Axiom of Power Set)**.** For any set A, there exists a set $\mathcal{P}(A)$ whose elements are exactly the subsets of A.

$$\forall A \exists B \forall x (x \in B \iff x \subseteq A).$$

Axiom 2.6 (Subset Axiom Schema/Axiom of Separation)**.** For any set A and any formula $\varphi(x, t_0, \ldots, t_n)$ not involving A, there exists a set whose elements are exactly the elements of A that satisfy $\varphi(x, t_0, \ldots, t_n)$.

$$\forall t_0, \ldots, t_n \forall A \exists B \forall x (x \in B \iff (x \in A \wedge \varphi(x, t_0, \ldots, t_n))).$$

Remark 2.7 (Intersection and Union)*.* We can define the intersection/union of a set of sets A as follows(Not strictly):

$$\bigcap_{B \in A} B := \{x : \forall B \in A (x \in B)\}, \quad \bigcup_{B \in A} B := \{x : \exists B \in A (x \in B)\}.$$

Remark 2.8 (Function)*.* A function f from a set A to a set B, denoted by $f : A \to B$, is a subset of $A \times B$ such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. The set A is called the domain of f and the set B is called the codomain of f. The element b is called the image of a under f, denoted by $f(a)$.

Axiom 2.9 (Axiom of Choice)**.** For any relation R, there exists a function f such that for every x, if there exists a y where $dom(f) = dom(R)$ and $ran(f) \subseteq ran(R)$, and $(x, f(x)) \in R$.

3. Natural Numbers

Definition 3.1. Let S be a set, we define the successor of S as $S^+ = S \cup \{S\}$.

Remark 3.2. Denote $0 = \emptyset$, $1 = 0^+ = \{\emptyset\}$, $2 = 1^+ = \{\emptyset, \{\emptyset\}\}$, $3 = 2^+ = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, and so on.

Definition 3.3 (Inductive Set)**.** A set A is called inductive if $\emptyset \in A$ and for every $x \in A$, $x^+ \in A$.

Axiom 3.4 (Axiom of Infinity)**.** There exists an inductive set.

$$\exists A (\emptyset \in A \wedge \forall x (x \in A \to x^+ \in A)).$$

Theorem 3.5. *There exists a set ω consisting of exactly the natural numbers.*

Proof. Suppose A is an inductive set. We use the subset axiom instead of the intersection because the set of all inductive sets may not exist. Let

$$\omega = \{x \in A : \forall B (B \text{ is inductive} \to x \in B)\}.$$

We can show that every element in ω is either $\emptyset$ or the successor of an element in ω. Thus, ω consists of exactly the natural numbers. □

Theorem 3.6. *The set ω is the smallest inductive set, i.e., ω is an inductive set and for any inductive set A, $\omega \subseteq A$.*

Definition 3.7. A set A is called transitive if it satisfies the following equivalent conditions:

1. for every $x \in A$, if $y \in x$, then $y \in A$;
2. $A \subseteq \mathcal{P}(A)$;
3. $\bigcup A \subseteq A$.

Theorem 3.8. *Every natural number is a transitive set.*

References