SERRE SPECTRAL SEQUENCE-I

XINGZHI HUANG

1. Review of the spectral sequence

Given a differential object $X \in (A, T)_d$ with a filtration $F^{\bullet}X$, we can construct a spectral sequence by the following steps:

Step 1. From the short exact sequence

$$0 \to (F^{p+1}X, d) \to (F^pX, d) \to (gr^pX, gr^pd) \to 0$$

we obtain the associated long exact sequence in cohomology (homology):

$$0 \to H(F^{p+1}X, d) \to H(F^pX, d) \to H(gr^pX, gr^pd) \xrightarrow{+1} H(F^{p+1}X, d) \to \cdots$$

which can be viewed as an exact couple \mathcal{C}_1 :

$$H(F^{\bullet}X, d) \xrightarrow{i, (0, -1)} H(F^{\bullet}X, d)$$

$$\downarrow i, (0, -1) \longrightarrow H(F^{\bullet}X, d)$$

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where the bidegree (n, p) encodes the shifting n of the differential object and the shifting p in filtration degree.

Step 2. From an initial exact couple C_1 , we can generate the derived exact couples C_r for $r \geq 1$:

$$D \xrightarrow{i,(0,-1)} D$$

$$k,(+1,+r) \qquad \qquad j,(0,0)$$

$$E$$

Step 3. We can define the spectral sequence $\{E_r^p, d_r^p\}$ associated to the exact couples C_r as follows:

$$E^p_r = E^p$$

$$d^p_r = j \circ k : E^p_r \to E^{p+r}_r$$

where d_r^p shifts in degree +1 as the differential of the differential object.

Theorem 1.1. If the filtration $F^{\bullet}X$ is bounded and exhaustive, then the spectral sequence $\{E_r^p, d_r^p\}$ converges to H(X), i.e.,

$$E^p_\infty \cong \operatorname{gr}^p H(X)$$

Remark 1.2. In the case when X is a \mathbb{Z} -graded differential object with a decreasing(increasing) filtration $F^{\bullet}X$, we can rewrite the above construction in the bidegree form. In that case, the spectral sequence becomes a bigraded spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ with q defined by n-p, where n is the graded degree of X.

In that case, $E^p_\infty \cong gr^pH(X)$ when $E^{p,q}_\infty \cong gr^pH^{p+q}(X)$. We will also denote it as

$$E^{p,q}_{\infty} \Rightarrow H^{p+q}(X)$$

Reveresed case

If we reverse the arrows of the short exact sequence in Step 1, i.e.

$$0 \to (gr^p X, gr^p d) \to (F^p X, d) \to (F^{p+1} X, d) \to 0$$

we will obtain another exact couple \mathcal{C}'_1 with all arrows reversed:

$$H(F^{\bullet}X, d) \xleftarrow{i, (0, -1)} H(F^{\bullet}X, d)$$

$$\downarrow j, (0, 0)$$

$$H(gr^{\bullet}X, gr^{\bullet}d)$$

Similarly, we can generate the derived exact couples C'_r for $r \geq 1$, and thus generate a spectral sequence $\{E'_r^p, d'_r^p\}$ associated to the exact couples C'_r where $E'_r^p = E'^p$ and $d'^p_r = k \circ j : E'^p_r \to E'^{p+r}_r$.

2. Spectral sequence of a filtered chain complex

Given $X \in \text{Top}$ with an increasing filtration $\{X^p\}_{p\geq 0}$, we can induce a filtration on the singular chain complex $C_*(X;G)$ with coefficients in an abelian group G:

$$F_pC_*(X;G) = C_*(X^p;G)$$

Notice that $H_n(C_*(X;G))$ is exactly the singular homology group $H_n(X;G)$. If the filtration is bounded and exhaustive, we have the convergence $E^{p,q}_{\infty} \cong gr^p H_{p+q}(X;G)$.

Example 2.1. Suppose X is a CW-complex with skeleta $\{X^n\}_{n\geq 0}$. Then we have a filtration on the singular chain complex $C_*(X;G)$ given by the structure of the skeleta:

$$F_pC_*(X;G) = C_*(X^p;G)$$

The first exact couple is given by

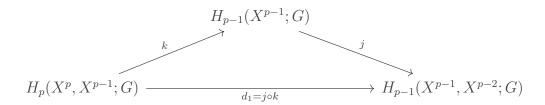
So the first page of the spectral sequence is given by:

$$E_1^{p,q} = \begin{cases} H_p(X^p, X^{p-1}; G), & q = 0\\ 0, & q \neq 0 \end{cases}$$

Notice then (p,0) term is exactly the cellular chain group $C_p^{\operatorname{Cell}}(X;G)$. Moreover, the morphism k is exactly the boundary map in the long exact sequence of the pair (X^p, X^{p-1}) , j is induced by the inclusion map. So the differential $d_1 = j \circ k$ at degree (p,q) is exactly the cellular differential:

$$d_{1,p,q}: H_p(X^p, X^{p-1}; G) \to H_{p-1}(X^{p-1}, X^{p-2}; G)$$

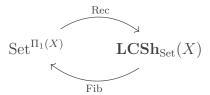
witnessing:



Thus we have $E_2^{p,q} \cong H_p(X;G)$ when q=0 and 0 otherwise. The spectral sequence collapses at the second page, and it converges to $H_*(X;G)$.

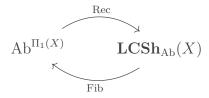
3. Local systems

Recall that in covering space theory, given a locally path-connected and semi-locally simply connected space X, there is an equivalence of categories:

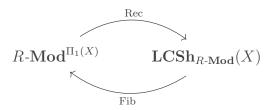


Here we write $LCSh_{Set}(X)$ for the category of locally constant sheaves of sets on X to represent Cov(X) for generalization purpose.

Theorem 3.1. Given X a locally connected space, the equivalence holds if we replace **Set** by Ab or R-**Mod**, i.e.,



and



So we can define the local system of a locally connected space X with coefficients R in the following equivalent ways:

Definition 3.2. A local system of X with coefficients in R is either

- (1) a functor $\mathcal{L}: \Pi_1(X) \to R\text{-}\mathbf{Mod}$; or
- (2) a locally constant sheaf of R-modules on X.

Remark 3.3. If X is path-connected, we can further define the local system as a group action of $\pi_1(X)$ on an R-module M, i.e., a representation $\rho: \pi_1(X) \to \operatorname{Aut}_R(M)$.

Example 3.4. Given a fibration $F \to E \xrightarrow{p} B$ where B is locally path-connected and semi-locally simply connected, we can define a local system $\mathcal{H}_q(F;R)$ on B with coefficients in R as follows:

For each $b \in B$, let $\mathcal{H}_q(F;R)(b) = H_q(F_b;R)$ where $F_b = p^{-1}(b)$ is the fiber over b. For each path class $[\gamma]: b \to b'$ in $\Pi_1(B)$, we can define the morphism $\tilde{\gamma}F_b \to F_{b'}$ induced by the homotopy lifting property of fibrations, which further induces a morphism $\mathcal{H}_q(F;R)([\gamma]): H_q(F_b;R) \to H_q(F_{b'};R)$. Notice that $F_b \simeq F$ everywhere, thus we have defined a functor $\mathcal{H}_q(F;R): \Pi_1(B) \to R$ -Mod, which is a local system on B with coefficients in R.

Remark 3.5. If $\Pi(X)$ acts trivially on the fiber homology $H_q(F;R)$, then the local system $\mathcal{H}_q(F;R)$ is constant with value $H_q(F;R)$. In particular, if X is path-connected and simply connected, then every local system on X is constant.

4. Construction of the Serre spectral sequence

Given a fibration $F \to E \xrightarrow{p} B$ where B is a CW-complex, we can induce a filtration on the singular chain complex $C_*(E;R)$ with coefficients in a ring R:

$$F_pC_*(E;R) = (C_*(p^{-1}(B^p);R))$$

where B^p is the p-skeleton of B. Notice that $H_n(C_*(E;R))$ is exactly the singular homology group $H_n(E;R)$.

The first exact couple is given by

$$H_{*}(p^{-1}(B^{\bullet});R) \xrightarrow{i,(0,+1)} H_{*}(p^{-1}(B^{\bullet});R)$$

$$H_{*}(p^{-1}(B^{\bullet}),p^{-1}(B^{\bullet-1});R)$$

where k is induced by the boundary map in the long exact sequence of the pair $(p^{-1}(B^p), p^{-1}(B^{p-1}))$, and j is induced by the inclusion map.

We want to show in the next lecture that $H_{p+q}(p^{-1}(B^p), p^{-1}(B^{p-1}); R) \cong \operatorname{Cell}_p(B; \mathcal{H}_q(F; R))$, while j and k correspond to the cellular differential. Thus the second page of the spectral sequence is given by:

$$E_2^{p,q} \cong H_p(B; \mathcal{H}_q(F; R))$$

If the filtration is bounded and exhaustive, we have the convergence $E^{p,q}_{\infty} \cong gr^p H_{p+q}(E;R)$ or $E^{p,q}_{\infty} \Rightarrow H_{p+q}(E;R)$.

References

- 1. Wenwei Li. Methods of Algebra, Vol. 2: Linear Algebra. Higher Education Press, 2024. (in Chinese)
- 2. McCleary, John. A User's Guide to Spectral Sequences, 2nd ed. Cambridge University Press, 2001.