Rigid Body Transformations and Displacements

MEAM 620, SPRING 2020

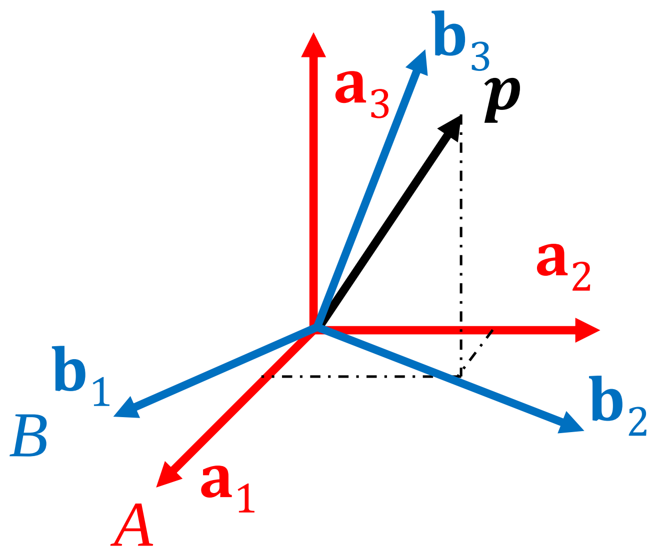
Last time:

1) Given one free vector $\boldsymbol{p}$, we thought about how the coordinate vectors ${}^{A}[\boldsymbol{p}]$ and ${}^{B}[\boldsymbol{p}]$ relate.

$$ {}^{A}[\boldsymbol{p}] = [\,{}^{A}\mathbf{R}_{B}\,]\,{}^{B}[\boldsymbol{p}] $$

2) We saw that this matrix can be viewed as a description of the orientation of B with respect to A.

$$ {}^{A}\mathbf{R}_{B} = [\,{}^{A}[\mathbf{b}_1] \quad {}^{A}[\mathbf{b}_2] \quad {}^{A}[\mathbf{b}_3]] $$



This time:

Rigid body coordinate transformations of position vectors.

Homogeneous transformation matrices.

Rigid body displacements.

Many more choices of coordinates for rotations.

Reference Frames and Points

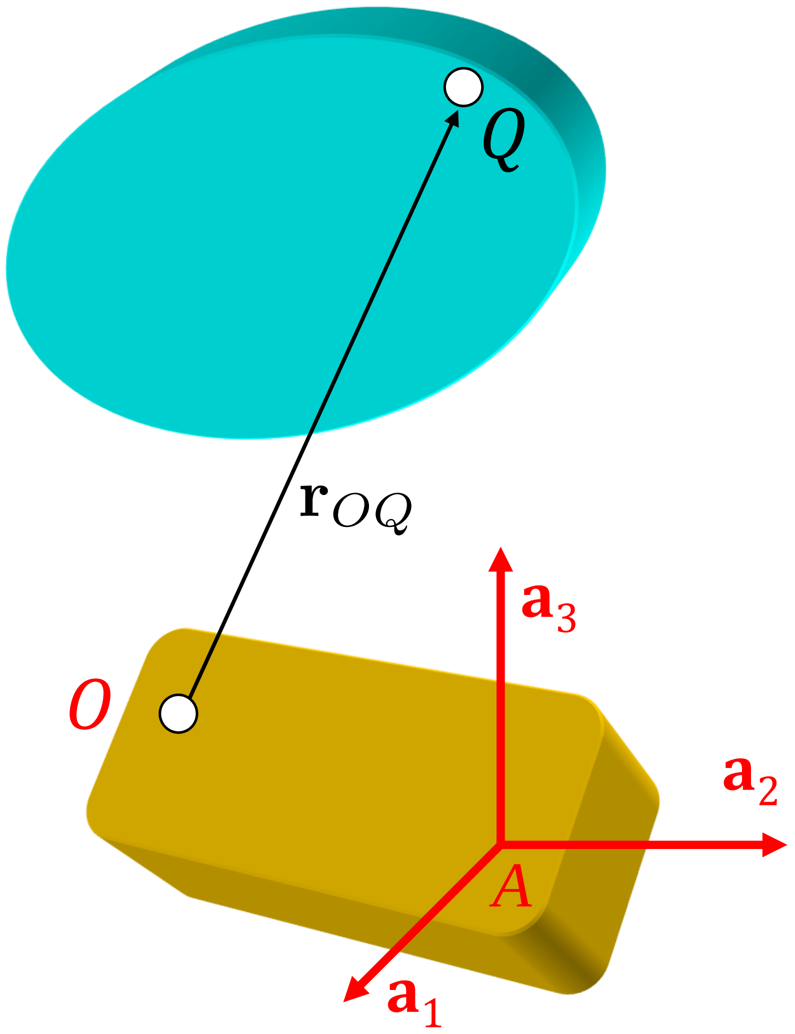
A reference frame A combines an

- origin point O, and
- basis vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

A new point Q can be identified by the position vector $\boldsymbol{r}_{OQ}$.

As a shorthand, let $\{{}^{A}q_1, {}^{A}q_2, {}^{A}q_3\}$ be the coordinates of $\boldsymbol{r}_{OQ}$ in the basis of A.

$$\boldsymbol{r}_{OQ} = {}^{A}q_1\mathbf{a}_1 + {}^{A}q_2\mathbf{a}_2 + {}^{A}q_3\mathbf{a}_3$$



Transformation of Position Vectors

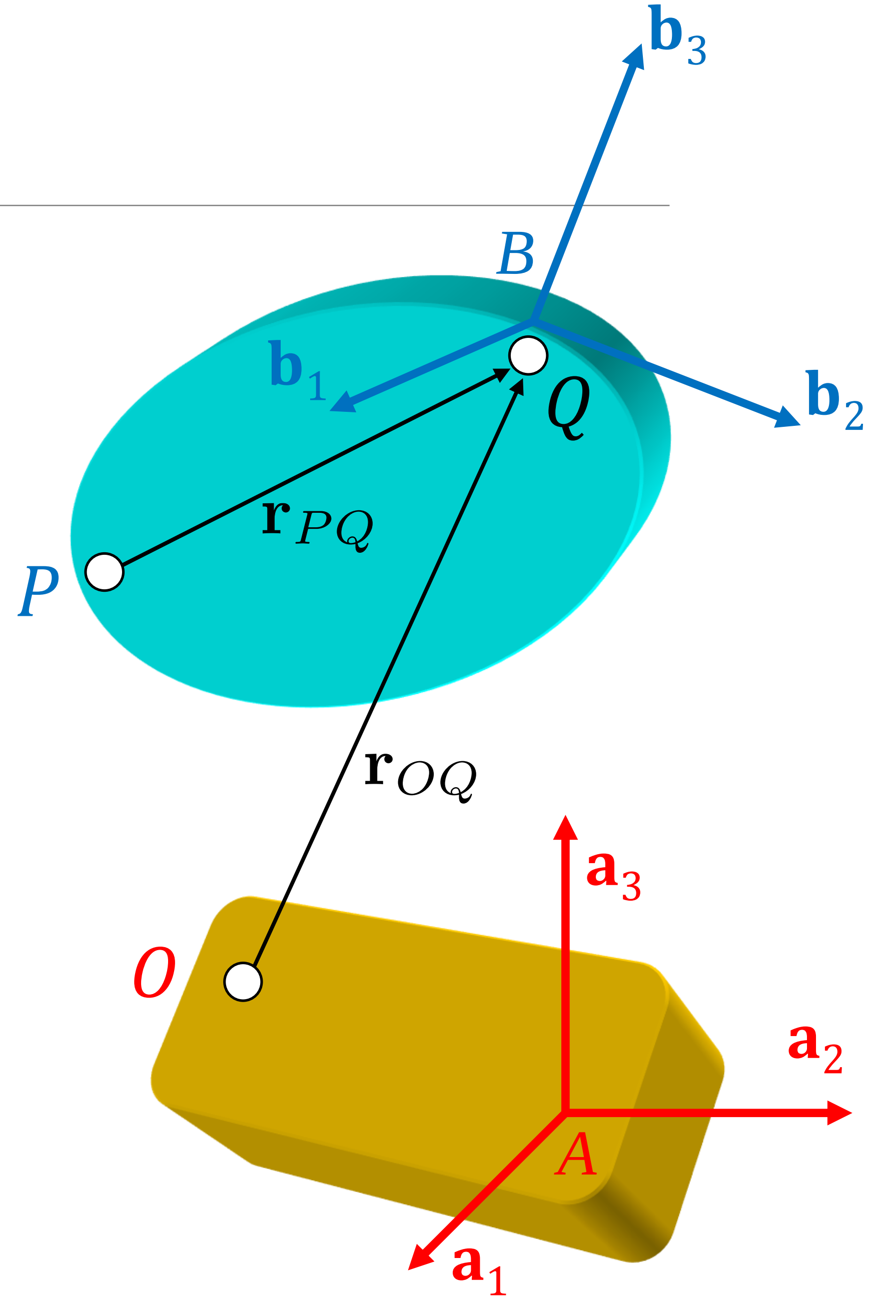
Introduce frame B with origin P and basis vectors $\{b_1, b_2, b_3\}$.

This gives us a new basis vector and associated coordinates.

$$\boldsymbol{r}_{OQ} = {}^{A}q_1\mathbf{a}_1 + {}^{A}q_2\mathbf{a}_2 + {}^{A}q_3\mathbf{a}_3$$

$$\boldsymbol{r}_{PQ} = {}^{B}q_1\boldsymbol{b}_1 + {}^{B}q_2\boldsymbol{b}_2 + {}^{B}q_3\boldsymbol{b}_3$$

How are these coordinates describing the point Q related?



Transformation of Position Vectors

Start with vector addition, and complete the triangle.

$$\boldsymbol{r}_{OQ} = \boldsymbol{r}_{OP} + \boldsymbol{r}_{PQ}$$

Expand the vectors in whatever bases seem convenient.

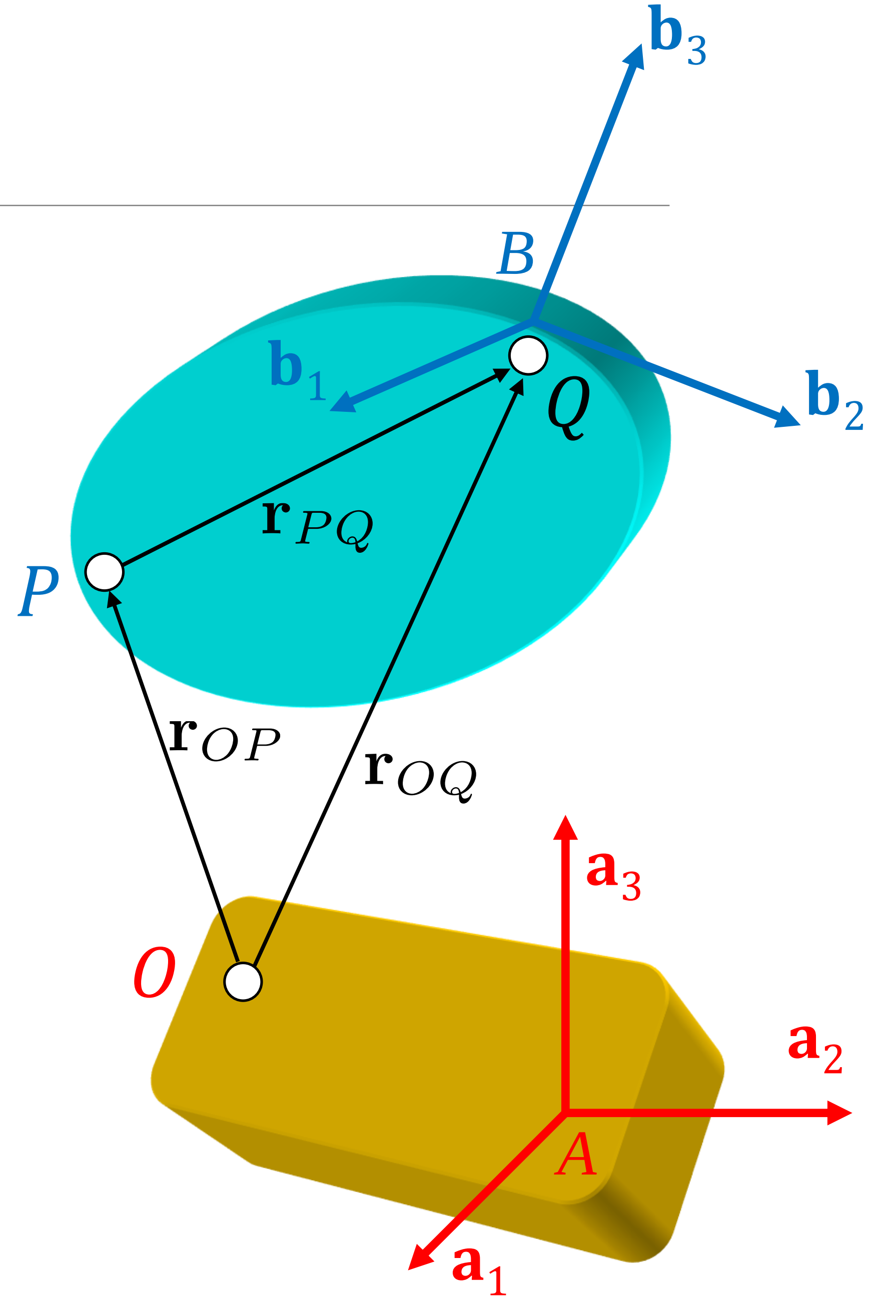
$$\boldsymbol{r}_{OQ} = {}^{A}q_1\mathbf{a}_1 + {}^{A}q_2\mathbf{a}_2 + {}^{A}q_3\mathbf{a}_3$$

$$\boldsymbol{r}_{OP} = {}^{A}p_1\mathbf{a}_1 + {}^{A}p_2\mathbf{a}_2 + {}^{A}p_3\mathbf{a}_3$$

$$\boldsymbol{r}_{PQ} = {}^{B}q_1\boldsymbol{b}_1 + {}^{B}q_2\boldsymbol{b}_2 + {}^{B}q_3\boldsymbol{b}_3$$

Can't compute on coordinates of different bases.

$$\cancel{\begin{bmatrix} {}^{A}q_1 \\ {}^{A}q_2 \\ {}^{A}q_3 \end{bmatrix} = \begin{bmatrix} {}^{A}p_1 \\ {}^{A}p_2 \\ {}^{A}p_3 \end{bmatrix} + \begin{bmatrix} {}^{B}q_1 \\ {}^{B}q_2 \\ {}^{B}q_3 \end{bmatrix}}$$



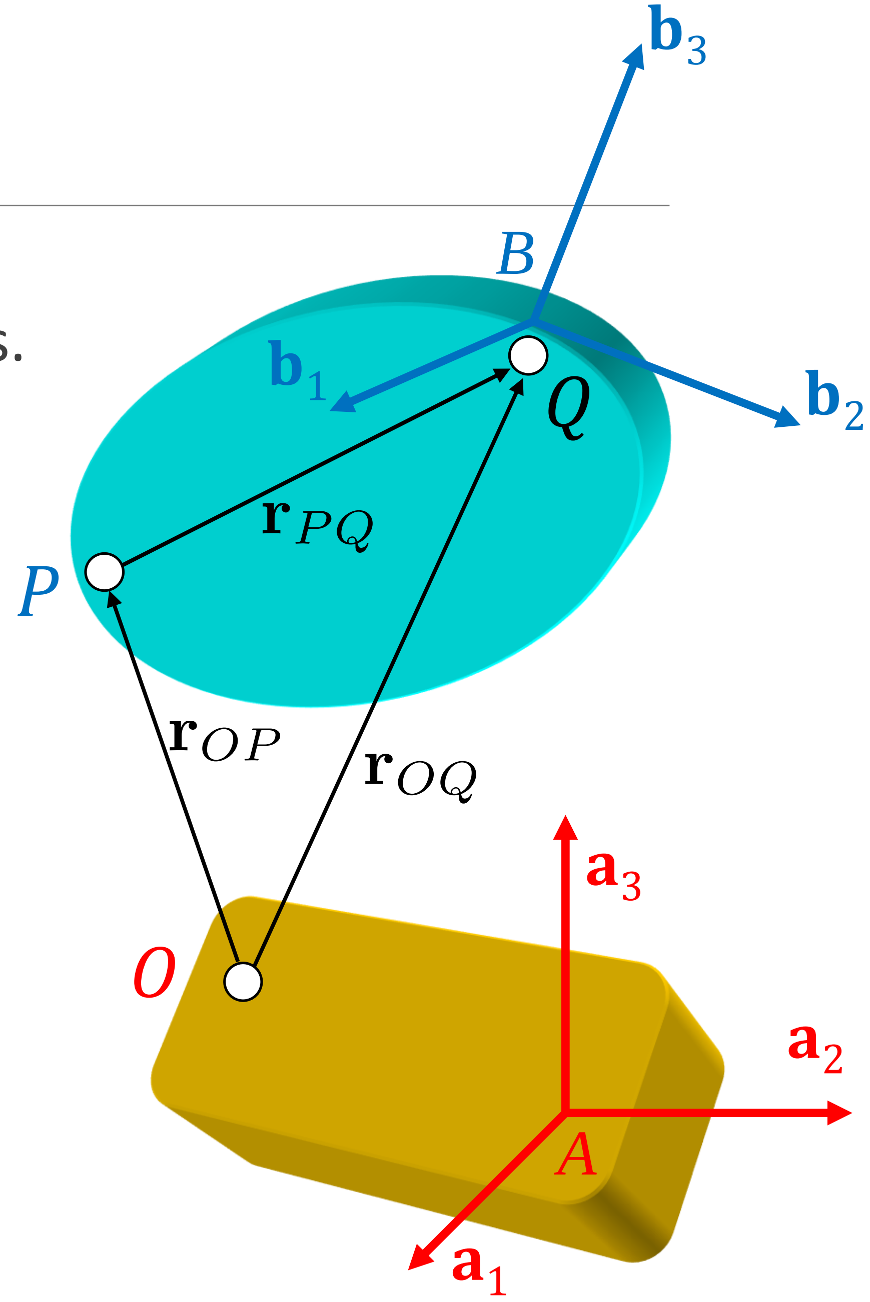
Rotation Matrix

Recall the rotation matrix transforms coordinates in B to coordinates in A. This brings the equation into consistent bases.

$$\boldsymbol{r}_{OQ} = \boldsymbol{r}_{OP} + \boldsymbol{r}_{PQ}$$

$$\begin{bmatrix} {}^{A}q_1 \\ {}^{A}q_2 \\ {}^{A}q_3 \end{bmatrix} = \begin{bmatrix} {}^{A}p_1 \\ {}^{A}p_2 \\ {}^{A}p_3 \end{bmatrix} + {}^{A}\mathbf{R}_{B} \begin{bmatrix} {}^{B}q_1 \\ {}^{B}q_2 \\ {}^{B}q_3 \end{bmatrix}$$

coordinates of Q in A | coordinates of P in A | coordinates of Q in B



Homogeneous Transformation Matrix

This rigid body transformation can be described by a 4x4 homogeneous transformation matrix.

$$\begin{bmatrix} {}^{A}q_1 \\ {}^{A}q_2 \\ {}^{A}q_3 \end{bmatrix} = \begin{bmatrix} {}^{A}p_1 \\ {}^{A}p_2 \\ {}^{A}p_3 \end{bmatrix} + {}^{A}\mathbf{R}_B \begin{bmatrix} {}^{B}q_1 \\ {}^{B}q_2 \\ {}^{B}q_3 \end{bmatrix}$$

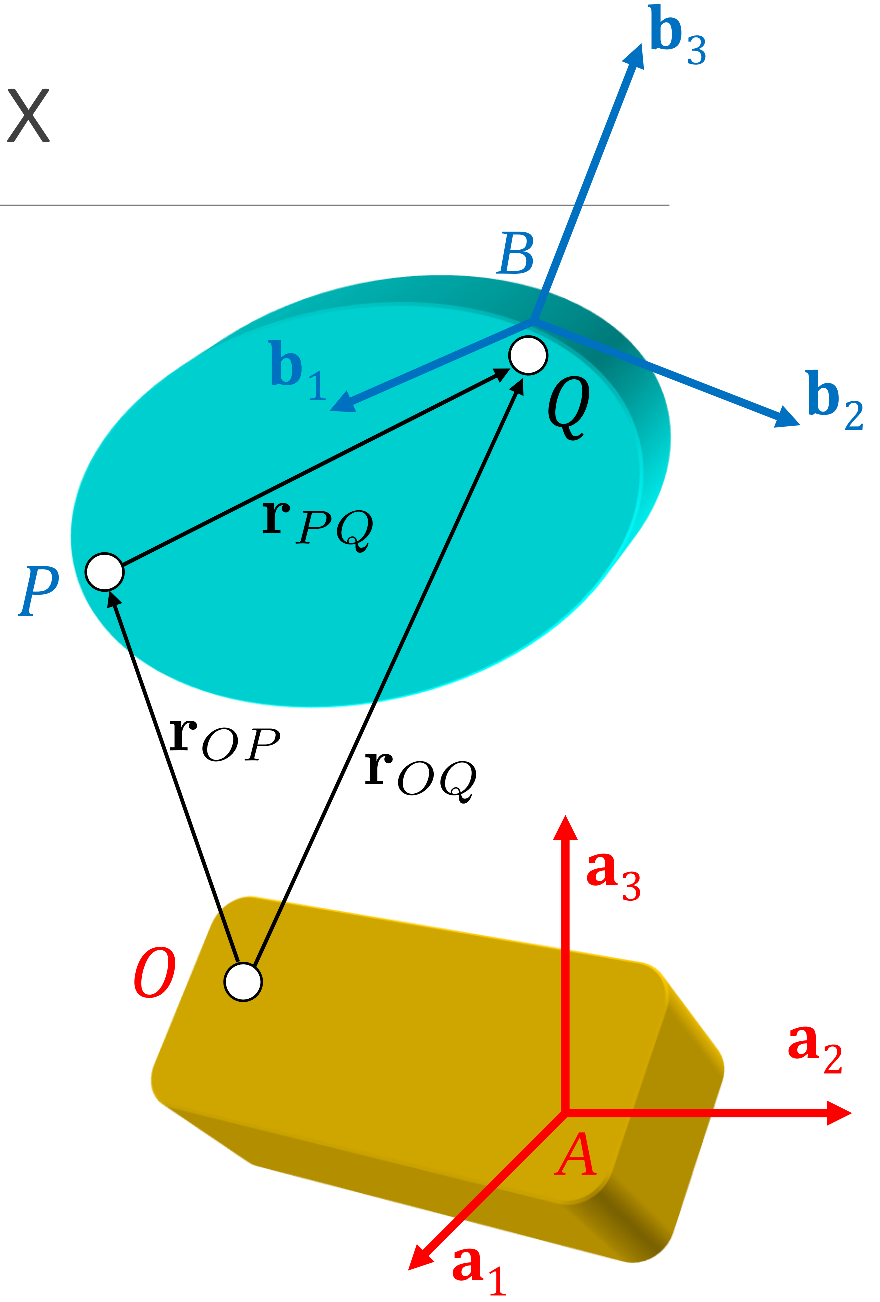
$$\underbrace{\begin{bmatrix} {}^{A}q_1 \\ {}^{A}q_2 \\ {}^{A}q_3 \\ 1 \end{bmatrix}}_{\text{coordinates of } Q \text{ in } A} = \underbrace{\begin{bmatrix} & {}^{A}\mathbf{R}_B & & {}^{A}p_1 \\ & & & {}^{A}p_2 \\ & & & {}^{A}p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{the 4x4 homogeneous transformation matrix } {}^{A}\mathbf{T}_B} \underbrace{\begin{bmatrix} {}^{B}q_1 \\ {}^{B}q_2 \\ {}^{B}q_3 \\ 1 \end{bmatrix}}_{\text{coordinates of } Q \text{ in } B}$$

coordinates of Q in A

coordinates of Q in B

the 4x4 homogeneous transformation matrix ${}^{A}\mathbf{T}_B$

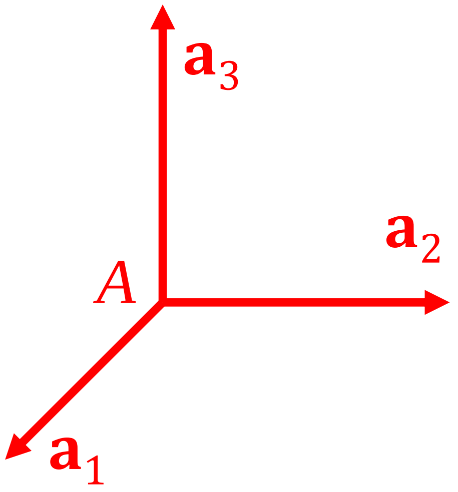
"homogeneous coordinates"

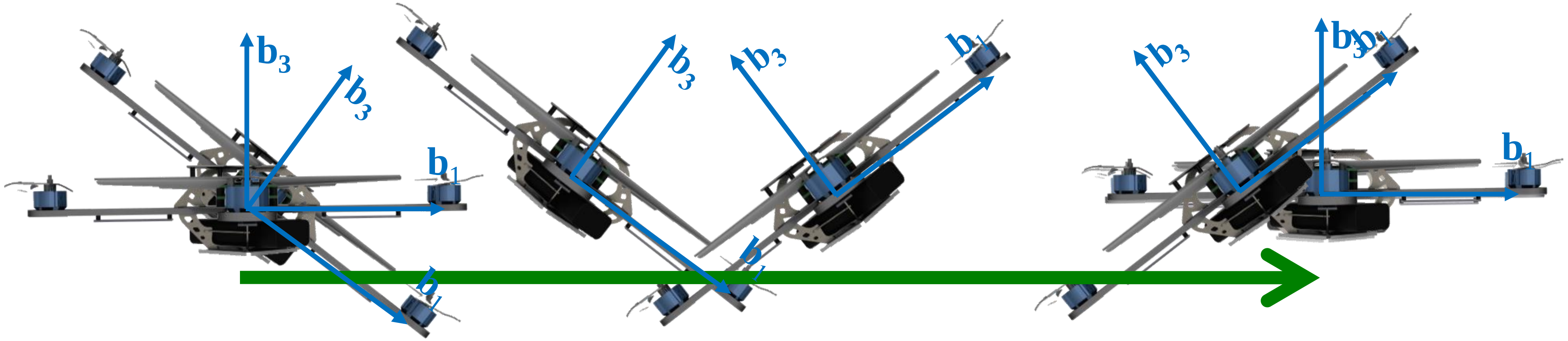


Rigid Body Transformations

So far we've been working with examples of rigid body transformations.

- Describe the relationship between reference frames attached to two different rigid bodies.
- Describe how to relate coordinates for the same point observed in two different reference frames.



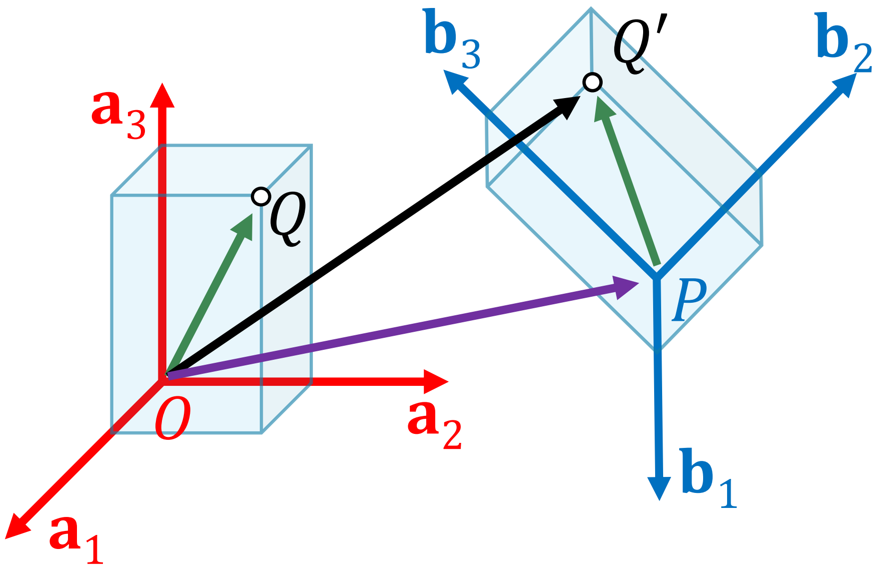


Rigid Body Displacements

Consider two distinct poses of the same rigid body.

Point Q is fixed to the rigid body, and moves with it. Where does it go?

$$\boldsymbol{r}_{OQ'} = \boldsymbol{r}_{OP} + \boldsymbol{r}_{PQ'}$$



Write in terms of the coordinate vectors

${}^{A}\boldsymbol{q}' = {}^{A}\boldsymbol{p} + {}^{A}\mathbf{R}_{B}\,{}^{B}\boldsymbol{q}'$ but now notice that ${}^{B}\boldsymbol{q}' = {}^{A}\boldsymbol{q}$

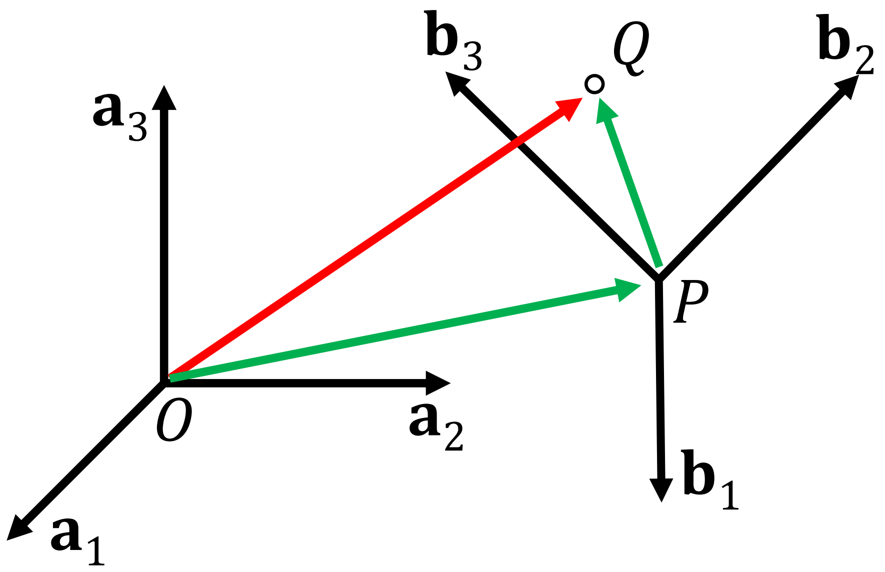
$$\begin{bmatrix} {}^{A}q'_1 \\ {}^{A}q'_2 \\ {}^{A}q'_3 \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}\mathbf{R}_{B} & & \begin{matrix} {}^{A}p_1 \\ {}^{A}p_2 \\ {}^{A}p_3 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}q_1 \\ {}^{A}q_2 \\ {}^{A}q_3 \\ 1 \end{bmatrix}$$

new point coordinates in A — transformation ${}^{A}\mathbf{T}_{B}$ — old point coordinates in A

The same transformation ${}^{A}\mathbf{T}_{B}$ can describe
1) Transform of points in B to points in A.
2) Rigid displacement of a point in A (point Q) to another point in A (point A’).

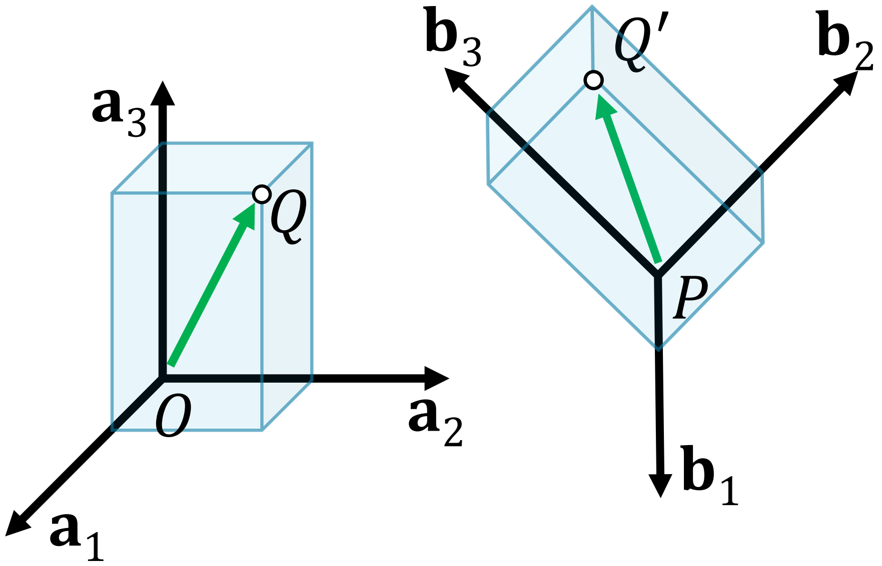
Two interpretations of the matrix ${}^{A}\mathbf{T}_{B}$

Rigid body transformation.



A transformation of points in B to points in A.

Rigid body displacement.



A rigid displacement of a point in A (point Q) to another point in A (point Q').

Two interpretations of the matrix ${}^{A}\mathbf{T}_{B}$

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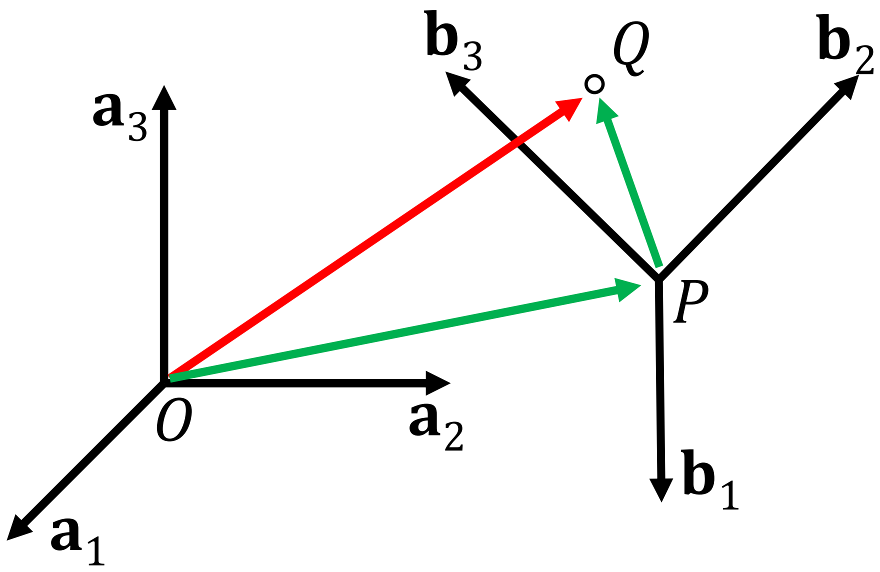
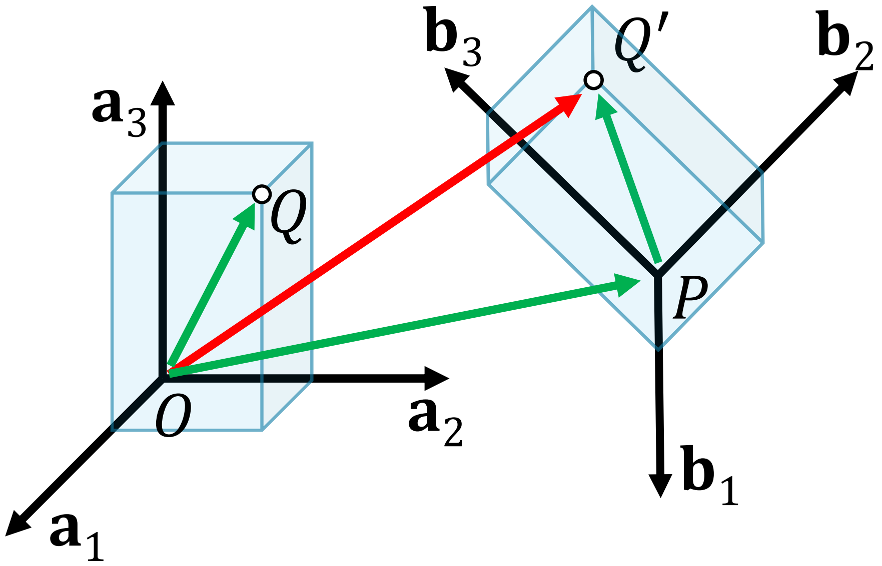


A transformation of points in B to points in A.

Rigid body displacement.



A rigid displacement of a point in A (point Q) to another point in A (point Q').

Composition of Displacements

Displacement from A to B

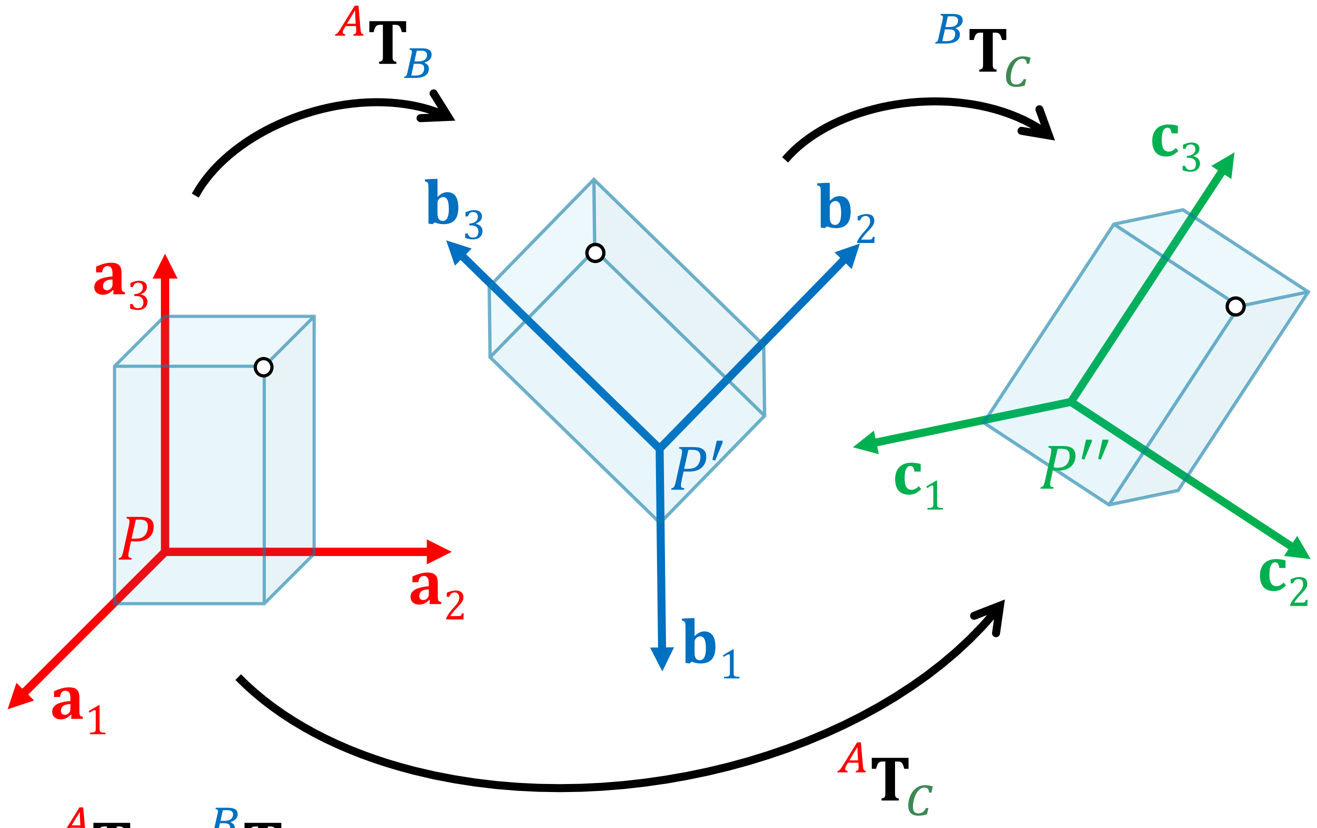
$$ {}^{A}\mathbf{T}_{B} = \begin{bmatrix} {}^{A}\mathbf{R}_{B} & {}^{A}\boldsymbol{p} \\ 0_{1\times 3} & 1 \end{bmatrix} $$

Displacement from B to C

$$ {}^{B}\mathbf{T}_{C} = \begin{bmatrix} {}^{B}\mathbf{R}_{C} & {}^{B}\boldsymbol{p}' \\ 0_{1\times 3} & 1 \end{bmatrix} $$

Displacement from A to C

$$ {}^{A}\mathbf{T}_{C} = \begin{bmatrix} {}^{A}\mathbf{R}_{C} & {}^{A}\boldsymbol{p}'' \\ 0_{1\times 3} & 1 \end{bmatrix} \quad \text{or} \quad {}^{A}\mathbf{T}_{C} = {}^{A}\mathbf{T}_{B}\ {}^{B}\mathbf{T}_{C} $$



Here ${}^{B}\boldsymbol{p}'$ denotes the position vector of P' in B, etc.

In each case, ${}^{X}\mathbf{T}_{Y}$ describes the displacement of body fixed frame X to Y in reference frame X.

Conventional Use

Composition of Displacements

- Displacements are generally described in a body-fixed frame.
- For example: ${}^{B}\mathbf{T}_{C}$ is the displacement of a rigid body from B to C described relative to the axes of the "first frame" B.

Composition of Transformations

- Treated similarly to displacements.
- ${}^{A}\mathbf{T}_{C} = {}^{A}\mathbf{T}_{B} \; {}^{B}\mathbf{T}_{C}$