

NOTES ON SET THEORY

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1. PRELIMINARIES

Definition 1.1 (Syntax of one-order logic). A **signature** σ is a set of relation symbols, function symbols and constant symbols, each with a specified arity. A **first-order language** \mathcal{L} over a signature σ consists of:

1. a countable set of variables x_0, x_1, x_2, \dots
2. the symbols in σ
3. the logical symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, =$
4. the punctuation symbols $(,), \text{ and } ,$.

The terms and formulas of \mathcal{L} are defined inductively as follows:

1. every variable and constant symbol is a term
2. if f is an n -ary function symbol and t_1, \dots, t_n are terms, then $f(t_1, \dots, t_n)$ is a term
3. if R is an n -ary relation symbol and t_1, \dots, t_n are terms, then $R(t_1, \dots, t_n)$ is an atomic formula
4. if φ and ψ are formulas, then so are $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$
5. if φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

A variable x is free in a formula φ if it is not bound by a quantifier in φ . A formula with no free variables is called a sentence.

2. AXIOMS

Axiom 2.1 (Axiom of Extensionality). Two sets are equal if and only if they have the same elements:

$$\forall A \forall B (\forall x (x \in A \iff x \in B) \iff A = B).$$

Axiom 2.2 (Axiom of Empty Set). There exists a set with no elements, denoted by \emptyset .

$$\exists A \forall x (x \notin A).$$

Axiom 2.3 (Axiom of Pairing). For any sets A and B , there exists a set $\{A, B\}$ whose elements are exactly A and B .

$$\forall A \forall B \exists C \forall x (x \in C \iff (x = A \vee x = B)).$$

Axiom 2.4 (Axiom of Union). For any sets A and B , there exists a set $A \cup B$ whose elements are exactly the elements of A or B .

$$\forall A \forall B \exists C \forall x (x \in C \iff (x \in A \vee x \in B)).$$

Axiom 2.5 (Axiom of Power Set). For any set A , there exists a set $\mathcal{P}(A)$ whose elements are exactly the subsets of A .

$$\forall A \exists B \forall x (x \in B \iff x \subseteq A).$$

Axiom 2.6 (Subset Axiom Schema/Axiom of Separation). For any set A and any formula $\varphi(x, t_0, \dots, t_n)$ not involving A , there exists a set whose elements are exactly the elements of A that satisfy $\varphi(x, t_0, \dots, t_n)$.

$$\forall t_0, \dots, t_n \forall A \exists B \forall x (x \in B \iff (x \in A \wedge \varphi(x, t_0, \dots, t_n))).$$

Remark 2.7 (Intersection and Union). We can define the intersection/union of a set of sets A as follows (Not strictly):

$$\bigcap_{B \in A} B := \{x : \forall B \in A (x \in B)\}, \quad \bigcup_{B \in A} B := \{x : \exists B \in A (x \in B)\}.$$

Remark 2.8 (Function). A function f from a set A to a set B , denoted by $f : A \rightarrow B$, is a subset of $A \times B$ such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. The set A is called the domain of f and the set B is called the codomain of f . The element b is called the image of a under f , denoted by $f(a)$.

REFERENCES