MEAM 620

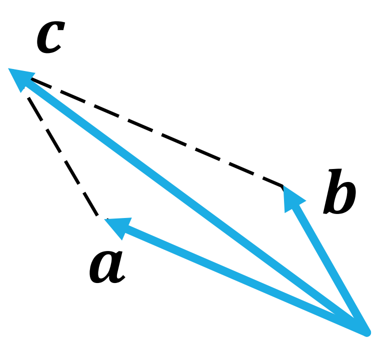
REVIEW: COORDINATE VECTORS AND ROTATIONS

Vectors and Vector Operations

A *free vector* represents a magnitude and direction.

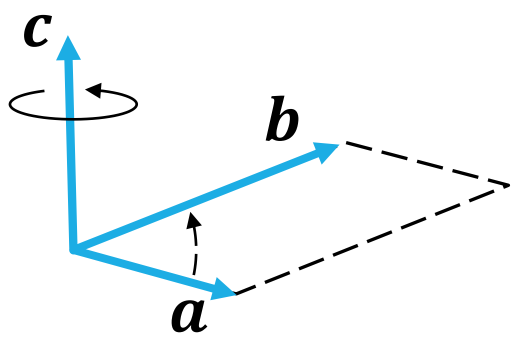
Vector Addition

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$$



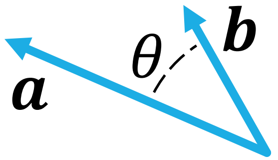
Vector (Cross) Product

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$$



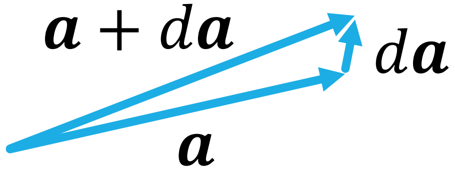
Scalar (Dot) Product

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$



Derivative

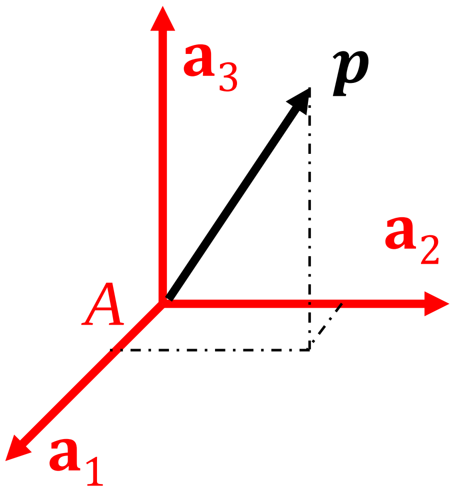
$$\frac{d\boldsymbol{a}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t + \Delta t) - \boldsymbol{a}(t)}{\Delta t}$$



No need refer to any reference frame or coordinate system!

Coordinates of a Vector

We can associate three orthonormal **basis vectors** $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ with a reference frame A.



$$\boldsymbol{p} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3$$

$\alpha_3 = \boldsymbol{p} \cdot \mathbf{a}_3$

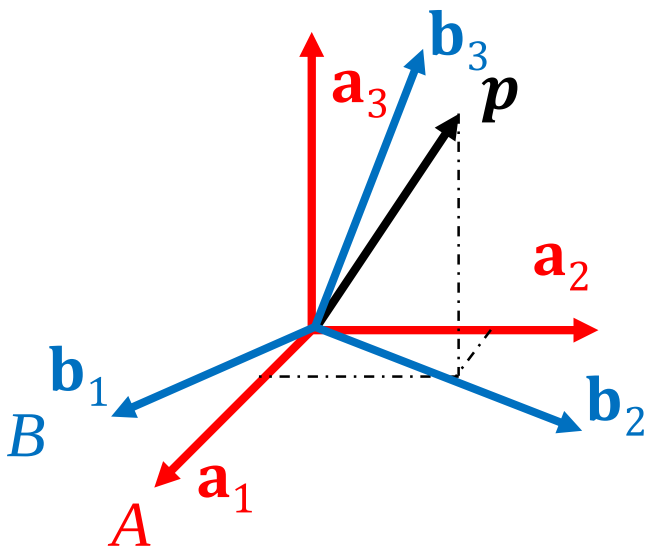
The *coordinates* of a vector depend on the choice of basis (A).

To remove ambiguity, the frame should be specified if writing the *coordinate vector*.

$$^{A}[\boldsymbol{p}] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Relating Components in Different Bases

The same vector can be described with respect to multiple bases.



$$\boldsymbol{p} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3$$

$$\boldsymbol{p} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3$$

here, ${}^{A}[\boldsymbol{p}]$ and ${}^{B}[\boldsymbol{p}]$ represent the same vector

How do the components in A relate to the components in B?

Coordinate Transformation

$$\boldsymbol{p} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3$$

$$\boldsymbol{p} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3$$

$$\boldsymbol{p} \cdot \mathbf{a}_1 \longrightarrow \quad \alpha_1 = \beta_1 \mathbf{a}_1 \cdot \mathbf{b}_1 + \beta_2 \mathbf{a}_1 \cdot \mathbf{b}_2 + \beta_3 \mathbf{a}_1 \cdot \mathbf{b}_3$$

$$\boldsymbol{p} \cdot \mathbf{a}_2 \longrightarrow \quad \alpha_2 = \beta_1 \mathbf{a}_2 \cdot \mathbf{b}_1 + \beta_2 \mathbf{a}_2 \cdot \mathbf{b}_2 + \beta_3 \mathbf{a}_2 \cdot \mathbf{b}_3$$

$$\boldsymbol{p} \cdot \mathbf{a}_3 \longrightarrow \quad \alpha_3 = \beta_1 \mathbf{a}_3 \cdot \mathbf{b}_1 + \beta_2 \mathbf{a}_3 \cdot \mathbf{b}_2 + \beta_3 \mathbf{a}_3 \cdot \mathbf{b}_3$$

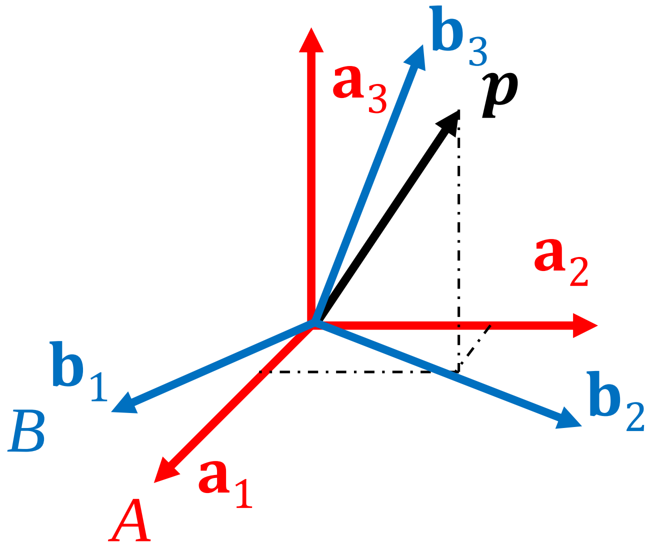
$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The Rotation Matrix

Columns of ${}^{A}\mathbf{R}_{B}$ are the basis vectors of B represented in the coordinates *of* A.

$$ {}^{A}\mathbf{R}_{B} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{bmatrix} $$

$$ {}^{A}\mathbf{R}_{B} = \left[\, {}^{A}[\mathbf{b}_1] \quad {}^{A}[\mathbf{b}_2] \quad {}^{A}[\mathbf{b}_3] \right] $$



1) Possibly the most useful and unimaginative way to represent a robot orientation.

2) In addition, ${}^{A}\mathbf{R}_{B}$ can be used to transform components in B to components in A.

$$ {}^{A}[\boldsymbol{p}] = \left[\, {}^{A}\mathbf{R}_{B} \right] {}^{B}[\boldsymbol{p}] $$

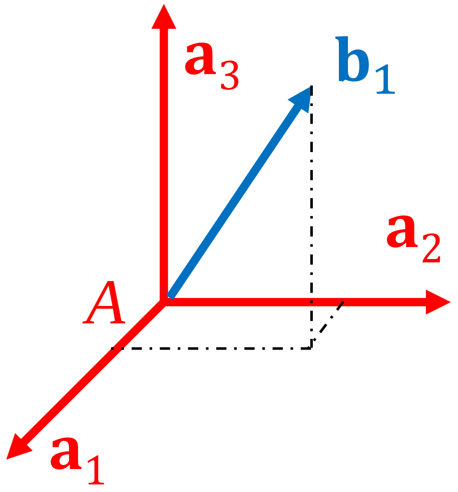
Recall how we got α's from β's.

Sometime called a “Direction Cosine Matrix”

Since $\mathbf{a}_1$ and $\mathbf{b}_1$ are both unit vectors, $(\mathbf{a}_1 \cdot \mathbf{b}_1)$ gives the cosine of the angle between them.

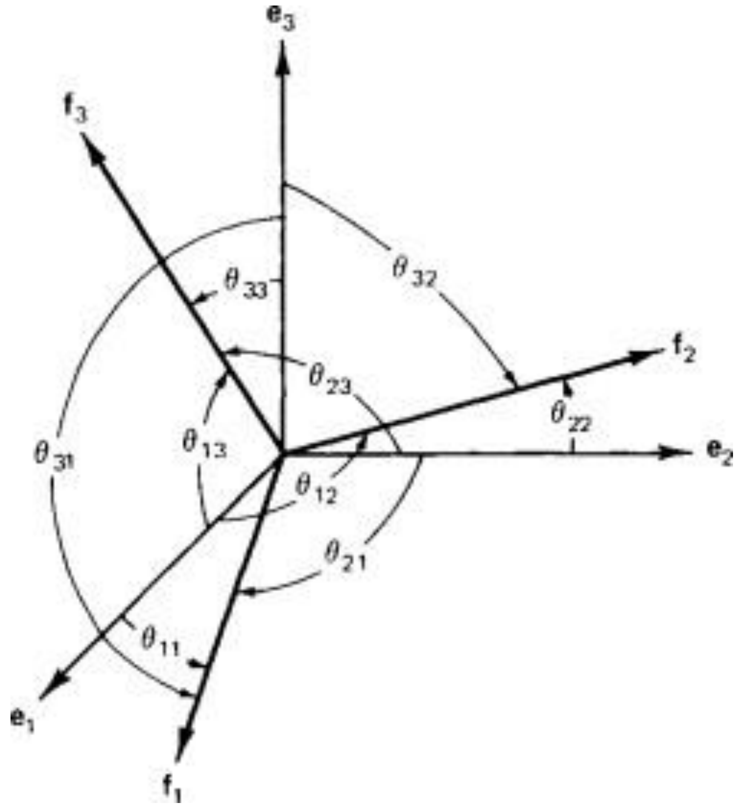
For this reason, the components of $\mathbf{b}_1$ are sometimes called the “direction cosines.”

The rotation matrix is also known as the “direction cosine matrix” or “DCM.”



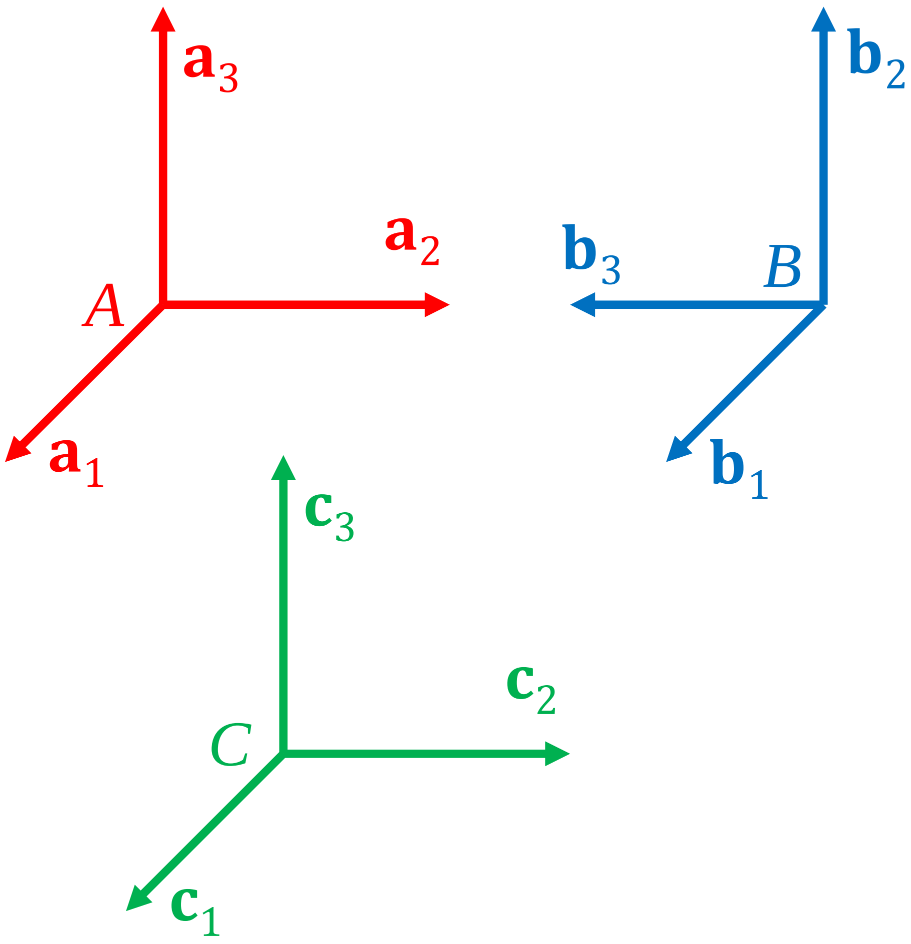
$$
{}^{A}\mathbf{R}_{B} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} \end{bmatrix}
$$



Describing Orientation

The rotation matrix ${}^{A}\mathbf{R}_{B}$ can be used to describe the orientation of coordinate frame B with respect to A.



Recall,

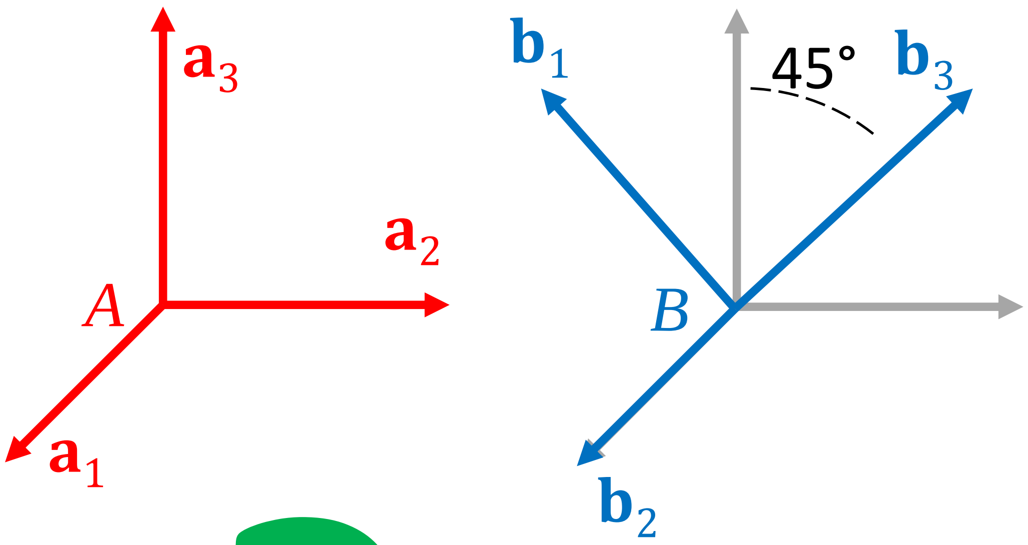
$${}^{A}\mathbf{R}_{B} = \begin{bmatrix} {}^{A}[\mathbf{b}_1] & {}^{A}[\mathbf{b}_2] & {}^{A}[\mathbf{b}_3] \end{bmatrix}$$

$${}^{A}\mathbf{R}_{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$${}^{A}\mathbf{R}_{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Describing Orientation

The rotation matrix ${}^{A}\mathbf{R}_{B}$ can be used to describe the orientation of coordinate frame B with respect to A.



Recall,

$$ {}^{A}\mathbf{R}_{B} = \left[\, {}^{A}[\mathbf{b}_1] \quad {}^{A}[\mathbf{b}_2] \quad {}^{A}[\mathbf{b}_3] \right] $$

$$ {}^{A}\mathbf{R}_{B} = \begin{bmatrix} 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} $$

$$ {}^{A}\mathbf{R}_{C} = \begin{bmatrix} 0.5609 & 0.3332 & -0.7578 \\ 0.0962 & 0.8830 & 0.4594 \\ 0.8222 & -0.3306 & 0.4633 \end{bmatrix} $$

Is my robot upside down? no.