NOTES ON MODEL THEORIES

XINGZHI HUANG

1. Structures

Definition 1.1. A structure $\mathcal{A} := (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$ consists of:

1. a non-empty set A, called the universe of $\mathcal{A}$
2. a relation $R_i^{\mathcal{A}}$ on A for each $i \in I$
3. a function $f_j^{\mathcal{A}} : A^{n_j} \to A$ for each $j \in J$
4. an element $c_k^{\mathcal{A}} \in A$ for each $k \in K$.

The sets I, J, K are disjoint index sets. The arity of $R_i^{\mathcal{A}}$ is some natural number m_i, and the arity of $f_j^{\mathcal{A}}$ is some natural number n_j.

Definition 1.2. Given a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$, the signature $L_{\mathcal{A}}$ of $\mathcal{A}$ is the collection of symbols $\{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}$, where R_i is a relation symbol of arity m_i, f_j is a function symbol of arity n_j, and c_k is a constant symbol.

Definition 1.3. Fix a signature L. An L-structure is a structure whose signature is L.

Definition 1.4. We can define a category of L-structures, denoted by $Str(L)$, as follows.

The objects of $Str(L)$ are all L-structures. If $\mathcal{A}$ and $\mathcal{B}$ are two L-structures, then a morphism from $\mathcal{A}$ to $\mathcal{B}$ is a function $h : A \to B$ such that:

1. for each relation symbol R_i of arity m_i in L, and for all $a_1, a_2, \ldots, a_{m_i} \in A$, if $(a_1, a_2, \ldots, a_{m_i}) \in R_i^{\mathcal{A}}$, then $(h(a_1), h(a_2), \ldots, h(a_{m_i})) \in R_i^{\mathcal{B}}$;
2. for each function symbol f_j of arity n_j in L, and for all $a_1, a_2, \ldots, a_{n_j} \in A$, we have $h(f_j^{\mathcal{A}}(a_1, a_2, \ldots, a_{n_j})) = f_j^{\mathcal{B}}(h(a_1), h(a_2), \ldots, h(a_{n_j}))$;
3. for each constant symbol c_k in L, we have $h(c_k^{\mathcal{A}}) = c_k^{\mathcal{B}}$.

Theorem 1.5. *The injective morphisms in $Str(L)$ are precisely the embeddings; the surjective morphisms in $Str(L)$ are precisely the surjective homomorphisms.*

Definition 1.6. Given two L-structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is a substructure of $\mathcal{B}$, denoted by $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and the inclusion map $i : A \to B$ is an embedding.

2. Terms and Formulas

2.1. Syntax.

Definition 2.1. Let L be a signature. The set of L-terms is defined inductively as follows:

1. Every variable is an L-term.

2. Every constant symbol in L is an L-term.
3. If f is an n-ary function symbol in L and $t_1, t_2, \ldots, t_n$ are L-terms, then $f(t_1, t_2, \ldots, t_n)$ is an L-term.
4. Nothing else is an L-term.

Remark 2.2. A term is called closed if it contains no variables.

Remark 2.3. The complexity of a term t, denoted by $c(t)$, is defined inductively as follows:

1. If t is a variable or a constant symbol, then $c(t) = 0$.
2. If t is of the form $f(t_1, t_2, \ldots, t_n)$, where f is an n-ary function symbol and $t_1, t_2, \ldots, t_n$ are L-terms, then $c(t) = 1 + \max(c(t_1), c(t_2), \ldots, c(t_n))$.

Definition 2.4. Let L be a signature. The set of atomic L-formulas is defined as follows:

1. If t_1 and t_2 are L-terms, then $(t_1 = t_2)$ is an atomic L-formula.
2. If R is an n-ary relation symbol in L and $t_1, t_2, \ldots, t_n$ are L-terms, then $R(t_1, t_2, \ldots, t_n)$ is an atomic L-formula.
3. Nothing else is an atomic L-formula.

2.2. **Semantics.**

Definition 2.5. Let $t(x_1, x_2, \ldots, x_n)$ be an L-term with variables among $x_1, x_2, \ldots, x_n$. Let $\mathcal{A}$ be an L-structure and let $a_1, a_2, \ldots, a_n \in A$. The interpretation of t in $\mathcal{A}$ at $(a_1, a_2, \ldots, a_n)$, denoted by $t^{\mathcal{A}}(a_1, a_2, \ldots, a_n)$, is defined inductively as follows:

1. If t is a variable x, then $t^{\mathcal{A}}(a) = a$.
2. If t is a constant symbol c, then $t^{\mathcal{A}}() = c^{A}$.(Here we use c^A to indicate c^A live in the semantics world.)
3. If t is of the form $f(t_1, t_2, \ldots, t_m)$, where f is an m-ary function symbol and $t_1, t_2, \ldots, t_m$ are L-terms, then

$$t^{\mathcal{A}}(a_1, a_2, \ldots, a_n) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n), \ldots, t_m^{\mathcal{A}}(a_1, a_2, \ldots, a_n)).$$

Definition 2.6. Let $\varphi(x_1, x_2, \ldots, x_n)$ be an atomic L-formula with variables among $x_1, x_2, \ldots, x_n$. Let $\mathcal{A}$ be an L-structure and let $a_1, a_2, \ldots, a_n \in A$. We say that φ is true in $\mathcal{A}$ at $(a_1, a_2, \ldots, a_n)$, or $(a_1, a_2, \ldots, a_n)$ satisfies φ in $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$, if one of the following conditions holds:

1. If φ is of the form $(t_1 = t_2)$, where t_1 and t_2 are L-terms, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n) = t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n)$.
2. If φ is of the form $R(t_1, t_2, \ldots, t_m)$, where R is an m-ary relation symbol and $t_1, t_2, \ldots, t_m$ are L-terms, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $(t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n), \ldots, t_m^{\mathcal{A}}(a_1, a_2, \ldots, a_n)) \in R^{\mathcal{A}}$.

Theorem 2.7. *Let A, B be two L-structures and let $f : A \to B$ be a morphism. Let $\varphi(x_1, x_2, \ldots, x_n)$ be an atomic L-formula with variables among $x_1, x_2, \ldots, x_n$. Then for all $a_1, a_2, \ldots, a_n \in A$, $f(t^A(a_1, a_2, \ldots, a_n)) = t^B(f(a_1), f(a_2), \ldots, f(a_n))$.*

Corollary 2.8. *Let A, B be two L-structures and let $f : A \to B$ be a morphism of* **Set**. *Then:*

1. f is a morphism of $Str(L)$ if and only if for every atomic L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and all $a_1, a_2, \ldots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n) \implies \mathcal{B} \models \varphi(f(a_1), f(a_2), \ldots, f(a_n)).$$

2. If f is an embedding, then for every atomic L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and all $a_1, a_2, \ldots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n) \iff \mathcal{B} \models \varphi(f(a_1), f(a_2), \ldots, f(a_n)).$$

3. Canonical Models

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures, and let $\bar{a} = (a_1, \ldots, a_n) \in A^n$, $\bar{b} = (b_1, \ldots, b_n) \in B^n$. Consider the expanded signatures $L(\bar{c})$ where $\bar{c} = (c_1, \ldots, c_n)$ are new constant symbols, and the $L(\bar{c})$-structures $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ interpreting c_i as a_i and b_i respectively.*

The following are equivalent:

(1) *For every atomic $L(\bar{c})$-sentence φ, if $(\mathcal{A}, \bar{a}) \models \varphi$ then $(\mathcal{B}, \bar{b}) \models \varphi$.*
(2) *There exists a homomorphism $f : \langle \bar{a} \rangle_{\mathcal{A}} \to \mathcal{B}$ such that $f(a_i) = b_i$ for all i.*

Moreover, the homomorphism f in (2) is unique if it exists, and f is an embedding if and only if

$$\text{for every atomic } L(\bar{c})\text{-sentence } \varphi,\ (\mathcal{A}, \bar{a}) \models \varphi \iff (\mathcal{B}, \bar{b}) \models \varphi.$$

Definition 3.2. Let L be a signature and A be a L-structure. A set T of atomic L-sentences is called a closed set of atomic L-sentences if the following conditions hold:

1. For every closed term t, the sentence $(t = t)$ is in T.
2. For every n-ary function symbol f and closed terms $t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n$, if the sentences $(t_1 = s_1), (t_2 = s_2), \ldots, (t_n = s_n)$ and $(f(t_1, t_2, \ldots, t_n) = r)$ are in T, then the sentence $(f(s_1, s_2, \ldots, s_n) = r)$ is also in T.

TBC

4. First-Order Logic

4.1. Syntax.

Definition 4.1. Let L be a signature. The class of L-formulas is defined inductively as follows:

1. Every atomic L-formula is an L-formula.
2. If φ is an L-formula, then $(\neg\varphi)$ is also an L-formula. If Φ is a set of atmoic L-formulas, then $(\bigwedge_{\varphi \in \Phi} \varphi)$ and $(\bigvee_{\varphi \in \Phi} \varphi)$ are also L-formulas.
3. If φ is an L-formula and x is a variable, then $(\forall x \varphi)$ and $(\exists x \varphi)$ are also L-formulas.

4.2. **Semantics.**

Definition 4.2. Let A be an L-structure. The satisfaction relation $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ for an L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and elements $a_1, a_2, \ldots, a_n \in A$ is defined inductively as follows:

1. If φ is an atomic L-formula, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ is defined as in the previous section.
2. If φ is of the form $(\neg\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $\mathcal{A} \not\models \psi(a_1, a_2, \ldots, a_n)$.
3. If φ is of the form $(\bigwedge_{\psi\in\Phi} \psi)$, where Φ is a set of L-formulas, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if for every $\psi \in \Phi$, $\mathcal{A} \models \psi(a_1, a_2, \ldots, a_n)$.
4. If φ is of the form $(\bigvee_{\psi\in\Phi} \psi)$, where Φ is a set of L-formulas, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if for some $\psi \in \Phi$, $\mathcal{A} \models \psi(a_1, a_2, \ldots, a_n)$.
5. If φ is of the form $(\forall x\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if for every $b \in A$, $\mathcal{A} \models \psi(b, a_1, a_2, \ldots, a_n)$.
6. If φ is of the form $(\exists x\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if there exists some $b \in A$ such that $\mathcal{A} \models \psi(b, a_1, a_2, \ldots, a_n)$.

Remark 4.3. We can define the (infinity) language $L_{\infty\omega}$ to be the collection of all L-formulas.

Definition 4.4. Let L be a signature, the first-order language over L, denoted by $L_{\omega\omega}$, is the collection of all L-formulas where the conjunctions and disjunctions are finite.

Definition 4.5. Fix a signature L. A sentence is an L-formula with no free variables. A class T of L-sentences is called a theory.

Definition 4.6. Let L be a signature and T be a theory of L. An L-structure $\mathcal{A}$ is called a model of T, denoted by $\mathcal{A} \models T$, if for every sentence $\varphi \in T$, we have $\mathcal{A} \models \varphi$.

Definition 4.7. Let T be a theory in $L_{\infty\omega}$ and K a class of L-structures. We say that T axiomatizes K if for every L-structure $\mathcal{A}$, $\mathcal{A} \in K$ if and only if $\mathcal{A} \models T$.

5. Compactness Theorem

5.1. **Teminology.**

Definition 5.1. A theory T in $L_{\infty\omega}$ is called a Hintikka set if it satisfies the following conditions:

1. For every closed term t, the sentence $(t = t)$ is in T.
2. If $\varphi \in T$, then $(\neg\varphi) \in T$.
3. If φ is an atmoic formula in T, s, t are closed terms, and $(s = t) \in T$, then the $\varphi(s)$ is in T if and only if $\varphi(t)$ is in T.
4. If $(\neg\neg\varphi) \in T$, then $\varphi \in T$.
5. If $(\bigwedge_{\varphi\in\Phi} \varphi) \in T$, then for every $\varphi \in \Phi$, we have $\varphi \in T$; if $\neg(\bigwedge_{\varphi\in\Phi} \varphi) \in T$, then there exists some $\varphi \in \Phi$ such that $(\neg\varphi) \in T$.
6. If $(\bigvee_{\varphi\in\Phi} \varphi) \in T$, then there exists some $\varphi \in \Phi$ such that $\varphi \in T$; if $\neg(\bigvee_{\varphi\in\Phi} \varphi) \in T$, then for every $\varphi \in \Phi$, we have $(\neg\varphi) \in T$.

7. If $(\forall x\varphi) \in T$, then for every closed term t, we have $\varphi(t) \in T$; if $(\neg\forall x\varphi) \in T$, then there exists some closed term t such that $(\neg\varphi(t)) \in T$.

8. If $(\exists x\varphi) \in T$, then there exists some closed term t such that $\varphi(t) \in T$; if $(\neg\exists x\varphi) \in T$, then for every closed term t, we have $(\neg\varphi(t)) \in T$.

Example 5.2. Let T be the class of all L-sentences that are true in some L-structure $\mathcal{A}$. Then T is a Hintikka set.

Theorem 5.3. *Let T be a Hintikka set in $L_{\omega\omega}$. Then there exists an L-structure $\mathcal{A}$ such that $\mathcal{A} \models T$.*

Theorem 5.4. *Let L be a first-order language and T be a theory in L. T is a Hintikka set if it satisfies the following conditions:*

1. *Every finite subset of T has a model;*
2. *For every sentence $\varphi \in L$, either $\varphi \in T$ or $(\neg\varphi) \in T$.*
3. *For every sentence $\exists x\varphi(x) \in T$, there exists some closed term $t \in L$ such that $\varphi(t) \in T$.*

5.2. Main Theorems.

Theorem 5.5 (Compactness Theorem)**.** *Let L be a first-order language and T be a theory in L. If every finite subset of T has a model, then T has a model.*

5.3. Filters and Ultrafilters.

Definition 5.6. Given a set S, a filter $\mathcal{F}$ is a subset of $\mathcal{P}(S)$ such that:

1. $\emptyset \notin \mathcal{F}$;
2. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{F}$.

Definition 5.7. A filter $\mathcal{U}$ on a set S is called an ultrafilter if for every $A \subseteq S$, either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$.

Theorem 5.8. *If $\mathcal{F}$ is a filter on a set S, $F_1, F_2, \ldots, F_n$ are subsets of S, then $F_1 \cap F_2 \cap \ldots \cap F_n \neq \emptyset$.*

Theorem 5.9. *Let S be a set and $\mathcal{F} \subseteq \mathcal{P}(S)$. If $\mathcal{F}$ has the finite intersection property, i.e., for every finite subset $\{F_1, F_2, \ldots, F_n\} \subseteq \mathcal{F}$, we have $F_1 \cap F_2 \cap \ldots \cap F_n \neq \emptyset$, then there exists an ultrafilter $\mathcal{U}$ on S such that $\mathcal{F} \subseteq \mathcal{U}$.*

Proof. Notice that given a subset of $\mathcal{P}(S)$ which satisfies the finite intersection property, we can find a filter containing it. Furthermore, we can add X or $S \setminus X$ for any $X \subseteq S$ to the subset so that the finite intersection property still holds. By Zorn's lemma(?Maybe transfinite induction, i have no idea why zorn's lemma fails), we can find a maximal such subset, which gives rise to an ultrafilter (Actually if a subset satisfying finite intersection property and contains every one of the pair of the complements, then itself is an ultrafilter). □

Definition 5.10. Given a collection of sets M_x indexed by a set X and an ultrafilter $\mathcal{U}$ on X, the ultraproduct $\prod_{x \in X} M_x/\mathcal{U}$ is defined as the quotient of the Cartesian product $\prod_{x \in X} M_x$ by the equivalence relation $\sim$ defined as follows: for $(m_x), (n_x) \in \prod_{x \in X} M_x$, we have $(m_x) \sim (n_x)$ if and only if $\{x \in X : m_x = n_x\} \in \mathcal{U}$.

Definition 5.11. Given a collection of structures $\mathcal{A}_x$ indexed by a set X and an ultrafilter $\mathcal{U}$ on X, we can define the ultraproduct $\prod_{x\in X}\mathcal{A}_x/\mathcal{U}$ as follows:

1. The universe of $\prod_{x\in X}\mathcal{A}_x/\mathcal{U}$ is the ultraproduct of the universes of $\mathcal{A}_x$, i.e., $\prod_{x\in X} A_x/\mathcal{U}$.
2. For each n-ary relation symbol R in the signature, we define $R^{\prod_{x\in X}\mathcal{A}_x/\mathcal{U}}$ as follows: for $[(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)] \in \prod_{x\in X} A_x/\mathcal{U}$, we have

$$([(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)]) \in R^{\prod_{x\in X}\mathcal{A}_x/\mathcal{U}} \iff \{x\in X : (a_x^1, a_x^2, \ldots, a_x^n) \in R^{\mathcal{A}_x}\} \in \mathcal{U}.$$

3. For each n-ary function symbol f in the signature, we define $f^{\prod_{x\in X}\mathcal{A}_x/\mathcal{U}}$ as follows: for $[(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)] \in \prod_{x\in X} A_x/\mathcal{U}$, we have

$$f^{\prod_{x\in X}\mathcal{A}_x/\mathcal{U}}([(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)]) = [(b_x)],$$

where $b_x = f^{\mathcal{A}_x}(a_x^1, a_x^2, \ldots, a_x^n)$ for each $x \in X$.

4. For each constant symbol c in the signature, we define $c^{\prod_{x\in X}\mathcal{A}_x/\mathcal{U}} = [(c^{\mathcal{A}_x})]$.

Theorem 5.12 (Łoś's Theorem)**.** *Let $\{\mathcal{A}_x : x \in X\}$ be a collection of L-structures and $\mathcal{U}$ be an ultrafilter on X. For any L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and elements $[(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)] \in \prod_{x\in X} A_x/\mathcal{U}$, we have*

$$\prod_{x\in X}\mathcal{A}_x/\mathcal{U} \models \varphi([(a_x^1)], [(a_x^2)], \ldots, [(a_x^n)]) \iff \{x\in X : \mathcal{A}_x \models \varphi(a_x^1, a_x^2, \ldots, a_x^n)\} \in \mathcal{U}.$$

Theorem 5.13 (Tarski-Vaught Test)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures with $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$, denoted by $\mathcal{A} \preceq \mathcal{B}$, if and only if for every L-formula $\varphi(x, y_1, y_2, \ldots, y_n)$ and all $a_1, a_2, \ldots, a_n \in A$, whenever $\mathcal{B} \models \exists x\varphi(x, a_1, a_2, \ldots, a_n)$, there exists some $b \in A$ such that $\mathcal{B} \models \varphi(b, a_1, a_2, \ldots, a_n)$.*

5.4. **Proof.**

Proof. Given $\varphi \in T$, we can define $T_\varphi = \{\text{finite subsets of } T \text{ containing } \varphi \text{ (that have a model)}\}$, $\mathcal{F} = \{T_\varphi : \varphi \in T\}$. It is easy to see that $\mathcal{F}$ has the finite intersection property. By the previous theorem, there exists an ultrafilter $\mathcal{U}$ on T such that $\mathcal{F} \subseteq \mathcal{U}$. □

References