SERRE SPECTRAL SEQUENCE-I

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1. Review of the spectral sequence

Given a differential object $X \in (\mathcal{A}, T)_d$ with a filtration $F^\bullet X$, we can construct a spectral sequence by the following steps:

Step 1. From the short exact sequence

$$0 \to (F^{p+1}X, d) \to (F^p X, d) \to (gr^p X, gr^p d) \to 0$$

we obtain the associated long exact sequence in cohomology (homology):

$$0 \to H(F^{p+1}X, d) \to H(F^p X, d) \to H(gr^p X, gr^p d) \xrightarrow{+1} H(F^{p+1}X, d) \to \cdots$$

which can be viewed as an exact couple $\mathcal{C}_1$:

$$\begin{tikzcd}
H(F^\bullet X, d) \arrow[rr, "{i,(0,-1)}"] & & H(F^\bullet X, d) \arrow[dl, "{j,(0,0)}"] \\
& H(gr^\bullet X, gr^\bullet d) \arrow[ul, "{k,(+1,+1)}"] &
\end{tikzcd}$$

where the bidegree (n, p) encodes the shifting n of the differential object and the shifting p in filtration degree.

Step 2. From an initial exact couple $\mathcal{C}_1$, we can generate the derived exact couples $\mathcal{C}_r$ for $r \geq 1$:

$$\begin{tikzcd}
D \arrow[rr, "{i,(0,-1)}"] & & D \arrow[dl, "{j,(0,0)}"] \\
& E \arrow[ul, "{k,(+1,+r)}"] &
\end{tikzcd}$$

Step 3. We can define the spectral sequence $\{E_r^p, d_r^p\}$ associated to the exact couples $\mathcal{C}_r$ as follows:

$$E_r^p = E^p$$

$$d_r^p = j \circ k : E_r^p \to E_r^{p+r}$$

where d_r^p shifts in degree $+1$ as the differential of the differential object.

Theorem 1.1. *If the filtration $F^\bullet X$ is bounded and exhaustive, then the spectral sequence $\{E_r^p, d_r^p\}$ converges to $H(X)$, i.e.,*

$$E_\infty^p \cong gr^p H(X)$$

Remark 1.2. In the case when X is a $\mathbb{Z}$-graded differential object with a decreasing(increasing) filtration $F^\bullet X$, we can rewrite the above construction in the bidegree form. In that case, the spectral sequence becomes a bigraded spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ with q defined by $n-p$, where n is the graded degree of X.

In that case, $E_\infty^p \cong gr^p H(X)$ when $E_\infty^{p,q} \cong gr^p H^{p+q}(X)$. We will also denote it as

$$E_\infty^{p,q} \Rightarrow H^{p+q}(X)$$

Reveresed case

If we reverse the arrows of the short exact sequence in Step 1, i.e.

$$0 \to (gr^p X, gr^p d) \to (F^p X, d) \to (F^{p+1} X, d) \to 0$$

we will obtain another exact couple $\mathcal{C}_1'$ with all arrows reversed:

$$\begin{tikzcd} H(F^\bullet X, d) \arrow[dr, "{k,(+1,+1)}"'] & & H(F^\bullet X, d) \arrow[ll, "{i,(0,-1)}"'] \\ & H(gr^\bullet X, gr^\bullet d) \arrow[ur, "{j,(0,0)}"'] & \end{tikzcd}$$

Similarly, we can generate the derived exact couples $\mathcal{C}_r'$ for $r \geq 1$, and thus generate a spectral sequence $\{E_r'^p, d_r'^p\}$ associated to the exact couples $\mathcal{C}_r'$ where $E_r'^p = E'^p$ and $d_r'^p = k \circ j : E_r'^p \to E_r'^{p+r}$.

2. Spectral sequence of a filtered chain complex

Given $X \in$ Top with an increasing filtration $\{X^p\}_{p \geq 0}$, we can induce a filtration on the singular chain complex $C_*(X; G)$ with coefficients in an abelian group G:

$$F_p C_*(X; G) = C_*(X^p; G)$$

Notice that $H_n(C_*(X; G))$ is exactly the singular homology group $H_n(X; G)$. If the filtration is bounded and exhaustive, we have the convergence $E_\infty^{p,q} \cong gr^p H_{p+q}(X; G)$.

Example 2.1. Suppose X is a CW-complex with skeleta $\{X^n\}_{n \geq 0}$. Then we have a filtration on the singular chain complex $C_*(X; G)$ given by the structure of the skeleta:

$$F_p C_*(X; G) = C_*(X^p; G)$$

The first exact couple is given by

$$\begin{tikzcd} H_*(X^\bullet) \arrow[rr, "{i,(0,+1)}"] & & H_*(X^\bullet) \arrow[dl, "{j,(0,0)}"] \\ & H_*(X^\bullet, X^{\bullet-1}) \arrow[ul, "{k,(-1,-1)}"] & \end{tikzcd}$$

So the first page of the spectral sequence is given by:

$$E_1^{p,q} = \begin{cases} H_p(X^p, X^{p-1}; G), & q = 0 \\ 0, & q \neq 0 \end{cases}$$

Notice then $(p, 0)$ term is exactly the cellular chain group $C_p^{\mathrm{Cell}}(X; G)$. Moreover, the morphism k is exactly the boundary map in the long exact sequence of the pair (X^p, X^{p-1}), j is induced by the inclusion map. So the differential $d_1 = j \circ k$ at degree (p, q) is exactly the cellular differential:

$$d_{1,p,q} : H_p(X^p, X^{p-1}; G) \to H_{p-1}(X^{p-1}, X^{p-2}; G)$$

witnessing:

$$\begin{tikzcd} & H_{p-1}(X^{p-1}; G) \arrow[dr, "j"] & \\ H_p(X^p, X^{p-1}; G) \arrow[ur, "k"] \arrow[rr, "d_1 = j \circ k"'] & & H_{p-1}(X^{p-1}, X^{p-2}; G) \end{tikzcd}$$

Thus we have $E_2^{p,q} \cong H_p(X; G)$ when $q = 0$ and 0 otherwise. The spectral sequence collapses at the second page, and it converges to $H_*(X; G)$.

3. Local systems

Recall that in covering space theory, given a locally path-connected and semi-locally simply connected space X, there is an equivalence of categories:

$$\begin{tikzcd} \mathrm{Set}^{\Pi_1(X)} \arrow[r, bend left, "\mathrm{Rec}"] & \mathbf{LCSh}_{\mathrm{Set}}(X) \arrow[l, bend left, "\mathrm{Fib}"] \end{tikzcd}$$

Here we write $LCSh_{\mathrm{Set}}(X)$ for the category of locally constant sheaves of sets on X to represent $Cov(X)$ for generalization purpose.

Theorem 3.1. *Given X a locally connected space, the equivalence holds if we replace* **Set** *by* Ab *or R-***Mod***, i.e.,*

$$\begin{tikzcd} \mathrm{Ab}^{\Pi_1(X)} \arrow[r, bend left, "\mathrm{Rec}"] & \mathbf{LCSh}_{\mathrm{Ab}}(X) \arrow[l, bend left, "\mathrm{Fib}"] \end{tikzcd}$$

and

$$
\begin{tikzcd}
R\text{-}\mathbf{Mod}^{\Pi_1(X)} \arrow[r, bend left, "\mathrm{Rec}"] & \mathbf{LCSh}_{R\text{-}\mathbf{Mod}}(X) \arrow[l, bend left, "\mathrm{Fib}"]
\end{tikzcd}
$$

So we can define the local system of a locally connected space X with coefficients R in the following equivalent ways:

Definition 3.2. A **local system** of X with coefficients in R is either
(1) a functor $\mathcal{L} : \Pi_1(X) \to R\text{-}\mathbf{Mod}$; or
(2) a locally constant sheaf of R-modules on X.

Remark 3.3. If X is path-connected, we can further define the local system as a group action of $\pi_1(X)$ on an R-module M, i.e., a representation $\rho : \pi_1(X) \to \mathrm{Aut}_R(M)$.

Example 3.4. Given a fibration $F \to E \xrightarrow{p} B$ where B is locally path-connected and semi-locally simply connected, we can define a local system $\mathcal{H}_q(F;R)$ on B with coefficients in R as follows:

For each $b \in B$, let $\mathcal{H}_q(F;R)(b) = H_q(F_b;R)$ where $F_b = p^{-1}(b)$ is the fiber over b. For each path class $[\gamma] : b \to b'$ in $\Pi_1(B)$, we can define the morphism $\tilde{\gamma}F_b \to F_{b'}$ induced by the homotopy lifting property of fibrations, which further induces a morphism $\mathcal{H}_q(F;R)([\gamma]) : H_q(F_b;R) \to H_q(F_{b'};R)$. Notice that $F_b \simeq F$ everywhere, thus we have defined a functor $\mathcal{H}_q(F;R) : \Pi_1(B) \to R\text{-}\mathbf{Mod}$, which is a local system on B with coefficients in R.

Remark 3.5. If $\Pi(X)$ acts trivially on the fiber homology $H_q(F;R)$, then the local system $\mathcal{H}_q(F;R)$ is constant with value $H_q(F;R)$. In particular, if X is path-connected and simply connected, then every local system on X is constant.

4. Construction of the Serre spectral sequence

Given a fibration $F \to E \xrightarrow{p} B$ where B is a CW-complex, we can induce a filtration on the singular chain complex $C_*(E;R)$ with coefficients in a ring R:

$$F_pC_*(E;R) = (C_*(p^{-1}(B^p);R))$$

where B^p is the p-skeleton of B. Notice that $H_n(C_*(E;R))$ is exactly the singular homology group $H_n(E;R)$.

The first exact couple is given by

$$
\begin{tikzcd}
H_*(p^{-1}(B^\bullet);R) \arrow[rr, "{i,(0,+1)}"] & & H_*(p^{-1}(B^\bullet);R) \arrow[dl, "{j,(0,0)}"] \\
& H_*(p^{-1}(B^\bullet), p^{-1}(B^{\bullet-1});R) \arrow[ul, "{k,(-1,-1)}"] &
\end{tikzcd}
$$

where k is induced by the boundary map in the long exact sequence of the pair $(p^{-1}(B^p), p^{-1}(B^{p-1}))$, and j is induced by the inclusion map.

We want to show in the next lecture that $H_{p+q}(p^{-1}(B^p), p^{-1}(B^{p-1}); R) \cong \mathrm{Cell}_p(B; \mathcal{H}_q(F; R))$, while j and k correspond to the cellular differential. Thus the second page of the spectral sequence is given by:

$$E_2^{p,q} \cong H_p(B; \mathcal{H}_q(F; R))$$

If the filtration is bounded and exhaustive, we have the convergence $E_\infty^{p,q} \cong gr^p H_{p+q}(E; R)$ or $E_\infty^{p,q} \Rightarrow H_{p+q}(E; R)$.

References

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2. McCleary, John. *A User's Guide to Spectral Sequences*, 2nd ed. Cambridge University Press, 2001.