Rotations

MEAM 620, SPRING 2020

Rotations

Rotations

Rotations are rigid body displacements, *i.e.*, must satisfy two important properties

1. A rotation preserves lengths

$$||Rp|| = ||p||$$

2. Cross products are preserved by a rotation

$$Rp \times Rq = R(p \times q)$$

Which implies

$$\boxed{RR^T = R^T R = I, \quad \det R = 1}$$

for all rotation matrices

The group of rotations

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I,\ \det R = 1\}$$

$SO(3)$ satisfies the four axioms that must be satisfied by the elements of an *algebraic group*:

- Closure under the binary operation
- Associativity
- $SO(3)$ includes the identity element
- $SO(3)$ includes the inverse of every element

Example: the integers $\mathbb{Z}$

- addition
- (a+b)+c = a + (b+c)
- a + 0 = a
- (a) + (-a) = 0

$SO(3)$ is a *continuous group*.

the binary operation above is a continuous operation

the inverse of any element is a continuous function of that element.

Manifold

Definition

A manifold of dimension n is a set M which is locally homeomorphic to R^n.

Homeomorphism:

A map f from M to N and its inverse, f^{-1}, are both continuous.

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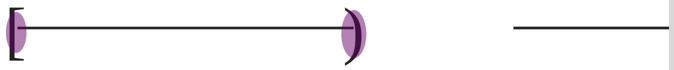
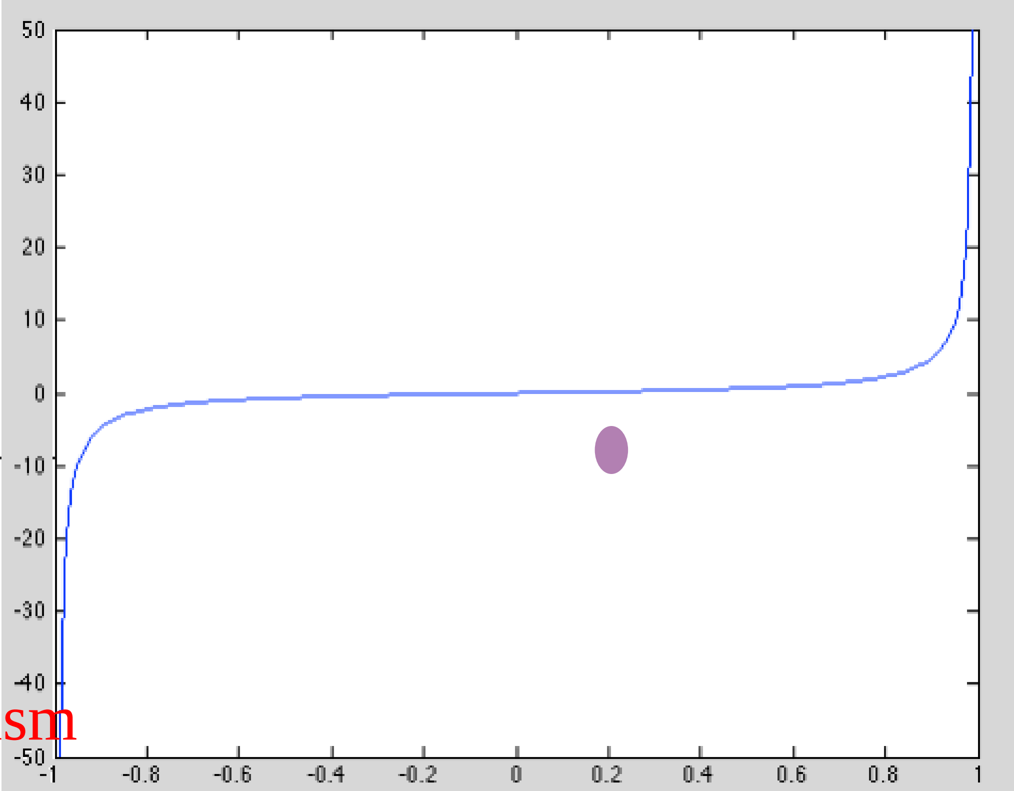
Examples

$$f:(-1,1)\rightarrow R,\quad f(x)=\frac{x}{1-x^{2}}$$

Homeomorphism



0 2π



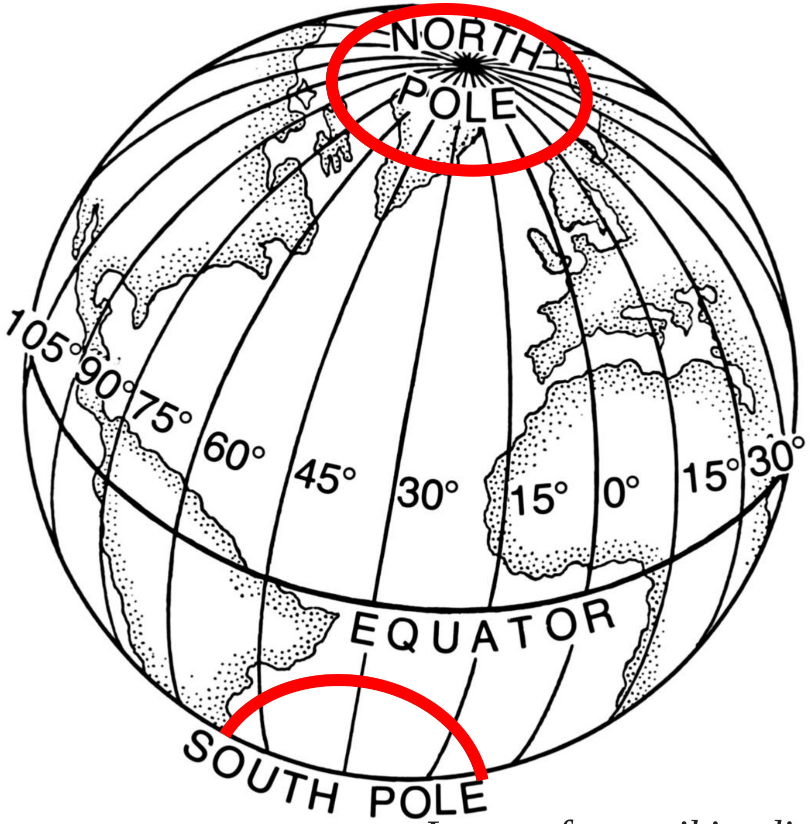
Not a homeomorphism

Sphere in three dimensions (S_2)

- Manifold is locally homeomorphic to R^2

- Parametrize using a set of local coordinate charts (latitude and longitude

- Need a collection of charts covering the surface of the earth
- A collection of charts is an "atlas."



Images from wikipedia

The group of rotations

$SO(3)$ is a *continuous group*.

the binary operation is a continuous operation

the inverse is a continuous function

$SO(3)$ is a *smooth manifold*.

$SO(3)$ is a Lie group

Continuous map from $SO(3)$ to R^n

Map is also differentiable

Coordinates for SO(3)

1 Rotation matrices

2 Euler angles

3 Axis angle parameterization

4 Exponential coordinates

5 Quaternions

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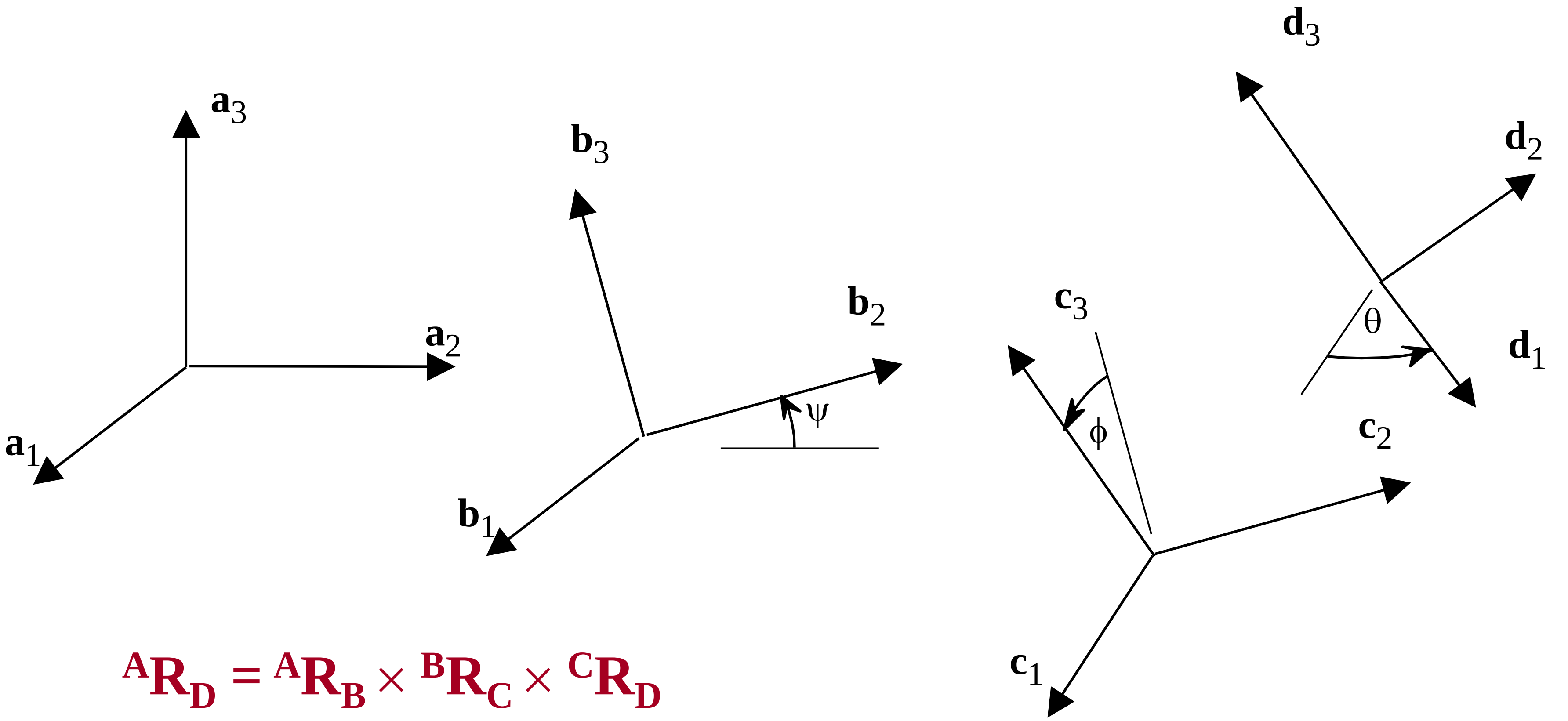
2 Euler Angles

Composition of Three Rotations



$$ {}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D} $$

$$ {}^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta) $$

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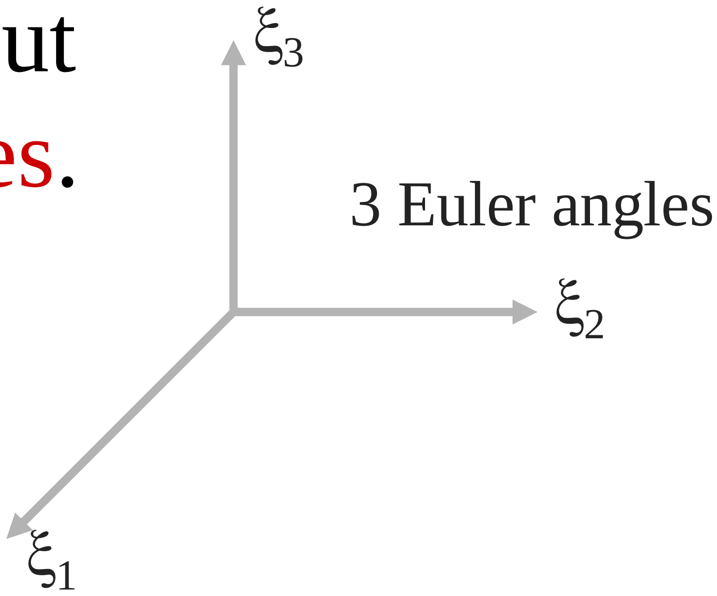
Euler Angles

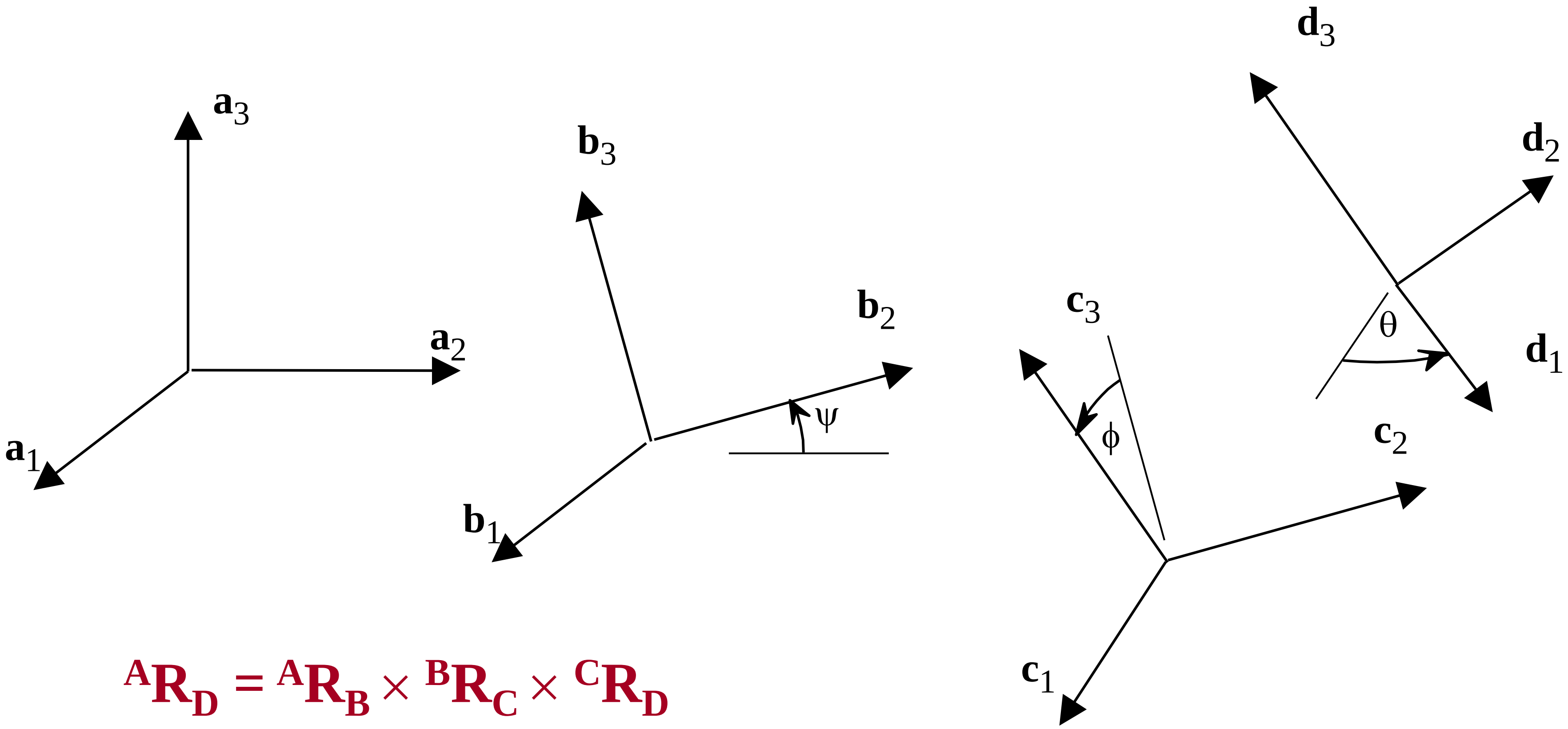
Any rotation can be described by three successive rotations about linearly independent axes.

$$\begin{pmatrix} 3 \times 3 \\ \text{rotation} \\ \text{matrix} \end{pmatrix}$$



Almost 1-1 transformation

X-Y-Z Euler Angles

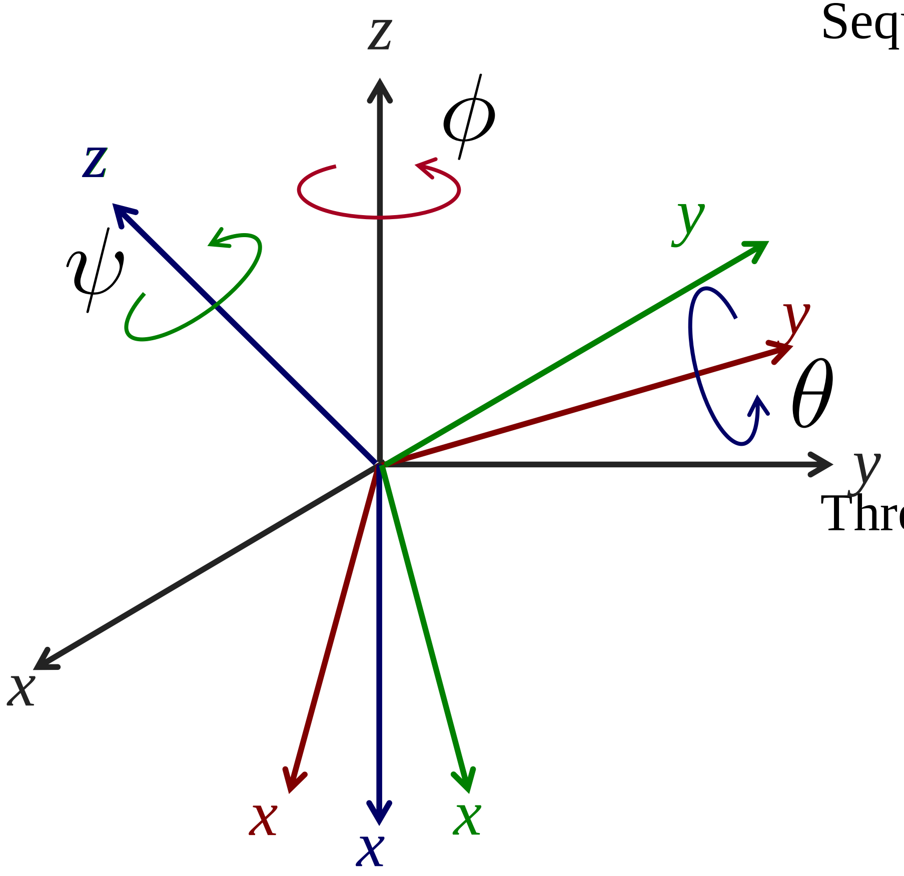


$$^{A}\mathbf{R}_{\mathbf{D}} = {}^{A}\mathbf{R}_{\mathbf{B}} \times {}^{B}\mathbf{R}_{\mathbf{C}} \times {}^{C}\mathbf{R}_{\mathbf{D}}$$

$$^{A}\mathbf{R}_{\mathbf{D}} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta)$$

roll *pitch* *yaw*

Z-Y-Z Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Are these linearly independent?

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- $\theta = 0$ is a special (singular) case

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

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Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \boxed{\cos\phi\sin\theta} \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \boxed{\cos\theta} \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} & \boxed{R_{13}} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & \boxed{R_{33}} \end{bmatrix}$$

$$R_{31} = -\sin\theta\cos\psi$$

$$R_{32} = \sin\theta\sin\psi$$

$$R_{33} = \cos\theta$$

$$R_{13} = \sin\theta\cos\phi$$

$$R_{23} = \sin\theta\sin\phi$$

known rotation matrix

Determination of Euler Angles

If $|R_{33}| < 1$,

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

$$\theta = \sigma \arccos(R_{33}), \quad \sigma = \pm 1$$

$$\psi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

Two sets of Euler angles for every **R** for almost all **R**'s!

If $R_{33} = 1$,

$f(\phi + \psi)$

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi + \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Infinite set of Euler Angles!

If $R_{33} = -1$,

$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$g(\psi - \theta)$

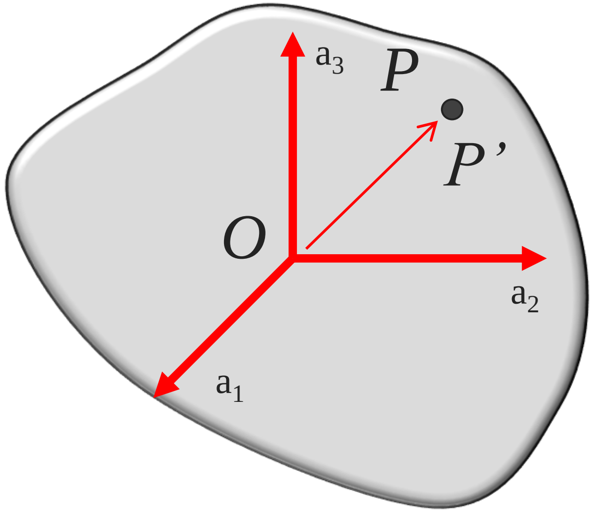
3 Axis/Angle Representation

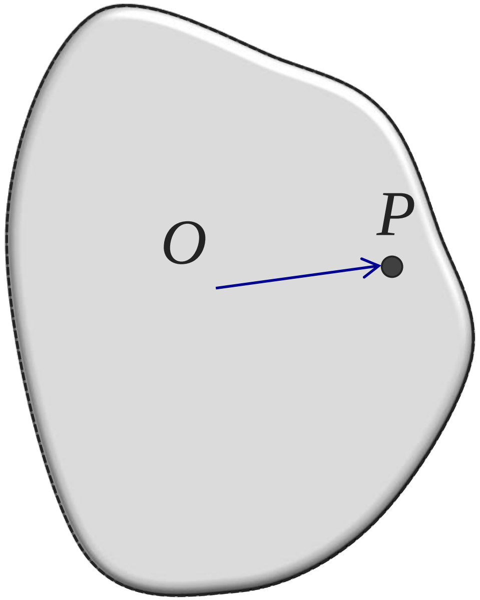
Euler's Theorem

Rotations

Any displacement of a rigid body such that a point on the rigid body, say *O*, remains fixed, is equivalent to a rotation about a fixed axis through the point *O*.

Rotation with O fixed





$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$

$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

coord. of P'   coord. of P

Sketch of Proof of Euler's Theorem

$$q = Rp$$

Is there a point p that maps onto itself?

If there were such a point p …

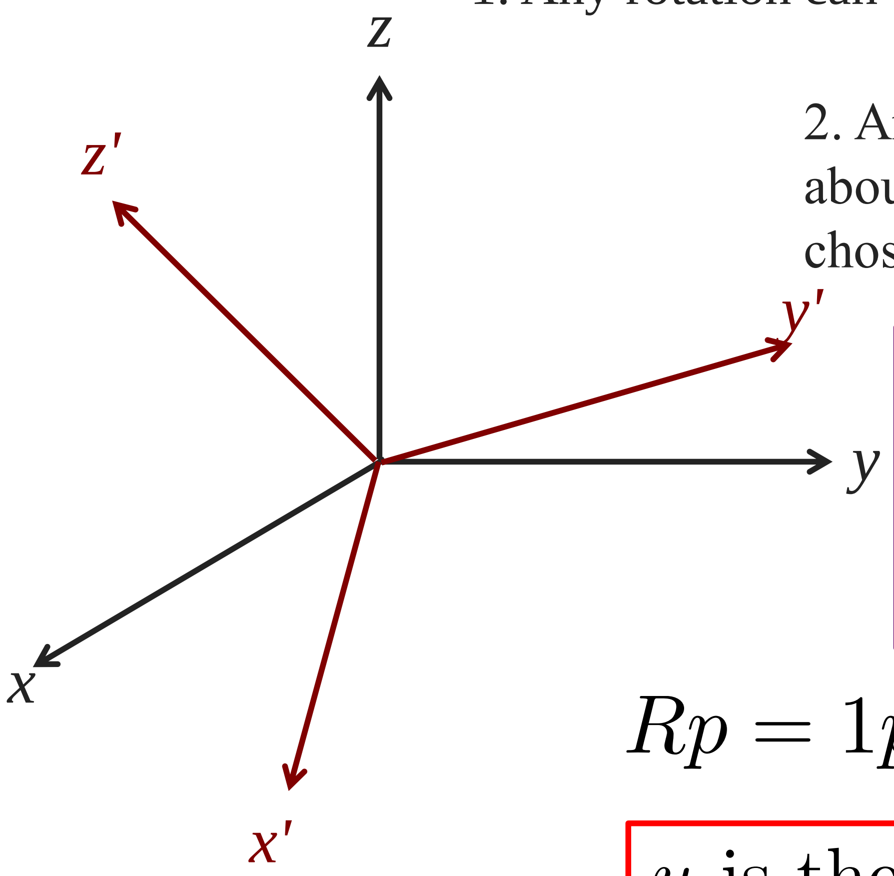
$$p = Rp$$

Eigenvalue problem

$$Rp = \lambda p$$

Rotation Axis and Angle

1. Any rotation can be described by an element of $SO(3)$



2. Any rotation is equivalent to a rotation about a fixed axis through a suitably chosen origin (Euler)

Axis of rotation	u
Rotation angle	ϕ

$$Rp = 1p$$

$$\boxed{u \text{ is the eigenvector for } \lambda = 1}$$

$$\tau = (R_{11} + R_{22} + R_{33})$$

$$\boxed{\cos\phi = \frac{\tau - 1}{2}}$$

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Are the axis and angle always uniquely defined for a rotation?

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How does one find the rotation matrix for a general axis and angle of rotation?

Note we already know the answer if the axis of rotation is one of the coordinate axes.

Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

1. Set u to be a unit vector along x (or y or z). Verify result is the same as $Rot(x, \phi)$.

2. Is the (axis, angle) to rotation matrix map *onto*? 1-1?

Euler's theorem

$Rot(u,\phi)$ and $Rot(-u, 2\pi-\phi)$?

restrict ϕ to the interval $[0,\pi]$?

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Rotation Matrix to Axis/Angle

Rotation of a generic vector *p* about *u* through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Lets extract the axis and the angle from the rotation matrix, *R*

Verify

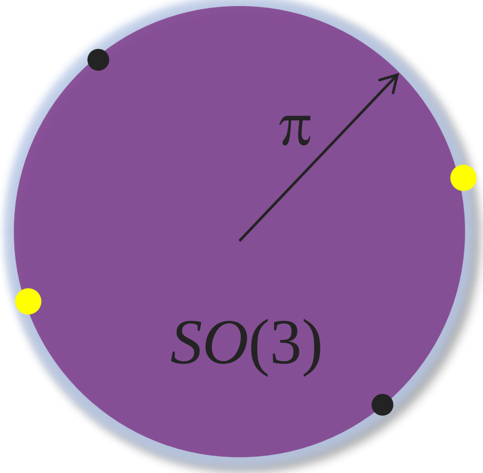
$$\cos\phi = \frac{\tau - 1}{2} \qquad \hat{u} = \frac{1}{2\sin\phi}(R - R^T)$$ (*u*, without solving for eigenvector)

1. (axis, angle) to rotation matrix map is many to 1

2. restricting angle to the interval [0,π] makes it 1-1 except for

$\tau = 3 \Rightarrow \phi = 0 \Rightarrow$ *no unique axis*

$\tau = -1 \Rightarrow \phi = \pi \Rightarrow$ *u* or -*u*



4 Exponential Coordinates

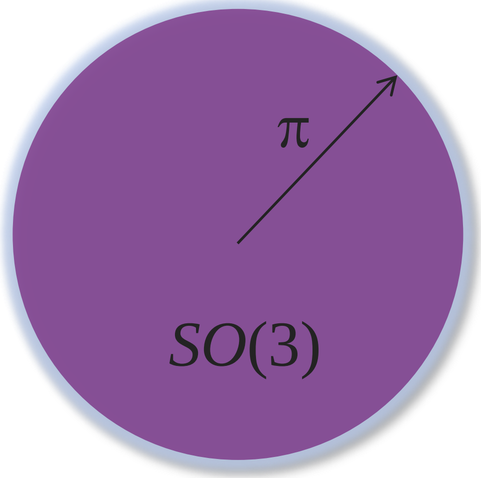
Exponential map onto *SO*(3)

Property 1: Exponentials of 3×3 skew symmetric matrices are rotation matrices

$$\forall a \in \mathbb{R}^3, \ \exp \hat{a} \in SO(3)$$

Property 2: The exponential map is surjective onto *SO*(3).

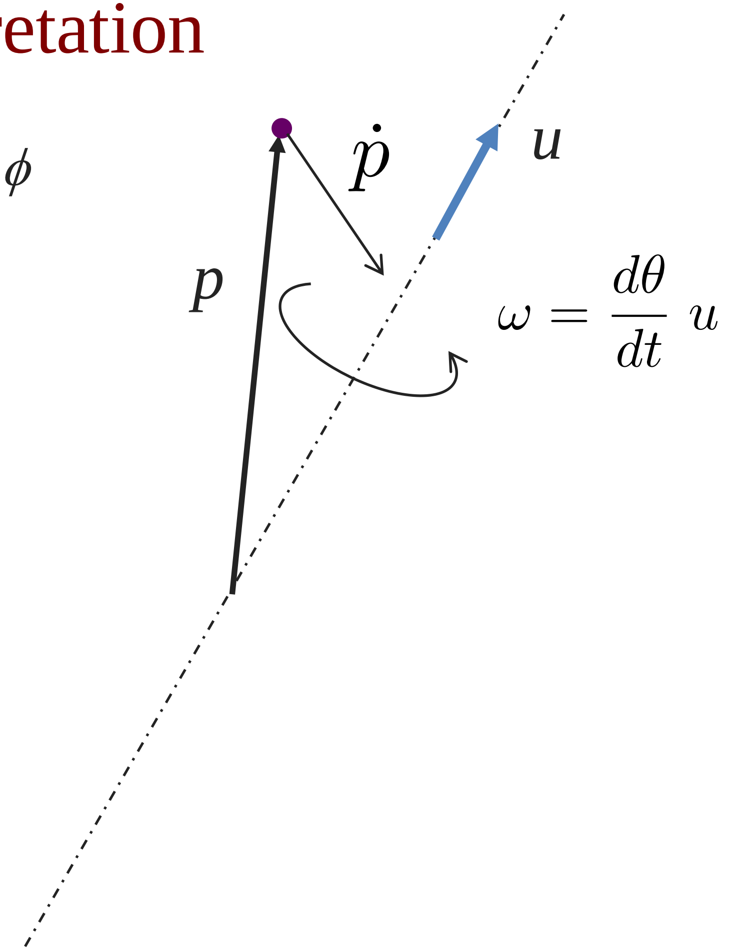
$$\forall R \in SO(3), \exists \boldsymbol{a} \in \mathbb{R}^3 \mid R = \exp \hat{\boldsymbol{a}}$$



Alternative interpretation

Rotation of a generic vector p about u through ϕ

Rotation of a generic vector p about u at a constant angular velocity ω through time Δt



$$\phi = \omega \Delta t$$

$$\dot{p} = \omega \times p$$

$$\dot{p} = \hat{\omega} p$$

$$p(t) = \exp(\hat{\omega}\Delta t) p(t = 0)$$

$$\exp \hat{\omega} \Delta T = I + \hat{\omega} \frac{\sin \omega \Delta T}{|\omega|} + \hat{\omega}^2 \frac{(1 - \cos \omega \Delta T)}{|\omega|^2}$$

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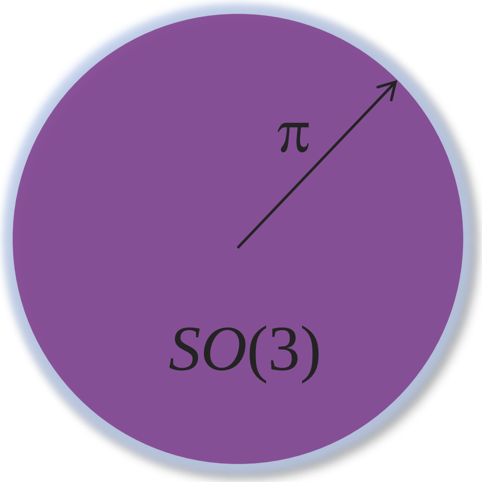
angular velocity vector (×Δt)



Definition: The set of all 3×3 skew symmetric matrices is a *Lie algebra*, denoted by *so*(3).

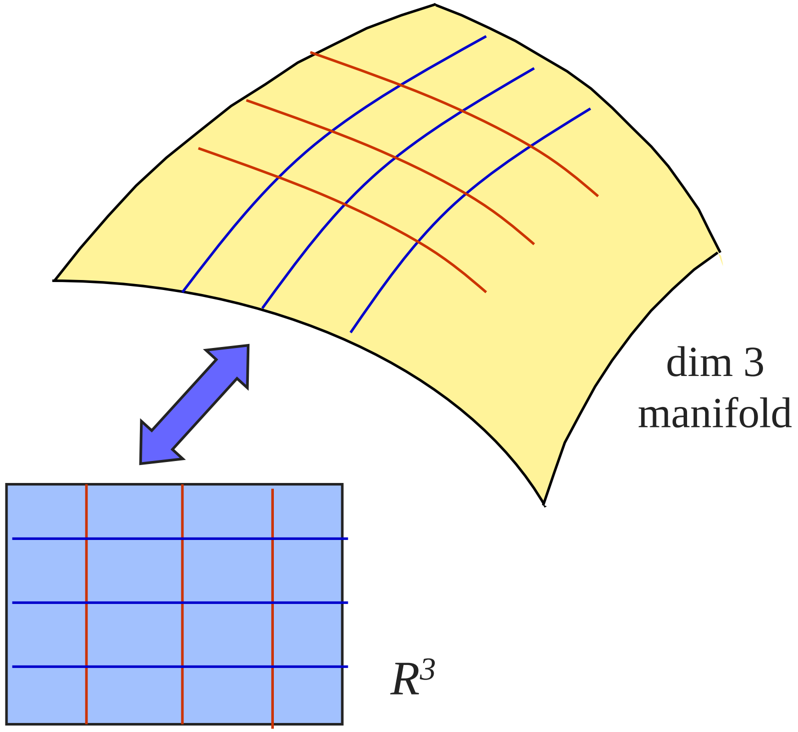
Summary: Coordinates for SO(3)

- Can write every rotation matrix as a function of a unit vector and a scalar

$$R = \exp \hat{u}\theta$$

Exponential coordinates are *canonical coordinates* of the first kind.

$$R = \exp\left(\xi_1 + \xi_2 + \xi_3\right)$$



Product of exponentials define *canonical coordinates* of the second kind.

See Euler angles

$$R = \exp\left(\xi_1'\right)\exp\left(\xi_2'\right)\exp\left(\xi_3'\right)$$

Quaternions

An extension of the complex numbers.

Basis

$$\{1, i,$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k$$

$$jk = i$$

$$ki = j$$

Represent with a (real) scalar part and (imaginary) vector part.

$$\underbrace{\mathbf{q} = s}_{} + \mathbf{v}$$

$s \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$

Products of quaternions can be (tediously) worked out using the rules of products of the basis vectors.

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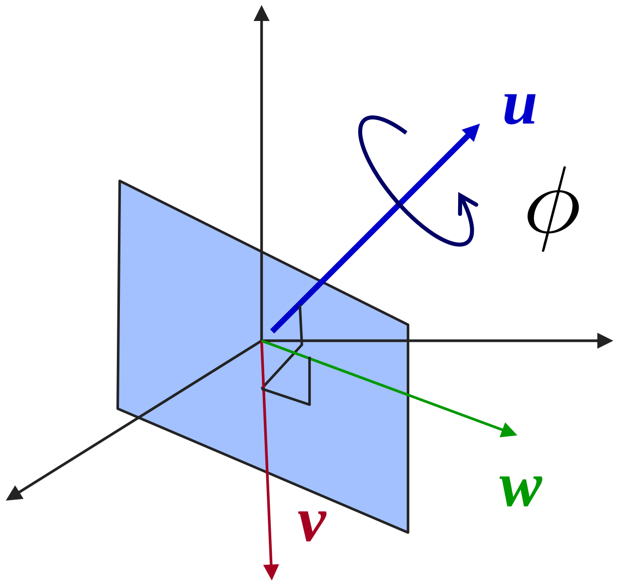
Unit Quaternion

Unit quaternions describe rotations by their axis and angle.

$$\mathbf{v} = \mathbf{u} \sin \frac{\phi}{2} \qquad \mathbf{q} = (s, \mathbf{v})$$

$$s = \cos \frac{\phi}{2}$$



The unit quaternions form a group called Spin(3).

They are a double cover of the group SO(3).

The negative of the quaternion represents the same rotation.

Summary

- Rotation matrices
- Euler angles (exponential coordinates of the second kind)
- Axis/angle parameterization (exponential coordinates)
- Quaternions

No free lunch; do you want:

- Compactness?
- Easy calculations?
- Unique representations?
- Geometric insight?

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