A brief proof of the homological equivalence of subchain complexes

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Contents

Lemma 0.0.1. *For a Grothendieck category* C*,* $X \in Ob(\mathsf{C})$*, if there exists* $\{A_i\}_{i \in I}$ *such that* $A_i \in Sub_X$*, define* α *as a functor induced by the subcategory, whose:*

Objects are $\cap_{j \in J} A_j, J \subset I$*,morphisms are* $\cap_{j \in J} A_j \hookrightarrow \cap_{k \in K} A_k, J \subset K$

Then we have $+_{i \in I} A_i \xrightarrow{\sim} \varinjlim \alpha$

Theorem 0.0.2. *For a space* X*, let* $\mathcal{U} = \{U_j\}$ *be a collection of subspaces of* X*. The inclusion* $\iota : C_n^u(X) \hookrightarrow C_n(X)$ *is a chain homotopy equivalence, that is, there is a chain map* $\rho : C_n(X) \to C_n^{\mathcal{X}}(X)$ *such that* $\rho\rho$ *and* $\rho\iota$ *are chain homotopic to the identity. Hence* $\imath$ *induces isomorphisms* $H_n^{\mathcal{U}}(X) \approx H_n(X)$ *for all* n*.*

Proof. There exists a quillen functor adjunction sequence as below:

$$\begin{tikzcd}
\mathsf{Ch}_{\geq}(\mathcal{A}\mathsf{b}) \arrow[r, bend left, "\Gamma"] & \mathsf{s}\mathcal{A}\mathsf{b} \arrow[l, bend left, "C"] \arrow[r, bend left, "F"] & \mathsf{s}\mathcal{S}\mathsf{et} \arrow[l, bend left, "U"] \arrow[r, bend left, "|\cdot|"] & \mathsf{Top} \arrow[l, bend left, "Sing"]
\end{tikzcd}$$

where

$\Gamma \dashv_{Qu} C : \mathsf{Ch}_{\geq}(\mathcal{A}\mathsf{b}) \leftrightarrows \mathsf{s}\mathcal{A}\mathsf{b}$

and

$|\cdot| \dashv_{Qu} \mathrm{Sing} : \mathsf{s}\mathcal{S}\mathsf{et} \leftrightarrows \mathsf{Top}$

are both quillen category equivalence ([1] Thm 3.6.7)

$U \dashv_{Qu} F : \mathsf{s}\mathcal{S}\mathsf{et} \leftrightarrows \mathsf{s}\mathcal{A}\mathsf{b}$

is a quillen category adjunction

so the functor $Ho(C) \circ Ho(U) \circ Ho(Sing) : Ho(Top) \to \mathsf{D}(\mathcal{A}\mathsf{b})$ preserves colimit. Since homology functor H_n depends on the derived category $\mathsf{D}(\mathcal{A}\mathsf{b})$, $H_n(X) = H_n(\varinjlim U_j) \xrightarrow{\sim} \varinjlim H_n(U_j) \xrightarrow{\sim} H_n^{\mathcal{U}}(X)$.

□

Remark 0.0.3. *Here we only discuss* Mod *as a Grothendieck category because we want to consider infinite situation. If we just need finite situation* *(which is actually what we really need for excision), the theorem can be generalized to Abel category.*

The motivation of this proof comes from there: We have known that there are Dold-Kan correspondece (as a category equivalence) between $\mathsf{Ch}_{\geq}(\mathcal{A}\mathsf{b})$ and $\mathsf{s}\mathcal{A}\mathsf{b}$, Free-Forgetful category adjunction between $\mathsf{s}\mathcal{S}\mathsf{et}$ and $\mathsf{s}\mathcal{A}\mathsf{b}$ and Geometric-Realization-Sing category adjunction $\mathsf{s}\mathcal{S}\mathsf{et}$ and Top. So it is naturally to think that we can get a functor which preserve colimit by composing these functor.

However, the sad thing is that the forgetful functor and the singular functor is not of the same direction. So we have to consider it on homotopy category, where the singular functor induces a quillen category equivalence, and on the same time the forgetful funtor and the funtor C induce quillen category adjunction and quillen category equivalence separately. Since forgetful functor is a right adjunction, the composite functor preserves colimit.

References

[1] Hovey, Mark. Model categories. No. 63. American Mathematical Soc., 2007.