MEAM 620 - Sensing

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What this section is about

- Study how the robot perceives the world
- More specifically
  - How it can estimate where it is in space
  - How it can estimate where other objects are around it

Flying Robots:
A different perspective

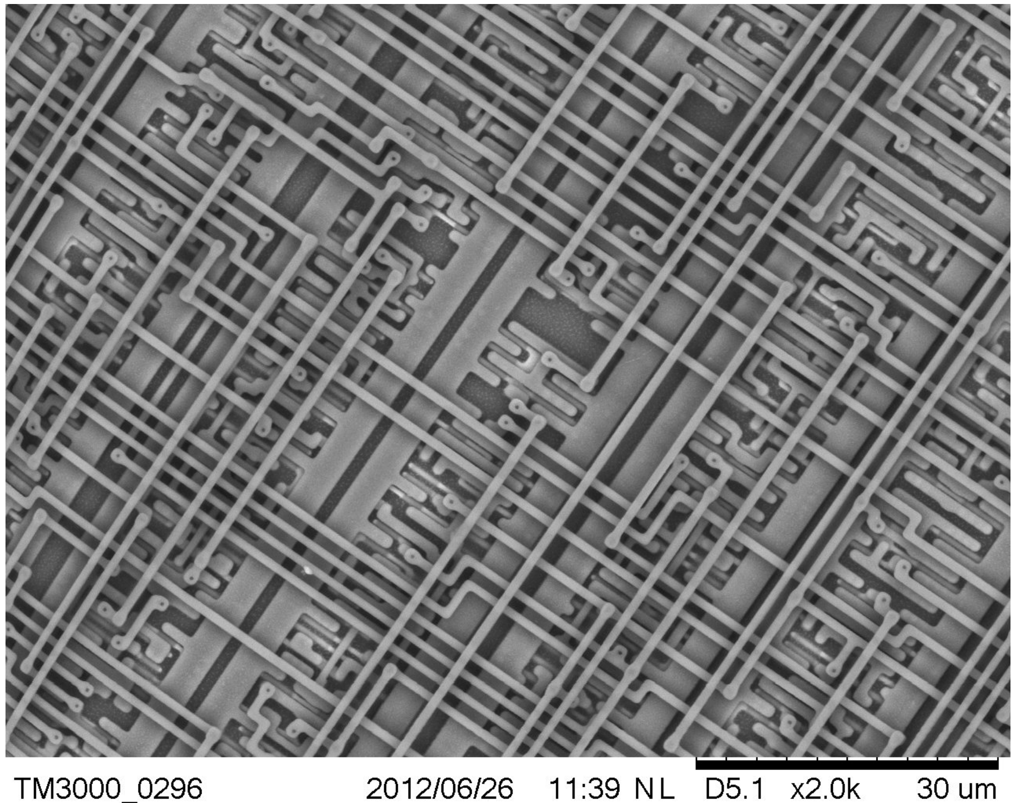
Taking a step back

- What makes the modern small scale flying robot possible?
- Microfabrication!

Microfabrication

- Modern microfabrication makes use of lithographic procedures where structures are ‘printed’ on a substrate one layer at a time using a light source and a mask.

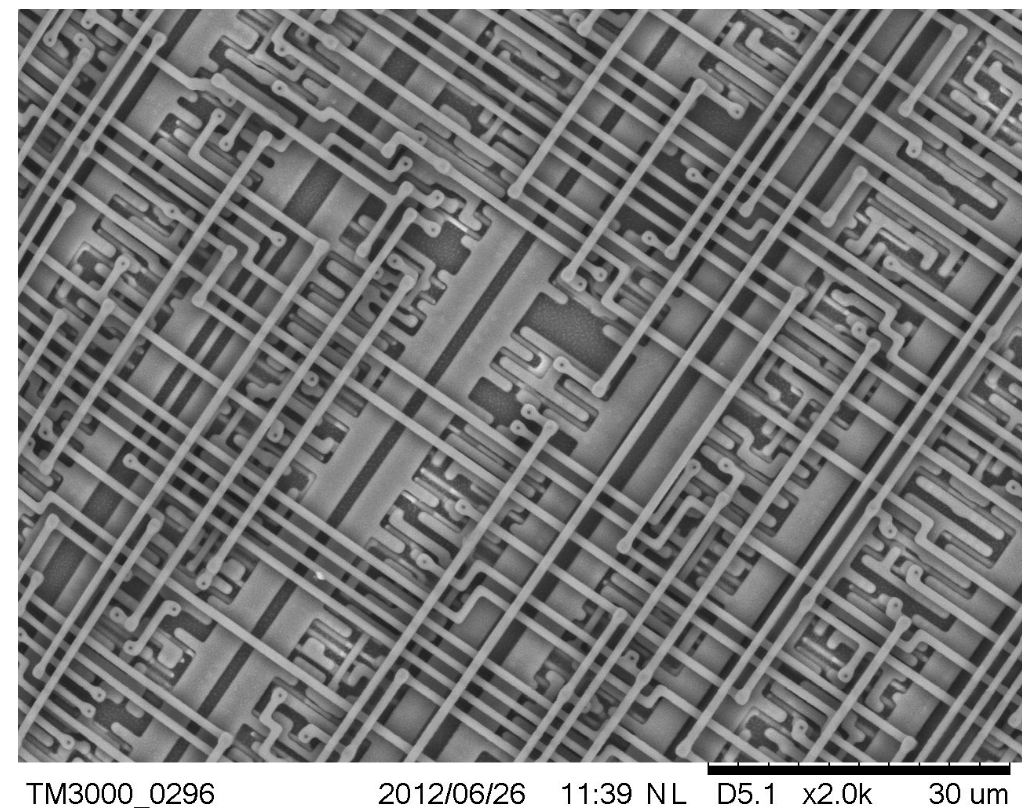
- Chemical process are then used to add or remove material to create the desired structure



Microfabrication

- Scanning Electron Microscope picture of a computer chip showing the layout of transistors and interconnect

- Moore's law reflects the fact that we have been able to build smaller and smaller structures, transistors, wires, mechanical devices etc. over time



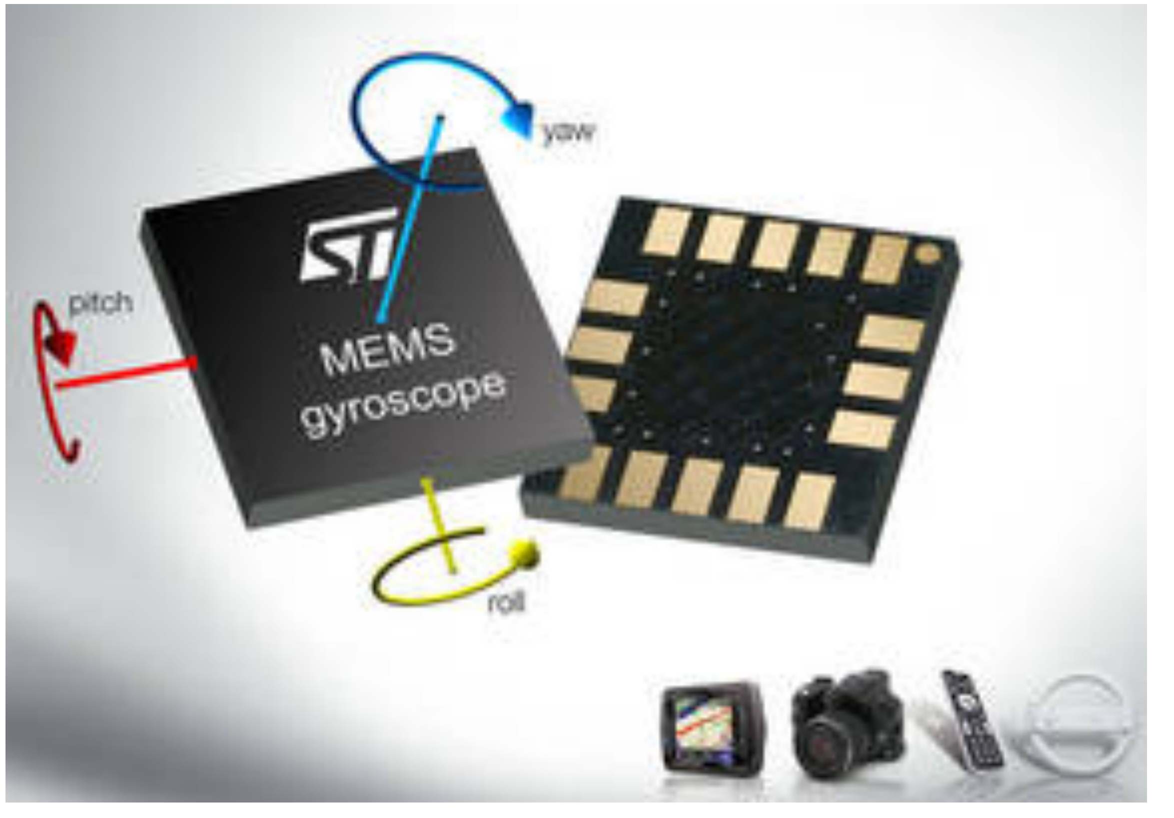
Small Computers

- One consequence of this is small computer chips which are capable of performing billions computations per second
- This is needed to run the sophisticated real-time control algorithms required for autonomous flight

Small Inertial Sensors

- Quadrotors also need to sense their attitude in real time so that the control systems can maintain balance.

- This is accomplished using micro electro mechanical (MEMS) systems which are fabricated cheaply using modern microfabrication techniques



Small Cameras

- Microfabrication also makes it possible to manufacture image sensors that are very small and very cheap. These are commonly used as a key sensor for flying robots

Microfabrication

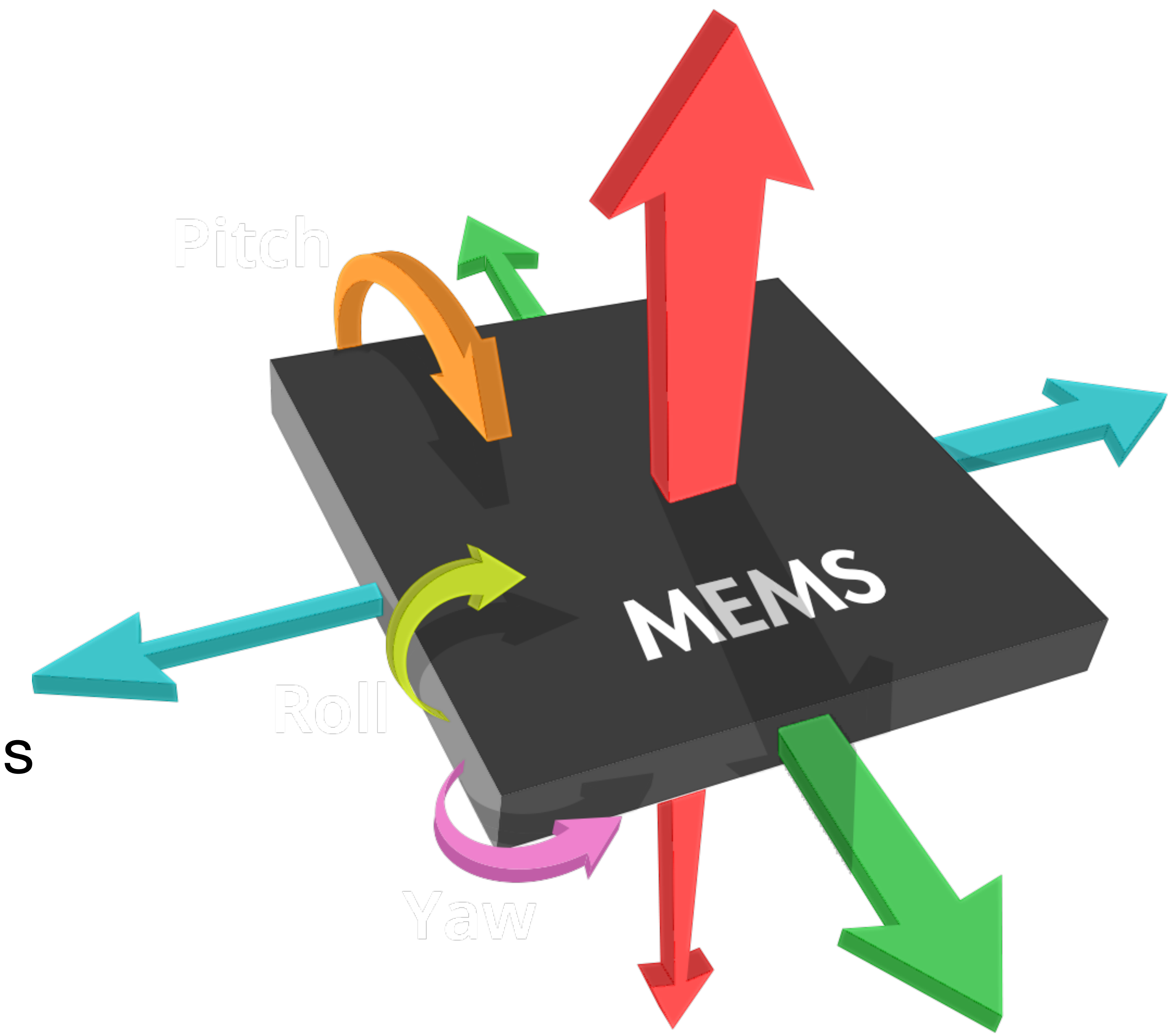
- The development of the modern autonomous flying robot has been made possible by the semiconductor industry which has focused on making small, powerful, inexpensive computing modules and sensors for cell phones.

Inertial Measurement Units

IMUs

- Inertial Measurement Units (IMUs) are devices for measuring the accelerations experienced by a moving platform.

- Small scale, inexpensive IMUs can be constructed using micro electromechanical devices (MEMS)

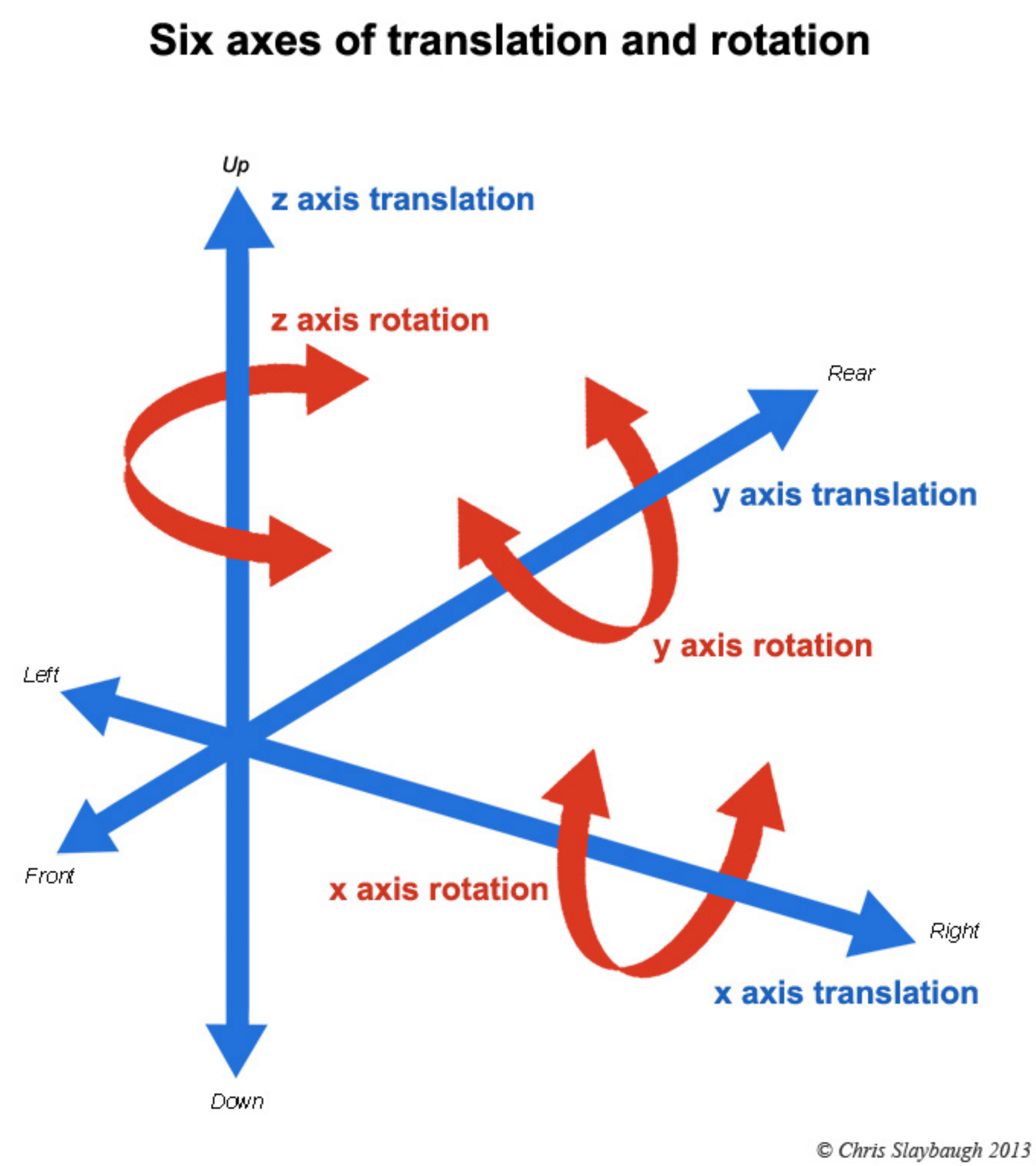


Video

https://www.youtube.com/watch?v=eqZgxR6eRjo

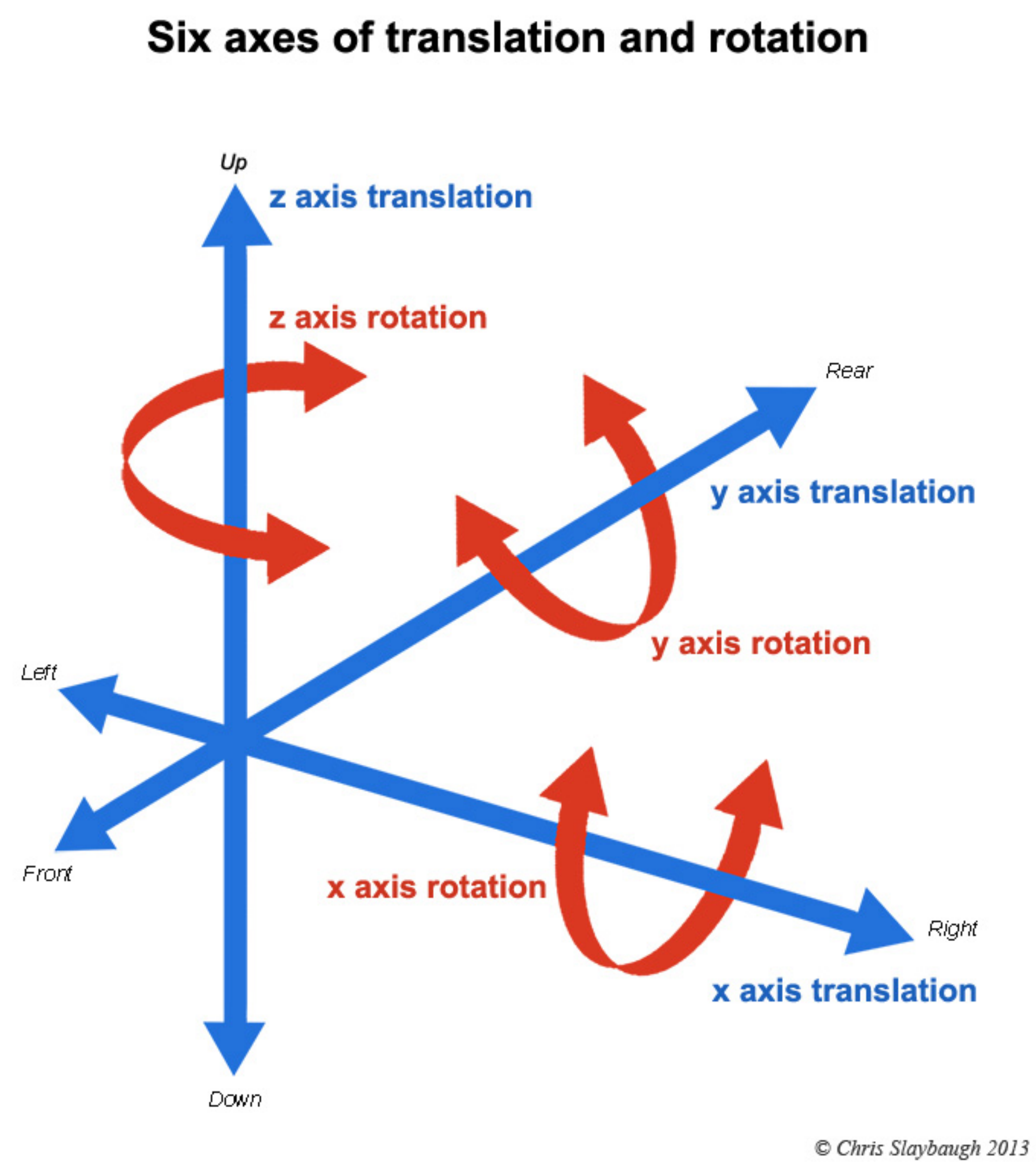
What do IMUs measure

- IMU units measure the following along each axis (x,y,z)
  - Linear acceleration
  - Angular velocity
  - Magnetic field strength
- A 6 axis sensor typically measures acceleration and rotation rate while a 9 axis device includes the magnetic field measurements



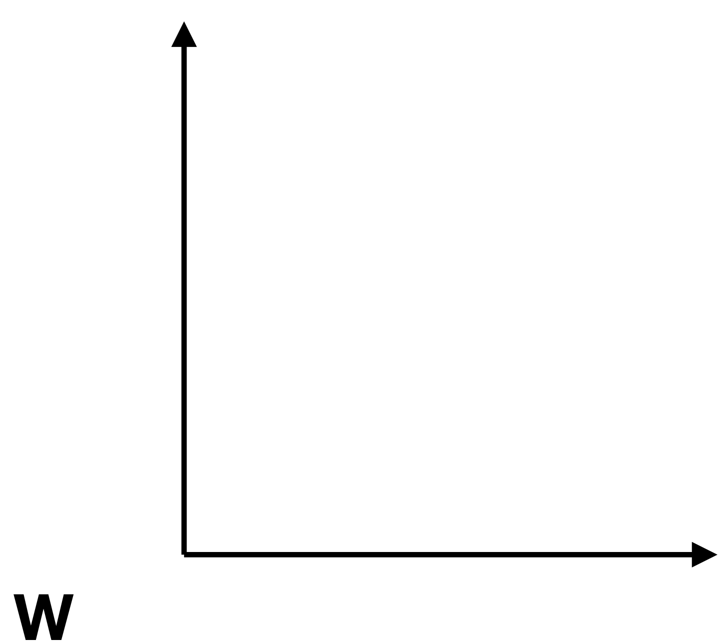
What do IMUs measure

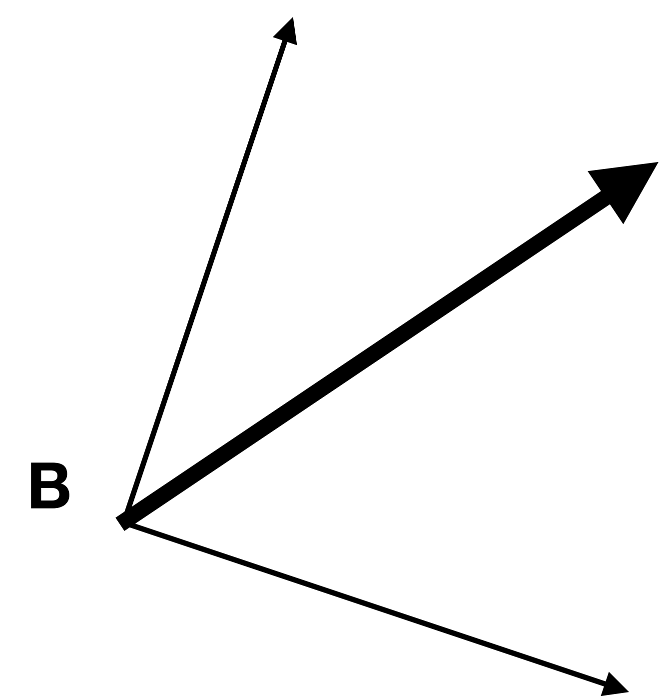
- Crucially, IMUs provide their measurements in the current reference frame of the device not the world frame



What IMUs measure

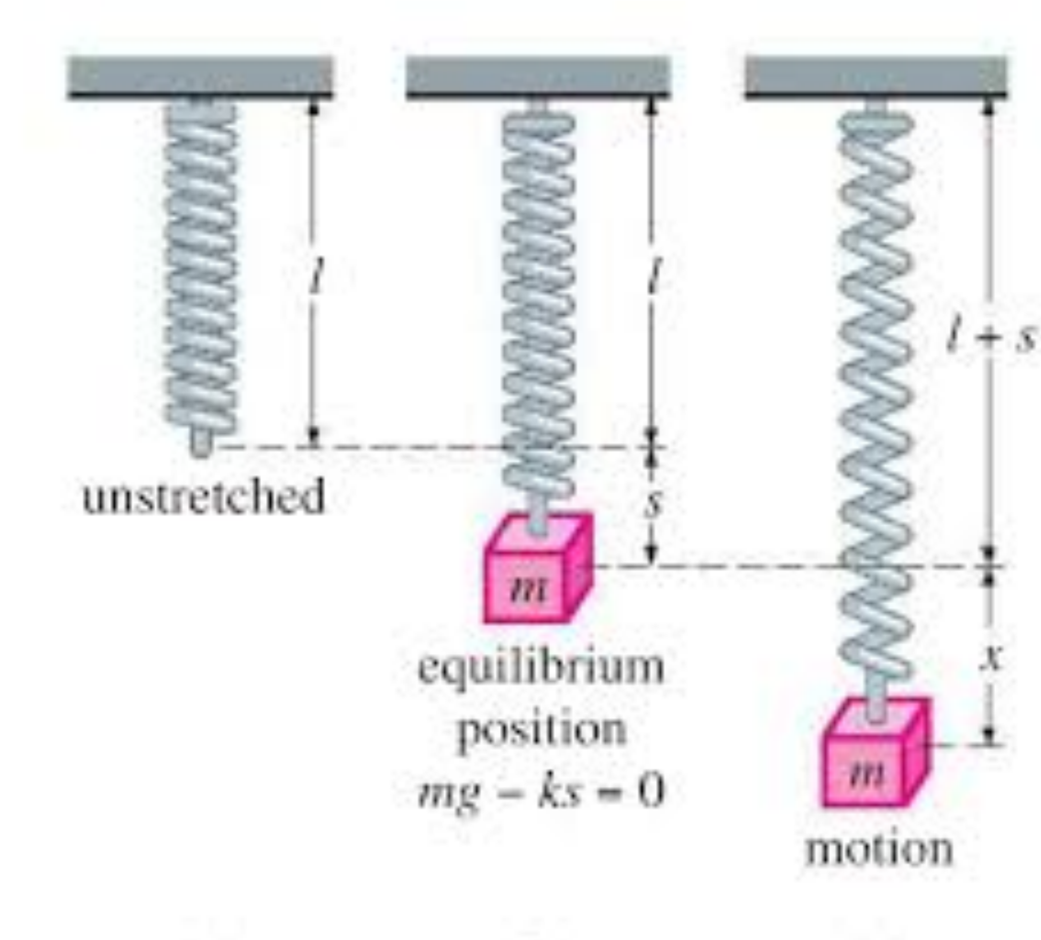
- Let $a_W, a_B \in \mathbb{R}^3$ denote the linear acceleration of the platform in the world and body frames respectively.
- Let $\omega_W, \omega_B \in \mathbb{R}^3$ denote the angular velocity of the platform in the world and body frames respectively.
- The IMU measures a_B and ω_B directly, not a_W and ω_W
- Let $R_{WB} \in SO(3)$ denote the rotation that relates the two frames.
- $a_W = R_{WB} a_B$
- $\omega_W = R_{WB} \omega_B$





What an IMU measures

- Imagine an IMU composed of a spring and a mass at rest with gravity acting down as shown in the cartoon
- By measuring the deflection of the spring you measure the force on the mass and hence the acceleration, in this case the acceleration due to gravity - g.
- If the assembly is falling freely there would be no deflection so the IMU would read zero



Rotations
A Review

Cross Product

- The cross product of two vectors $u, v \in \mathbb{R}^3$ produces a new vector orthogonal to the first two by the right hand rule with magnitude given by $\sin\theta \|u\| \|v\|$. Where θ denotes the angle between the two vectors
- Algebraically the cross product in $\mathbb{R}^3$ can be computed as follows.

$$a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \hat{a} b$$

- $\hat{a}$ denotes the skew symmetric matrix in $\mathbb{R}^{3\times 3}$ derived from the entries of the vector $a \in \mathbb{R}^3$. Note $\mathbb{R}^3$ is the only vector space where there is a one to one mapping from the vector space to the set of corresponding skew symmetric matrices.

$$\hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

Skew Symmetric Matrices

- Where symmetric matrices are defined by the equation

$$A^T = A, \;\; A \in \mathbb{R}^{n \times n}$$

Skew symmetric matrices are defined by the equation.

$$A^T = -A, \;\; A \in \mathbb{R}^{n \times n}$$

- You can think of them as the symmetric matrices evil twin with equal and opposite properties

- Note that every square matrix $B \in \mathbb{R}^{n \times n}$ can be expressed as a sum of symmetric and skew symmetric parts as follows.

$$B = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T)$$

SO(3)

- The set of all orthonormal matrices in $\mathbb{R}^{3\times 3}$ with determinant $+1$ is referred to as $SO(3)$

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I,\ \det(R) = +1\}$$

- Elements of $SO(3)$ represent rotations about an axis.
- $SO(3)$ is a group with respect to the operation of matrix multiplication.

Alternative view of Rotations

- Consider a point $r(t) \in \mathbb{R}^3$ that's being rotated around an axis denoted by a unit vector $\omega \in \mathbb{R}^3$ with an angular velocity of 1 radian per second.

- The instantaneous linear velocity of the point is given by

$$\dot{r}(t) = \omega \times r(t) = \hat{\omega} r(t)$$

- This is a linear differential equation which can be solved as follows.

$$r(t) = \exp(\hat{\omega} t) r(0)$$

  Substituting θ for t yields.

$$r(\theta) = \exp(\hat{\omega}\theta) r(0)$$

- Definition of matrix exponential

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

Alternative view of Rotations

- A rotation about a fixed axis ω by an angle θ can be represented by the matrix

$$R = \exp(\hat{\omega}\theta)$$

- Proving that $R \in SO(3)$

$$\begin{aligned}
R^T R &= \exp(\hat{\omega}\theta)^T \exp(\hat{\omega}\theta) \\
&= \exp(\hat{\omega}^T\theta) \exp(\hat{\omega}\theta) \\
&= \exp(-\hat{\omega}\theta) \exp(\hat{\omega}\theta) \\
&= \exp(-\hat{\omega}\theta + \hat{\omega}\theta) \\
&= \exp(0) \\
&= I
\end{aligned}$$

Note that the last step is possible because $\exp(A)\exp(B) = \exp(A + B)$ iff $AB = BA$ which is clearly true for $\hat{\omega}$ and $-\hat{\omega}$

Alternative view of Rotations

- A rotation about a fixed axis ω by an angle θ can be represented by the matrix

$$R = \exp(\hat{\omega}\theta)$$

- Proving that $R \in SO(3)$ part 2.

  Since $R^T R = I$ we can easily conclude that $\det(R)^2 = 1$. To show that $\det(R) = +1$ we note that $\det(\exp(\hat{\omega}\theta))$ is a continuous function of θ and that when $\theta = 0$, $R = I$ and $\det(R) = +1$. So we conclude that it must remain $+1$ for all θ by continuity.

Alternative view of Rotations

- Expanding the exponential

$$R = \exp(\hat{\omega}\theta) = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2} + \frac{\hat{\omega}^3\theta^3}{3!} + \cdots$$

- In general

$$\hat{a}\hat{b} = ba^t - a^t bI$$

Note that $\omega^t\omega = 1$ so

$$\hat{\omega}^2 = \omega\omega^T - I$$

and

$$\hat{\omega}^3 = \hat{\omega}(\omega\omega^T - I) = -\hat{\omega}$$

- More generally

$$\hat{\omega}^{2i} = \hat{\omega}^2(-1)^{i+1}$$

$$\hat{\omega}^{2i+1} = \hat{\omega}(-1)^{i+1}$$

- This means that R can be expressed as follows grouping odd and even powers of $\hat{\omega}$.

$$R = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} \cdots)\hat{\omega}^2$$

Rodrigues Formula

- This means that $R = \exp(\hat{\omega}\theta)$ can be expressed as follows grouping odd and even powers of $\hat{\omega}$.

$$R = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} \cdots)\hat{\omega}^2$$

- This expansion leads to the Rodrigues formula that relates an angle θ and axis ω to the corresponding rotation matrix.

$$R = I + \sin\theta\,\hat{\omega} + (1 - \cos\theta)\,\hat{\omega}^2$$

Recovering angle axis

- Given a rotation matrix $R \in SO(3)$ we can recover the corresponding angle and axis using the Rodrigues formula as follows.

$$R = I + \sin\theta\,\hat{\omega} + (1 - \cos\theta)\,\hat{\omega}^2$$

- $\mathrm{trace}(R) = 1 + 2\cos\theta$: We can compute the angle θ from this.
- $R - R^T = 2\sin\theta\,\hat{\omega}$: We can recover the axis ω from this skew symmetric matrix.

Euler's Theorem

- Euler's theorem states that every matrix $R \in SO(3)$ can be written as $\exp(\hat{\omega}\theta)$ for some choice of ω and θ.

Unit Quaternions

- Unit quaternions provide an alternative representation of rotations.
- The set of unit quaternions can be thought of as the set of tuples (u_0, u) where $u_0 \in \mathbb{R}$ and $u \in \mathbb{R}^3$ such that $u_0^2 + u^t u = 1$.
- This set forms a group under the operation of quaternion multiplication defined as follows.

$$(u_0, u) \cdot (v_0, v) = (u_0 v_0 - u^T v, u_0 v + v_0 u + u \times v)$$

  This group is referred to as the Symplectic Group $Sp(1)$.

- There is a 2 to 1 mapping between elements of $Sp(1)$ and $SO(3)$ which can be defined by the following mapping.

$$H(u_0, u) = (u_0^2 - u^T u)I + 2u_0 \hat{u} + 2uu^T$$

  You can verify that $H(u_0, u) \in SO(3)$. Note that (u_0, u) and $(-u_0, -u)$ map to the same matrix.

Quaternions

- Given a quaternion, $q = (u_0, u)$, we can define its conjugate as follows $q^* = (u_0, -u)$.
- Given a vector $x \in \mathbb{R}^3$ we can form a quaternion $(0, x)$.
- You can show that
$$q \cdot (0, x) \cdot q^* = (0, H(q)x)$$
- You can also show that
$$H(q_1 \cdot q_2) = H(q_1)H(q_2)$$
- This means that we can represent rotations using quaternions, 4 numbers instead of 9, and easier to normalize. We can then perform quaternion multiplications instead of matrix multiplications.
- Lastly we have the following important relationship.
$$q = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\omega) \Rightarrow H(q) = \exp(\hat{\omega}\theta)$$

Summary

- We have discussed three different ways of representing rotations
  - As matrices $R \in SO(3) \subset R^{3\times 3}$, $R^T R = I$, $\det(R) = 1$
  - Angle axis representations $\theta \in \mathbb{R}, \omega \in \mathbb{R}^3$
  - Unit quaternions (u_0, u) such that $u_0^2 + u^T u = 1$
- These representations are related by the Rodrigues formula

$$R = \exp(\hat{\omega}\theta) = I + \sin\theta\hat{\omega} + (1 - \cos\theta)\hat{\omega}^2$$

and by the equations

$$q = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\omega) \Rightarrow H(q) = \exp(\hat{\omega}\theta)$$

and

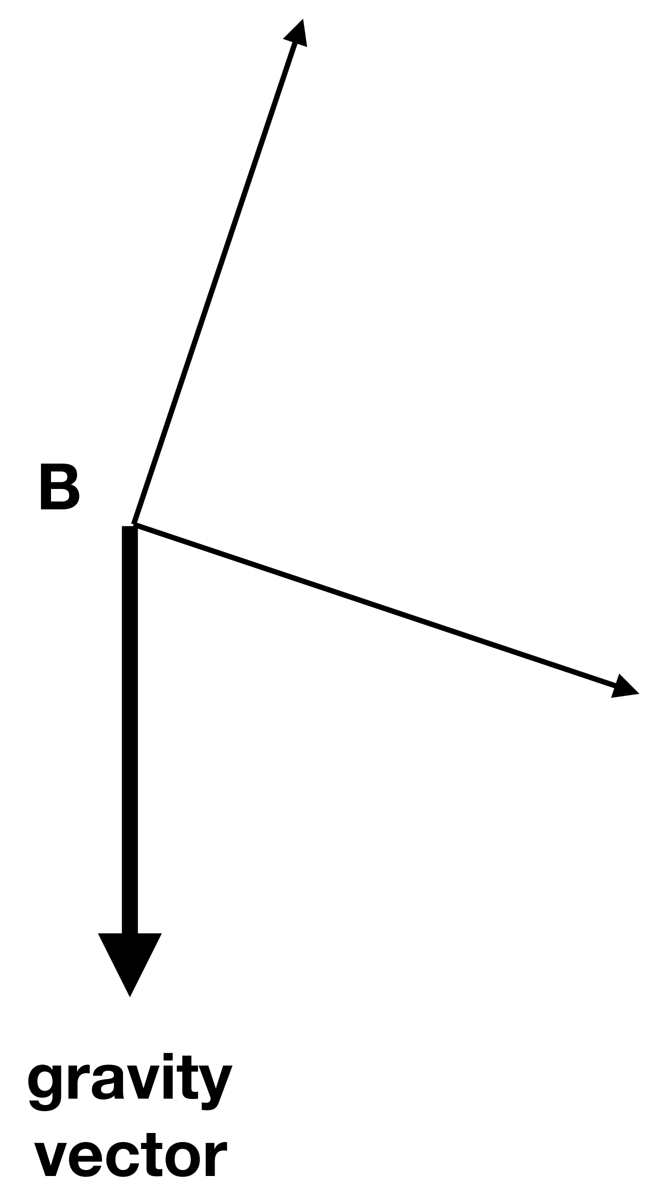
$$H(u_0, u) = (u_0^2 - u^T u)I + 2u_0\hat{u} + 2uu^T$$

Estimating Rotation

The Complementary Filter

Estimating orientation from gravity

- Imagine an inertial measurement unit sitting at rest
- In this case the accelerometer will measure the direction of the gravity vector. This measurement constrains the orientation of the device with respect to a fixed world frame of reference.
- Informally it defines the pitch and roll of the platform but not the yaw



Estimating orientation from gyros

- Imagine that the IMU starts from rest and then is moved around, translated and rotated.

- We can estimate the orientation of the device with respect to its original pose by integrating the angular rate measurements

Complementary Filter idea

- The accelerometer and the gyro are complementary in the sense that the accelerometer provides information about the absolute orientation with respect to a fixed frame **when the platform is moving at a constant velocity**
- The gyro can be integrated to provide information about orientation as the platform moves **but this estimate will drift over time due to the integration of error**
- **The complementary filter seeks to combine these two sources of information to form a better estimate than either alone. (This is a theme we will return to)**

Integrating Gyro Measurements

- Imagine an IMU that provides measurements at regular intervals, say 100Hz. In this case the time between measurements, Δt, would be 10 milliseconds

- Imagine that at time instant number 1 the angular velocity measurement is $\omega_1 \in \mathbb{R}^3$ where the units of the entries in this vector are radians per second. Similarly at time instant 2 the angular velocity measurement is $\omega_2 \in \mathbb{R}^3$

- We can estimate the rotation that relates frames 1 and 2 as follows

$$R_{12} = \exp(\hat{\omega}_1 \Delta t)$$

- This rotation can be easily expressed in the form of a quaternion since the IMU essentially provides it to us in angle axis form.

- These rotation estimates can be integrated over time to estimate the orientation of the IMU with respect to its original position.

$$R_{1k} = R_{12} R_{23} R_{34} \cdots R_{(k-1)k}$$

Correcting Rotation with Accelerometer

- Assume wlog that at time 1 the IMU is at rest and that the gravity vector is aligned with the z-axis so that the reading from the accelerometer is as follows: $a_1 = e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Note that we assume that we are measuring acceleration in gs.

- Let's assume that at time instant k the IMU is moving with constant (or near constant) velocity and the accelerometer reading is a_k and our rotation estimate is R_{1k}. In theory $R_{1k}a_k = e_z$ but in practice due to drift and other errors $R_{1k}a_k = g'$ where g' is close to but not equal to e_z. We will assume that g' has been normalized to unit length, $\|g'\| = 1$

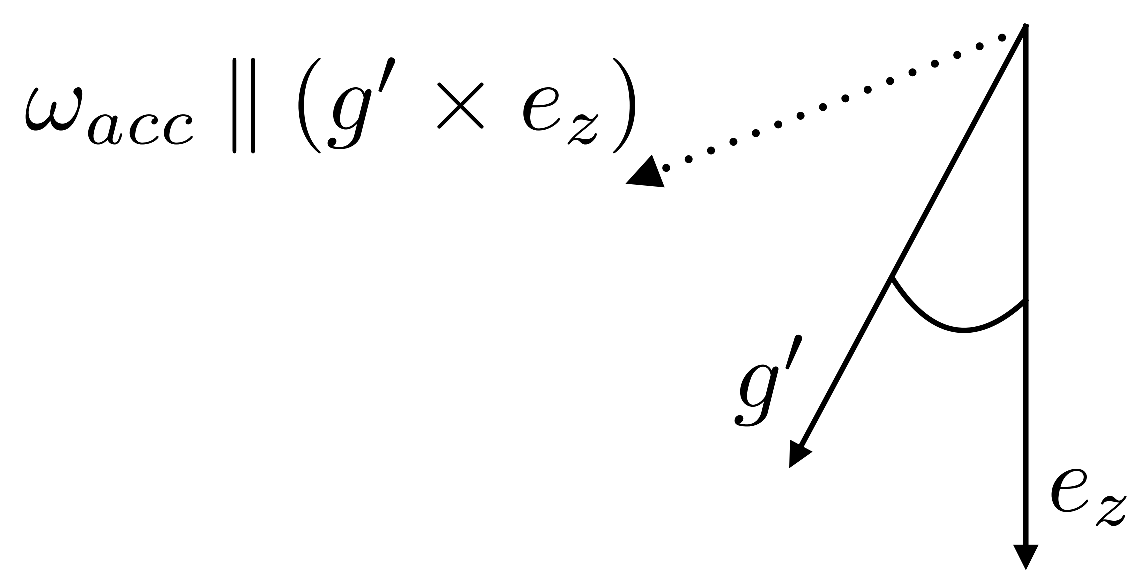
- Our idea is to construct a correction rotation R_{acc} which is designed to align the g' with e_z. That is $R_{acc}g' = e_z$. Once we have done this we could apply this rotation to correct R_{1k} as follows: $R_{acc}R_{1k}$.

Constructing a rotation correction

- Our recipe for constructing the rotation correction R_{acc} is actually quite simple. We consider a rotation axis, ω_{acc}, which is parallel to the cross product $g' \times e_z$ and choose the rotation angle to be the angle between g' and e_z which is easily recovered from the inner product between these unit vectors.

- A quaternion that does the job can be computed explicitly as follows.

$$\Delta q_{acc} = \left(\sqrt{\frac{g'_z+1}{2}}, \quad \frac{g'_y}{\sqrt{2(g'_z+1)}}, \quad \frac{-g'_x}{\sqrt{2(g'_z+1)}}, \quad 0 \right)$$



Applying the rotation correction

- In practice we don't completely trust the accelerometer either so instead of applying the complete rotation correction, Δq_{acc}, we choose to apply a version that is mid way between that rotation and the null rotation given by the quaternion $q_I = (1, 0, 0, 0)$
- This blended quaternion is modulated by a scalar parameter α which ranges between 0 and 1. When $\alpha = 1$ the blended quaternion is Δq_{acc}, when $\alpha = 0$ the correction is ignored and the applied quaternion is q_I.
- A simple way to blend the two quaternions is by computing a simple weighted sum and then normalizing the result to make it a unit quaternion

$$\Delta q'_{acc} = (1 - \alpha) q_I + \alpha \Delta q_{acc}$$

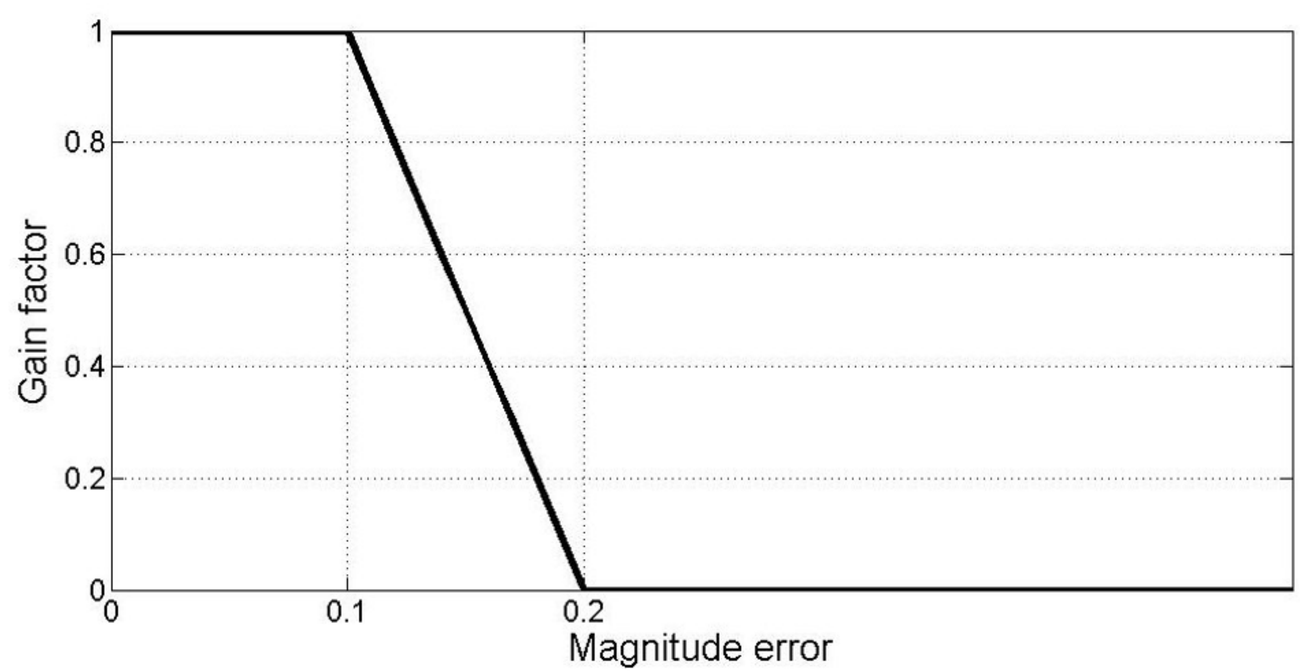
Adaptive Gain

- Note that the acceleration should really only be applied when the platform is moving with constant velocity. In this case the only measured acceleration will be that due to gravity.

- One way to try to check for this condition is to compare the magnitude of the sensed acceleration vector to 1. If it differs from 1 that indicates that there is definitely additional translational acceleration.

$$e_m = |\|a_k\| - 1|$$

- The larger e_m, the less likely the platform is at rest. Given this measure we adapt the blending parameter α as shown in the graph below



Summary of Complementary Filter

- In summary the complementary filter performs the following steps every time it receives a measurement packet which includes angular velocity and acceleration measurements.
  - Update the Rotation estimate using the measured angular velocity
  - Consider the measured acceleration vector, compute the error measure e_m, and use that to compute the gain α
  - Compute the rotation correction based on the accelerometer reading Δq_{acc}
  - Apply the rotation correction to the rotation estimate taking into account the gain α