Rotations and Angular Velocities

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Time Derivatives of Rotations

Rotation matrix

$$R(t)$$

Orthogonality

$$R^T(t)R(t) = I \quad \xrightarrow{\frac{d}{dt}(\,.\,)} \quad \dot{R}^T R + R^T \dot{R} = 0$$

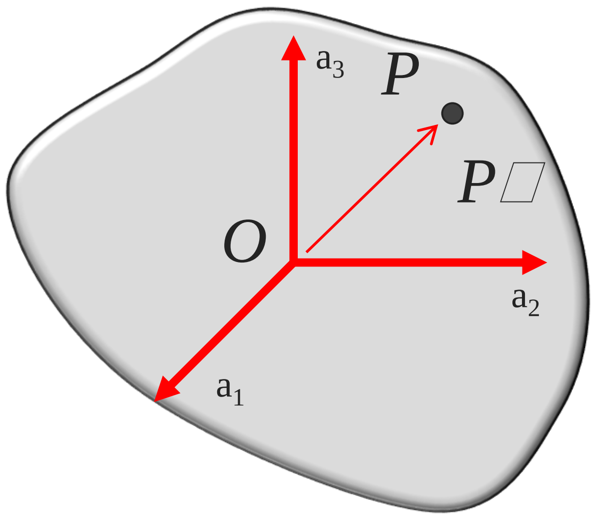
$$R(t)R^T(t) = I \qquad\qquad R\dot{R}^T + \dot{R}R^T = 0$$

$R^T \dot{R}$ and $\dot{R}R^T$ are skew symmetric

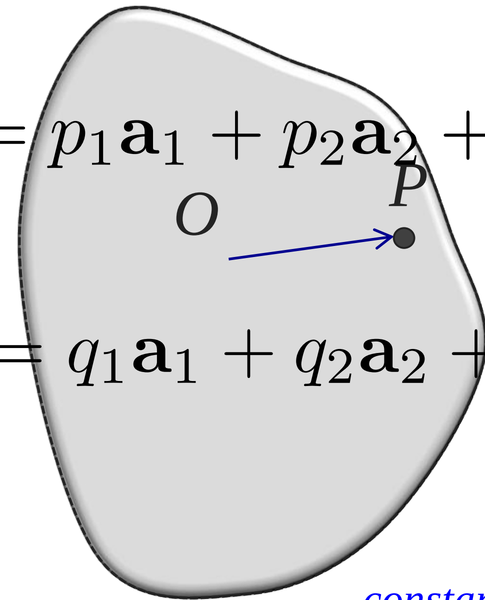
$$\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} \times \boldsymbol{b} = [\widehat{\boldsymbol{\omega}}][\boldsymbol{b}]$$

Rotation with O fixed



$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$



$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

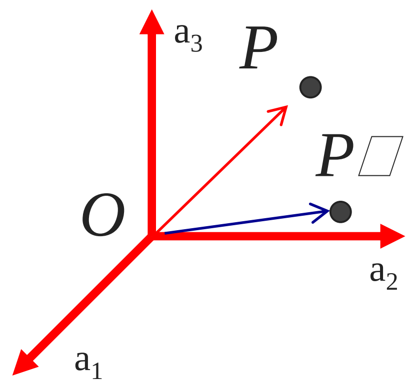
$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{q} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{p}$$

constant coordinates of P in body-fixed frame

$$\boxed{q(t) = R(t)p}$$

changing coordinates of P as the rigid body rotates

Rotation with O fixed



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\boxed{q(t) = R(t)p}$$

$$R^T \dot{q} = \boxed{R^T \dot{R}} p$$

velocity in body-fixed frame

encodes angular velocity in body-fixed frame

$$\dot{q} = \boxed{\dot{R} R^T} q$$

velocity in inertial frame

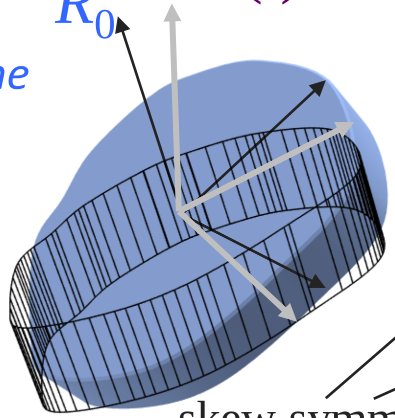
encodes angular velocity in inertial frame

$$\dot{q} = \dot{R} p$$

velocity in inertial frame

position in body-fixed frame

Angular Velocities



$$\hat{\omega}^b = R^T \dot{R}$$ *angular velocity in body-fixed frame*

$$\hat{\omega}^s = \dot{R} R^T$$ *angular velocity in inertial or spatial frame*

angular velocity in body-fixed frame

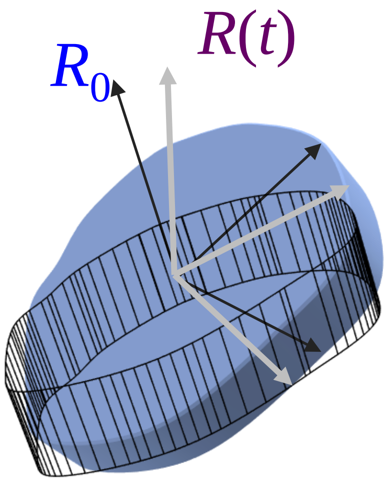
$$\dot{R} = R\hat{\omega}^b \qquad R(t+\Delta t) \sim R(t) + \Delta t R(t)\hat{\omega}^b$$

angular velocity in inertial frame

$$\dot{R} = \hat{\omega}^s R \qquad R(t+\Delta t) \sim R(t) + \Delta t \hat{\omega}^s R(t)$$

If angular velocities are constant …

Angular velocity in body-fixed frame



$$\dot{R} = R\hat{\omega}_0^b$$

$$R(t+\Delta t) = R(t) + \Delta t R(t)\hat{\omega}_0^b$$

$$R(t) = R_0 \exp(\hat{\omega}_0^b t)$$

Angular velocity in inertial frame

$$\dot{R} = \hat{\omega}_0^s R$$

$$R(t+\Delta t) = R(t) + \Delta t \hat{\omega}_0^s R(t)$$

$$R(t) = \exp(\hat{\omega}_0^s t) R_0$$

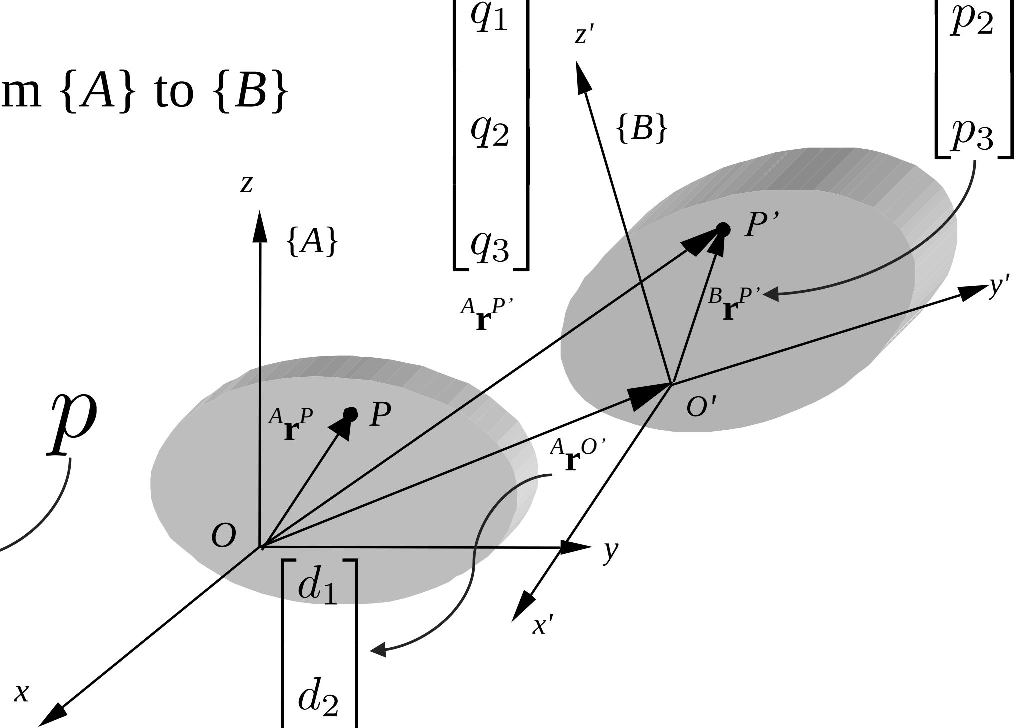
Displacements and Twists

Rigid Body Motion

Displacement of the rigid body from $\{A\}$ to $\{B\}$

$$\begin{bmatrix} {}^{A}\mathbf{r}^{P'}(t) \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B}(t) & {}^{A}\mathbf{r}^{O'}(t) \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \begin{bmatrix} {}^{B}\mathbf{r}^{P} \\ \hline 1 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\underset{q}{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix}} = \underset{A(t)}{\begin{bmatrix} R_{11} & R_{12} & R_{13} & d_1 \\ R_{21} & R_{22} & R_{23} & d_2 \\ R_{31} & R_{32} & R_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{p}{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}}$$

changing coordinates of P as the rigid body moves

homogeneous transformation matrix (function of time)

constant coordinates of P in body-fixed frame

$$\dot{q} = \dot{A} p$$

velocity in inertial frame

position in body-fixed frame

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Differentiation

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & d_1 \\ R_{21} & R_{22} & R_{23} & d_2 \\ R_{31} & R_{32} & R_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

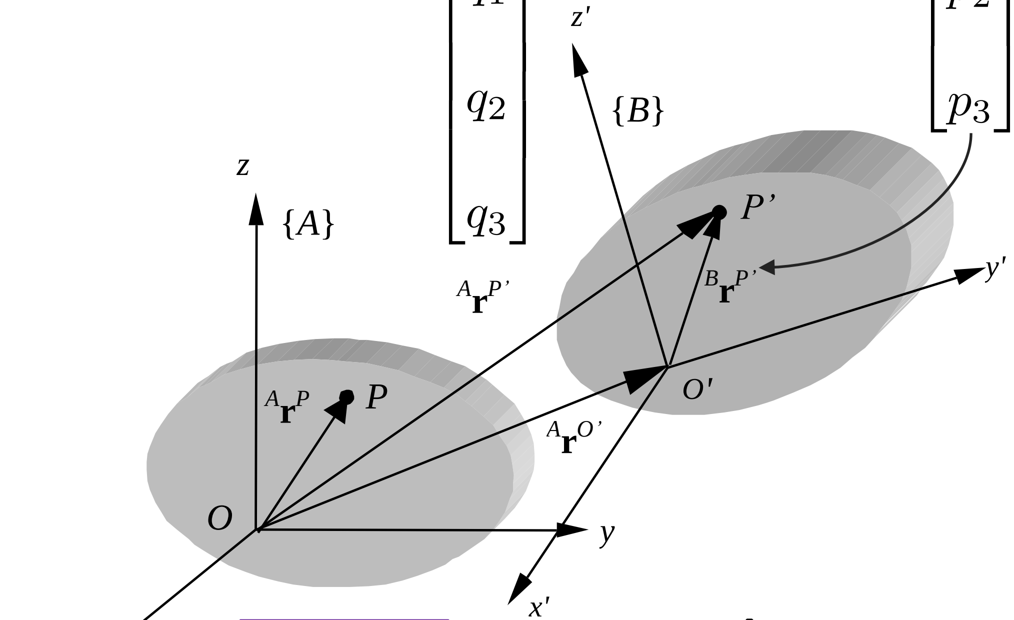
function of time — *function of time* — *constant*

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot R_{11} & \dot R_{12} & \dot R_{13} & \dot d_1 \\ \dot R_{21} & \dot R_{22} & \dot R_{23} & \dot d_2 \\ \dot R_{31} & \dot R_{32} & \dot R_{33} & \dot d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

$$\dot q = \dot A p$$

velocity in inertial frame — *position in body-fixed frame*

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$



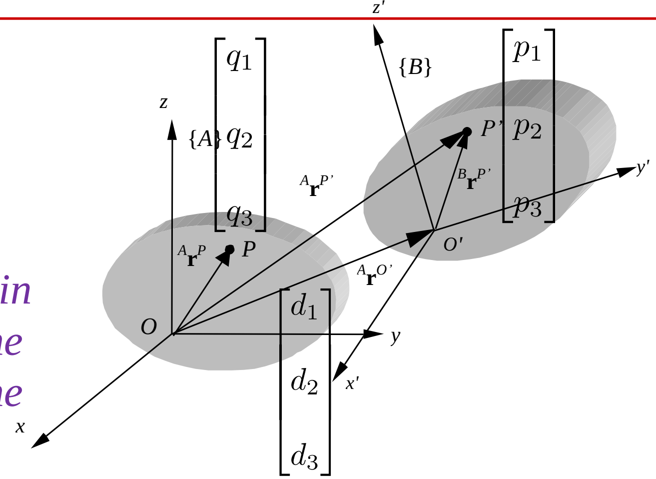
$$\boxed{A^{-1}\dot q} = \boxed{A^{-1}\dot A}\boxed{p}$$

velocity of P in body-fixed frame — *position of P in body-fixed frame*

$$\boxed{\dot q} = \boxed{\dot A A^{-1}}\boxed{q}$$

velocity of P in inertial frame — *position of P in inertial frame*

Twist



$$A^{-1}\dot{A} = \begin{bmatrix} R^T & -R^T d \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{d} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

v^s, velocity of origin of body-fixed frame in body-fixed frame

$$A^{-1}\dot{A} = \begin{bmatrix} \hat{\omega}^b & \boxed{R^T \dot{d}} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

$$\hat{\omega}^b = R^T \dot{R}$$ *angular velocity in body-fixed frame*

$$\widehat{\xi^b} = A^{-1}\dot{A}$$ *twist in body-fixed frame, or body velocity*
used to find the velocity of a point in body coordinates

$$\dot{A}A^{-1} = \begin{bmatrix} \dot{R} & \dot{d} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T d \\ \mathbf{0}^T & 1 \end{bmatrix}$$

v^b, velocity of point at the origin of inertial frame in the inertial frame

$$\dot{A}A^{-1} = \begin{bmatrix} \hat{\omega}^s & \boxed{\dot{d} - \hat{\omega}^s d} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

$$\hat{\omega}^s = \dot{R} R^T$$ *angular velocity in inertial or spatial frame*

$$\widehat{\xi^s} = \dot{A}A^{-1}$$ *twist in inertial frame, or spatial velocity*
used to find the velocity of a point in spatial coordinates

Twist Matrix to Twist Vector

Twist vectors are a more compact way of representing twist matrices

$$\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \omega \\ v \end{bmatrix}$$

The set of all angular velocity matrices

$$so(3) = \left\{\hat{\omega}, \hat{\omega} \in \mathbb{R}^{3\times 3},\ \hat{\omega}^T = -\hat{\omega}\right\}$$

The set of all twists

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}, \quad \hat{\omega} \in so(3),\ v \in \mathbb{R}^3 \right\}$$

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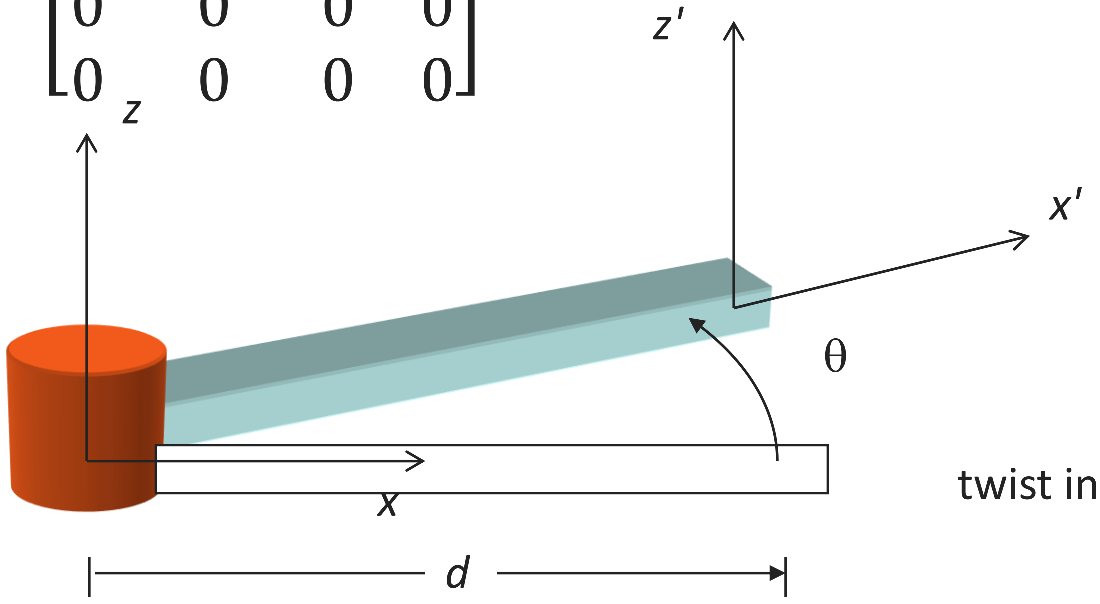
Example: Rotation about a single joint

twist in frame attached to base

$$\widehat{\xi^s} = \begin{bmatrix} 0 & -\dot{\theta} & 0 & 0 \\ \dot{\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & \cos(\theta)\,d \\ \sin(\theta) & \cos(\theta) & 0 & \sin(\theta)\,d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



twist in frame attached to arm

$$\widehat{\xi^b} = \begin{bmatrix} 0 & -\dot{\theta} & 0 & 0 \\ \dot{\theta} & 0 & 0 & \dot{\theta}d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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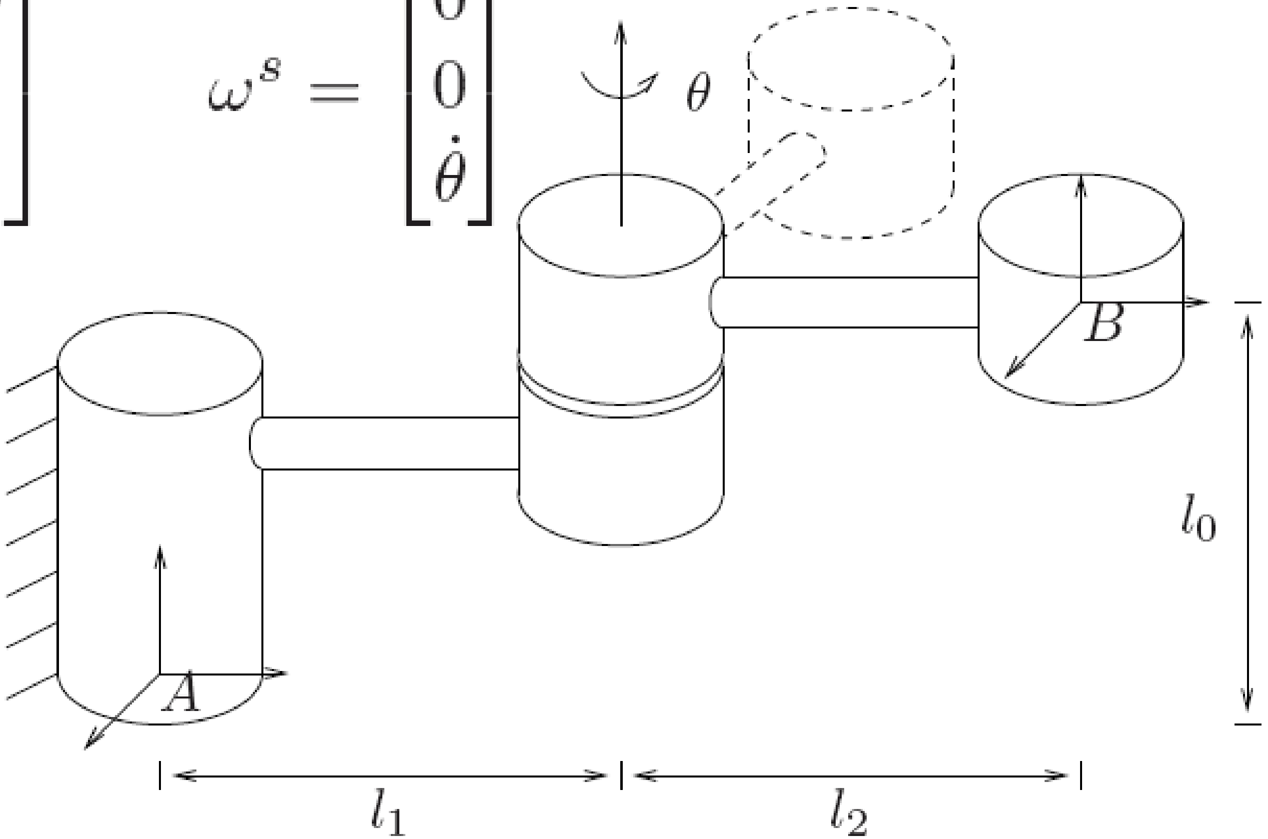
Example

$$v^s = \begin{bmatrix} l_1\dot{\theta} \\ 0 \\ 0 \end{bmatrix} \qquad \omega^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$



$$v^b = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \end{bmatrix} \qquad \omega^b = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

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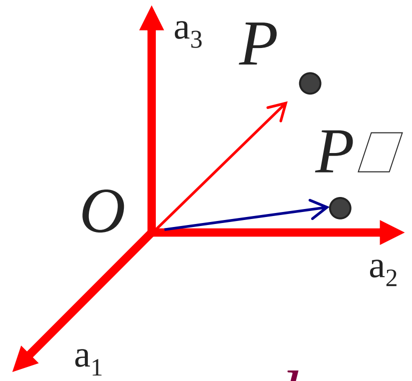
General Derivatives

Derivative of a Vector Attached (Rigidly) to a Moving Frame



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$R^T \dot{q} = \overset{\hat{\omega}^b}{\boxed{R^T \dot{R}}}\, p$$

velocity in body-fixed frame

angular velocity in body-fixed frame

$$\dot{q} = \overset{\hat{\omega}^s}{\boxed{\dot{R} R^T}}\, q$$

velocity in inertial frame

angular velocity in inertial (spatial) frame

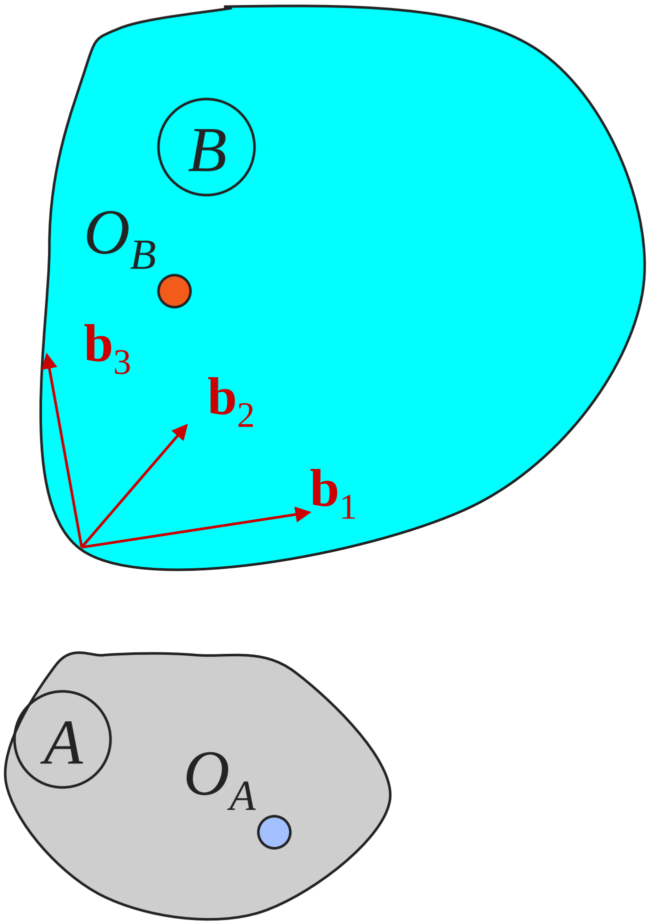
Important: The derivative of a vector attached to a moving frame is equal to the cross product of the angular velocity and the vector.

This result is independent of whether A is an inertial frame or moving!

Replace "body-fixed" and "inertial frames" with frame "B" and frame "A"

$\hat{\omega}^s \rightarrow {}^A\hat{\omega}^B$ in frame A

$\hat{\omega}^b \rightarrow {}^A\hat{\omega}^B$ in frame B



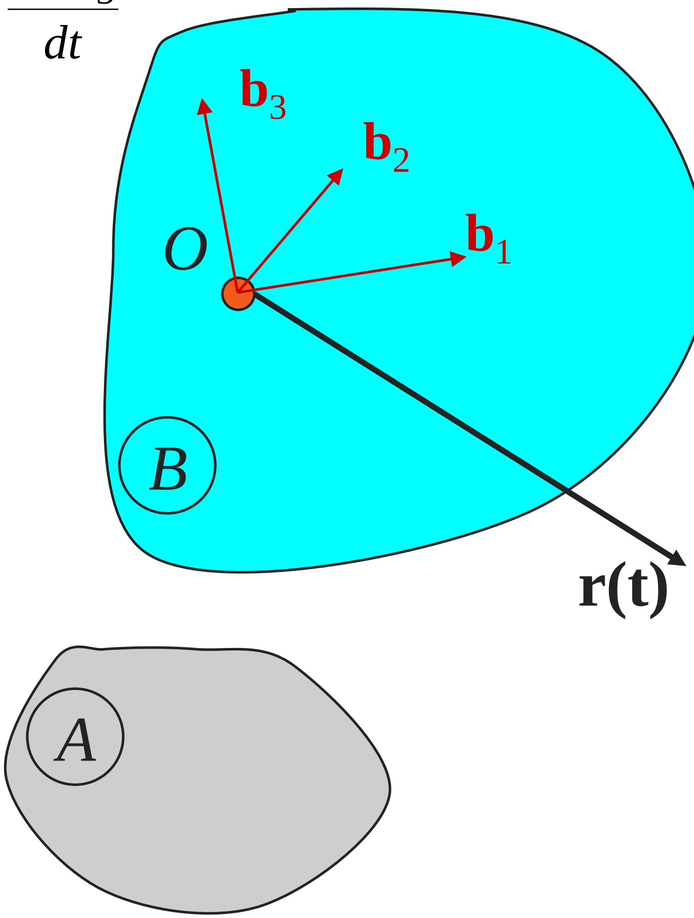
Derivative of a Vector in a Moving Frame

$$\frac{{}^A d\mathbf{r}}{dt} = \frac{dr_1}{dt}\mathbf{b}_1 + \frac{dr_2}{dt}\mathbf{b}_2 + \frac{dr_3}{dt}\mathbf{b}_3 + r_1\frac{{}^A d\mathbf{b}_1}{dt} + r_2\frac{{}^A d\mathbf{b}_2}{dt} + r_3\frac{{}^A d\mathbf{b}_3}{dt}$$

$$= \frac{{}^B d\mathbf{r}}{dt} + r_1\,{}^A\omega^B \times \mathbf{b}_1 + r_2\,{}^A\omega^B \times \mathbf{b}_2 + r_3\,{}^A\omega^B \times \mathbf{b}_3$$

$$= \frac{{}^B d\mathbf{r}}{dt} + {}^A\omega^B \times \mathbf{r}$$

$$\boxed{\frac{{}^A d\mathbf{r}}{dt} = \frac{{}^B d\mathbf{r}}{dt} + {}^A\omega^B \times \mathbf{r}}$$



r can be *any vector*

e.g., velocity of *P* in *B*

e.g., angular momentum

Angular Acceleration

The *angular acceleration of B in A*, denoted by ${}^{A}\boldsymbol{\alpha}^{B}$, is defined as the first time-derivative in A of the angular velocity of B in A:

$$ {}^{A}\hat{\alpha}^{B} = \frac{d\ {}^{A}\hat{\omega}^{B}}{dt} \text{ in frame } A $$

Notice consistency in leading superscripts!