

ADAMS SPECTRAL SEQUENCE-II

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1. REVIEW

We first review some basic preliminaries on the lecture of Adams spectral sequence-I.

1.1. Spectrum.

Definition 1.1. We can define a **spectrum** E as the following two equivalent ways:

- A sequence of pointed spaces $\{E_n\}_{n \in \mathbb{Z}}$ together with morphisms $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$.
- A sequence of pointed spaces $\{E_n\}_{n \in \mathbb{Z}}$ together with isomorphisms $\tilde{\sigma}_n : E_n \xrightarrow{\sim} \Omega E_{n+1}$.

Definition 1.2 (Generalized Cohomology & Homology Theory). We call a functor $h^* : h\mathcal{S}_*^{op} \rightarrow \text{Ab}^{\mathbb{Z}}$ a **generalized cohomology theory** if it satisfies:

1. Suspension Axiom: For any space X , there is a natural isomorphism $h^*(X) \cong h^{*+1}(\Sigma X)$.
2. Exactness Axiom: For any cofiber sequence $X \rightarrow Y \rightarrow Z$, the induced sequence

$$\cdots \rightarrow h^*(Z) \rightarrow h^*(Y) \rightarrow h^*(X) \rightarrow \cdots$$

is exact.

3. Wedge Axiom: For any family of spaces $\{X_\alpha\}_{\alpha \in A}$, the natural map

$$h^* \left(\bigvee_{\alpha \in A} X_\alpha \right) \rightarrow \prod_{\alpha \in A} h^*(X_\alpha)$$

is an isomorphism.

We call a functor $h_* : h\mathcal{S}_* \rightarrow \text{Ab}^{\mathbb{Z}}$ a **generalized homology theory** if it satisfies:

1. Suspension Axiom: For any space X , there is a natural isomorphism $h_*(X) \cong h_{*+1}(\Sigma X)$.
2. Exactness Axiom: For any cofiber sequence $X \rightarrow Y \rightarrow Z$, the induced sequence

$$\cdots \rightarrow h_*(X) \rightarrow h_*(Y) \rightarrow h_*(Z) \rightarrow \cdots$$

is exact.

3. Wedge Axiom: For any family of spaces $\{X_\alpha\}_{\alpha \in A}$, the natural map

$$h_* \left(\bigvee_{\alpha \in A} X_\alpha \right) \rightarrow \bigoplus_{\alpha \in A} h_*(X_\alpha)$$

is an isomorphism.

Theorem 1.3 (Brown Representability Theorem). *Any generalized cohomology theory h^* is representable, i.e. there exists a spectrum E such that for any pointed space X , there is a natural isomorphism*

$$h^n(X) \cong [X, E_n].$$

Any generalized homology theory h_ is representable, i.e. there exists a spectrum E such that for any pointed space X , there is a natural isomorphism*

$$h_n(X) \cong \varinjlim_k [S^k, X \wedge E_{n+k}].$$

Remark 1.4. A more natural way to present looks like:

$$h^n(X) \cong \pi_{-n} \text{Maps}(\Sigma^\infty X, E)$$

$$h_n(X) \cong \pi_n \text{Maps}(\Sigma^\infty X \wedge E)$$

Theorem 1.5 (Symmetric Monoidal Structure on Spectra).

Definition 1.6 (Ring Spectrum).

1.2. Eilenberg-MacLane Spectrum.

Definition 1.7 (Delooping).

Definition 1.8 (Eilenberg-MacLane Space).

Definition 1.9 (Eilenberg-MacLane Spectrum).

Theorem 1.10 (Representability of Singular Cohomology).

Theorem 1.11 (Ring Structure on Eilenberg-MacLane Spectrum).

1.3. Cohomology Operations and Stable Cohomology Operations.

Definition 1.12 (Cohomology Operation). A **cohomology operation** of type $(n, G) \rightarrow (m, G')$ is a natural transformation between two cohomology functors, i.e. a collection of maps

$$\{\theta_X : H^n(X; G) \rightarrow H^m(X; G')\}_{X \in \text{Top}_*}$$

such that for any pointed map $f : X \rightarrow Y$, the following diagram commutes:

$$\begin{array}{ccc} H^n(Y; G) & \xrightarrow{\theta_Y} & H^m(Y; G') \\ \downarrow f^* & & \downarrow f^* \\ H^n(X; G) & \xrightarrow{\theta_X} & H^m(X; G') \end{array}$$

Notice that by the representability of singular cohomology, $H^n(-; G)$ as a functor is represented by $[-, K(G, n)]$, thus a cohomology operation θ of type $(n, G) \rightarrow (m, G')$ corresponds to a natural transformation of functors

$$[-, K(G, n)] \rightarrow [-, K(G', m)]$$

, thus by Yoneda lemma (in the homotopy category of pointed spaces), it corresponds to $H^m(K(G, n); G')$.

Definition 1.13 (Stable Cohomology Operation). A **stable cohomology operation** of degree k from G to G' is a collection of cohomology operations

$$\{\theta^n : H^n(-; G) \rightarrow H^{n+k}(-; G')\}_{n \in \mathbb{Z}}$$

such that for any pointed space X , the following diagram commutes:

$$\begin{array}{ccc} H^n(X; G) & \xrightarrow{\theta_X^n} & H^{n+k}(X; G') \\ \downarrow \Sigma & & \downarrow \Sigma \\ H^{n+1}(\Sigma X; G) & \xrightarrow{\theta_{\Sigma X}^{n+1}} & H^{n+k+1}(\Sigma X; G') \end{array}$$

2. HOPF ALGEBRA

Definition 2.1 (Algebra and Coalgebra over a Monoidal Category).

Definition 2.2 (Bialgebra).

Example 2.3 (k -module).

Definition 2.4 (Hopf Algebra).

3. STEENROD ALGEBRA

Definition 3.1 (Steenrod Squares).

4. E-ADAMS SPECTRAL SEQUENCE

4.1. Classic Adams Spectral Sequence.

REFERENCES