NOTES ON SET THEORY

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1. Preliminaries

Definition 1.1 (Syntax of one-order logic)**.** A **signature** σ is a set of relation symbols, function symbols and constant symbols, each with a specified arity. A **first-order language** $\mathcal{L}$ over a signature σ consists of:

1. a countable set of variables $x_0, x_1, x_2, \ldots$
2. the symbols in σ
3. the logical symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, =$
4. the punctuation symbols (,), and ,.

The terms and formulas of $\mathcal{L}$ are defined inductively as follows:

1. every variable and constant symbol is a term
2. if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term
3. if R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is an atomic formula
4. if φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$
5. if φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

A variable x is free in a formula φ if it is not bound by a quantifier in φ. A formula with no free variables is called a sentence.

2. Axioms

Axiom 2.1 (Axiom of Extensionality)**.** Two sets are equal if and only if they have the same elements:

$$\forall A \forall B (\forall x (x \in A \iff x \in B) \iff A = B).$$

Axiom 2.2 (Axiom of Empty Set)**.** There exists a set with no elements, denoted by $\emptyset$.

$$\exists A \forall x (x \notin A).$$

Axiom 2.3 (Axiom of Pairing)**.** For any sets A and B, there exists a set $\{A, B\}$ whose elements are exactly A and B.

$$\forall A \forall B \exists C \forall x (x \in C \iff (x = A \vee x = B)).$$

Axiom 2.4 (Axiom of Union)**.** For any sets A and B, there exists a set $A \cup B$ whose elements are exactly the elements of A or B.

$$\forall A\forall B\exists C\forall x(x\in C \iff (x\in A\vee x\in B)).$$

Axiom 2.5 (Axiom of Power Set)**.** For any set A, there exists a set $\mathcal{P}(A)$ whose elements are exactly the subsets of A.

$$\forall A\exists B\forall x(x\in B \iff x\subseteq A).$$

Axiom 2.6 (Subset Axiom Schema/Axiom of Separation)**.** For any set A and any formula $\varphi(x,t_0,\dots,t_n)$ not involving A, there exists a set whose elements are exactly the elements of A that satisfy $\varphi(x,t_0,\dots,t_n)$.

$$\forall t_0,\dots,t_n\forall A\exists B\forall x(x\in B \iff (x\in A\wedge\varphi(x,t_0,\dots,t_n))).$$

Remark 2.7 (Intersection and Union). We can define the intersection/union of a set of sets A as follows(Not strictly):

$$\bigcap_{B\in A} B := \{x : \forall B\in A(x\in B)\},\qquad \bigcup_{B\in A} B := \{x : \exists B\in A(x\in B)\}.$$

Remark 2.8 (Function). A function f from a set A to a set B, denoted by $f : A\to B$, is a subset of $A\times B$ such that for every $a\in A$, there exists a unique $b\in B$ such that $(a,b)\in f$. The set A is called the domain of f and the set B is called the codomain of f. The element b is called the image of a under f, denoted by $f(a)$.

Axiom 2.9 (Axiom of Choice)**.** For any relation R, there exists a function f such that for every x, if there exists a y where $dom(f) = dom(R)$ and $ran(f)\subseteq ran(R)$, and $(x,f(x))\in R$.

3. Natural Numbers

Definition 3.1. Let S be a set, we define the successor of S as $S^+ = S\cup\{S\}$.

Remark 3.2. Denote $0=\emptyset$, $1=0^+=\{\emptyset\}$, $2=1^+=\{\emptyset,\{\emptyset\}\}$, $3=2^+=\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$, and so on.

Definition 3.3 (Inductive Set)**.** A set A is called inductive if $\emptyset\in A$ and for every $x\in A$, $x^+\in A$.

Axiom 3.4 (Axiom of Infinity)**.** There exists an inductive set.

$$\exists A(\emptyset\in A\wedge\forall x(x\in A\to x^+\in A)).$$

Theorem 3.5. *There exists a set ω consisting of exactly the natural numbers.*

Proof. Suppose A is an inductive set. We use the subset axiom instead of the intersection because the set of all inductive sets may not exist. Let

$$\omega = \{x\in A : \forall B(B \text{ is inductive} \to x\in B)\}.$$

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References