

Motivic Homotopy Theory

Xingzhi Huang

November 4, 2025

Contents

1	Introduction	3
1.1	Foreword	3
1.2	The main content of this paper	3
2	Language of ∞-categories	4
2.1	Models of ∞ -categories	4
2.2	Mapping Spaces	5
2.3	Functors of ∞ -categories	7
2.4	Fibrations of ∞ -categories	9
2.5	Limits and Colimits	10
2.6	Adjoint functors	11
2.7	Presentable categories	11
2.8	Stable categories	12
2.9	$(\infty, 1)$ -Operads and symmetric monoidal categories	13
3	Preliminaries	14
3.1	Higher topos theory	14
3.1.1	Localization	14
3.1.2	Properties of presentable categories	15
3.2	Stable Homotopy Theory	15
3.3	Six functor formalism	16
4	Unstable Motivic Homotopy Theory	17
4.1	Nisnevich topology	17
4.2	Excision property	18
4.3	A1-equivalent	18
4.4	Unstable motivic homotopy category	19
4.5	Tate sphere	20
5	Stable Motivic Homotopy Theory	20

5.1	Stabilization	21
5.2	Motivic cohomology	22
5.3	Motivic Spectra	22
5.4	Universal Property	23
5.5	Six functor formalism	24

1 Introduction

1.1 Foreword

The concept of “motive” was introduced by Grothendieck who noticed a series of similarities in Weil cohomology theories. In his idea, he thought there should be an universal cohomology theory which every cohomology theory taking values in a \mathbb{Q} -category should pass through it.

This idea soon became so important in the research about Weil conjecture that people believe once the “motive” is discovered, then the problem can be done.

Thanks to the effort of many mathematicians, we succeeded in constructing “pure motives” for smooth projective varieties as well as “mixed motives” for a more general cases.

One important observation in algebraic topology is that when we talk about cohomology theories, what really matters is the derived category. It was Voevodsky who introduced the “motivic homotopy category” and prove the universal property of it, which is what we are interested in this paper.

1.2 The main content of this paper

In short, motivic homotopy theory concerns how to construct a algebraic topology theory on algebraic geometry. To give the readers an intuition, we include a corresponding diagram below.

Table 1: Comparison between classic algebraic topology world and motivic world

Classic World	Motivic World
\mathcal{S} (Spaces)	$\mathcal{H}(S)$
I	\mathbb{A}^1
I -homotopy equivalence	\mathbb{A}^1 -homotopy equivalence
S^1	\mathbb{P}^1
\mathbf{Sp}	$\mathcal{SH}(S)$

In Chapter 2, we’ll recommend the language of ∞ -categories. Any reader who is already familiar with it can skip this chapter.

In Chapter 3, we will firstly recommend some preliminaries that are important to understand the topic we’ll discuss. It mainly contains higher topos theory, stable homotopy theory and six functor formalism.

In Chapter 4, we will construct the unstable motivic homotopy category $\mathcal{H}(S)$ as a motivic localization of the Nisnevich presheaf category. We will also see the reasonability of such construction.

In Chapter 5, we will construct the stable motivic homotopy category $\mathcal{SH}(S)$, as a variant of the classic stabilization of the $\mathcal{H}(S)$. We will also see how Robalo unifies them into a general framework and proves the universal property of $\mathcal{SH}(S)$.

2 Language of ∞ -categories

The motivation of ∞ -categories is to construct a language where we can talk about higher morphisms. A naive approach towards this is enriched category. However, the coherence data soon become too complicated to describe as the level increases. So we will construct some models to describe ∞ -categories.

It should be noted what we call ∞ - categories here are in fact $(\infty, 1)$ -categories, which means all n -morphisms ($n \geq 2$) are invertible.

Readers are assumed to have been familiar with the knowledge of simplicial sets and model categories.

2.1 Models of ∞ -categories

When we say a “model” for ∞ - categories, we mean a model category, whose objects(up to weak equivalence) are ∞ - categories and the weak equivalence between two objects represents equivalence between two ∞ - categories. These models are “equivalent”, namely Quillen equivalent.

In short,

The theory of $(\infty, 1)$ -categories lies in the homotopy category of a model category which “models” the ∞ -categories. (2.1)

Although for practical reasons we have to construct those definitions and propositions in a specific category, they should be invariant under weak equivalences.

In this subsection, we will introduce two models of ∞ -categories, which are quasi-categories and simplicially enriched categories.

Definition 2.1 (Boardman-Vogt, 1973). A simplicial set X is called a quasi-category if it satisfies the right lifting property aganist inner horn inclusions:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \exists h & \downarrow \\ \Delta^n & \longrightarrow & * \end{array} \quad (0 < k < n)$$

Take $\Lambda_k^n = \Lambda_1^2$, then the lifting property provides a lifting $h : \Delta^2 \rightarrow X$. If we denote f as the image of $\{0 \rightarrow 1\}$ in X , g as the image of $\{1 \rightarrow 2\}$ in X , then we can view h as the “homotopy” witnessing the composition of f and g .

A question arising here is that the “homotopy” h witnessing the composition is not unique. However, if we understand the “uniqueness” following the philosophy in 2.1, it is “unique”, or precisely, $\text{Fun}(\Delta^2, C) \times_{\text{Fun}(\Lambda_1^2, C)} (g, \bullet, f)$ is contractible (2.24).

Remark 2.2. We denote the full subcategory of Set_Δ generated by quasi-categories as QCat .

Definition 2.3. We define Cat_Δ as the category of Set_Δ -enriched categories, whose morphisms are given by enriched functors. We also define Cat_{Kan} as the full subcategory of Cat_Δ generated by those Kan complex-enriched(which we will introduce in 2.8) categories.

We will then show two model categories which model the ∞ -categories, or more precisely, whose fibrant full subcategoryes model the $(\infty, 1)$ -categories.

Theorem 2.4 (Joyal, 2008). There exists a model structure called Joyal model structure on Set_Δ where fibrant objects are exactly quasi-categories.

Theorem 2.5 (Bergner, 2004). There exists a model structure called Bergner model structure on Cat_Δ , whose fibrant objects are exactly Kan complex-enriched categories.

Joyal and Tierney pointed out that these two model categories are Quillen equivalent, thus whose fibrant subcategoryes are also Quillen equivalent.

Theorem 2.6 (Joyal & Tierney, 2007). There exists a Quillen equivalence: $\mathfrak{C} : \text{Set}_\Delta^{\text{Joyal}} \rightleftarrows \text{Cat}_\Delta : \mathfrak{N}$.

Corollary 2.7. QCat and Cat_{Kan} are Quillen equivalent.

2.2 Mapping Spaces

The idea of ∞ -categories is that for any two objects in the category, there exists a “mapping space” between such that the “mapping space” is an $(\infty, 0)$ -category (Just as the case in \mathcal{CG} -enriched categories). That gives rise to the concept of Kan complexes where every morphism is “invertible”.

We still define the mapping space as the fibrant object in a homotopy category of a model category which we call as “Kan-Quillen” model category. This model category is Quillen equivalent to Top , whose fibrant objects are CW complexes, that is, “good spaces”.

Definition 2.8. A simplicial set X is called a Kan complex if it satisfies the right lifting property against horn inclusions:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \exists h & \downarrow \\ \Delta^n & \longrightarrow & * \end{array} \quad (0 \leq k \leq n)$$

We make it clear why we say “any morphism in Kan complex is invertible” here.

Remark 2.9. Let C be a quasi-category, the expression $x \in C$ means x is a 0-simplex in C (We also call x as a 0-morphism); the expression $f : x \rightarrow y$ means f is a 1-simplex in C satisfying $d_0 f = x, d_1 f = y$ (We also call f as a 1-morphism).

Theorem 2.10. ([2], 1.4.3.6) Let C be a quasi-category, $x, y \in C, f, g : x \rightarrow y$, The following are equivalent:

1. There exists a 2-simplex σ satisfying $d_0 \sigma = id_y, d_1 \sigma = g, d_2 \sigma = f$, i.e.

$$\begin{array}{ccccc} & & y & & \\ & \nearrow f & & \searrow id_y & \\ x & \xrightarrow{g} & y & & \end{array}$$

2. There exists a 2-simplex σ satisfying $d_0\sigma = f, d_1\sigma = g, d_2\sigma = id_x$, i.e.

$$\begin{array}{ccc} & x & \\ \text{id}_x \nearrow & & \searrow f \\ x & \xrightarrow{g} & y \end{array}$$

Remark 2.11. Obviously the condition is a equivalence relationship, we will say f is homotopic to g if one holds.

Definition 2.12. Let C be an quasi-category, The homotopy category hC of C is defined as the ordinary category as below:

$$\text{Ob}(hC) = \text{Ob}(C);$$

$$\text{Hom}_{hC}(x, y) = \text{Hom}_C(x, y)/\sim, \text{ where } \sim \text{ is defined by homotopy};$$

For $f : x \rightarrow y, g : y \rightarrow z \in hC$, the composition $g \circ f$ is defined as $d_1(h)$, where h is a 2-morphism witenessing the composition of f and g . This is well-defined because $\text{Fun}(\Delta^2, C) \times_{\text{Fun}(\Delta_1^2, C)} (g, \bullet, f)$ is contractible.

Remark 2.13. Let $f : x \rightarrow y \in C$, we will call f an isomorphism if the image $[f]$ of f in hC is an isomorphism.

Theorem 2.14. ([3]) A quasi-category C is a Kan complex if and only if all morphisms in C are isomorphisms.

Before formally giving the consturction of mapping space, we have to clairify the third model category whose fibrant objects are exactly Kan complexes. The following results are classic in algebraic topology.

Theorem 2.15. There exists a model structure called Kan-Quillen model structure on Set_Δ where fibrant objects are Kan complexes. We call the weak equivalences in Kan-Quillen model as "weak homotopy equivalences".

Theorem 2.16. There exists a Quillen equivalence:

$$|\cdot| : \text{Set}_\Delta^{\text{Kan-Quillen}} \rightleftarrows \text{Top} : \text{Sing}.$$

Corollary 2.17. Kan and CW are Quillen equivalent.

It is well-known that CW complexes are "good spaces", that is, whose "mapping space" between two CW complexes itself is still a CW complex.

We will then formally give the consturction of mapping spaces.

Definition 2.18. Let X be a simplicial set, $X^\triangleleft := \Delta^0 \star X; X^\triangleright := X \star \Delta^0$, where \star represents join in simplicial set.

Definition 2.19. Let X be a simplicial set, $x \in X$, the slice simplicial set $X_{/x}$ of X over x is defined by:

$$\text{Hom}(Y, X_{/x}) := \text{Hom}(Y^\triangleright, X) \underset{\text{Hom}(\Delta^0, X)}{\times} \{x\};$$

the coslice simplicial set $X_{x/}$ of X under x is defined by:

$$\text{Hom}(Y, X_{x/}) := \text{Hom}(Y^\triangleleft, X) \underset{\text{Hom}(\Delta^0, X)}{\times} \{x\};$$

Definition 2.20. Let C be a quasi-category, $x, y \in C$. We will define:

The left pinched complex $\text{Hom}_C^L(x, y) := C_{x/} \times_C \{y\}$;

The right pinched complex $\text{Hom}_C^R(x, y) := \{x\} \times_C C_{/y}$;

The balanced complex $\text{Hom}_C^B(x, y) := \{x\} \times_{\text{Fun}(\{0\}, C)} \text{Fun}(\Delta^1, C)) \times_{\text{Fun}(\{1\}, C)} \{y\}$.

Theorem 2.21. ([2], 4.6.5.5) $\text{Hom}_C^L(x, y)$, $\text{Hom}_C^R(x, y)$, and $\text{Hom}_C^B(x, y)$ are Kan complexes which are weak homotopy equivalent.

As a result, we can well-defined the mapping space as a kan complex up to weak equivalence (in the sense of Kan-Quillen model structure).

Definition 2.22. Let C be a quasi-category, $x, y \in C$, we can define the mapping space $\text{Maps}(x, y)$ from x to y as the object in $\text{Set}_\Delta[W^{-1}]$ defined by 2.21.

In fact, Maps can be extended to an $(\infty, 1)$ -functor $C^{op} \times C \rightarrow \mathcal{S}$, which we will show in 2.43.

Based on this observation, we can illustrate the idea that mapping spaces are well-defined up to weak equivalence (following the philosophy in 2.1) by stating a stronger theorem in 2.49.

At the end of this subsection, we will make it clear what we mean by “contractible”.

Definition 2.23. A Kan complex X is called contractible if it satisfies the right lifting property aganist boundary inclusions:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow \exists h & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

Theorem 2.24. ([2], 1.5.6.2) Let C be a quasi-category, $x, y, z \in C$, $f : x \rightarrow y, g : y \rightarrow z$, then $\text{Fun}(\Delta^2, C) \times_{\text{Fun}(\Lambda_1^2, C)} \{(g, \cdot, f)\}$ is a contractible Kan complex.

2.3 Functors of ∞ -categories

Functors between two quasi-categories are just morphisms in Set_Δ . Two types of weak equivalences provide two types of “equivalences”. We’ll show their meaning in the $(\infty, 1)$ -category of $(\infty, 1)$ -categories.

Definition 2.25. Let C, D be two quasi-categories, we will call the morphism $F : C \rightarrow D$ in Set_Δ a functor from C to D .

Proposition 2.26. Set_Δ admits a Cartesian closed structure. With a little abuse of the notation, we will denote the corresponding internal hom $[X, Y]$ as $\text{Fun}(X, Y) \in \text{Set}_\Delta$.

By Yoneda lemma clearly the 0-morphisms in $\text{Fun}(X, Y)$ are exactly the functors from X to Y . We call this category the $(\infty, 1)$ -functor categories from X to Y .

Corollary 2.27. Let X, Y, Z be three simplicial sets, then there exists a natural composition $\text{Fun}(X, Y) \times \text{Fun}(Y, Z) \rightarrow \text{Fun}(X, Z)$.

Theorem 2.28. ([4], 5.4.5) Let C, D be two quasi-categories, $F : C \rightarrow D$ be a functor from C to D . Then the following are equivalent:

1. F is a weak equivalence in $\text{Set}_\Delta^{\text{Joyal}}$;
2. For any quasi-category E , the induced morphism $\text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$ is a weak equivalence in $\text{Set}_\Delta^{\text{Joyal}}$.

We will call this weak equivalence a “homotopy equivalence” or “equivalence”.

Following the philosophy in 2.1, this is the right “isomorphism” in the $(\infty, 1)$ -category of $(\infty, 1)$ -categories, that is, Cat_∞ .

Lemma 2.29. ([2], 1.5.3.7) Let $X \in \text{Set}_\Delta, Y \in \text{QCat}$, then $\text{Fun}(X, Y) \in \text{QCat}$.

Lemma 2.30. ([2], 1.3.5.4) Let $X \in \text{QCat}$, then there exists a maximal Kan complex contained in X , which we denote as X^\simeq , or the core of X .

Definition 2.31. Let QCat be a Set_Δ -enriched category which is defined as:

Objects are all quasi-categories;

$\text{Hom}(X, Y) := \text{Fun}(X, Y)^\simeq$;

The composition map is given by 2.27.

Definition 2.32. We define the $(\infty, 1)$ -category of $(\infty, 1)$ -categories as $\mathfrak{N}(\text{QCat})$, which we denote as Cat_∞ .

Similarly we have:

Definition 2.33. Let Kan be a Set_Δ -enriched category which is defined as:

Objects are all Kan complexes;

$\text{Hom}(X, Y) := \text{Fun}(X, Y)^\simeq$;

The composition map is given by 2.27.

Definition 2.34. We define the $(\infty, 1)$ -category of $(\infty, 1)$ -groupoids as $\mathfrak{N}(\text{Kan})$, which we denote as Grpd_∞ or \mathcal{S} .

Proposition 2.35. ([4], 5.7.6) The homotopy category hCat_∞ is equivalent to the ordinary category $\text{hQC}\dashv\sqcup$ defined as:

$\text{Ob}(\text{hQC}\dashv\sqcup) :=$ all small quasi-categories;

$\text{Hom}_{\text{hQC}\dashv\sqcup}(C, D) := \pi_0(\text{Fun}(C, D)^\simeq)$;

$[G] \circ [F] = [G \circ F]$.

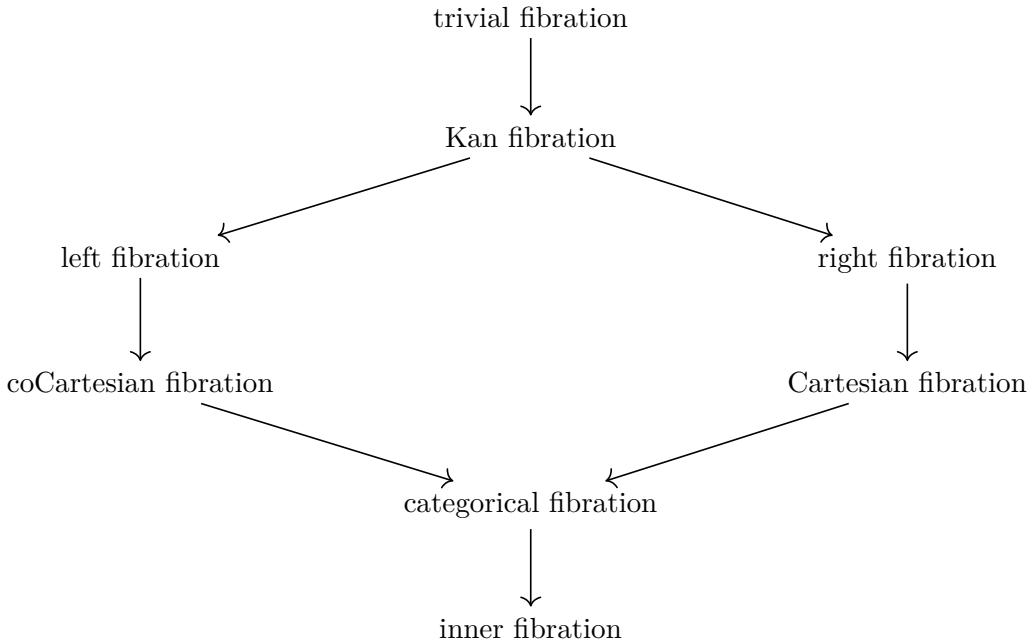
Theorem 2.36. ([4], 5.4.1) Let C, D be two quasi-categories, $F : C \rightarrow D$, F is a homotopy equivalence iff $[F]$ is an isomorphism in $\text{hQC}\dashv\sqcup$.

This explains the reasonability of “homotopy equivalence”.

2.4 Fibrations of ∞ -categories

In ordinary category theory, we are already familiar with the Grothendieck construction with respect to a kind of fibration. These concepts still have their correspondence in ∞ -categories.

The relationship of those kinds of fibrations can be represented by the diagram below[1].



While the Grothendieck-Lurie construction[1] gives the equivalence of $(\infty, 1)$ -categories:

The $(\infty, 1)$ -category of Fibrations	The ∞ -category of Functors
$\text{LFib}_{/S}$	$\text{Fun}(S, \text{Grpd}_\infty)$
$\text{RFib}_{/S}$	$\text{Fun}(S^{\text{op}}, \text{Grpd}_\infty)$
$\text{coCart}_{/S}$	$\text{Fun}(S, \text{Cat}_\infty)$
$\text{Cart}_{/S}$	$\text{Fun}(S^{\text{op}}, \text{Cat}_\infty)$

In this subsection we will not illustrate everything in these two pictures. Instead, we only define left fibration and present the related Grothendieck-Luire construction to give a “genuine” definition of Maps.

Definition 2.37. Let C, D be two quasi-categories, $p : C \rightarrow D$ is a functor, we say p is an inner fibration if it satisfies the right lifting property against inner horn inclusions:

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\hspace{1cm}} & C \\ \downarrow \int \nearrow \exists h & & \downarrow \\ \Lambda^n & \xrightarrow{\hspace{1cm}} & D \end{array} \quad (0 < k < n)$$

Definition 2.38. Let C, D be two $(\infty, 1)$ -categories, $p : C \rightarrow D$ be an inner fibration¹. We say

¹ As the word “fibration” implies, inner fibration is stable under pullback, thus given $x \in C$ the fiber $C_{p(x)}$ is a quasi-category. In a model-independent theory this condition is not required.

p is a left fibration if it satisfies the right lifting property:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & C \\ \downarrow \swarrow \exists h & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & D \end{array} \quad (0 \leq k < n)$$

Definition 2.39. Let S be an $(\infty, 1)$ -category, we define the $(\infty, 1)$ -category LFib_S as the full subcategory of Cat_∞ generated by those left fibrations.

Theorem 2.40. ([1] 3.2.0.1) Given an $(\infty, 1)$ -category S , there exists an equivalence of $(\infty, 1)$ -categories:

$$\text{LFib}_S \rightleftarrows \text{Fun}(S, \text{Grpd}_\infty).$$

Finally we can extend the definition of Maps.

Definition 2.41. Let C be an $(\infty, 1)$ -category, we define the $(\infty, 1)$ -category of twisted arrow category $\text{TwArr}(C) := \text{Hom}_{\text{Set}_\Delta}(\text{N}_\bullet((-)^{op} \star (-)), C)$.

Proposition 2.42. ([2], 8.1.1.15) The projection $\text{TwArr}(C) \rightarrow C^{op} \times C$ induced by $[n]^{op} \rightarrow [n]^{op} \star [n] \leftarrow [n]$ is a left fibration.

Consider The projection as an object in LFib_S , we can formally give our definition of Maps

Definition 2.43. We define the functor $\text{Maps} : C^{op} \times C \rightarrow \text{Grpd}_\infty$ as the correspondence of the projection $\text{TwArr}(C) \rightarrow C^{op} \times C$ via 2.40.

We'll use the following definition in 2.9.

Definition 2.44. Let C, D be two $(\infty, 1)$ -categories, $p : C \rightarrow D$ be an inner fibration, $f : x \rightarrow y \in C$. We say f is p -coCartesian if it satisfies the right lifting property:

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\phi} & C \\ \downarrow \swarrow \exists h & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & D \end{array}$$

where $n \geq 2, \phi|_{\{0,1\}} = f$.

2.5 Limits and Colimits

Definition 2.45. Let C be an $(\infty, 1)$ -category, $x \in C$, we say x is initial if $\forall y \in C, \text{Maps}(x, y)$ is contractible; x is final if $\forall y \in C, \text{Maps}(y, x)$ is contractible.

Definition 2.46. Let K be a simplicial set, C be an $(\infty, 1)$ -category, $u : K \rightarrow C$. The slice category of C over u is defined by:

$$\text{Hom}(Y, C_{/u}) := \text{Hom}(Y \star K, C) \underset{\text{Hom}(K, C)}{\times} \{u\};$$

The coslice category of C under u is defined by:

$$\text{Hom}(Y, C_{u/}) := \text{Hom}(K \star Y, C) \underset{\text{Hom}(K, C)}{\times} \{u\}.$$

Definition 2.47. Let K be a simplicial set, C be an $(\infty, 1)$ -category, $u : K \rightarrow C$. We say a diagram $\bar{u} : K^\Delta \rightarrow C$ is a limit diagram if it is a final object in $C_{/u}$; We say a diagram $\bar{u} : K^\triangleright \rightarrow C$ is a colimit diagram if it is an initial object in $C_{u/}$.

We also say that \bar{u} exhibits $\bar{u}(*)$ as a limit/colimit of u .

Definition 2.48. We say a limit diagram $\bar{u} : K^\Delta \rightarrow C$ is preserved by a functor $F : C \rightarrow C'$ if $F \circ \bar{u} : K^\Delta \rightarrow C'$ is a limit diagram; A colimit diagram \bar{u} is preserved by F if $F \circ \bar{u}$ is a colimit diagram.

Theorem 2.49. ([2] 4.6.4.21) Let C, D be two $(\infty, 1)$ -categories, $F : C \rightarrow D$ be an equivalence, $u : K \rightarrow C$ be a diagram, then the induced functor $C_{/u} \rightarrow D_{/F \circ u}$ is an equivalence.

2.6 Adjoint functors

The definition we give here is slightly different from Lurie's classic one, which explicitly gives the equivalence.

Definition 2.50. Let C, D be two $(\infty, 1)$ -categories. We say a pair of functors $F : C \rightleftarrows D : G$ is an adjunction pair if there exists an equivalence $\varphi : \text{Maps}(F(-), -) \rightarrow \text{Maps}(-, G(-))$ in $\text{Fun}(C^{op} \times D, \mathcal{S})$.

We say that F is a left adjoint functor which admits a right adjoint functor G , and vice versa.

Proposition 2.51. Let $F : C \rightleftarrows D : G$ be an adjunction pair, then F preserves all small colimits while G preserves all small limits.

2.7 Presentable categories

Roughly speaking, we will see a series of correspondences between the filtered version and the normal version (just as in the case of 1-categories).

Definition 2.52. Let K be a simplicial set, κ be a regular cardinal. We say K is κ -small if $|S| < \kappa$, where S is the set of nondegenerate simplexes in K .

We say call a diagram $u : K \rightarrow C$ is κ -small if K is κ -small.

Definition 2.53. Let C be an $(\infty, 1)$ -category, we say C is a κ -filtered category if any κ -small diagram $u : K \rightarrow C$ can be extended to $K^\triangleright \rightarrow C$.

The corresponding limit/colimit diagram is called as κ -filtered limit/colimit diagram.

Definition 2.54. Let C be an $(\infty, 1)$ -category, we define the presheaf category $\text{Psh}(C)$ of C as $\text{Fun}(C^{op}, \mathcal{S})$.

We call the functor $h_C : C \rightarrow \text{Psh}(C)$, $X \mapsto \text{Maps}(-, X)$ as the "Yoneda embedding".

Definition 2.55. Let C be an $(\infty, 1)$ -category, we define the κ -ind category $\text{Ind}_\kappa(C)$ of C as the full subcategory of $\text{Psh}(C)$ generated by theose objects which are small filtered colimits in $\text{Psh}(C)$ where $K \subset h_C(C)$.

Remark 2.56. Clearly $C \subset \text{Ind}_\kappa(C)$, thus we have another embedding functor from C to $\text{Ind}_\kappa(C)$, which is denoted as ι .

We then show that Ind category is just the filtered version of presheaf category.

Proposition 2.57. The Yoneda functor h_C preserves small colimits while ι preserves κ -filtered small colimits.

Thus given a functor F from C to a cocomplete/ κ -filtered cocomplete category D , it can be extended naturally to $\tilde{F} : \text{PSh}(C)/\text{Ind}_\kappa(C) \rightarrow D$.

To be precise, that is:

Theorem 2.58. There is an equivalence of category from $\text{LFun}(\text{PSh}(C), D) \simeq \text{Fun}(C, D)$, where $\text{LFun}(\text{PSh}(C), D)$ is the full subcategory of $\text{Fun}(\text{PSh}(C), D)$ generated by those functors which preserve small colimits.

Theorem 2.59. There is an equivalence of category from $\text{Fun}(\text{Ind}(C), D)_{\kappa\text{-cont}} \simeq \text{Fun}(C, D)$, where $\text{Fun}(\text{Ind}(C), D)_{\kappa\text{-con}}$ is the full subcategory of $\text{Fun}(\text{Ind}(C), D)$ generated by those functors which preserve κ -small filtered colimits.

Definition 2.60. Let C be a $(\infty, 1)$ -category, κ be a regular cardinal. We let $\text{Ind}_\kappa(C)$ denote the full subcategory of $\text{Psh}(C)$ spanned by these functors $F : C^{\text{op}} \rightarrow \mathcal{S}$ which classifies all right Kan fibrations $f : \tilde{C} \rightarrow C$ where \tilde{C} is κ -regular.

Definition 2.61. Let C be a $(\infty, 1)$ -category, κ be a regular cardinal. We say C is κ -accessible if \exists an essentially small $(\infty, 1)$ -category C_0 such that

$$C \simeq \text{Ind}_\kappa(C_0).$$

We say C is accessible if there exists a regular cardinal κ such that C is κ -accessible.

Definition 2.62. Let C be a $(\infty, 1)$ -category. We say C is locally presentable or presentable if C satisfies:

1. C is accessible;
2. C admits all small colimits.

There is an equivalent definition of presentable categories, which may give a more intuitive understanding of the name.

Definition 2.63. Let C be an κ -accessible category, D be a category. We say a functor $F : C \rightarrow D$ is κ -accessible if it preserves κ -filtered small colimits.

We also say a functor F is accessible if it is κ -accessible for some regular cardinal κ .

2.8 Stable categories

Definition 2.64. Let C be an ∞ -category. We say C is a pointed category if there exists a zero object $0 \in C$.

Theorem 2.65. Let C be an ∞ -category with a terminal object $*$, then the coslice category $C_{*/}$ is a pointed category. This process induces a functor which we denote as $(_)_+$.

Definition 2.66. Let C be a pointed ∞ -category that admits finite colimits, $X, Y \in C$, $f : X \rightarrow Y$, then we define the cofiber $\text{cof}(f) := 0 \sqcup_X Y$;

If C admits finite limits, we can define the fiber $\text{fib}(f) := 0 \times_Y X$.

Definition 2.67. Let C be a pointed ∞ -category, a triangle is a diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

We say the triangle is a fiber sequence if it is a pullback square, and a cofiber sequence if it is a pushout square.

Definition 2.68. Let C be a pointed ∞ -category, we say C is stable if:

1. $\forall f \in \text{Mor}(C)$, f admits a fiber and a cofiber;
2. A triangle is a fiber sequence if and only if it is a cofiber sequence.

Definition 2.69. Let C be a pointed ∞ -category that admits finite colimits, we can define the suspension functor $\Sigma : C \rightarrow C$ which maps

$$X \mapsto 0 \sqcup_X 0;$$

Let C be a pointed ∞ -category that admits finite limits, we can define the suspension functor $\Omega : C \rightarrow C$ which maps

$$X \mapsto 0 \times_X 0.$$

Proposition 2.70. Let C be a pointed ∞ -category that admits finite limits and colimits, then the following are equivalent:

1. C is stable
2. The adjunction pair $\Sigma : C \rightleftarrows C : \Omega$ are equivalences.
3. A square in C is a pushout square if and only if it is a pullback square.

2.9 $(\infty, 1)$ -Operads and symmetric monoidal categories

Definition 2.71. We denote the pointed finite set category as Fin_* . We denote $[n] \amalg \{\ast\}$ as $\langle n \rangle$.

Definition 2.72. Let $I \rightarrow J$ be a morphism in Fin_* , $\alpha : I \rightarrow J$. We say α is inert if $\forall j \in J \setminus \{\ast\}$, $\alpha^{-1}(j)$ is a singleton; We say α is active if $\alpha^{-1}(\{\ast\}) = \{\ast\}$.

Definition 2.73. An $(\infty, 1)$ -operad O is a pair (O^\otimes, p) where O^\otimes is an $(\infty, 1)$ -category and p is a functor $O^\otimes \rightarrow \text{Fin}_*$ such that (We denote $p^{-1}(\langle n \rangle)$ as $O_{\langle n \rangle}^\otimes$):

1. Any inert morphism $f : \langle m \rangle \rightarrow \langle n \rangle$ can be lifted to a p -coCartesian morphism \tilde{f} . In particular, the lifting induces the $(\infty, 1)$ -functor $f_! : O_{\langle m \rangle}^\otimes \rightarrow O_{\langle n \rangle}^\otimes$;
2. The ∞ -functor $\text{Maps}_f^\otimes(C_1, C_2) \rightarrow \prod_{1 \leq i \leq n} \text{Maps}_{\rho^i \circ f}(C_1, C_3^i)$ induced by morphism $f : \langle m \rangle \rightarrow \langle n \rangle$ is an equivalence, where $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$ denotes the morphism which maps everything but i to \ast ;

3. For any $C_1, C_2, \dots, C_n \in \mathcal{O}_{\langle 1 \rangle}^{\otimes}$, there exists $C \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ such that there exists a collection of the p -CoCartesian lifting from C to C_i of ρ^i .

Remark 2.74. In fact Fin_* is an ∞ -operad which we denote as Comm^{\otimes} .

Example 2.75. $(\text{Fin}_*, \text{Id})$ iteself is an $(\infty, 1)$ -operad. We denote it as Comm^{\otimes} or \mathbb{E}_{∞} .

Example 2.76. Let Assoc^{\otimes} denotes the subcategory of Fin_* whose objects are those of Fin_* but the morphisms are generated by those $f : \langle m \rangle \rightarrow \langle n \rangle$ such that for each $j \in \langle n \rangle \setminus \{*\}, \alpha|_{\alpha^{-1}(j)}$ preserves ordering.

We also denote $(\text{Assoc}^{\otimes}, i)$ as \mathbb{E}_1 .

Definition 2.77. Let $(\mathcal{O}^{\otimes}, p)$ be an $(\infty, 1)$ -operad, we call a pair (C^{\otimes}, p') where C^{\otimes} is an $(\infty, 1)$ -category and p' is a coCartesian fibration is an \mathcal{O} -monoidal $(\infty, 1)$ -category if the composite $p' \circ p$ exhibits C^{\otimes} as an ∞ -operad.

Remark 2.78. We will call the Comm^{\otimes} -monoidal categories as symmetric monoidal categories; We will call the Assoc^{\otimes} -monoidal categories as monoidal categories.

3 Preliminaries

3.1 Higher topos theory

3.1.1 Localization

In algebraic geometry, it is common to consider sheafification with respect to a Grothendieck topology. This process can be put in a more general framework called localization. It gives a characterization of presentable categories.

Definition 3.1. ([1] 5.2.7.2) Let C, D be two ∞ -categories, $F : C \rightarrow D$ be a functor. We say F exhibits D as a localization of C if F admits a fully faithful right adjoint $G : D \rightarrow C$.

Definition 3.2. Let $C \subset D$ be two ∞ -categories, we say a localization functor $L : D \rightarrow C$ is accessible if the composite functor $D \xrightarrow{L} C \hookrightarrow D$ is an accessible functor.

Warning: A localization functor is always accessible, so the definition won't cause confusion.

Then we have the following characterization of presentable categories.

Theorem 3.3. ([1] 5.5.1.1) Let C be an ∞ -category, the following are equivalent:

1. C is presentable;
2. There exists a small ∞ -category D such that C is an accessible localization of $\text{PSh}(D)$.

These immediately give us an important observation in algebraic geometry.

Example 3.4. Let (C, τ) be a small Grothendieck site, then the sheafification functor $\text{PSh}(C) \rightarrow \text{Shv}(C, \tau)$ exhibits $\text{Shv}(C, \tau)$ as an accessible localization of $\text{PSh}(C)$. Thus $\text{Shv}(C, \tau)$ is presentable.

3.1.2 Properties of presentable categories

Presentable categories are very useful in practice, we could firstly show some important properties of presentable categories here.

Theorem 3.5. Let C be a presentable ∞ -category, then C admits all small limits.

Corollary 3.6. $\mathrm{PSh}(C)$ and $\mathrm{Shv}(C, \tau)$ are complete and cocomplete.

Theorem 3.7 (Adjoint functor theorem). ([1] 5.5.2.9) Let C, D be two presentable ∞ -categories, $F : C \rightarrow D$ be a functor. Then:

1. F is a left adjoint if and only if it preserves all small colimits;
2. F is a right adjoint if and only if it is accessible and preserves all small limits.

Thus we can consider a category which has really good properties (especially in stable homotopy theory) Pr^L .

Definition 3.8. ([1] 5.5.3.1) We define Pr^L (resp. Pr^R) as the subcategory of Cat_∞ whose objects are presentable ∞ -categories while the morphisms are left (resp. right) adjoint functors. For Pr^L , that's exactly the functors which preserve all small colimits.

3.2 Stable Homotopy Theory

To stabilize an ∞ -category, we can formally invert the suspension functor. If we view the suspension as the smash product with S^1 , then we can view the stabilization as a "localization" with respect to S^1 . This philosophy is generated by Robalo to explain the universal property of the stable motivic homotopy category.

Definition 3.9. Let C be a pointed ∞ -category which admits finite colimits, we define the stabilization $\mathrm{Sp}(C)$ of C as the limit of the tower:

$$\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C.$$

We will denote the stabilization functor from C to $\mathrm{Sp}(C)$ as $\Sigma^\infty : C \rightarrow \mathrm{Sp}(C)$.

Here the limit is taken in $\hat{\mathrm{Cat}}_\infty$. However, if C is a presentable category, we can take the limit in Pr^R . Furthermore, we have duality in Pr^L and Pr^R .

Theorem 3.10. Let C be a presentable pointed ∞ -category which admits finite colimits, then the following are equivalent:

1. The limit of the tower in Pr^R : $\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C$.
2. The colimit of the tower in Pr^L : $C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots$.
3. $\mathrm{Sp}(C)$.

If C is \mathcal{S}_* , then $\mathrm{Sp}(C)$ is exactly the classic stable homotopy category Sp . Notice that \mathcal{S}_* itself has a symmetric monoidal structure induced by the smash product \wedge , then the universal property of Sp can be stated as follows:

Definition 3.11. We denote $\text{Pr}^{\otimes, L}$ as the subcategory of Pr^L whose objects are presentably symmetric monoidal ∞ -categories while the morphisms are symmetric monoidal left adjoint functors.

Theorem 3.12 (Hovey). For every $C^\otimes \in \text{Pr}^{\otimes, L}$ and morphism $F : \mathcal{S}_*^\wedge \rightarrow C^\otimes$ in $\text{Pr}^{\otimes, L}$ for which the object $F(S^1)$ is invertible (admits an inverse with respect to \otimes), there exists an essentially unique morphism $\hat{F} : \text{Sp}^\otimes \rightarrow C^\otimes$ in $\text{Pr}^{\otimes, L}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_*^\wedge & \xrightarrow{F} & C^\otimes \\ \Sigma^\infty \downarrow & \nearrow \hat{F} & \\ \text{Sp}^\otimes & & \end{array}$$

That is, $F \simeq \hat{F} \circ \Sigma^\infty$.

3.3 Six functor formalism

Six functor theories, or six operations, are a formalization of some normal operations in algebraic topology. It was Grothendieck who first introduced the theory to explain *étale* cohomology.

The contents of this section come from [6] and [7].

Definition 3.13. (Naive) A six functor theory contains the following data:

- (1) An ∞ -category C which admits finite limits.
- (2) A class of morphisms E contains all isomorphisms, and is stable under pullback and composition.
- (3) An association $X \mapsto D(X)$ from C to ∞ -categories $D(X)$.
- (4) A symmetric monoidal closed structure on $D(X)$, where the tensor functor and the internal hom functor are denoted as \otimes and $\mathcal{H}\text{om}$.
- (5) For each $f : X \rightarrow Y$, a pullback functor $f^* : D(Y) \rightarrow D(X)$ and a pushforward functor $f_* : D(X) \rightarrow D(Y)$.
- (6) For each $f : X \rightarrow Y$ in E , a proper pullback functor $f^! : D(Y) \rightarrow D(X)$ and a proper pushforward functor $f_! : D(Y) \rightarrow D(X)$.

which satisfies the following compatible conditions:

$$\begin{aligned} f^* : D(Y) &\rightarrow D(X) & f_* : D(X) &\rightarrow D(Y); \\ f_! : D(X) &\rightarrow D(Y) & f^! : D(Y) &\rightarrow D(X); \end{aligned}$$

For a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

there exists isomorphism of functors: $g^* f_! \simeq f'_! g'^*$;

Let $f : Y \rightarrow X \in E$, $M \in D(X)$, $N \in D(Y)$, then we have $M \otimes f_! N \simeq f_! (f^* M \otimes N)$.

f^* and $f_!$ are compatible with composition.

Definition 3.14. Let (C, E) be a six functor theory, we can define the cohomology $\Gamma(X, \Lambda) = p_* 1$ for $X \in C$, where $p : X \rightarrow *$, 1 is the unit in $D(X)$.

For $p \in E$, we can define the cohomology with compact support $\Gamma_c(X, \Lambda) = p_! 1$.

We define $\omega_X := p^! 1$, where 1 is the unit of $D(*)$.

Proposition 3.15. (Küneth formula)

Let $X, Y \in C$, then we have:

$$\Gamma_c(X \times Y, \Lambda) \simeq \Gamma_c(X, \Lambda) \otimes \Gamma_c(Y, \Lambda).$$

Proposition 3.16. (Poincaré duality)

$$p_* \omega_X \simeq \mathcal{H}om(\Gamma_c(X, \Lambda), \Lambda).$$

4 Unstable Motivic Homotopy Theory

The unstable motivic homotopy theory was first introduced by Voevodsky [8] in 1998 to construct a homotopy theory in algebraic geometry. The basic intuition is that in such homotopy theory, the affine line \mathbb{A}^1 should play the same role as I in classic algebraic topology, so it was also called \mathbb{A}^1 -homotopy theory.

In Voevodsky's original work, he used the simplicial sheaves to capture the higher information. However, as we've mentioned in the prelimaries, the development of ∞ -categories supply a more convenient framework to capture them.

In this chapter, we will give the definition of motivic homotopy category $\mathcal{H}(S)$ related to a Noethrian scheme S , which plays the role of "spaces" where algebraic topology works. With this intuition, we'll soon see the reasonability of its definition.

In short, the motivic homotopy category $\mathcal{H}(S)$ is just the homotopy localization at \mathbb{A}^1 of the ∞ presheaves over the Nisnevich site Sm/S , or equivalently, the homotopy localization at \mathbb{A}^1 of the ∞ presheaves over the Nisnevich site Sm/S which satisfies excision property.

In the motivic homotopy category $\mathcal{H}(S)$, there are two trivial examples: the yoneda embedding from Sm/S and the constant sheaf from \mathcal{S} . These two "embedding" provide two kinds of spheres, i.e. the tate sphere and the standard sphere, thus inducing two types of stabilization in stable motivic homotopy theory.

4.1 Nisnevich topology

In algebraic geometry, we were very familiar with two types of topology, i.e., the Zariski topology and the étale topology. However, the Zariski topology is too coarse while the étale topology is too fine for algebraic topology theories to work.

To be concrete, we can see the following two examples from [9].

Example 4.1. Let $X \in Sm/\mathbb{C}$ with Zariski topology, \mathcal{F} is a constant sheaf, then $\dim H^1(X(\mathbb{C}), \mathcal{F}) = b_1(X(\mathbb{C}))$ while $H^1(X, \mathcal{F}) = 0$, where b_1 denotes the first betti number.

Example 4.2. Let $X = Spec(\mathbb{F}_q)$, then $\dim X = 0$ while $H^1(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$, where $(p, q) = 1$.

The right topology suitable for algebraic topology works ends up to be the Nisnevich topology, developed by Yevsey Nisnevich in 1989[10].

Definition 4.3. Given a Noethrian scheme S , consider in Sm/S , the category of smooth schemes of finite type over S , a family of $\{\varphi_i : X_i \rightarrow Y\}_{i \in I}$ is called a Nisnevich cover if:

1. φ_i is étale;
2. $\forall y \in Y, \exists i \in I, x \in \varphi_i^{-1}(y)$ such that the induced map of residue field $\tilde{\varphi}_i : k(x) \rightarrow k(y)$ is an isomorphism.

The collection of Nisnevich covers form a Grothendieck pretopology and the topology generated by it is called the Nisnevich topology. The site defined by it is called the Nisnevich site.

4.2 Excision property

The so called “excision property” can be understood in two intuitive ways, which motivate the two equivalent definitions below.

1. We expect the “motivic spaces” should satisfy a kind of Mayer-Vietoris property.
2. Sheaves are presheaves which preserve colimits to limits, considering in stable cases a pullback square is also a pushout square.

Definition 4.4. Let \mathcal{F} be a ∞ -presheaf on the Nisnevich site Sm/S . We say \mathcal{F} has the excision property if:

1. $\mathcal{F}(\emptyset)$ is contractible, i.e., isomorphic to the terminal object $* \in \mathcal{S}$.

2. \mathcal{F} satisfies one of the equivalent[11] conditions below:

- \mathcal{F} is a sheaf;
- $\forall X \in Sm/S, U$ is a open subscheme of $X, X' \rightarrow X$ is a morphism in Sm/S , then

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \times_X X') \end{array}$$

is a pullback square.

4.3 A1-equivalent

As we've mentioned, the affine line \mathbb{A}^1 should play the role as I in calssic algebraic topology world. We need every object is isomorphic to its “cylinder object”, thus we can define its “left homotopy”, as the case in model category.

Remark 4.5. We will simply write \mathbb{A}^1 in replace of \mathbb{A}_S^1 in Sm/S .

Definition 4.6. Let \mathcal{F} be a ∞ -presheaf on the Nisnevich site Sm/S . We say \mathcal{F} is \mathbb{A}^1 -equivalent if $\forall X \in Sm/S, \mathcal{F}(X) \rightarrow \mathcal{F}(X \times \mathbb{A}^1)$ is an isomorphism.

The following theorem provides a functor called “ \mathbb{A}^1 -localization functor”, which make it possible to construct a map from Sm/S to its motivic category.

Theorem 4.7. [12] Let $\mathrm{Sh}(Sm/S)$ be the category of ∞ -sheaves on the Nisnevich site Sm/S , $\mathrm{Sh}(Sm/S)_{\mathbb{A}^1}$ be the subcategory of $\mathrm{Sh}(Sm/S)$ which satisfies \mathbb{A}^1 -equivalent property, then the forgetful functor $U : \mathrm{Sh}(Sm/S)_{\mathbb{A}^1} \rightarrow \mathrm{Sh}(Sm/S)$ admits a right functor, which we denote as $L_{\mathbb{A}^1}$.

Corollary 4.8. The category $\mathrm{Sh}(Sm/S)_{\mathbb{A}^1}$ is a reflective subcategory of $\mathrm{Sh}(Sm/S)$, with the reflective localization functor $L_{\mathbb{A}^1}$.

Definition 4.9. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(Sm/S)$, $f, g : \mathcal{F} \rightarrow \mathcal{G}$, we say f and g are \mathbb{A}^1 -homotopy via h if $h : \mathcal{F} \times \mathbb{A}^1 \rightarrow \mathcal{G}$ satisfies $h \circ i_0 = f, h \circ i_1 = g$. This is denoted as $f \simeq_{\mathbb{A}^1} g$

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{i_0} & \mathcal{F} \times \mathbb{A}^1 & \xleftarrow{i_1} & \mathcal{F} \\ & \searrow f & \downarrow h & \swarrow g & \\ & & \mathcal{G} & & \end{array}$$

Definition 4.10. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(Sm/S)$, we say \mathcal{F} and \mathcal{G} are \mathbb{A}^1 -homotopy equivalent if $\exists f, g : \mathcal{F} \rightarrow \mathcal{G}$ such that $f \circ g \simeq_{\mathbb{A}^1} \mathrm{id}_{\mathcal{G}}$ and $g \circ f \simeq_{\mathbb{A}^1} \mathrm{id}_{\mathcal{F}}$.

Proposition 4.11. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(Sm/S)$, \mathcal{F} and \mathcal{G} are \mathbb{A}^1 -homotopy equivalent if and only if $L_{\mathbb{A}^1}(\mathcal{F})$ is isomorphic to $L_{\mathbb{A}^1}(\mathcal{G})$.

4.4 Unstable motivic homotopy category

After illustrating the reasonability of Nisnevich topology, excision property, \mathbb{A}^1 -equivalent, we can finally give our definition of unstable motivic homotopy category.

Definition 4.12. Let S be a Noethrian scheme, the (unstable) motivic homotopy category $\mathcal{H}(S)$ is defined as the following equivalent category:

1. The subcategory of $\mathrm{Psh}(Sm/S)$ which satisfies excision property and \mathbb{A}^1 -equivalent property;
2. The subcategory of $\mathrm{Sh}(Sm/S)$ which satisfies \mathbb{A}^1 -equivalent property.

We will call the objects in the motivic homotopy category as “motivic spaces”.

We use presheaves of Sm/S instead of Sm/S because we expect the motivic homotopy category to be complete cocomplete and presentable, as \mathcal{S} . It is known that yoneda embedding is exactly the “free cocompletion” and the presheaf category is complete and presentable. (This idea also motivates the definition of algebraic spaces) After sheafification and \mathbb{A}^1 -localization, this is still true because:

Lemma 4.13. For any ∞ -category C , its sheaf category $\mathrm{Sh}(C)$ is a reflective subcategory of $\mathrm{Psh}(C)$, with sheafification functor the reflective localization functor.

Since the reflective localization functor is stable under composition, we have:

Proposition 4.14. $\mathcal{H}(S)$ is complete, cocomplete, and locally presentable.

Here are two trivial examples of objects in motivic homotopy category $\mathcal{H}(S)$.

Example 4.15. Let $K \in \mathcal{S}$, then the constant sheaf over Sm/S is a motivic space. We will still denote it as K .

Example 4.16. Let $X/S \in Sm/S$, then the yoneda embedding of X/S is a motivic space. With a little abuse of the notation, we still denote it as X .

These two examples provide two embedding from \mathcal{S} and Sm/S to \mathcal{S} . The two circles embedded from each category is what we call as “standard sphere” and “tate sphere”.

Definition 4.17. We call the constant sheaf in $\mathcal{H}(S)$ with value on $\Delta[1]/\partial\Delta[1]$ the standard sphere.

4.5 Tate sphere

We will talk in pointed case in this section. Since the functor $(\cdot)_+$ has functoriality in Cat_∞ , it commutes with any functor. Thus with an abuse of notation, we will not distinguish the theory in pointed case with that of unpointed case.

Remark 4.18. We denote the product in $Sh_\bullet(Sm/S)$ as \wedge , i.e. the smash product.

Proposition 4.19. The following objects (or more precisely, the yoneda embedding of them) in $Sh_\bullet(Sm/S)$ are \mathbb{A}^1 -equivalent.

1. \mathbb{P}^1 ;
2. cofiber of $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$;
3. $S^1 \wedge \mathbb{G}_m$.

As a result, these objects in $Sh_\bullet(Sm/S)$ well-define an object in $\mathcal{H}(S)_\bullet$, which we call “tate sphere” or “motivic sphere”.

As in classic algebraic topology, the “sphere” induces the infinite loop functor, which gives rise to the stable motivic homotopy theory.

5 Stable Motivic Homotopy Theory

As we’ve shown in ??, their is a standard process to get a stable ∞ -category from a pointed ∞ -category. The process happens in this chapter is a little different.

Instead of using the ∞ -pushout of $0 \leftarrow X \rightarrow 0$ (which concides with $X \wedge S^1$ in \mathcal{S}_\bullet), we use the original definition of suspension functor, which is just the smash product with “circle”.

However, we have two kinds of circles in motivic homotopy category, so correspondingly there are two types of suspension functors, which naturally lead to bispectrum as well as bicohomology.

The stabilization process gives rise to the stable motivic homotopy category $\mathcal{SH}(S)$. Robalo shows[15] the universal property of it, which show the reasonability of “motivic” in its name.

5.1 Stabilization

This section is just a generalized version for biindexes of stabilization process in higher algebra.

Remark 5.1. Denote S_t^1 as the tate sphere, S_s^1 as the standard sphere.

Definition 5.2. The s-suspension functor $\Sigma_s : \mathcal{H}(S)_\bullet \rightarrow \mathcal{H}(S)_\bullet$ is defined as the functor: $- \wedge S_s^1$, while the t-suspension functor $\Sigma_t : \mathcal{H}(S)_\bullet \rightarrow \mathcal{H}(S)_\bullet$ is defined as the functor: $- \wedge S_t^1$.

Definition 5.3. The s-loop functor is defined as $\text{Maps}(S_s^1, -)$, while the t-loop functor is defined as $\text{Maps}(S_t^1, -)$.

Proposition 5.4. There are two adjunctions:

$$\begin{aligned} \Sigma_s : \mathcal{H}(S)_\bullet &\rightleftarrows \mathcal{H}(S)_\bullet : \Omega_s; \\ \Sigma_t : \mathcal{H}(S)_\bullet &\rightleftarrows \mathcal{H}(S)_\bullet : \Omega_t. \end{aligned}$$

Definition 5.5. The stable motivic homotopy category $\mathcal{SH}_s(S)$ (or simply denoted as $\mathcal{SH}(S)$) is defined as:

$$\text{colim}[\mathcal{H}(S)_\bullet \xrightarrow{\Sigma_s} \mathcal{H}(S)_\bullet \xrightarrow{\Sigma_s} \dots] \in \text{Pr}^L$$

While the $\mathcal{SH}_t(S)$ can be defined as:

$$\text{colim}[\mathcal{H}(S)_\bullet \xrightarrow{\Sigma_t} \mathcal{H}(S)_\bullet \xrightarrow{\Sigma_t} \dots] \in \text{Pr}^L$$

The object in motivic homotopy category is called “motivic spectra”.

In duality, we can also define $\lim[\mathcal{H}(S)_\bullet \xleftarrow{\Omega_s} \mathcal{H}(S)_\bullet \xleftarrow{\Omega_s} \dots] \in \text{Pr}^R$, $\lim[\mathcal{H}(S)_\bullet \xleftarrow{\Omega_t} \mathcal{H}(S)_\bullet \xleftarrow{\Omega_t} \dots] \in \text{Pr}^R$. Since $\mathcal{H}(S)_\bullet$ is presentable, we have similar propositions like ??.

Proposition 5.6. The colimit $\text{colim}[\mathcal{H}(S)_\bullet \xrightarrow{\Sigma_s} \mathcal{H}(S)_\bullet \xrightarrow{\Sigma_s} \dots]$ corresponds to the limit $\lim[\mathcal{H}(S)_\bullet \xleftarrow{\Omega_s} \mathcal{H}(S)_\bullet \xleftarrow{\Omega_s} \dots]$ via $\text{Pr}^L \simeq (\text{Pr}^R)^{op}$;

The colimit $\text{colim}[\mathcal{H}(S)_\bullet \xrightarrow{\Sigma_t} \mathcal{H}(S)_\bullet \xrightarrow{\Sigma_t} \dots]$ corresponds to the limit $\lim[\mathcal{H}(S)_\bullet \xleftarrow{\Omega_t} \mathcal{H}(S)_\bullet \xleftarrow{\Omega_t} \dots]$ via $\text{Pr}^L \simeq (\text{Pr}^R)^{op}$.

So we will not distinguish the colimit and limit and call them all as $\mathcal{SH}_s(S)$ and $\mathcal{SH}(S)$.

Remark 5.7. $\Sigma_* : \mathcal{H}(S)_\bullet \rightarrow \mathcal{SH}_*(S)$ and $\Omega_* : \mathcal{H}(S)_\bullet \rightarrow \mathcal{SH}_*(S)$ each naturally induce functor from $\mathcal{SH}_*(S)$ to $\mathcal{SH}_*(S)$ ($*$ $\in \{s, t\}$). With a little abuse of notation, we still denote them as Σ_* and Ω_* .

Remark 5.8. Since the stabilization process we've mentioned in 5.5 has functoriality, the process can be considered as a functor from $\mathcal{H}(S)_\bullet$ to $\mathcal{SH}(S)$, which we denoted as Σ_*^∞ , $* \in \{s, t\}$.

Duality, we denote the functor from $\mathcal{SH}(S)$ to $\mathcal{H}(S)_\bullet$ which maps a spectrum to its first term.

Remark 5.9. We will denote $\Sigma_*[k]$ as Σ_*^k , $\Omega_*[k]$ as Ω_*^k , $* \in \{s, t\}$;

We will denote $\Sigma_*^\infty \circ \Sigma_*[k]$ as $\Sigma_*^{\infty-k}$, $\Omega_*^\infty \circ \Omega_*[k]$ as $\Omega_*^{\infty-k}$, $* \in \{s, t\}$.

It's easy to verify the commutativity of these functors, which is why we use additivity to represent its index.

Proposition 5.10. There are two adjunctions:

$$\Sigma_*^{\infty-k} : \mathcal{H}(S)^\bullet \rightleftarrows \mathcal{SH}_*(S) : \Omega_*^{\infty-k}, * \in \{s, t\};$$

As for bigraded cases, the situation become a little more complicated.

Definition 5.11. The category $\mathcal{SH}_{s,t}(S)$ can be defined as:

$$\varinjlim_{\Sigma_s, \Sigma_t} \left[\begin{array}{ccccccc} \dots & \xrightarrow{\Sigma_s} & \dots & \xrightarrow{\Sigma_s} & \dots & \xrightarrow{\Sigma_s} & \dots \\ \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t \\ \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \dots \\ \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t \\ \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \dots \\ \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t & & \uparrow \Sigma_t \\ \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \mathcal{H}(S)_\bullet & \xrightarrow{\Sigma_s} & \dots \end{array} \right] \in \text{Pr}^L$$

Similarly we can define $\Sigma_{s,t}^{\infty,\infty}$ as the stabilization functor and $\Omega_{s,t}^{\infty,\infty}$ as taking the $(1, 1)$ -term of the spectrum. The definition of $\Sigma_{s,t}^{i,j}$ as well as $\Omega_{s,t}^{i,j}$ is similar.

Proposition 5.12. $\Sigma_{s,t}^{\infty-i, \infty-j} : \mathcal{H}(S)^\bullet \rightleftarrows \mathcal{SH}_{s,t}(S) : \Omega_{s,t}^{\infty-i, \infty-j}$.

5.2 Motivic cohomology

In classic algebraic topology, Brown representation theorem tell us that there exists a 1-1 corresponding between cohomology theories and spectra. Naumann and Spitzweck proved[13] that the theorem still holds in motivic world, after giving the axioms which a bigraded cohomology theory should satisfy.

Definition 5.13. We call a sequence of functor $(F^{p,q})_{(p,q \in \mathbb{Z}_{\geq 0}^2)} : (Sm/S)^{op} \rightarrow \mathcal{E}ns$ is a motivic cohomology if:

1. $F^{p+1,q}(S_s^1 \wedge X) = F^{p,q}(X)$;
2. $F^{p,q+1}(S_t^1 \wedge X) = F^{p,q}(X)$;
3. $F^{p,q}$ satisfies Nisnevich excision property and \mathbb{A}^1 -invariant property.

Theorem 5.14. A motivic cohomology is represented by a motivic spectrum E by $F^{p,q}(X) := [X, \Omega_{s,t}^{\infty-(p-2q), \infty-q} E] = [\Sigma_{s,t}^{\infty} X, \Omega_{s,t}^{p-2q, q} E]$, and vice versa.

We will show some classic examples of motivic spectra together with their corresponding motivic cohomology.

5.3 Motivic Spectra

We firstly see the motivic Eilenberg-MacLane spectrum. The construction below come from [14].

Definition 5.15. Let $X \in Sm/S$, define $L(X) : (Sm/S)^{op} \rightarrow Ab$ which maps:

$$U \mapsto F(\text{Irr}(U, X)),$$

where $\text{Irr}(U, X)$ is the set of closed irreducible subscheme Y of $U \times X$ such that $p_0 : Y \rightarrow U$ is of finite type and $\text{im}(p_0)$ covers one of the connected components of U , F is the free abelianization functor.

Proposition 5.16. The functor $U \circ L$ is a Nisnevich sheaf which satisfies \mathbb{A}^1 -equivalent category, where U is the forgetful functor. Thus $U \circ L \in \mathcal{H}(S)$.

Definition 5.17. The Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}^{p,q} := L((S_s^1)^{\wedge p} \wedge (S_t^1)^{\wedge q})$.

We don't forget its group structure here because we want to transfer the cohomology theory.

Just as in classic algebraic topology, Eilenberg-MacLane spectrum represents the "singular cohomology", that is, the Bloch's higher Chow group.

Proposition 5.18. $[X, \mathbf{H}\mathbb{Z}^{p,q}] = CH^q(X, 2q - p)$.

The next example we're cared about is the algebraic K-theory spectrum.

Definition 5.19. Define $\text{BGL}_n := \text{colim}(\dots \rightarrow \text{Gr}_n(\mathbb{A}^{n+m}) \rightarrow \text{Gr}_n(\mathbb{A}^{n+m+1}) \rightarrow \dots)$;
 $\text{KGL}_n := \mathbb{Z} \times \text{BGL}_n$.
 $\text{KGL} := \text{colim}[\dots \rightarrow \text{KGL}_{n-1} \rightarrow \text{KGL}_n \rightarrow \text{KGL}_{n+1} \rightarrow \dots]$

Proposition 5.20. There exists $\beta : \mathbb{P}^1 \wedge \text{KGL} \rightarrow \text{KGL}$.

Definition 5.21. We call the spectrum $(\text{KGL}, \text{KGL}, \dots)$ together with β on each degree the algebraic K-theory spectrum.

Proposition 5.22. The algebraic K-theory spectrum represents algebraic K-theory.

5.4 Universal Property

In this section, we will show that the stable motivic homotopy category $\mathcal{SH}(S)$ is indeed "motivic", i.e. has the universal property, which is proved by Robalo in 2012[16].

Definition 5.23. Let $X, Y \in \mathcal{SH}(S)$, $X \otimes Y := \lim_n n \lim_m (\Sigma^\infty (X_n \wedge Y_m) [-(n+m)])$

Definition 5.24. Let $X, Y \in \mathcal{H}(S)$, $\text{map}(X, Y) := ([X, Y[n]], [[X, Y[n]] \rightarrow [X, \Omega Y[n+1]]] \rightarrow [\Omega[X, Y[n+1]]])$.

Proposition 5.25. The \otimes functor and the map functor define a symmetric monoidal closed structure on $\mathcal{SH}(S)$.

Since $\mathcal{H}(S)$ is presentable, we have:

Proposition 5.26. $\mathcal{SH}(S)$ is pointed and presentable.

Proposition 5.27. The composition of functor $L_{\mathbb{A}^1} \circ (_)_+ \circ h_C : Sm/S \rightarrow \mathcal{SH}(S)$ is a symmetric monoidal, colimit-preserving functor.

Theorem 5.28. [16] Let D^\otimes be a pointed presentable symmetric monoidal $(\infty, 1)$ -category, then the composition map:

$$Fun^{\otimes, L}(\mathcal{SH}(S)^\otimes, D^\otimes) \rightarrow Fun^{\otimes, L}(Sm/S^\times, D^\otimes)$$

is fully faithful and its image satisfies:

- 1.Nisnevich excision property;
2. \mathbb{A}^1 -invariant;
- 3.send Tate sphere to an invertible object.

This theorem completes the original motivation for our topic, i.e. find a universal cohomology theory such that every cohomology theory passes through it.

5.5 Six functor fomalism

In 2007, Ayoub [17] found that the stable motivic homotopy category $\mathcal{SH}(S)$ admits a six functor theory, which can be seen as an enhance of cohomology theory.

Theorem 5.29. [17] There is a six functor theory on $(Sm/S, \mathcal{SH}(S))$, where E is the set of separated morphisms of finite type.

References

- [1] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, Princeton, NJ, 2009.
- [2] Jacob Lurie. Kerodon: An Online Resource for Higher Category Theory. <https://kerodon.net>, 2023.
- [3] André Joyal. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.
- [4] Chen Lin. Infinite Category Theory. <https://windshower.github.io/linchen/teaching/f2024.html>, 2024.
- [5] Jacob Lurie. Higher Algebra. <https://people.math.harvard.edu/~lurie/papers/HA.pdf>, 2017.
- [6] Peter Scholze. Six-Functor Formalisms. <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>, 2023.
- [7] Christian Heyer and Lucas Mann. 6-Functor Formalisms and Smooth Representations. *arXiv preprint*, 2024.
- [8] Vladimir Voevodsky. \mathbb{A}^1 -homotopy theory. In *Proceedings of the international congress of mathematicians*, volume 1, pages 579–604, 1998.
- [9] Joseph Hlavinka. Motivic Homotopy Theory and Cellular Schemes. 2022.

- [10] Yevsey A. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In *Algebraic K-theory: connections with geometry and topology*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, volume 279, pages 241–342. Kluwer Academic Publishers Group, Dordrecht, 1989.
- [11] Fabien Morel and Vladimir Voevodsky. \mathbb{A}^1 -homotopy theory of schemes. *Publications Mathématiques de l'IHÉS*, 90:45–143, 1999.
- [12] Fabien Morel. \mathbb{A}^1 -Algebraic Topology over a Field. Lecture Notes in Mathematics, volume 2052. Springer, Berlin, Heidelberg, 2012.
- [13] N. Axel Naumann and Markus Spitzweck. Brown representability in \mathbb{A}^1 -homotopy theory. *arXiv: Algebraic Geometry*, 2009.
- [14] Bjørn Ian Dundas, Marc Levine, Paul Arne Østvær, Oliver Röndigs, and Vladimir Voevodsky. *Motivic Homotopy Theory: Lectures at a Summer School in Nordfjordeid, Norway, August 2002*. Universitext. Springer-Verlag, Berlin, Heidelberg, 2007.
- [15] Marco Robalo. K-theory and the bridge from motives to noncommutative motives. *Advances in Mathematics*, 269:399–550, 2015.
- [16] Marco Robalo. Noncommutative motives I: A universal characterization of the motivic stable homotopy theory of schemes. *arXiv e-prints*, 2012.
- [17] Joseph Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I)*. Astérisque, volume 314. Société Mathématique de France, 2007. 476 pages.