SEMINAR ON SPECTRAL SEQUENCES IN ALGEBRAIC TOPOLOGY

XINGZHI HUANG

1. Goals

This seminar is mainly based on the book of Allen Hatcher [HatSS] focusing on the Serre spectral sequence and Adams spectral sequence, but also includes some other materials to give some preliminaries, applications and generations.

I expect the leading question of the seminar to be computing the homotopy groups of spheres, so we might focus more on that.

2. Preliminaries

We assume the participants have taken MATH 750, MATH 751 and MATH 752 so that they are familiar with cohomology theoreis and spectral sequences. We do not assume participants have prior knowedgement of the topics covered in [Hat02] but not covered in MATH 751 and MATH 752 (such as Hopf Algebra, Steenrod Algebra and Postnikov towers). Hoewver, we encourge the participants to self-study these topics ahead because they are not the central focus of this seminar.

3. Outline

We expect each chapter to be covered in one presentation and to include the listed topics. The topics marked with $*$ are optional, which means: In principle, we would like them to be covered, but the speaker may skip them due to time constraints or a lack of interest. We also expect that at least the examples necessary for understanding the key concepts should be included in the presentation.

§1: Serre Spectral Sequence. Constrcution and properties of Serre spectral sequences, **Serre classes*, Reference: [HatSS], [May11]

§2: Hopf Algebra, H-Space and H-Group. Given a braided monoidal category $\mathcal{C}$, we will see $\mathrm{Alg}(\mathcal{C})$ still has monoidal structure, thus we can define the coalgebra of it, which we call as the bialgebra of $\mathcal{C}$. Furthermore, if we have the antipodal structure, we can call it the Hopf algebra.

Given a space X with coefficient R, the ring structure of cohomology naturally gives the algebra structure of $H^*(X;R)$. If we give X a kind of "weak" algebra structure on Top, i.e. H-space structure, then we can get the coalgebra structure of $H^*(X;R)$; If we further give X a kind of inverse structure, i.e. H-group sturcture, then we can get the antipodal structure of $H^*(X;R)$. Thus we have the Hopf algebra structure of $H^*(X;R)$.

Reference: [OAMM], [Hat02]

§3: Localization of Spaces and Rational Homotopy Groups of Spheres. Given a set of primes $\mathcal{P}$ and a finitely generated abelian group G, we can construct the $\mathcal{P}$-localization of G, $G_{\mathcal{P}}$, to kill the torsion part of G whose order is not divided by any prime in $\mathcal{P}$. For a space X, the localization of its homotopy groups can be lifted to the space level, i.e. for a space X, there is a space $X_{\mathcal{P}}$ such that $\pi_i(X_{\mathcal{P}}) \cong \pi_i(X)_{\mathcal{P}}$.

One of the applications of localization lies in the case where $\mathcal{P}$ is empty(which we call as rational localization) and X is a H-group with finitely generated homotopy groups. In this case, we have $H^*(X;\mathbb{Q}) = \mathrm{Sym}(\pi_*(X)\otimes\mathbb{Q})$.

Take $X = S^n$, we have $\pi_i(S^n)\otimes\mathbb{Q} = \begin{cases}\mathbb{Q} & i = n \text{ or } i = 2n-1 \text{ when } n \text{ is even}\\ 0 & \text{otherwise}\end{cases}$

This encodes the the non-torsion part of homotopy groups of spheres.

Reference: [May11], [HatSS]

§4: Cohomology Operation, Steenrod Square and Steenrod Algebra. A cohomology operation of type (π, n, G, m) is a family of morphisms

$$\theta_X : H^n(X;\pi) \to H^m(X;G)$$

which is natural with respect to the base space. We further call a cohomology operation stable if it commutes with the suspension. The Steenrod algebra is the algebra of stable cohomology operations with coefficient $\mathbb{Z}_2$.

We could establish a bijection between the set of cohomology operations of type (π, n, G, m) and $H^m(K(\pi, n); G)$. Thus when $\pi = G = \mathbb{Z}_2$, we will see the only nontrivial cohomology operations are the Steenrod squares $Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$. The Steenrod algebra $\mathcal{A}_2$ is isomorphic to the algebra generated by all Steenrod squares with the Adem relations:

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2\rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k, \quad i < 2j$$

We will see that the Steenrod algebra $\mathcal{A}_2$ plays an important role in calculating the Eilenberg-MacLane spaces with coefficient $\mathbb{Z}_2$, which gives rise to some important results of the 2-torsion part of homotopy groups of spheres. We will finally see that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Reference: [Hat02], [MT08]

§5: Adams Spectral Sequence and the Stable Homotopy Groups of Spheres. Given a spectrum E under mild assumptions, there is a spectral sequence with

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(E;\mathbb{Z}_p), \mathbb{Z}_p) \Rightarrow \pi_{t-s}(E)\otimes\mathbb{Z}_p$$

for any prime p. This spectral sequence is called the Adams spectral sequence. Take E as the sphere spectrum, we can use this spectral sequence to compute the stable homotopy groups of spheres.

To compute the E_2-page, a useful tool is the May spectral sequence, which is a spectral sequence with

$$E_1^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathrm{Gr}(\mathcal{A}_p)}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{Z}_p, \mathbb{Z}_p)$$

where $\mathrm{Gr}(\mathcal{A}_p)$ is the associated graded algebra of $\mathcal{A}_p$ with respect to the May filtration.

Reference: [HatSS], [Koc96]

§6: Chromatic Homotopy Theory and Chromatic Spectral Sequence. [Attention: I have no idea of this section yet, this part is from wiki]

A chromatic spectral sequence is a spectral sequence of a filtered stable homotopy type for the case of a filtering given by a chromatic tower.This is a type of spectral sequence useful for computing the E_1 term of the Adams-Novikov spectral sequence.

Reference: [Rog23]

References

HatSS. Hatcher, Allen. *Spectral Sequences.* preprint, 2004. https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf

May11. May, J. Peter, and Ponto, Kate. *More Concise Algebraic Topology: Localization, Completion, and Model Categories.* University of Chicago Press, 2011. https://webhomes.maths.ed.ac.uk/~v1ranick/papers/mayponto.pdf

OAMM. Kriz, Igor and May, J. Peter, *Operads, algebras, modules and motives*, Astérisque, vol. 233, Société Mathématique de France, Paris, 1995. https://www.numdam.org/issue/AST_1995__233__1_0.pdf

Hat02. Hatcher, Allen, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf

MT08. Mosher, Robert E. and Tangora, Martin C., *Cohomology operations and applications in homotopy theory*, Courier Corporation, Mineola, NY, 2008. https://webhomes.maths.ed.ac.uk/~v1ranick/papers/moshtang.pdf

Koc96. Kochman, Stanley O., *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs, vol. 7, American Mathematical Society, Providence, RI, 1996.

Rog23. Rognes, John, *Chromatic homotopy theory*, Lecture notes, Algebraic Topology III (MAT4580/MAT9580), University of Oslo, Oslo, Norway, Spring 2023. https://www.uio.no/studier/emner/matnat/math/MAT9580/v23/documents/chromatic.pdf