NOTES ON MODEL THEORIES

XINGZHI HUANG

1. Structures

Definition 1.1. A structure $\mathcal{A} := (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$ consists of:

1. a non-empty set A, called the universe of $\mathcal{A}$
2. a relation $R_i^{\mathcal{A}}$ on A for each $i \in I$
3. a function $f_j^{\mathcal{A}} : A^{n_j} \to A$ for each $j \in J$
4. an element $c_k^{\mathcal{A}} \in A$ for each $k \in K$.

The sets I, J, K are disjoint index sets. The arity of $R_i^{\mathcal{A}}$ is some natural number m_i, and the arity of $f_j^{\mathcal{A}}$ is some natural number n_j.

Definition 1.2. Given a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$, the signature $L_{\mathcal{A}}$ of $\mathcal{A}$ is the collection of symbols $\{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}$, where R_i is a relation symbol of arity m_i, f_j is a function symbol of arity n_j, and c_k is a constant symbol.

Definition 1.3. Fix a signature L. An L-structure is a structure whose signature is L.

Definition 1.4. We can define a category of L-structures, denoted by $Str(L)$, as follows.

The objects of $Str(L)$ are all L-structures. If $\mathcal{A}$ and $\mathcal{B}$ are two L-structures, then a morphism from $\mathcal{A}$ to $\mathcal{B}$ is a function $h : A \to B$ such that:

1. for each relation symbol R_i of arity m_i in L, and for all $a_1, a_2, \ldots, a_{m_i} \in A$, if $(a_1, a_2, \ldots, a_{m_i}) \in R_i^{\mathcal{A}}$, then $(h(a_1), h(a_2), \ldots, h(a_{m_i})) \in R_i^{\mathcal{B}}$;
2. for each function symbol f_j of arity n_j in L, and for all $a_1, a_2, \ldots, a_{n_j} \in A$, we have $h(f_j^{\mathcal{A}}(a_1, a_2, \ldots, a_{n_j})) = f_j^{\mathcal{B}}(h(a_1), h(a_2), \ldots, h(a_{n_j}))$;
3. for each constant symbol c_k in L, we have $h(c_k^{\mathcal{A}}) = c_k^{\mathcal{B}}$.

Theorem 1.5. *The injective morphisms in $Str(L)$ are precisely the embeddings; the surjective morphisms in $Str(L)$ are precisely the surjective homomorphisms.*

Definition 1.6. Given two L-structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is a substructure of $\mathcal{B}$, denoted by $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and the inclusion map $i : A \to B$ is an embedding.

2. Terms and Formulas

2.1. Syntax.

Definition 2.1. Let L be a signature. The set of L-terms is defined inductively as follows:

1. Every variable is an L-term.

2. Every constant symbol in L is an L-term.
3. If f is an n-ary function symbol in L and $t_1, t_2, \ldots, t_n$ are L-terms, then $f(t_1, t_2, \ldots, t_n)$ is an L-term.
4. Nothing else is an L-term.

Remark 2.2. A term is called closed if it contains no variables.

Remark 2.3. The complexity of a term t, denoted by $c(t)$, is defined inductively as follows:
1. If t is a variable or a constant symbol, then $c(t) = 0$.
2. If t is of the form $f(t_1, t_2, \ldots, t_n)$, where f is an n-ary function symbol and $t_1, t_2, \ldots, t_n$ are L-terms, then $c(t) = 1 + \max(c(t_1), c(t_2), \ldots, c(t_n))$.

Definition 2.4. Let L be a signature. The set of atomic L-formulas is defined as follows:
1. If t_1 and t_2 are L-terms, then $(t_1 = t_2)$ is an atomic L-formula.
2. If R is an n-ary relation symbol in L and $t_1, t_2, \ldots, t_n$ are L-terms, then $R(t_1, t_2, \ldots, t_n)$ is an atomic L-formula.
3. Nothing else is an atomic L-formula.

2.2. **Semantics.**

Definition 2.5. Let $t(x_1, x_2, \ldots, x_n)$ be an L-term with variables among $x_1, x_2, \ldots, x_n$. Let $\mathcal{A}$ be an L-structure and let $a_1, a_2, \ldots, a_n \in A$. The interpretation of t in $\mathcal{A}$ at $(a_1, a_2, \ldots, a_n)$, denoted by $t^{\mathcal{A}}(a_1, a_2, \ldots, a_n)$, is defined inductively as follows:
1. If t is a variable x, then $t^{\mathcal{A}}(a) = a$.
2. If t is a constant symbol c, then $t^{\mathcal{A}}() = c^{A}$.(Here we use c^A to indicate c^A live in the semantics world.)
3. If t is of the form $f(t_1, t_2, \ldots, t_m)$, where f is an m-ary function symbol and $t_1, t_2, \ldots, t_m$ are L-terms, then

$$t^{\mathcal{A}}(a_1, a_2, \ldots, a_n) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n), \ldots, t_m^{\mathcal{A}}(a_1, a_2, \ldots, a_n)).$$

Definition 2.6. Let $\varphi(x_1, x_2, \ldots, x_n)$ be an atomic L-formula with variables among $x_1, x_2, \ldots, x_n$. Let $\mathcal{A}$ be an L-structure and let $a_1, a_2, \ldots, a_n \in A$. We say that φ is true in $\mathcal{A}$ at $(a_1, a_2, \ldots, a_n)$, or $(a_1, a_2, \ldots, a_n)$ satisfies φ in $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$, if one of the following conditions holds:
1. If φ is of the form $(t_1 = t_2)$, where t_1 and t_2 are L-terms, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n) = t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n)$.
2. If φ is of the form $R(t_1, t_2, \ldots, t_m)$, where R is an m-ary relation symbol and $t_1, t_2, \ldots, t_m$ are L-terms, then $\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $(t_1^{\mathcal{A}}(a_1, a_2, \ldots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \ldots, a_n), \ldots, t_m^{\mathcal{A}}(a_1, a_2, \ldots, a_n)) \in R^{\mathcal{A}}$.

Theorem 2.7. *Let A, B be two L-structures and let $f : A \to B$ be a morphism. Let $\varphi(x_1, x_2, \ldots, x_n)$ be an atomic L-formula with variables among $x_1, x_2, \ldots, x_n$. Then for all $a_1, a_2, \ldots, a_n \in A$, $f(t^A(a_1, a_2, \ldots, a_n)) = t^B(f(a_1), f(a_2), \ldots, f(a_n))$.*

Corollary 2.8. *Let A, B be two L-structures and let $f : A \to B$ be a morphism of* **Set**. *Then:*

1. f is a morphism of $Str(L)$ if and only if for every atomic L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and all $a_1, a_2, \ldots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n) \implies \mathcal{B} \models \varphi(f(a_1), f(a_2), \ldots, f(a_n)).$$

2. If f is an embedding, then for every atomic L-formula $\varphi(x_1, x_2, \ldots, x_n)$ and all $a_1, a_2, \ldots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \ldots, a_n) \iff \mathcal{B} \models \varphi(f(a_1), f(a_2), \ldots, f(a_n)).$$

References