

NOTES ON MODEL THEORIES

XINGZHI HUANG

1. STRUCTURES

Definition 1.1. A structure $\mathcal{A} := (A, \{R_i^{\mathcal{A}}\}_{i \in I}, \{f_j^{\mathcal{A}}\}_{j \in J}, \{c_k^{\mathcal{A}}\}_{k \in K})$ consists of:

1. a non-empty set A , called the universe of \mathcal{A}
2. a relation $R_i^{\mathcal{A}}$ on A for each $i \in I$
3. a function $f_j^{\mathcal{A}} : A^{n_j} \rightarrow A$ for each $j \in J$
4. an element $c_k^{\mathcal{A}} \in A$ for each $k \in K$.

The sets I, J, K are disjoint index sets. The arity of $R_i^{\mathcal{A}}$ is some natural number m_i , and the arity of $f_j^{\mathcal{A}}$ is some natural number n_j .

Definition 1.2. Given a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i \in I}, \{f_j^{\mathcal{A}}\}_{j \in J}, \{c_k^{\mathcal{A}}\}_{k \in K})$, the signature $L_{\mathcal{A}}$ of \mathcal{A} is the collection of symbols $\{R_i\}_{i \in I}, \{f_j\}_{j \in J}, \{c_k\}_{k \in K}$, where R_i is a relation symbol of arity m_i , f_j is a function symbol of arity n_j , and c_k is a constant symbol.

Definition 1.3. Fix a signature L . An L -structure is a structure whose signature is L .

Definition 1.4. We can define a category of L -structures, denoted by $Str(L)$, as follows.

The objects of $Str(L)$ are all L -structures. If \mathcal{A} and \mathcal{B} are two L -structures, then a morphism from \mathcal{A} to \mathcal{B} is a function $h : A \rightarrow B$ such that:

1. for each relation symbol R_i of arity m_i in L , and for all $a_1, a_2, \dots, a_{m_i} \in A$, if $(a_1, a_2, \dots, a_{m_i}) \in R_i^{\mathcal{A}}$, then $(h(a_1), h(a_2), \dots, h(a_{m_i})) \in R_i^{\mathcal{B}}$;
2. for each function symbol f_j of arity n_j in L , and for all $a_1, a_2, \dots, a_{n_j} \in A$, we have $h(f_j^{\mathcal{A}}(a_1, a_2, \dots, a_{n_j})) = f_j^{\mathcal{B}}(h(a_1), h(a_2), \dots, h(a_{n_j}))$;
3. for each constant symbol c_k in L , we have $h(c_k^{\mathcal{A}}) = c_k^{\mathcal{B}}$.

Theorem 1.5. *The injective morphisms in $Str(L)$ are precisely the embeddings; the surjective morphisms in $Str(L)$ are precisely the surjective homomorphisms.*

Definition 1.6. Given two L -structures \mathcal{A} and \mathcal{B} , we say that \mathcal{A} is a substructure of \mathcal{B} , denoted by $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and the inclusion map $i : A \rightarrow B$ is an embedding.

2. TERMS AND FORMULAS

2.1. Syntax.

Definition 2.1. Let L be a signature. The set of L -terms is defined inductively as follows:

1. Every variable is an L -term.

2. Every constant symbol in L is an L -term.
3. If f is an n -ary function symbol in L and t_1, t_2, \dots, t_n are L -terms, then $f(t_1, t_2, \dots, t_n)$ is an L -term.
4. Nothing else is an L -term.

Remark 2.2. A term is called closed if it contains no variables.

Remark 2.3. The complexity of a term t , denoted by $c(t)$, is defined inductively as follows:

1. If t is a variable or a constant symbol, then $c(t) = 0$.
2. If t is of the form $f(t_1, t_2, \dots, t_n)$, where f is an n -ary function symbol and t_1, t_2, \dots, t_n are L -terms, then $c(t) = 1 + \max(c(t_1), c(t_2), \dots, c(t_n))$.

Definition 2.4. Let L be a signature. The set of atomic L -formulas is defined as follows:

1. If t_1 and t_2 are L -terms, then $(t_1 = t_2)$ is an atomic L -formula.
2. If R is an n -ary relation symbol in L and t_1, t_2, \dots, t_n are L -terms, then $R(t_1, t_2, \dots, t_n)$ is an atomic L -formula.
3. Nothing else is an atomic L -formula.

2.2. Semantics.

Definition 2.5. Let $t(x_1, x_2, \dots, x_n)$ be an L -term with variables among x_1, x_2, \dots, x_n . Let \mathcal{A} be an L -structure and let $a_1, a_2, \dots, a_n \in A$. The interpretation of t in \mathcal{A} at (a_1, a_2, \dots, a_n) , denoted by $t^{\mathcal{A}}(a_1, a_2, \dots, a_n)$, is defined inductively as follows:

1. If t is a variable x , then $t^{\mathcal{A}}(a) = a$.
2. If t is a constant symbol c , then $t^{\mathcal{A}}() = c^{\mathcal{A}}$. (Here we use $c^{\mathcal{A}}$ to indicate $c^{\mathcal{A}}$ live in the semantics world.)
3. If t is of the form $f(t_1, t_2, \dots, t_m)$, where f is an m -ary function symbol and t_1, t_2, \dots, t_m are L -terms, then

$$t^{\mathcal{A}}(a_1, a_2, \dots, a_n) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, a_2, \dots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \dots, a_n), \dots, t_m^{\mathcal{A}}(a_1, a_2, \dots, a_n)).$$

Definition 2.6. Let $\varphi(x_1, x_2, \dots, x_n)$ be an atomic L -formula with variables among x_1, x_2, \dots, x_n . Let \mathcal{A} be an L -structure and let $a_1, a_2, \dots, a_n \in A$. We say that φ is true in \mathcal{A} at (a_1, a_2, \dots, a_n) , or (a_1, a_2, \dots, a_n) satisfies φ in \mathcal{A} , denoted by $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$, if one of the following conditions holds:

1. If φ is of the form $(t_1 = t_2)$, where t_1 and t_2 are L -terms, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if $t_1^{\mathcal{A}}(a_1, a_2, \dots, a_n) = t_2^{\mathcal{A}}(a_1, a_2, \dots, a_n)$.
2. If φ is of the form $R(t_1, t_2, \dots, t_m)$, where R is an m -ary relation symbol and t_1, t_2, \dots, t_m are L -terms, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if $(t_1^{\mathcal{A}}(a_1, a_2, \dots, a_n), t_2^{\mathcal{A}}(a_1, a_2, \dots, a_n), \dots, t_m^{\mathcal{A}}(a_1, a_2, \dots, a_n)) \in R^{\mathcal{A}}$.

Theorem 2.7. Let A, B be two L -structures and let $f : A \rightarrow B$ be a morphism. Let $\varphi(x_1, x_2, \dots, x_n)$ be an atomic L -formula with variables among x_1, x_2, \dots, x_n . Then for all $a_1, a_2, \dots, a_n \in A$, $f(t^{\mathcal{A}}(a_1, a_2, \dots, a_n)) = t^{\mathcal{B}}(f(a_1), f(a_2), \dots, f(a_n))$.

Corollary 2.8. Let A, B be two L -structures and let $f : A \rightarrow B$ be a morphism of **Set**. Then:

1. f is a morphism of $\text{Str}(L)$ if and only if for every atomic L -formula $\varphi(x_1, x_2, \dots, x_n)$ and all $a_1, a_2, \dots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n) \implies \mathcal{B} \models \varphi(f(a_1), f(a_2), \dots, f(a_n)).$$

2. If f is an embedding, then for every atomic L -formula $\varphi(x_1, x_2, \dots, x_n)$ and all $a_1, a_2, \dots, a_n \in A$,

$$\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n) \iff \mathcal{B} \models \varphi(f(a_1), f(a_2), \dots, f(a_n)).$$

3. CANONICAL MODELS

Lemma 3.1. Let \mathcal{A} and \mathcal{B} be L -structures, and let $\bar{a} = (a_1, \dots, a_n) \in A^n$, $\bar{b} = (b_1, \dots, b_n) \in B^n$. Consider the expanded signatures $L(\bar{c})$ where $\bar{c} = (c_1, \dots, c_n)$ are new constant symbols, and the $L(\bar{c})$ -structures (\mathcal{A}, \bar{a}) and (\mathcal{B}, \bar{b}) interpreting c_i as a_i and b_i respectively.

The following are equivalent:

- (1) For every atomic $L(\bar{c})$ -sentence φ , if $(\mathcal{A}, \bar{a}) \models \varphi$ then $(\mathcal{B}, \bar{b}) \models \varphi$.
- (2) There exists a homomorphism $f : \langle \bar{a} \rangle_{\mathcal{A}} \rightarrow \mathcal{B}$ such that $f(a_i) = b_i$ for all i .

Moreover, the homomorphism f in (2) is unique if it exists, and f is an embedding if and only if

$$\text{for every atomic } L(\bar{c})\text{-sentence } \varphi, (\mathcal{A}, \bar{a}) \models \varphi \iff (\mathcal{B}, \bar{b}) \models \varphi.$$

Definition 3.2. Let L be a signature and A be a L -structure. A set T of atomic L -sentences is called a closed set of atomic L -sentences if the following conditions hold:

1. For every closed term t , the sentence $(t = t)$ is in T .
2. For every n -ary function symbol f and closed terms $t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_n$, if the sentences $(t_1 = s_1), (t_2 = s_2), \dots, (t_n = s_n)$ and $(f(t_1, t_2, \dots, t_n) = r)$ are in T , then the sentence $(f(s_1, s_2, \dots, s_n) = r)$ is also in T .

TBC

4. FIRST-ORDER LOGIC

4.1. Syntax.

Definition 4.1. Let L be a signature. The class of L -formulas is defined inductively as follows:

1. Every atomic L -formula is an L -formula.
2. If φ is an L -formula, then $(\neg \varphi)$ is also an L -formula. If Φ is a set of atomic L -formulas, then $(\bigwedge_{\varphi \in \Phi} \varphi)$ and $(\bigvee_{\varphi \in \Phi} \varphi)$ are also L -formulas.
3. If φ is an L -formula and x is a variable, then $(\forall x \varphi)$ and $(\exists x \varphi)$ are also L -formulas.

4.2. Semantics.

Definition 4.2. Let A be an L -structure. The satisfaction relation $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ for an L -formula $\varphi(x_1, x_2, \dots, x_n)$ and elements $a_1, a_2, \dots, a_n \in A$ is defined inductively as follows:

1. If φ is an atomic L -formula, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ is defined as in the previous section.
2. If φ is of the form $(\neg\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if $\mathcal{A} \not\models \psi(a_1, a_2, \dots, a_n)$.
3. If φ is of the form $(\bigwedge_{\psi \in \Phi} \psi)$, where Φ is a set of L -formulas, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if for every $\psi \in \Phi$, $\mathcal{A} \models \psi(a_1, a_2, \dots, a_n)$.
4. If φ is of the form $(\bigvee_{\psi \in \Phi} \psi)$, where Φ is a set of L -formulas, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if for some $\psi \in \Phi$, $\mathcal{A} \models \psi(a_1, a_2, \dots, a_n)$.
5. If φ is of the form $(\forall x\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if for every $b \in A$, $\mathcal{A} \models \psi(b, a_1, a_2, \dots, a_n)$.
6. If φ is of the form $(\exists x\psi)$, then $\mathcal{A} \models \varphi(a_1, a_2, \dots, a_n)$ if and only if there exists some $b \in A$ such that $\mathcal{A} \models \psi(b, a_1, a_2, \dots, a_n)$.

Remark 4.3. We can define the (infinity) language $L_{\infty\omega}$ to be the collection of all L -formulas.

Definition 4.4. Let L be a signature, the first-order language over L , denoted by $L_{\omega\omega}$, is the collection of all L -formulas where the conjunctions and disjunctions are finite.

Definition 4.5. Fix a signature L . A sentence is an L -formula with no free variables. A class T of L -sentences is called a theory.

Definition 4.6. Let L be a signature and T be a theory of L . An L -structure \mathcal{A} is called a model of T , denoted by $\mathcal{A} \models T$, if for every sentence $\varphi \in T$, we have $\mathcal{A} \models \varphi$.

Definition 4.7. Let T be a theory in $L_{\infty\omega}$ and K a class of L -structures. We say that T axiomatizes K if for every L -structure \mathcal{A} , $\mathcal{A} \in K$ if and only if $\mathcal{A} \models T$.

5. COMPACTNESS THEOREM

Definition 5.1. A theory T in $L_{\infty\omega}$ is called a Hintikka set if it satisfies the following conditions:

1. For every closed term t , the sentence $(t = t)$ is in T .
2. If $\varphi \in T$, then $(\neg\varphi) \in T$.
3. If φ is an atomic formula in T , s, t are closed terms, and $(s = t) \in T$, then the $\varphi(s)$ is in T if and only if $\varphi(t)$ is in T .
4. If $(\neg\neg\varphi) \in T$, then $\varphi \in T$.
5. If $(\bigwedge_{\varphi \in \Phi} \varphi) \in T$, then for every $\varphi \in \Phi$, we have $\varphi \in T$; if $\neg(\bigwedge_{\varphi \in \Phi} \varphi) \in T$, then there exists some $\varphi \in \Phi$ such that $(\neg\varphi) \in T$.
6. If $(\bigvee_{\varphi \in \Phi} \varphi) \in T$, then there exists some $\varphi \in \Phi$ such that $\varphi \in T$; if $\neg(\bigvee_{\varphi \in \Phi} \varphi) \in T$, then for every $\varphi \in \Phi$, we have $(\neg\varphi) \in T$.
7. If $(\forall x\varphi) \in T$, then for every closed term t , we have $\varphi(t) \in T$; if $(\neg\forall x\varphi) \in T$, then there exists some closed term t such that $(\neg\varphi(t)) \in T$.

8. If $(\exists x\varphi) \in T$, then there exists some closed term t such that $\varphi(t) \in T$; if $(\neg\exists x\varphi) \in T$, then for every closed term t , we have $(\neg\varphi(t)) \in T$.

Example 5.2. Let T be the class of all L -sentences that are true in some L -structure \mathcal{A} . Then T is a Hintikka set.

Theorem 5.3. Let T be a Hintikka set in $L_{\omega\omega}$. Then there exists an L -structure \mathcal{A} such that $\mathcal{A} \models T$.

Theorem 5.4. Let L be a first-order language and T be a theory in L . T is a Hintikka set if it satisfies the following conditions:

1. Every finite subset of T has a model;
2. For every sentence $\varphi \in L$, either $\varphi \in T$ or $(\neg\varphi) \in T$.
3. For every sentence $\exists x\varphi(x) \in T$, there exists some closed term $t \in L$ such that $\varphi(t) \in T$.

Theorem 5.5 (Compactness Theorem). Let L be a first-order language and T be a theory in L . If every finite subset of T has a model, then T has a model.

Definition 5.6. Given a set S , a filter \mathcal{F} is a subset of $\mathcal{P}(S)$ such that:

1. $\emptyset \notin \mathcal{F}$;
2. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{F}$.

Definition 5.7. A filter \mathcal{U} on a set S is called an ultrafilter if for every $A \subseteq S$, either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$.

Theorem 5.8. If \mathcal{F} is a filter on a set S , F_1, F_2, \dots, F_n are subsets of S , then $F_1 \cap F_2 \cap \dots \cap F_n \neq \emptyset$.

Theorem 5.9. Let S be a set and $\mathcal{F} \subseteq \mathcal{P}(S)$. If \mathcal{F} has the finite intersection property, i.e., for every finite subset $\{F_1, F_2, \dots, F_n\} \subseteq \mathcal{F}$, we have $F_1 \cap F_2 \cap \dots \cap F_n \neq \emptyset$, then there exists an ultrafilter \mathcal{U} on S such that $\mathcal{F} \subseteq \mathcal{U}$.

Proof. Notice that given a subset of $\mathcal{P}(S)$ which satisfies the finite intersection property, we can find a filter containing it. Furthermore, we can add X or $S \setminus X$ for any $X \subseteq S$ to the subset so that the finite intersection property still holds. By Zorn's lemma, we can find a maximal such subset, which gives rise to an ultrafilter (Actually if a subset satisfying finite intersection property and contains every one of the pair of the complements, then itself is an ultrafilter). \square

REFERENCES