Motivic Homotopy Theory

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Contents

1 Introduction

1.1 Foreword

The concept of “motive" was introduced by Grothendieck who noticed a series of similarities in Weil cohomology theories. In his idea, he thought there should be an universal cohomology theory which every cohomolgy theory taking values in a $\mathbb{Q}$-category should pass through it.

This idea soon became so important in the research about Weil conjecture that people belive once the “motive" is discovered, then the problem can be done.

Thanks to the effort of many mathematicians, we succeeded in constructing “pure motives" for smooth projective varieties as well as “mixed motives" for a more general cases.

One important observation in algebraic topology is that when we talk about cohomology theories, what really matters is the derived category. It was Voevodksy who introduced the “motivic homotopy category" and prove the universal property of it, which is what we are interested in this paper.

1.2 The main content of this paper

In short, motivic homotopy theory concerns how to construct a algebraic topology theory on algebraic geometry. To give the readers an intuition, we include a correponding diagram below.

Table 1: Comparison between classic algebraic topology world and motivic world

Classic World	**Motivic World**
$\mathcal{S}$ (Spaces)	$\mathcal{H}(S)$
I	$\mathbb{A}^1$
I-homotopy equivalence	$\mathbb{A}^1$-homotopy equivalence
S^1	$\mathbb{P}^1$
Sp	$\mathcal{SH}(S)$

In Chapter 2, we'll recommend the language of $\infty-$categories. Any reader who is already familiar with it can skip this chapter.

In Chapter 3, we will firstly recommend some preliminaries that are important to understand the topic we'll discuss. It mainly contains higher topos theory, stable homotopy theory, six functor formalism and algebraic K-theory.

In Chapter 4, we will construct the unstable motivic homotopy category $\mathcal{H}(S)$ as a motivic localization of the Nisnevich presheaf category. We will also see the reasonability of such construction.

In Chapter 5, we will construct the stable motivic homotopy category $\mathcal{SH}(S)$, as a variant of the classic stabilization of the $\mathcal{H}(S)$. We will also see how Robalo unifies them into a general framework and proves the universal property of $\mathcal{SH}(S)$.

2 Language of ∞-categories

The motivation of ∞-categories is to construct a language where we can talk about higher morphisms. A naive approach towards this is enriched category. However, the coherence data soon become too complicated to describe as the level increases. So we will construct some models to describe ∞-categories.

It should be noted what we call ∞- categories here are in fact $(\infty, 1)$-categories, which means all n-morphisms ($n \geq 2$) are invertible.

Readers are assumed to have been familiar with the knowledge of simplicial sets and model categories.

2.1 Models of ∞-categories

When we say a "model" for ∞- categories, we mean a model category, whose objects(up to weak equivalence) are ∞- categories and the weak equivalence between two objects represents equivalence between two ∞- categories. These models are "equivalent", namely Quillen equivalent.

In short,

The theory of $(\infty, 1)$-categories lies in the homotopy category of a model category which "models" the ∞-categories. (2.1)

Although for practical reasons we have to construct those definitions and propositions in a specific category, they should be invariant under weak equivalences.

In this subsection, we will introduce two models of ∞-categories, which are quasi-categories and simplicially enriched categories.

Definition 2.1 (Boardman-Vogt, 1973)**.** A simplical set X is called a quasi-category if it satisfies the right lifting property aganist inner horn inclusions:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & * \end{tikzcd} \qquad (0 < k < n)$$

Take $\Lambda^n_k = \Lambda^2_1$, then the lifting property provides a lifting $h : \Delta^2 \to X$. If we denote f as the image of $\{0 \to 1\}$ in X, g as the image of $\{1 \to 2\}$ in X, then we can view h as the "homotopy" witnessing the composition of f and g.

A question arising here is that the "homotopy" h witnessing the composition is not unique. However, if we understand the "uniqueness" following the philosophy in 2.1, it is "unique", or precisely, $\mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Lambda^2_1, C)} (g, \bullet, f)$ is contractible (2.24).

Remark 2.2. We denote the full subcategory of Set_Δ generated by quasi-categories as QCat.

Definition 2.3. We define Cat_Δ as the category of Set_Δ-enriched categories, whose morphisms are given by enriched functors. We also define Cat_{Kan} as the full subcategory of Cat_Δ generated by those Kan complex-enriched(which we will introduce in 2.8) categories.

We will then show two model categories which model the ∞-categories, or more precisely, whose fibrant full subcatgoreis model the $(\infty, 1)-$categories.

Theorem 2.4 (Joyal, 2008)**.** There exists a model structure called Joyal model structure on Set_Δ where fibrant objects are exactly quasi-categories.

Theorem 2.5 (Bergner, 2004)**.** There exists a model structure called Bergner model structure on Cat_Δ, whose fibrant objects are exactly Kan complex-enriched categories.

Joyal and Tierney pointed out that these two model categories are Quillen equivalent, thus whose fibrant subcategoreis are also Quillen equivalent.

Theorem 2.6 (Joyal & Tierney, 2007)**.** There exists a Quillen equivalence: $\mathfrak{C} : \mathrm{Set}_\Delta^{Joyal} \rightleftarrows \mathrm{Cat}_\Delta : \mathfrak{N}$.

Corollary 2.7. QCat and Cat_{Kan} are Quillen equivalent.

2.2 Mapping Spaces

The idea of ∞-categories is that for any two objects in the category, there exists a "mapping space" between such that the "mapping space" is an $(\infty, 0)$-category (Just as the case in $\mathcal{CG}$-enriched categories). That gives rise to the concept of Kan complexes where every morphism is "invertible".

We still define the mapping space as the fibrant object in a homotoy category of a model category which we call as "Kan-Quillen" model category. This model category is Quillen equivalent to Top, whose fibrant objects are CW complexes, that is, "good spaces".

Definition 2.8. A simplical set X is called a Kan complex if it satisfies the right lifting property against horn inclusions:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & * \end{tikzcd} \qquad (0 \le k \le n)$$

We make it clear why we say "any morphism in Kan complex is invertible" here.

Remark 2.9. Let C be a quasi-category, the expression $x \in C$ means x is a 0-simplex in C (We also call x as a 0-morphism); the expression $f : x \to y$ means f is a 1-simplex in C satisfying $d_0 f = x, d_1 f = y$ (We also call f as a 1-morphism).

Theorem 2.10. ([2], 1.4.3.6) Let C be a quasi-category, $x, y \in C, f, g : x \to y$, The following are equivalent:

1. There exists a 2-simplex σ satisfying $d_o\sigma = id_y, d_1\sigma = g, d_2\sigma = f$,i.e.

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

2. There exists a 2-simplex σ satisfying $d_o\sigma = f, d_1\sigma = g, d_2\sigma = id_x$,i.e.

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

Remark 2.11. Obviously the condition is a equivalence ralationship, we will say f is homotopic to g if one holds.

Definition 2.12. Let C be an quasi-category, The homotopy category $\mathsf{h}C$ of C is defined as the ordinary category as below:

$\mathrm{Ob}(\mathsf{h}C) = \mathrm{Ob}(C)$;

$\mathrm{Hom}_{\mathsf{h}C}(x, y) = \mathrm{Hom}_C(x, y)/\sim$, where $\sim$ is defined by homotopy;

For $f : x \to y, g : y \to z \in \mathsf{h}C$, the composition $g \circ f$ is defined as $d_1(h)$, where h is a 2-morphism witenessing the composition of f and g. This is well-defined because $\mathrm{Fun}(\Delta^2, C)\times_{\mathrm{Fun}(\Lambda^2_1, C)} (g, \bullet, f)$ is contractible.

Remark 2.13. Let $f : x \to y \in C$, we will call f an isomorphism if the image $[f]$ of f in $\mathsf{h}C$ is an isomorphism.

Theorem 2.14. ([3]) A quasi-category C is a Kan complex if and only if all morphisms in C are isomorphisms.

Before formally giving the consturction of mapping space, we have to clairify the third model category whose fibrant objects are exactly Kan complexes. The following results are classic in algebraic topology.

Theorem 2.15. There exists a model structure called Kan-Quillen model structure on Set_Δ where fibrant objects are Kan complexes. We call the weak equivalences in Kan-Quillen model as "weak homotopy equivalences".

Theorem 2.16. There exists a Quillen equivalence:

$|\cdot| : \mathrm{Set}_\Delta^{Kan-Quillen} \rightleftarrows \mathrm{Top} : \mathrm{Sing}.$

Corollary 2.17. Kan and CW are Quillen equivalent.

It is well-known that CW complexes are "good spaces", that is, whose "mapping space" between two CW complexes itself is still a CW complex.

We will then formally give the consturction of mapping spaces.

Definition 2.18. Let X be a simplicial set, $X^{\triangleleft} := \Delta^0 \star X$; $X^{\triangleright} := X \star \Delta^0$, where $\star$ represents join in simplicial set.

Definition 2.19. Let X be a simplicial set, $x \in X$, the slice simplicial set $X_{/x}$ of X over x is defined by:

$Hom(Y, X_{/x}) := Hom(Y^{\triangleright}, X) \underset{Hom(\Delta^0,X)}{\times} \{x\};$

the coslice simplicial set $X_{x/}$ of X under x is defined by:

$Hom(Y, X_{x/}) := Hom(Y^{\triangleleft}, X) \underset{Hom(\Delta^0,X)}{\times} \{x\};$

Definition 2.20. Let C be a quasi-category, $x, y \in C$. We will define:

The left pinched complex $\mathrm{Hom}^L_C(x, y) := C_{x/} \underset{C}{\times} \{y\}$;

The right pinched complex $\mathrm{Hom}^R_C(x, y) := \{x\} \underset{C}{\times} C_{/y}$;

The balanced complex $\mathrm{Hom}^B_C(x, y) := \{x\} \underset{\mathrm{Fun}(\{0\},C)}{\times} \mathrm{Fun}(\Delta^1, C)) \underset{\mathrm{Fun}(\{1\},C)}{\times} \{y\}$.

Theorem 2.21. ([2], 4.6.5.5) $\mathrm{Hom}^L_C(x, y)$, $\mathrm{Hom}^R_C(x, y)$, and $\mathrm{Hom}^B_C(x, y)$ are Kan complexes which are weak homotopy equivalent.

As a result, we can well-defined the mapping space as a kan complex up to weak equivalence (in the sense of Kan-Quillen model structure).

Definition 2.22. Let C be a quasi-category, $x, y \in C$, we can define the mapping space $\mathrm{Maps}(x, y)$ from x to y as the object in $\mathrm{Set}_\Delta[W^{-1}]$ defined by 2.21.

In fact, Maps can be extended to an $(\infty, 1)-$functor $C^{op} \times C \to \mathcal{S}$, which we will show in 2.43.

Based on this observation, we can illustrate the idea that mapping spaces are well-defined up to weak equivalance (following the philosophy in 2.1) by stating a stronger theorem in 2.49.

At the end of this subsection, we will make it clear what we mean by “contractible".

Definition 2.23. A Kan complex X is called contractible if it satisfies the right lifting property aganist boundary inclusions:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & * \end{tikzcd}$$

Theorem 2.24. ([2], 1.5.6.2) Let C be a quasi-category, $x, y, z \in C, f : x \to y, g : y \to z$, then $\mathrm{Fun}(\Delta^2, C) \underset{\mathrm{Fun}(\Lambda^2_1,C)}{\times} \{(g, \cdot, f)\}$ is a contractible Kan complex.

2.3 Functors of ∞-categories

Functors between two quasi-categories are just morphisms in Set_Δ. Two types of weak equivalences provide two types of “equivalences". We'll show their meaning in the $(\infty, 1)$-category of $(\infty, 1)$-categories.

Definition 2.25. Let C, D be two quasi-categories, we will call the morphism $F : C \to D$ in Set_Δ a functor from C to D.

Proposition 2.26. Set_Δ admits a Cartesian closed structure. With a little abuse of the notation, we will denote the corresponding internal hom $[X, Y]$ as $\mathrm{Fun}(X, Y) \in \mathrm{Set}_\Delta$.

By Yoneda lemma clearly the 0-morphisms in $\mathrm{Fun}(X, Y)$ are exactly the functors from X to Y. We call this category the $(\infty, 1)-$functor categories from X to Y.

Corollary 2.27. Let X, Y, Z be three simplicial sets, then there exists a natural composition $\mathrm{Fun}(X, Y) \times \mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$.

Theorem 2.28. ([4], 5.4.5) Let C, D be two quasi-categories, $F : C \to D$ be a functor from C to D. Then the following are equivalent:

1. F is a weak equivalence in $\mathrm{Set}_\Delta^{Joyal}$;

2. For any quasi-category E, the induced morphism $\mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$ is a weak equivalence in $\mathrm{Set}_\Delta^{Joyal}$.

We will call this weak equivalence a "homotopy equivalence" or "equivalence".

Following the philosophy in 2.1, this is the right "isomorphism" in the $(\infty,1)-$category of $(\infty,1)-$categories, that is, Cat_∞.

Lemma 2.29. ([2], 1.5.3.7) Let $X \in \mathrm{Set}_\Delta, Y \in \mathrm{QCat}$, then $\mathrm{Fun}(X, Y) \in \mathrm{QCat}$.

Lemma 2.30. ([2], 1.3.5.4) Let $X \in \mathrm{QCat}$, then there exists a maximal Kan complex contained in X, which we denote as $X^{\simeq}$, or the core of X.

Definition 2.31. Let $\mathbb{QCat}$ be a $\mathrm{Set}_\Delta-$enriched category which is defined as:

Objects are all quasi-categories;

$\mathrm{Hom}(X, Y) := \mathrm{Fun}(X, Y)^{\simeq}$;

The composition map is given by 2.27.

Definition 2.32. We define the $(\infty,1)-$category of $(\infty,1)-$categories as $\mathfrak{N}(\mathbb{QCat})$, which we denote as Cat_∞.

Similarily we have:

Definition 2.33. Let $\mathbb{Kan}$ be a $\mathrm{Set}_\Delta-$enriched category which is defined as:

Objects are all Kan complexes;

$\mathrm{Hom}(X, Y) := \mathrm{Fun}(X, Y)^{\simeq}$;

The composition map is given by 2.27.

Definition 2.34. We define the $(\infty,1)-$category of $(\infty,1)-$groupoids as $\mathfrak{N}(\mathbb{Kan})$, which we denote as Grpd_∞ or $\mathcal{S}$.

Proposition 2.35. ([4], 5.7.6) The homotopy category $\mathsf{h}\mathrm{Cat}_\infty$ is equivalent to the ordinary category $\mathsf{h}QC\dashv\sqcup$ defined as:

$\mathrm{Ob}(\mathsf{h}QC\dashv\sqcup) :=$ all small quasi-categories;

$\mathrm{Hom}_{\mathsf{h}QC\dashv\sqcup}(C, D) := \pi_0(\mathrm{Fun}(C, D)^{\simeq})$;

$[G] \circ [F] = [G \circ F]$.

Theorem 2.36. ([4], 5.4.1) Let C, D be two quasi-categories, $F : C \to D$, F is a homotopy equivalence iff $[F]$ is an isomorphism in $\mathsf{h}QC\dashv\sqcup$.

This explains the reasonability of "homotopy equivalence".

2.4 Fibrations of ∞-categories

In ordinary category theory, we are already familiar with the Grothendieck construction with respect to a kind of fibration. These concepts still have their correspondence in ∞-categories.

The relationship of those kinds of fibrations can be represented by the diagram below[1].

$$\begin{tikzcd}
 & \text{trivial fibration} \arrow[d] & \\
 & \text{Kan fibration} \arrow[dl] \arrow[dr] & \\
\text{left fibration} \arrow[d] & & \text{right fibration} \arrow[d] \\
\text{coCartesian fibration} \arrow[dr] & & \text{Cartesian fibration} \arrow[dl] \\
 & \text{categorical fibration} \arrow[d] & \\
 & \text{inner fibration} &
\end{tikzcd}$$

While the Grothendieck-Lurie construction[1] gives the equivalence of $(\infty, 1)-$categories:

The $(\infty, 1)-$category of Fibrations	**The ∞-category of Functors**
$\mathrm{LFib}_{/S}$	$\mathrm{Fun}(S, \mathrm{Grpd}_\infty)$
$\mathrm{RFib}_{/S}$	$\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Grpd}_\infty)$
$\mathrm{coCart}_{/S}$	$\mathrm{Fun}(S, \mathrm{Cat}_\infty)$
$\mathrm{Cart}_{/S}$	$\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat}_\infty)$

In this subsection we will not illustrate everything in these two pictures. Instead, we only define left fibration and present the related Grothendieck-Luire construction to give a “genuine" definition of Maps.

Definition 2.37. Let C, D be two quasi-cateogries, $p : C \to D$ is a functor, we say p is an inner fibration if it satisfies the right lifting property against against inner horn inclusions:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & C \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & D
\end{tikzcd} \qquad (0 < k < n)$$

Definition 2.38. Let C, D be two $(\infty, 1)-$categories, $p : C \to D$ be an inner fibration[1]. We say

[1] As the word “fibration" implies, inner fibration is stable under pullback, thus given $x \in C$ the fiber $C_{p(x)}$ is a quasi-category. In a model-independent theory this condition is not required.

p is a left fibration if it satisfies the right lifting property:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & C \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & D \end{tikzcd} \qquad (0 \leq k < n)$$

Definition 2.39. Let S be an $(\infty,1)-$category, we define the $(\infty,1)-$category $\mathrm{LFib}_{/S}$ as the full subcategory of Cat_∞ generated by those left fibrations.

Theorem 2.40. ([1] 3.2.0.1) Given an $(\infty,1)$-category S, there exists an equivalence of $(\infty,1)$-categories:

$\mathrm{LFib}_{/S} \rightleftarrows \mathrm{Fun}(S, \mathrm{Grpd}_\infty)$.

Finally we can extend the definiton of Maps.

Definition 2.41. Let C be an $(\infty,1)-$category, we define the $(\infty,1)-$category of twisted arrow category $\mathrm{TwArr}(C) := \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet((-)^{op} \star (-)), C)$.

Proposition 2.42. ([2], 8.1.1.15) The projection $\mathrm{TwArr}(C) \to C^{op} \times C$ induced by $[n]^{op} \to [n]^{op} \star [n] \leftarrow [n]$ is a left fibration.

Consider The projection as an object in $\mathrm{LFib}_{/S}$, we can formly give our definition of Maps

Definition 2.43. We define the functor $\mathrm{Maps} : C^{op} \times C \to \mathrm{Grpd}_\infty$ as the correspondence of the projection $\mathrm{TwArr}(C) \to C^{op} \times C$ via 2.40.

We'll use the following definition in 2.9.

Definition 2.44. Let C, D be two $(\infty,1)-$categories, $p : C \to D$ be an inner fibration, $f : x \to y \in C$. We say f is p-coCartesian if it satisfies the right lifting property:

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\phi"] \arrow[d, hook] & C \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists h"] & D \end{tikzcd}$$

where $n \geq 2, \phi|_{\{0,1\}} = f$.

2.5 Limits and Colimits

Definition 2.45. Let C be an $(\infty,1)-$category, $x \in C$, we say x is initial if $\forall y \in C$, $\mathrm{Maps}(x, y)$ is contractible; x is final if $\forall y \in C$, $\mathrm{Maps}(y, x)$ is contractible.

Definition 2.46. Let K be a simplicial set, C be an $(\infty,1)$-category, $u : K \to C$. The slice category of C over u is defined by:

$\mathrm{Hom}(Y, C_{/u}) := \mathrm{Hom}(Y \star K, C) \underset{\mathrm{Hom}(K,C)}{\times} \{u\}$;

The coslice category of C under u is defined by:

$\mathrm{Hom}(Y, C_{u/}) := \mathrm{Hom}(K \star Y, C) \underset{\mathrm{Hom}(K,C)}{\times} \{u\}$.

Definition 2.47. Let K be a simplicial set, C be an $(\infty, 1)$-category, $u : K \to C$. We say a diagram $\bar{u} : K^{\triangleleft} \to C$ is a limit diagram if it is a final object in $C_{/u}$; We say a diagram $\bar{u} : K^{\triangleright} \to C$ is a colimit diagram if it is an initial object in $C_{u/}$.

We also say that $\bar{u}$ exhibits $\bar{u}(*)$ as a limit/colimit of u.

Definition 2.48. We say a limit diagram $\bar{u} : K^{\triangleleft} \to C$ is preserved by a functor $F : C \to C'$ if $F \circ \bar{u} : K^{\triangleleft} \to C'$ is a limit diagram; A colimit diagram $\bar{u}$ is preserved by F if $F \circ \bar{u}$ is a colimit diagram.

Theorem 2.49. ([2] 4.6.4.21) Let C, D be two $(\infty, 1)-$categories, $F : C \to D$ be an equivalence, $u : K \to C$ be a diagram, then the induced functor $C_{/u} \to C_{/F \circ u}$ is an equivalence.

2.6 Adjoint functors

The definition we give here is slightly different from Lurie's classic one, which explicitly gives the equivalence.

Definition 2.50. Let C, D be two $(\infty, 1)$-categories. We say a pair of functors $F : C \rightleftarrows D : G$ is an adjunction pair if there exists an equivalence φ : $\mathrm{Maps}(F(-), -) \to \mathrm{Maps}(-, G(-))$ in $\mathrm{Fun}(C^{op} \times D, \mathcal{S})$.

We say that F is a left adjoint functor which admits a right adjoint functor G, and vice versa.

Proposition 2.51. Let $F : C \rightleftarrows D : G$ be an adjunction pair, then F preserves all small colimits while G preserves all small limits.

2.7 Presentable categories

Roughly speaking, we will see a series of correspondences between the filtered version and the normal version (just as in the case of 1-categories).

Definition 2.52. Let K be a simplicial set, κ be a regular cardinal. We say K is κ-small if $|S| < \kappa$, where S is the set of nondegenerate simplexes in K.

We say call a diagram $u : K \to C$ is $\kappa-$small if K is $\kappa-$small.

Definition 2.53. Let C be an $(\infty, 1)-$category, we say C is a $\kappa-$filtered category if any κ-small diagram $u : K \to C$ can be extended to $K^{\triangleright} \to C$.

The corresponding limit/colimit diagram is called as $\kappa-$filtered limit/colimit diagram.

Definition 2.54. Let C be an $(\infty, 1)-$category, we define the presheaf category $\mathrm{Psh}(C)$ of C as $\mathrm{Fun}(C^{op}, \mathcal{S})$.

We call the functor $h_C : C \to \mathrm{PSh}(C)$, $X \mapsto \mathrm{Maps}(-, X)$ as the "Yoneda embedding".

Definition 2.55. Let C be an $(\infty, 1)-$category, we define the κ-ind category $\mathrm{Ind}_\kappa(C)$ of C as the full subcategory of $\mathrm{PSh}(C)$ generated by theose objects which are small filtered colimits in $\mathrm{PSh}(C)$ where $K \subset h_C(C)$.

Remark 2.56. Clearly $C \subset \mathrm{Ind}_\kappa(C)$, thus we have another embedding functor from C to $\mathrm{Ind}_\kappa(C)$, which is denoted as ι.

We then show that Ind category is just the filtered version of presheaf category.

Proposition 2.57. The Yoneda functor h_C preserves small colimits while ι preserves $\kappa-$filtered small colimits.

Thus given a functor F from C to a cocomplete/$\kappa-$filtered cocomplete category D, it can be extended naturally to $\widetilde{F} : \mathrm{PSh}(C)/\mathrm{Ind}_\kappa(C) \to D$.

To be precise, that is:

Theorem 2.58. There is an equivalence of category from $\mathrm{LFun}(\mathrm{PSh}(C), D) \simeq \mathrm{Fun}(C, D)$, where $\mathrm{LFun}(\mathrm{PSh}(C), D)$ is the full subcategory of $\mathrm{Fun}(\mathrm{PSh}(C), D)$ generatied by those functors which preserve small colimits.

Theorem 2.59. There is an equivalence of category from $\mathrm{Fun}(\mathrm{Ind}(C), D)_{\kappa-cont} \simeq \mathrm{Fun}(C, D)$, where $\mathrm{Fun}(\mathrm{Ind}(C), D)_{\kappa-con}$ is the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(C), D)$ generatied by those functors which preserve κ-small filtered colimits.

Definition 2.60. Let C be a $(\infty, 1)-$category, κ be a regular cardinal. We let $\mathrm{Ind}_\kappa(C)$ denote the full subcategory of $\mathrm{Psh}(C)$ spanned by these functors $F : C^{op} \to \mathcal{S}$ which classifies all right Kan fibrations $f : \tilde{C} \to C$ where $\tilde{C}$ is $\kappa-$regular.

Definition 2.61. Let C be a $(\infty, 1)-$category, κ be a regular cardinal. We way C is κ-accesible if $\exists$ an essentially small $(\infty, 1)-$ category C_0 such that

$C \simeq \mathrm{Ind}_\kappa(C_0)$.

We say C is accessible if there exists a regular cardinal κ such that C is $\kappa-$accesible.

Definition 2.62. Let C be a $(\infty, 1)-$category. We way C is locally presentable or presentable if C satisfies:

1. C is accessible;
2. C admits all small colimits.

There is an equivalent definition of presentbale categories, which may give a more intuitive understanding of the name.

Definition 2.63. Let C be an $\kappa-$accessible category, D be a category. We say a functor $F : C \to D$ is $\kappa-$accessible if it presereves $\kappa-$filtered small colimits.

We also say a functor F is accessible if it is $\kappa-$accessible for some regular cardinal κ.

2.8 Stable categories

Definition 2.64. Let C be an ∞-category. We say C is a pointed category if there exists a zero object $0 \in C$.

Theorem 2.65. Let C be an ∞-category with a terminal object $*$, then the coslice category $C_{*/}$ is a pointed category. This process induces a functor which we denote as $()_+$.

Definition 2.66. Let C be a pointed ∞-category that admits finite colimits, $X, Y \in C, f : X \to Y$, then we define the cofiber $\mathrm{cof}(f) := 0 \sqcup_X Y$;

If C admits finite limits, we can define the fiber $\mathrm{fib}(f) := 0 \times_Y X$.

Definition 2.67. Let C be a pointed ∞-category, a triangle is a diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

We say the triangle is a fiber sequence if it is a pullback square, and a cofiber sequence if it is a pushout square.

Definition 2.68. Let C be a pointed ∞-category, we say C is stable if:

1. $\forall f \in Mor(C)$, f admits a fiber and a cofiber;
2. A triangle is a fiber sequence if and only if it is a cofiber sequence.

Definition 2.69. Let C be a pointed ∞-category that admits finite colimits, we can define the suspension functor $\Sigma : C \to C$ which maps

$X \mapsto 0 \sqcup_X 0$;

Let C be a pointed ∞-category that admits finite limits, we can define the suspension functor $\Omega : C \to C$ which maps

$X \mapsto 0 \times_X 0$.

Proposition 2.70. Let C be a pointed ∞-category that admits finite limits and colimits, then the following are equivalent:

1. C is stable
2. The adjunction pair $\Sigma : \mathrm{C} \rightleftarrows \mathrm{C} : \Omega$ are equivalences.
3. A square in C is a pushout square if and only if it is a pullback square.

2.9 $(\infty, 1)-$Operads and symmetric monoidal categories

Definition 2.71. We denote the pointed finite set category as Fin_*. We denote $[n] \coprod \{*\}$ as $\langle n \rangle$.

Definition 2.72. Let $I \to J$ be a morphism in $\mathrm{Fin}_*, \alpha : I \to J$. We say α is inert if $\forall j \in J \backslash \{*\}, \alpha^{-1}(j)$ is a singleton; We say α is active if $\alpha^{-1}(\{*\}) = \{*\}$.

Definition 2.73. An $(\infty, 1)-$operad O is a pair $(O^{\otimes}, p)$ where $O^{\otimes}$ is an $(\infty, 1)-$category and p is a functor $O^{\otimes} \to \mathrm{Fin}_*$ such that(We denote $p^{-1}(\langle n \rangle)$ as $O^{\otimes}_{\langle n \rangle}$):

1. Any inert morphism $f : \langle m \rangle \to \langle n \rangle$ can be lifted to a p-coCartesian morphism $\tilde{f}$. In particular, the lifting induces the $(\infty, 1)-$functor $f_! : O^{\otimes}_{\langle m \rangle} \to O^{\otimes}_{\langle n \rangle}$;

2. The ∞-functor $\mathrm{Maps}^{\otimes}_f(C_1, C_2) \to \prod_{1 \le i \le n} \mathrm{Maps}_{\rho^i \circ f}(C_1, C^i_3)$ induced by morphism $f : \langle m \rangle \to \langle n \rangle$ is an equivalence, where $\rho^i : \langle n \rangle \to \langle 1 \rangle$ denotes the morphism which maps everthing but i to $*$;

3.For any $C_1, C_2, ..., C_n \in O^{\otimes}_{\langle 1\rangle}$, there exists $C \in O^{\otimes}_{\langle n\rangle}$ such that there exists a collection of the $p-$CoCartesian lifting from C to C_i of ρ^i.

Remark 2.74. In fact Fin_* is an ∞-operad which we denote as $\mathrm{Comm}^{\otimes}$.

Example 2.75. $(\mathrm{Fin}_*, \mathsf{Id})$ iteself is an $(\infty, 1)-$operad. We denote it as $\mathrm{Comm}^{\otimes}$ or $\mathbb{E}_{\infty}$.

Example 2.76. Let $\mathrm{Assoc}^{\otimes}$ denotes the subcategory of Fin_* whose objects are those of Fin_* but the morphisms are generated by those$f : \langle m\rangle \to \langle n\rangle$ such that for each $j \in \langle n\rangle\backslash\{*\}, \alpha|_{\alpha^{-1}(j)}$ preserves ordering.

We also denote $(\mathrm{Assoc}^{\otimes}, i)$ as $\mathbb{E}_1$.

Definition 2.77. Let $(O^{\otimes}, p)$ be an $(\infty, 1)-$operad, we call a pair $(C^{\otimes}, p')$ where $C^{\otimes}$ is an $(\infty, 1)-$category and p' is a coCartesian fibration is an $O-$monoidal $(\infty, 1)-$category if the composite $p' \circ p$ exhibits $C^{\otimes}$ as an ∞-operad.

Remark 2.78. We will call the $\mathrm{Comm}^{\otimes}$-monoidal categories as symmetric monoidal categories; We will call the $\mathrm{Assoc}^{\otimes}-$monoidal categories as monoidal categories.

3 Preliminaries

3.1 Higher topos theory

3.1.1 Localization

In algebraic geometry, it is common to consider sheafification with respect to a Grothendieck topology. This process can be put in a more general framework called localization. It gives a characterzation of presentable categories.

Definition 3.1. ([1] 5.2.7.2) Let C, D be two $\infty-$categories, $F : C \to D$ be a functor. We say F exhibits D as a localization of C if F admits a fully faithful right adjoint $G : D \to C$.

Definition 3.2. Let $C \subset D$ be two $\infty-$categories, we say a localization functor $L : D \to C$ is accessible if the composite functor $D \xrightarrow{L} C \hookrightarrow D$ is an accessible functor.

Warning: A localization functor is always accessible, so the definition won't cause confusion.

Theorem 3.3. ([1] 5.5.1.2) Let $F : D \to C$ be a localization functor, then the following are equivalent:

1. F is accessible;
2. C is accessible.

Then we have the following characterization of presentable categories.

Theorem 3.4. Let D be an accessible $\infty-$category and $L : D \to D$ be an accessible localization functor, then the essential image of L is presentable.

Theorem 3.5. ([1] 5.5.1.1) Let C be an $\infty-$category, the following are equivalent:

1. C is presentable;
2. There exists a small $\infty-$category D such that C is an accessible localization of $\mathrm{PSh}(D)$.

These immediately give us an important observation in algebraic geometry.

Example 3.6. Let (C,τ) be a small Grothendieck site, then the sheafification functor $\mathrm{PSh}(C) \to \mathrm{Shv}(C,\tau)$ exhibits $\mathrm{Shv}(C,\tau)$ as an accessible localization of $\mathrm{PSh}(C)$. Thus $\mathrm{Shv}(C,\tau)$ is presentable.

3.1.2 S-Localization

In this subsection we will introduce a special case of localization, which is called S-localization. This is the case we will use in the construction of unstable motivic homotopy category.

Definition 3.7. Let C be an $\infty-$category, S be a set of morphisms in C. We say an object $X \in C$ is S-local if $\forall f : A \to B \in S$, the induced map:

$$\mathrm{Maps}(B,X) \to \mathrm{Maps}(A,X)$$

is an equivalence in $\mathcal{S}$.

Theorem 3.8. ([1] 5.5.4.15) Let C be a presentable $\infty-$category, S be a small set of morphisms in C. Let C' be the full subcategory of C spanned by S-local objects, then the inclusion $C' \hookrightarrow C$ admits a left adjoint $L_S : C \to C'$.

A classic example of S-localization is sheafification. Recall that for a Grothendieck site (C,τ), a presheaf F is a sheaf if for any covering $\mathcal{U}$, the map:

$$F(X) \to \lim_{\Delta} F(\check{C}(\mathcal{U}))$$

is an equivalence, where $\check{C}(\mathcal{U})$ is the Cech nerve of the covering $\mathcal{U}$.

By yoneda lemma and the left exactness of $\mathrm{Maps}(-,F)$, this is equivalent to say that for any covering $\mathcal{U}$, the map:

$$\mathrm{Maps}(X,F) \to \mathrm{Maps}(\check{C}(\mathcal{U}),F)$$

is an equivalence. Thus we have:

Example 3.9. Let (C,τ) be a small Grothendieck site, S be the set of morphisms $\{\check{C}(\mathcal{U}) \to X | \mathcal{U}$ is a covering of $X\}$. Then the sheaf category $\mathrm{Shv}(C,\tau)$ is exactly the full subcategory of $\mathrm{PSh}(C)$ spanned by S-local objects.

Since $\mathrm{PSh}(C)$ is presentable, by the above theorem we have:

Corollary 3.10. The inclusion $\mathrm{Shv}(C,\tau) \hookrightarrow \mathrm{PSh}(C)$ admits a left adjoint, i.e. the sheafification functor. We denote it as $L_\tau : \mathrm{PSh}(C) \to \mathrm{Shv}(C,\tau)$.

Finally, we have the following universal property of S-localization, which explains the reasonability of the name.

Theorem 3.11. ([1] 5.5.4.20) Let C be a presentable ∞-category, S be a small set of morphisms in C. Let $L_S : C \to C'$ be the S-localization functor. Then for any functor $F : C \to \mathcal{E}$ that sends morphisms in S to equivalences in $\mathcal{E}$, there exists an essentially unique functor $\widetilde{F} : C' \to \mathcal{E}$ such that

$$\begin{tikzcd} C \arrow[r, "L_S"] \arrow[dr, "F"'] & C' \arrow[d, dashed, "\widetilde{F}"] \\ & \mathcal{E} \end{tikzcd}$$

commutes.

3.1.3 Properties of presentable categories

Presentable categories are very useful in practice, we could firstly show some important properties of presentable categories here.

Theorem 3.12. Let C be a presentable $\infty-$category, then C admits all small limits.

Corollary 3.13. $\mathrm{PSh}(C)$ and $\mathrm{Shv}(C, \tau)$ are complete and cocomplete.

Theorem 3.14 (Adjoint functor theorem)**.** ([1] 5.5.2.9) Let C, D be two presentable $\infty-$categories, $F : C \to D$ be a functor. Then:

1. F is a left adjoint if and only if it preserves all small colimits;
2. F is a right adjoint if and only if it is accessible and preserves all small limits.

Thus we can consider a category which has really good properties (especially in stable homotopy theory) Pr^L.

Definition 3.15. ([1] 5.5.3.1) We define Pr^L (resp. Pr^R) as the subcategory of Cat_∞ whose objects are presentable $\infty-$categories while the morphisms are left (resp. right) adjoint functors. For Pr^L, that's exactly the functors which preserve all small colimits.

3.2 Stable Homotopy Theory

To stabilize an $\infty-$category, we can formally invert the suspension functor. If we view the suspension as the smash product with S^1, then we can view the stabilization as a "localization" with respect to S^1. This philosophy is generated by Robalo to explain the universal property of the stable motivic homotopy category.

3.2.1 Stabilization and Spectra

Definition 3.16. Let C be a pointed $\infty-$category which admits finite colimits, we define the stabilization $\mathrm{Sp}(C)$ of C as the limit of the tower:

$$\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C.$$

We will denote the stabilization functor from C to $\mathrm{Sp}(C)$ as $\Sigma^\infty : C \to \mathrm{Sp}(C)$.

Here the limit is taken in $\widehat{\mathrm{Cat}}_\infty$. However, if C is a presentable category, we can take the limit in Pr^R. Furthermore, we have duality in Pr^L and Pr^R.

Theorem 3.17. Let C be a presentable pointed ∞–category which admits finite colimits, then the following are equivalent:

1. The limit of the tower in Pr^R: $\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C$.
2. The colimit of the tower in Pr^L: $C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots$.
3. $\mathrm{Sp}(C)$.

If C is $\mathcal{S}_*$, then $\mathrm{Sp}(C)$ is exactly the classic stable homotopy category Sp. Notice that $\mathcal{S}_*$ itself has a symmetric monoidal structure induced by the smash product $\wedge$, we'll show that Sp has a symmetric monoidal structure as well, which satisfies a universal property.

3.2.2 Symmetric monoidal structure and universal property of spectra

Definition 3.18. We denote $\mathrm{Pr}^L_{\mathrm{St}}$ as the subcategory of Pr^L whose objects are presentable stable ∞–categories while the morphisms are left adjoint functors.

Definition 3.19. We denote $\mathrm{Pr}^{\otimes,L}$ as the subcategory of Pr^L whose objects are presentably symmetric monoidal ∞–categories while the morphisms are symmetric monoidal left adjoint functors.

Theorem 3.20 (Lurie tensor product)**.** There exists a closed symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{St}}$, where the tensor product $C \otimes \mathcal{D}$ is given by:

$C \otimes \mathcal{D} := \mathrm{Fun}^R(C^{op}, \mathcal{D})$;

Theorem 3.21. ([5] 4.8.2.19) There exists a symmetric monoidal structure on Pr^L.

Thus we can talk about the universal property in $\mathrm{Pr}^{\otimes,L}$.

Theorem 3.22 (Hovey)**.** For every $C^\otimes \in \mathrm{Pr}^{\otimes,L}$ and morphism $F : \mathcal{S}_*^\wedge \to C^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ for which the object $F(S^1)$ is invertible (admits an inverse with respect to $\otimes$), there exists an essentially unique morphism $\hat{F} : \mathrm{Sp}^\otimes \to C^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{S}_*^\wedge \arrow[r, "F"] \arrow[d, "\Sigma^\infty"'] & C^\otimes \\ \mathrm{Sp}^\otimes \arrow[ur, dashed, "\hat{F}"'] & \end{tikzcd}$$

That is, $F \simeq \hat{F} \circ \Sigma^\infty$.

3.3 Six functor formalism

Six functor formalism is a synthetic framework to study homology/cohomology theories, allowing us to define homology and cohomology in its internal language. This section is mainly based on [6] and [7].

3.3.1 Definition of six functor formalism

Roughly speaking, given a category C of geometric objects, a six functor formalism is a system that assigns to each object X in C a corresponding category $\mathcal{D}(X)$ and provides six fundamental operations (functors) between these categories as follows:

1. Pullback Functors: $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ for a morphism $f : X \to Y$ in C.
2. Pushforward Functors: $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ for a morphism $f : X \to Y$ in C.
3. Exceptional Pullback Functors: $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$ for a morphism $f : X \to Y$ in C.
4. Exceptional Pushforward Functors: $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ for a morphism $f : X \to Y$ in C.
5. Tensor Product Functor: $\otimes : \mathcal{D}(X) \times \mathcal{D}(X) \to \mathcal{D}(X)$.
6. Internal Hom Functor: $\mathcal{H}om : \mathcal{D}(X)^{op} \times \mathcal{D}(X) \to \mathcal{D}(X)$.

These functors are required to satisfy:

- Adjunctions: There are adjunctions between pairs (f^*, f_*), $(f_!, f^!)$ and $(\otimes, \mathcal{H}om)$.
- Compatibility: The pullback functor f^* commutes with the tensor product $\otimes$.
- Proper Base change formula: For a pullback square in C:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

there is a natural isomorphism: $g^! f_* \simeq f'_* g'^!$.

- Projection formula: For any morphism $f : X \to Y$ in C, objects $M \in \mathcal{D}(Y)$ and $N \in \mathcal{D}(X)$, there is a natural isomorphism:

$$f_!(f^* M \otimes N) \simeq M \otimes f_! N$$

However, as Scholze pointed out in [6], this naive framework fails to capture the compatibility information. For example, the base change formula should be compatible with composition of base change squares. So we need to consider the six functor formalism in a higher categorical setting.

Definition 3.23. A geometric setup is a pair $(C, \mathcal{E})$ where C is an ∞-category admitting finite limits and $\mathcal{E}$ is a wide subcategory of C which satisfies:

1. $\mathcal{E}$ is stable under pullbacks;
2. $\mathcal{E}$ admits fiber products and the inclusion $\mathcal{E} \hookrightarrow C$ preserves fiber products.

Definition-Proposition 3.24. ([7] 2.2.10) Given a geometric setup $(C, \mathcal{E})$, there exists a symmetric monoidal ∞-category $\mathrm{Corr}(C, \mathcal{E})^{\otimes}$ called the correspondence category of $(C, \mathcal{E})$, which satisfies:

- Objects are the same as C;

- A morphism from X to Y is given by a correspondence:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

  where $f \in C$ and $g \in \mathcal{E}$;

- The composite of two correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ and $Y \xleftarrow{f'} Z' \xrightarrow{g'} W$ is given by the pullback square:

$$\begin{tikzcd} & & Z \times_Y Z' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & & Y & & W \end{tikzcd}$$

Definition 3.25. Let $(C, \mathcal{E})$ be a geometric setup, a 3-functor formalism on $(C, \mathcal{E})$ is a lax symmetric monoidal functor:

$$\mathcal{D} : \mathrm{Corr}(C, \mathcal{E})^{\otimes} \to \mathrm{Cat}_{\infty}^{\otimes}$$

Notice that $\mathcal{D}$ maps objects X in C to ∞-categories $\mathcal{D}(X)$. For a morphism $f : X \to Y$ in C, there is a correspondence $Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X$, while for a morphism $g : X \to Y$ in $\mathcal{E}$, there is a correspondence $X \xleftarrow{\mathrm{id}} X \xrightarrow{g} Y$. Thus $\mathcal{D}$ induces three functors:

- $f^* := \mathcal{D}(Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X) : \mathcal{D}(Y) \to \mathcal{D}(X)$
- $g_! := \mathcal{D}(X \xleftarrow{\mathrm{id}} X \xrightarrow{g} Y) : \mathcal{D}(X) \to \mathcal{D}(Y)$
- $\otimes : \mathcal{D}(X) \times \mathcal{D}(X) \to \mathcal{D}(X)$, which is given by the composite:

$$\mathcal{D}(X) \times \mathcal{D}(X) \xrightarrow{\boxtimes} \mathcal{D}(X \times X) \xrightarrow{\Delta^*} \mathcal{D}(X)$$

Definition 3.26. Let $(C, \mathcal{E})$ be a geometric setup, a six functor formalism on $(C, \mathcal{E})$ is a 3-functor formalism $\mathcal{D} : \mathrm{Corr}(C, \mathcal{E})^{\otimes} \to \mathrm{Cat}_{\infty}^{\otimes}$ such that:

1. For every morphism $f : X \to Y$ in C, the functor $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ admits a right adjoint $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$;
2. For every morphism $g : X \to Y$ in $\mathcal{E}$, the functor $g_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ admits a right adjoint $g^! : \mathcal{D}(Y) \to \mathcal{D}(X)$;
3. For every object X in C, the symmetric monoidal structure on $\mathcal{D}(X)$ is closed, i.e., the functor $\otimes M$ admits a right adjoint $\mathcal{H}om(M, -)$ for every $M \in \mathcal{D}(X)$.

3.3.2 Construction of six functor formalism

In practice, usually we only have the functor $\mathcal{D}' : C^{op} \to \mathrm{CMon}(\mathrm{Cat}_{\infty})$, which encodes the functors f^* and $\otimes$. To construct a six functor formalism from $\mathcal{D}'$, we need to check some conditions.

Theorem 3.27. ([20] A.5.10) Given a functor $\mathcal{D}' : C^{op} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$ with I, P, E the classes of morphisms containing id in C such that:

1. The classes of morphisms I and P are stable under pullback, composition and contain all isomorphisms. Any $f \in E$ is a composite $f = \bar{f} \circ j$ with $j \in I$ and $\bar{f} \in P$.

2. For any $f \in I$, the functor f^* admits a left adjoint $f_!$ which satisfies base change and the projection formula.

3. For any $f \in P$, the functor f^* admits a right adjoint f_* which satisfies base change and the projection formula.

4. For any Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ Y' \arrow[r, "j"] & Y \end{tikzcd}$$

with $j \in I$ (hence $j' \in I$) and $g \in P$ (hence $g' \in P$), the natural map $j_! g'_* \to g_* j'_!$ is an isomorphism.

Then the functor $\mathcal{D}'$ extends to a six functor formalism on $(C, \mathcal{E})$.

We call the set of morphisms I as open immersions while the set of morphisms P as proper morphisms.

Actually, we can extend the correspondence to higher sense.

Definition-Proposition 3.28. ([19]) Let $(C, \mathcal{E})$ be a geometric setup. Let E, I and P be classes of morphisms of C as in the previous theorem, and assume that all morphisms in $\mathcal{E}$ are n-truncated for some n.

Then there exists an ∞-categoriey of functors

$$\mathcal{D}_0 : C^{\mathrm{op}} \to \mathrm{CMon}\left(\mathrm{Cat}_\infty\right)$$

satisfying property 1 from theorem 3.27; and where morphisms $\mathcal{D}_0 \to \mathcal{D}'_0$ are those natural transformations that also commute with the functors $j_!$ for $j \in I$ and f_* for $f \in P$.

Actually there is a equivalence between the ∞-category of functor $\mathcal{D}_0$ and the ∞-category of six functor formalisms on $(C, \mathcal{E})$ which satisfies some other conditions. Furthermore, there is a presentable version of this equivalence.

3.4 Algebraic K-theory

3.4.1 Some more higher algebra

3.4.2 Construction of algebraic K-theory

In topological sense, we could define the topological K-theory on a compact space X as the colimit of the vector bundles on X(i.e. the stable equivalence classes of vector bundles on X), then by classifying space we could represent the topological K-theory by a spectrum KU called the K-theory spectrum. Similarily, Grothendieck defined the algebraic K-theory $K_0(X)$

of a ring X as the group completion of the monoid of isomorphism classes of finitely generated projective modules over X(in 1957, actually before the topological K-theory was defined). Later, Quillen generalized the definition of algebraic K-theory to higher K-theory $K_n(X)$ in the 1970s. From the modern perspective, we could view algebraic K-theory as the ∞-group completion of $\mathrm{Vect}(\mathrm{Spec}(R))$.

In general, we could also replace the $\mathrm{Vect}(\mathrm{Spec}(R))$ by any stable ∞-category. Thus we can define the algebraic K-theory as a functor $K : \mathrm{Cat}_\infty^{st} \to \mathrm{CGrp}(\mathcal{S})$.

The content of this section is mainly based on Wagner's notes [8]. To avoid symbol confusion, we use Grpd_∞ instead of $\mathcal{S}$ to denote the ∞-groupoids in this section.

Definition 3.29 (S-construction)**.** We define the S-construction as the functor $S : \mathrm{Cat}_\infty^{st} \to \mathrm{sCat}_\infty^{st}$ as follows:

Let $S_n(C)$ be the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], C)$ spanned by those functors $F : \mathrm{Ar}[n] \to C$ such that all square in $\mathrm{Ar}[n]$ of the form:

$$\begin{tikzcd}
(i \leq j) \arrow[r] \arrow[d] & (i \leq j+1) \arrow[d] \\
(i+1 \leq j) \arrow[r] & (i+1 \leq j+1)
\end{tikzcd}$$

are sent to pushout-pullback squares in C.

While the face and degeneracy maps are natural.

Theorem 3.30. There is an equivalence of ∞-categories:

$B : \mathrm{Grp}(\mathrm{Grpd}_\infty) \simeq (*/\mathrm{Grpd}_\infty)_{\geqslant 1} : \Omega$.

Lemma 3.31. $|(\mathrm{Core} S(C))|$ is connected.

Definition 3.32. We define the algebraic K-theory functor as the composite $\Omega \circ |\cdot| \circ \mathrm{Core} \circ S$.

4 Unstable Motivic Homotopy Theory

The unstable motivic homotopy theory was first introduced by Voevodskey [9] in 1998 to construct a homotopy theory in algebraic geometry. The basic intuition is that in such homotopy theory, the affine line $\mathbb{A}^1$ should play the same role as I in classic algebraic topology, so it was also called $\mathbb{A}^1$-homotopy theory.

In Voevodsiky's original work, he used the simplicial sheaves to capture the higher information. However, as we've mentioned in the prelimaries, the development of ∞-categories supply a more convenient framework to capture them.

In this chapter, we will give the definition of motivic homotopy category $\mathcal{H}(S)$ related to a Noethrian scheme S, which plays the role of "spaces" where algebraic topology works. With this intuition, we'll soon see the reasonability of its definition.

In short, the motivc homotopy category $\mathcal{H}(S)$ is just a S−localization of $\mathrm{PSh}(Sm/S)$, where S is generated by $\mathbb{A}^1$ and Nisnevich descent.

In the motivic homotopy category $\mathcal{H}(S)$, there are two trivial examples: the yoneda embedding from Sm/S and the constant sheaf from $\mathcal{S}$. These two “embedding" provide two kinds of spheres, i.e. the tate sphere and the standard sphere, thus inducing two types of stabilization in stable motivic homotopy theory.

4.1 Nisnevich topology

In algebraic geometry, we were very familiar with two types of topology, i.e., the Zariski topology and the $\acute{e}$tale topology. However, the Zariksi topology is too coarse while the $\acute{e}$tale topology is too fine for algebtraic topology theories to work.

To be concrete, we can see the following two examples from [10].

Example 4.1. Let $X \in Sm/\mathbb{C}$ with Zariski topology, $\mathcal{F}$ is a constant sheaf, then $\dim H^1(X(\mathbb{C}), \mathcal{F}) = b_1(X(\mathbb{C}))$ while $H^1(X, \mathcal{F}) = 0$, where b_1 denotes the first betti number.

Example 4.2. Let $X = Spec(\mathbb{F}_q)$, then $\dim X = 0$ while $H^1(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$, where $(p, q) = 1$.

The right topology suitable for algebraic topology works ends up to be the Nisnevich topology, developed by Yevsey Nisnevich in 1989[11].

Definition 4.3. Given a Noethrian scheme S, consider in Sm/S, the category of smooth schemes of finite type over S, a familiy of $\{\varphi_i : X_i \to Y\}_{i \in I}$ is called a Nisnevich cover if:

1.φ_i is $\acute{e}$tale;

2.$\forall y \in Y, \exists i \in I, x \in \varphi_i^{-1}(y)$ such that the induced map of residule field $\widetilde{\varphi_i} : k(x) \to k(y)$ is an isomorphism.

Theorem 4.4. The above definition indeed defines a Grothendieck (pre)topology on Sm/S, which we call the Nisnevich topology.

Now we can talk about the presheaf category $\mathrm{PSh}(Sm/S)$ and the sheaf category $\mathrm{Shv}_{Nis}(Sm/S)$ with respect to the Nisnevich topology. By Corollary 3.10, we have the sheafification functor $L_{Nis} : \mathrm{PSh}(Sm/S) \to \mathrm{Shv}_{Nis}(Sm/S)$.

4.2 Excision property

In algebraic geometry, usually it is hard to check the descent condition directly. Sometimes we can find an excision property to replace it.

Definition 4.5. Let $\mathcal{F}$ be a ∞-presheaf on the Nisnevich site Sm/S. We say $\mathcal{F}$ has the excision property if:

1. $\mathcal{F}(\emptyset) = *$;
2. For any Nisnevich distinguished square:

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "p'"'] & X' \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where j is an open immersion, p is an $\acute{e}$tale morphism, and the induced map $p^{-1}(X\backslash U) \to X\backslash U$ is an isomorphism, the induced square:

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "j^*"] \arrow[d, "p^*"'] & \mathcal{F}(U) \arrow[d, "p'^*"] \\ \mathcal{F}(X') \arrow[r, "j'^*"] & \mathcal{F}(U') \end{tikzcd}$$

is a pullback square in $\mathcal{S}$.

Theorem 4.6. Let $\mathcal{F}$ be a ∞-presheaf on the Nisnevich site Sm/S, then $\mathcal{F}$ is a Nisnevich sheaf if and only if it satisfies the excision property.

4.3 $\mathbb{A}^1$-equivalent

As we've mentioned, the affine line $\mathbb{A}^1$ should play the same role as the unit interval I in classical algebraic topology. We need every object to be weakly equivalent to its "cylinder object", so that we can define "left homotopy" relations, similar to the construction in model categories.

Remark 4.7. We will simply write $\mathbb{A}^1$ in replace of $\mathbb{A}^1_S$ in Sm/S.

Definition 4.8. Let $\mathcal{F}$ be a ∞-presheaf on the Nisnevich site Sm/S. We say $\mathcal{F}$ is $\mathbb{A}^1-$equivalent if $\forall X \in Sm/S$, $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$ is an equivalence in $\mathcal{S}$.

By Yoneda lemma, we can view $\mathbb{A}^1$ as an object in $\mathrm{Psh}(Sm/S)$. Thus we can consider the S-localization with respect to the set $S = \{X \times \mathbb{A}^1 \to X | X \in \mathrm{PSh}(Sm/S)\}$. Since $\mathrm{Psh}(Sm/S)$ is presentable, we can define the $\mathbb{A}^1-$localization functor.

Definition 4.9. Let $L_{\mathbb{A}^1} : \mathrm{PSh}(Sm/S) \to \mathrm{PSh}(Sm/S)$ be the S-localization functor with respect to the set $S = \{X \times \mathbb{A}^1 \to X | X \in \mathrm{PSh}(Sm/S)\}$. We call $L_{\mathbb{A}^1}$ the $\mathbb{A}^1-$localization functor.

Warning: A Naive idea arising here is that one might try to define the unstable motivic homotopy category as $\mathbb{A}^1$-localization of $\mathrm{Sh}(Sm/S)$, so that every Nisnevich sheaf becomes $\mathbb{A}^1$-equivalent. Sadly, this try fails because $L_{\mathbb{A}^1}$ will destroy the sheaf structure.

4.4 Unstable motivic homotopy category

The correct way to define the unstable motivic homotopy category is to combine the Nisnevich descent and the $\mathbb{A}^1$-equivalence together, but in the localization set.

Definition 4.10. Let S be a Noethrian scheme. We define the unstable motivic homotopy category $\mathcal{H}(S)$ as the S-localization of $\mathrm{PSh}(Sm/S)$, where S is the set of morphisms consisting of:

1. $\{\mathcal{F} \times \mathbb{A}^1 \to \mathcal{F} | \mathcal{F} \in \mathrm{PSh}(Sm/S)\}$;
2. $\{\check{C}(\mathcal{U}) \to X | \mathcal{U}$ is a Nisnevich covering of $X \in Sm/S\}$.

Since $\mathrm{PSh}(Sm/S)$ is presentable, we can define the motivic localization functor.

Definition 4.11. Let $L_{\text{mot}} : \text{PSh}(Sm/S) \to \mathcal{H}(S)$ be the S-localization functor with respect to the set S defined above. We call L_{mot} the motivic localization functor.

Since $\text{PSh}(Sm/S)$ is accessible, $\mathcal{H}(S)$ is accessible; since $\text{PSh}(Sm/S)$ is cocomplete and L_{mot} preserves colimits, $\mathcal{H}(S)$ is cocomplete.

Theorem 4.12. The unstable motivic homotopy category $\mathcal{H}(S)$ is presentable, thus complete and cocomplete.

Here are two trivial examples of objects in motivic homotopy category $\mathcal{H}(S)$.

Example 4.13. Let $K \in \mathcal{S}$, then the constant sheaf over Sm/S is a motivic space. We will still denote it as K.

Example 4.14. Let $X/S \in Sm/S$, then the yoneda embedding of X/S is a motivic space. With a little abuse of the notation, we still denote it as X.

These two examples provide two embedding from $\mathcal{S}$ and Sm/S to $\mathbf{S}$. The two circles embedded from each category is what we call as "standard sphere" and "tate sphere".

Definition 4.15. We call the constant sheaf valued on $\Delta[1]/\partial\Delta[1]$ the standard sphere, denoted as S^1_s. We call the yoneda embedding of $\mathbb{P}^1$ the tate sphere, denoted as S^1_t.

Actually, here are several $\mathbb{A}^1$-equivalent object in $\text{PSh}(Sm/S)$, thus equivalent in $\mathcal{H}(S)$, which all give equivalent definitions of the tate sphere.

Proposition 4.16. The following objects in $\mathcal{H}(S)_\bullet$ are equivalent:

1. The projective line $\mathbb{P}^1$;
2. The cofiber of the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$;
3. The smash product $S^1 \wedge \mathbb{G}_m$.

All of these give equivalent definitions of the tate sphere S^1_t.

4.5 Symmetric monoidal structure on $\mathcal{H}(S)_\bullet$

Theorem 4.17. The unstable motivic homotopy category $\mathcal{H}(S)_\bullet$ admits a symmetric monoidal structure, where the monoidal product is given by the smash product $\wedge$.

Sketch of proof. The construction follows from the general principle that if we have a presentable symmetric monoidal ∞-category $\mathcal{M}$ and a set of morphisms S that is closed under the monoidal product, then the localization $\mathcal{M}[S^{-1}]$ inherits a symmetric monoidal structure.

In our case, we start with $\text{PSh}(Sm/S)_\bullet$ which has a symmetric monoidal structure given by the smash product. The localization set S (consisting of $\mathbb{A}^1$-homotopy equivalences and Nisnevich descent morphisms) is closed under the smash product operation.

More precisely, if $(s,t) \in S \times S$, then the composition of the localization functor $L_{\text{mot}} : \text{PSh}(Sm/S)_\bullet \to \mathcal{H}(S)_\bullet$ with the smash product sends the pair (s,t) to an equivalence in $\mathcal{H}(S)_\bullet$. Therefore, the composition $L_{\text{mot}} \circ \wedge$ factors through $L_{\text{mot}} \times L_{\text{mot}}$, defining the desired monoidal product on $\mathcal{H}(S)_\bullet$. $\square$

Remark 4.18. The unit object for this symmetric monoidal structure is S^0, the sphere spectrum in $\mathcal{H}(S)_\bullet$. The smash product satisfies the usual properties: associativity, commutativity, and unitality up to coherent equivalences.

This symmetric monoidal structure is crucial for the stabilization process that leads to the stable motivic homotopy category $\mathcal{SH}(S)$.

5 Stable Motivic Homotopy Theory

As we've shown in, their is a standard process to get a stable ∞-category from a pointed ∞-category. In this section however, we will see a more generalized version of stabilization process, in Robalo's sense.

5.1 Robalo's Stabilization

The classic stabilization process depend ont the suspension functor defined by ∞-pushout against the base point. In Sp, that corresponds to the smash product with S^1. The generalization we make here is to replace S^1 by an arbitrary object X in a presentable symmetric monoidal pointed ∞-category.

Definition 5.1. Let $C^\otimes$ be a presentable symmetric monoidal pointed ∞-category, $X \in C$ be an object. The stable ∞-category $C[X^{-1}]$ is defined as:

$\operatorname{colim}(C \xrightarrow{X\otimes -} C \xrightarrow{X\otimes -} C \xrightarrow{X\otimes -} \cdots) \in \mathrm{Pr}^L$.

We will denote the stabilization functor from C to $C[X^{-1}]$ as $\Sigma^\infty_X : C \to C[X^{-1}]$.

Theorem 5.2 (Robalo)**.** Let $C^\otimes$ be a presentable symmetric monoidal pointed ∞-category, $X \in C$ be an object. Then the stabilization $C[X^{-1}]$ inherits a symmetric monoidal structure from $C^\otimes$, making the stabilization functor $\Sigma^\infty_X : C^\otimes \to C[X^{-1}]^\otimes$ a symmetric monoidal functor.

We also have the universal property of such stabilization.

Theorem 5.3 (Robalo)**.** Let $C^\otimes$ be a presentable symmetric monoidal pointed ∞-category, $X \in C$ be an object. For every $\mathcal{D}^\otimes \in \mathrm{Pr}^{\otimes,L}_{\mathrm{St}}$ and morphism $F : C^\otimes \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ for which the object $F(X)$ is invertible (admits an inverse with respect to $\otimes$), there exists an essentially unique morphism $\hat{F} : C[X^{-1}]^\otimes \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ such that the following diagram commutes:

$$\begin{tikzcd} C^\otimes \arrow[r, "F"] \arrow[d, "\Sigma^\infty_X"'] & \mathcal{D}^\otimes \\ C[X^{-1}]^\otimes \arrow[ur, dashed, "\hat{F}"'] & \end{tikzcd}$$

That is, $F \simeq \hat{F} \circ \Sigma^\infty_X$.

5.2 Stable motivic homotopy category

Definition 5.4. Let S be a Noethrian scheme. We define the stable motivic homotopy category $\mathcal{SH}(S)$ as the stabilization of $\mathcal{H}(S)_\bullet$ with respect to the tate sphere S^1_t:

$\mathcal{SH}(S) := \mathcal{H}(S)_\bullet[S^1_t]$.

As an immediate corollary of theorem 5.3, we have the universal property of stable motivic homotopy category.

Theorem 5.5. Let S be a Noetherian scheme. For every $\mathcal{D}^\otimes \in \mathrm{Pr}^{\otimes,L}_{\mathrm{St}}$ and morphism $F : \mathcal{H}(S)^\otimes_\bullet \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ for which the object $F(\mathbb{P}^1)$ is invertible (admits an inverse with respect to $\otimes$), there exists an essentially unique morphism $\hat{F} : \mathcal{SH}(S)^\otimes \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}(S)^\otimes_\bullet \arrow[r, "F"] \arrow[d, "\Sigma^\infty_{S^1_t}"'] & \mathcal{D}^\otimes \\
\mathcal{SH}(S)^\otimes \arrow[ur, dashed, "\hat{F}"'] &
\end{tikzcd}$$

That is, $F \simeq \hat{F} \circ \Sigma^\infty_{S^1_t}$.

Combine the universal property in theorem 3.11 and theorem 5.5, we can easily get the universal property of stable motivic homotopy category as we expected.

Theorem 5.6. Let S be a Noetherian scheme. For every $\mathcal{D}^\otimes \in \mathrm{Pr}^{\otimes,L}_{\mathrm{St}}$ and morphism $F : Sm/S^\times \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ for which the following conditions hold:

1. F satisfies Nisnevich descent;
2. F is $\mathbb{A}^1$−invariant;
3. The object $F(\mathbb{P}^1)$ is invertible (admits an inverse with respect to $\otimes$);

then there exists an essentially unique morphism $\hat{F} : \mathcal{SH}(S)^\otimes \to \mathcal{D}^\otimes$ in $\mathrm{Pr}^{\otimes,L}$ such that the following diagram commutes:

$$\begin{tikzcd}
Sm/S^\times \arrow[r, "F"] \arrow[d, "\Sigma^\infty_{S^1_t} \circ ()_+ \circ L_{\mathrm{mot}}"'] & \mathcal{D}^\otimes \\
\mathcal{SH}(S)^\otimes \arrow[ur, dashed, "\hat{F}"'] &
\end{tikzcd}$$

That is, $F \simeq \hat{F} \circ \Sigma^\infty_{S^1_t} \circ L_{\mathrm{mot}}$.

5.3 Motivic cohomology

To denote the topological dimension and the motivic dimension separately, we usually use the bi-graded notation in motivic homotopy theory. This means we could view S^1 as $S^{1,0}$ while $\mathbb{G}_m$ as $S^{1,1}$.

Definition 5.7. We denote the sphere $S^{p,q} := (S^1)^{\wedge p-q} \wedge (\mathbb{G}_m)^{\wedge q}$ for $p \geq q \geq 0$.

As we've shown in 4.16, $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. Thus we could view $\mathbb{P}^1 \simeq S^{2,1}$.

Definition 5.8. Given a motivic spectrum $E \in \mathcal{SH}(S)$, we define its motivic cohomology as the bi-graded family of abelian groups:

$E^{p,q}(X) := [\Sigma^\infty_{S^1_t} X_+, \Sigma^{p,q} E]_{\mathcal{SH}(S)}$ for $X \in Sm/S$;

$E_{p,q}(X) := [\Sigma^\infty S^{p,q}, \Sigma^\infty_{S^1_t} X_+ \wedge E]_{\mathcal{SH}(S)}$ for $X \in Sm/S$.

Definition 5.9. Given a motivic spectrum $E \in \mathcal{SH}(S)$, we define its bi-graded homotopy groups as the family of abelian groups:

$\pi_{p,q}(E) := [\Sigma^\infty S^{p,q}, E]_{\mathcal{SH}(S)}$.

5.4 Six functor formalism

It was Ayoub who first found the six functor formalism structure on stable motivic homotopy category in 2007. In 2020, Drew and Gallauer proved that such six functor formalism is universal among all six functor formalisms on schemes satisfying certain conditions. Thus, we give another explanation of the universal property of stable motivic homotopy category.

For the universal property, here we introduce the version of Scholze who improved Drew and Gallauer's work.

Theorem 5.10. ([6] Theorem 11.1, [21]) Let C be the category of separated finite type schemes over a Noetherian base ring k where $Spec(k)$ is of finite Krull dimension, with open immersions I and proper maps P. Consider the ∞-category of functors

$$D : C^{\mathrm{op}} \to \mathrm{CAlg}\left(\mathrm{Pr}^L_{\mathrm{st}}\right)$$

satisfying the following properties:

(0) Pullbacks along open immersions I admit left adjoints and pullbacks along proper maps admit right adjoints satisfying projection formula and base change.

(1) All smooth maps are cohomologically smooth.

(2) Excision holds: If X has a closed subset $i : Z \subset X$ with open complement $j : U \to X$, then

$$j_! 1 \to 1 \to i_* 1$$

is exact in $D(X)$.

(3) The homology of $\mathbb{A}^1$ is trivial, i.e. for $\pi : \mathbb{A}^1 \to \mathrm{Spec}(k)$ the adjunction map

$$1 \to \pi_! \pi^! 1$$

is an isomorphism.

Then this ∞-category has an initial object given by the stable motivic homotopy theory $\mathcal{SH}(k)$.

5.5 Algebraic K-theory

Definition 5.11. Given a scheme S, we define the algebraic K-theory $K := K' \circ \mathrm{Perf}(-)$, where $\mathrm{Perf}(-)$ is the functor sending a scheme to its stable ∞-category of perfect complexes, and K' is the algebraic K-theory functor defined in 3.32. Thus we can view algebraic K-theory as a presheaf on Sm/S.

Theorem 5.12. The algebraic K-theory K satisfies Nisnevich descent and is $\mathbb{A}^1$-invariant for regular Noethrian scheme S.

Sketch of proof. Since K' preserve the pullback squares, we only need to show that $\mathrm{Perf}(-)$ satisfies Nisnevich descent. $\mathbb{A}^1$-equivalence follows the statement of algebraic K-theory on rings. □

Definition-Proposition 5.13. There exists an equivalence $K(S) \to \Omega_{\mathbb{P}^1} K(S)$ in $\mathcal{H}(S)$ for regular Noetherian scheme S with finite dimension, thus by the universal property of stable motivic homotopy category, there exists a motivic spectrum $\mathbf{KGL}_S \in \mathcal{SH}(S)$ of the form $(K(S), K(S), \ldots)$.

Sketch of proof. The assumption allows us to consider on ring level. □

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