NOTES ON MODEL THEORIES

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1. Structures

Definition 1.1. A structure $\mathcal{A} := (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$ consists of:

1. a non-empty set A, called the universe of $\mathcal{A}$
2. a relation $R_i^{\mathcal{A}}$ on A for each $i \in I$
3. a function $f_j^{\mathcal{A}} : A^{n_j} \to A$ for each $j \in J$
4. an element $c_k^{\mathcal{A}} \in A$ for each $k \in K$.

The sets I, J, K are disjoint index sets. The arity of $R_i^{\mathcal{A}}$ is some natural number m_i, and the arity of $f_j^{\mathcal{A}}$ is some natural number n_j.

Definition 1.2. Given a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K})$, the signature $L_{\mathcal{A}}$ of $\mathcal{A}$ is the collection of symbols $\{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}$, where R_i is a relation symbol of arity m_i, f_j is a function symbol of arity n_j, and c_k is a constant symbol.

Definition 1.3. Fix a signature L. An L-structure is a structure whose signature is L.

Definition 1.4. We can define a category of L-structures, denoted by $Str(L)$, as follows.

The objects of $Str(L)$ are all L-structures. If $\mathcal{A}$ and $\mathcal{B}$ are two L-structures, then a morphism from $\mathcal{A}$ to $\mathcal{B}$ is a function $h : A \to B$ such that:

1. for each relation symbol R_i of arity m_i in L, and for all $a_1, a_2, \ldots, a_{m_i} \in A$, if $(a_1, a_2, \ldots, a_{m_i}) \in R_i^{\mathcal{A}}$, then $(h(a_1), h(a_2), \ldots, h(a_{m_i})) \in R_i^{\mathcal{B}}$;
2. for each function symbol f_j of arity n_j in L, and for all $a_1, a_2, \ldots, a_{n_j} \in A$, we have $h(f_j^{\mathcal{A}}(a_1, a_2, \ldots, a_{n_j})) = f_j^{\mathcal{B}}(h(a_1), h(a_2), \ldots, h(a_{n_j}))$;
3. for each constant symbol c_k in L, we have $h(c_k^{\mathcal{A}}) = c_k^{\mathcal{B}}$.

Theorem 1.5. *The injective morphisms in $Str(L)$ are precisely the embeddings; the surjective morphisms in $Str(L)$ are precisely the surjective homomorphisms.*

Definition 1.6. Given two L-structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is a substructure of $\mathcal{B}$, denoted by $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and the inclusion map $i : A \to B$ is an embedding.

References