The Motion Field Equations

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1 Definition

Here I derive the motion field equations of the optical flow $\dot{\mathbf{p}} = (u, v, 0)^T$ at a point $\mathbf{p} = (x, y, 1)^T$, they can be written as a function of the linear velocity $\mathbf{V}$, the angular velocity $\mathbf{\Omega}$, and the depth at that point Z. They can be written as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{Z} \begin{pmatrix} -1 & 0 & x \\ 0 & -1 & y \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} + \begin{pmatrix} xy & -(1+x^2) & y \\ 1+y^2 & -xy & -x \end{pmatrix} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} \tag{1}$$

Written in matrix form:

$$\dot{\mathbf{p}} = \frac{1}{Z} A(\mathbf{p})\mathbf{V} + B(\mathbf{p})\mathbf{\Omega} \tag{2}$$

It is assumed in these equations that the point p is calibrated (i.e. the focal length is 1 and center is (0,0)). If it is not, you will need to subtract off the center and divide out the focal length.

2 Derivation

We use a purely algabraic way to derive this. We have our point $\mathbf{P} = (X, Y, Z)$ in the camera frame, and its projection to the image $\mathbf{p} = \mathbf{P}/Z$. Taking the derivative of this we get:

$$\dot{\mathbf{p}} = \frac{1}{Z}\dot{\mathbf{P}} + \frac{\dot{Z}}{Z^2}\mathbf{P} = \frac{1}{Z}\dot{\mathbf{P}} + \frac{\dot{Z}}{Z}\mathbf{p}$$

Now we will need an additional equation to describe the motion of $\mathbf{P}$ in a nonrotating frame, specifically $\dot{\mathbf{P}} = -\mathbf{V} - \mathbf{\Omega} \times \mathbf{P}$, which describes the velocity of a point in a rotating reference frame, like the camera. To get $\dot{Z}$ in a managble form we write $\dot{Z} = \mathbf{e}_3^T \dot{\mathbf{P}}$ with $\mathbf{e}_3 = (0,0,1)^T$

$$
\begin{aligned}
\frac{1}{Z}\dot{\mathbf{P}} - \frac{\dot{Z}}{Z^2}\mathbf{P} &= \frac{1}{Z}\left(\dot{\mathbf{P}} + (\mathbf{e}_3^T \dot{\mathbf{P}})\mathbf{p}\right) \\
&= \frac{1}{Z}\left(I - \mathbf{p}\mathbf{e}_3^T\right)\dot{\mathbf{P}} \\
&= \frac{1}{Z}\left(I - \mathbf{p}\mathbf{e}_3^T\right)(-\mathbf{V} - \boldsymbol{\Omega} \times \mathbf{P}) \\
&= \frac{1}{Z}\left(-I + \mathbf{p}\mathbf{e}_3^T\right)\mathbf{V} + \left(I - \mathbf{p}\mathbf{e}_3^T\right)\left(-\boldsymbol{\Omega} \times \frac{\mathbf{P}}{Z}\right) \\
&= \frac{1}{Z}\left(-I + \mathbf{p}\mathbf{e}_3^T\right)\mathbf{V} + \left(I - \mathbf{p}\mathbf{e}_3^T\right)[\mathbf{p}]_\times \boldsymbol{\Omega}
\end{aligned}
$$

Denoting

$$
\begin{aligned}
A(\mathbf{p}) &= -I + \mathbf{p}\mathbf{e}_3^T && (3) \\
B(\mathbf{p}) &= \left(I - \mathbf{p}\mathbf{e}_3^T\right)[\mathbf{p}]_\times && (4)
\end{aligned}
$$

We get our origal equation Equation 2 in matrix form. Now for completeness we find the coordinate representations of these matrices using $\mathbf{p} = (x, y, 1)$

$$
\begin{aligned}
A(\mathbf{p}) &= -I + \mathbf{p}\mathbf{e}_3^T \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & x \\ 0 & -1 & y \\ 0 & 0 & 0 \end{pmatrix} \\
B(\mathbf{p}) &= \left(I - \mathbf{p}\mathbf{e}_3^T\right)[\mathbf{p}]_\times \\
&= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & -1 & y \\ 1 & 0 & -x \\ -y & x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & y \\ 1 & 0 & -x \\ -y & x & 0 \end{pmatrix} \\
&= \begin{pmatrix} xy & -(1+x^2) & y \\ 1+y^2 & -xy & -x \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

This agrees with Equation 1 except we get rid of the third row since it is redundant.

3 Application

We can manipulate this equation in several ways to get either $\mathbf{V}$ or $\mathbf{\Omega}$. We assume we are given a set of points with their corresponding optical flow measurements, $\{\mathbf{p}_i, \dot{\mathbf{p}}_i\}_{i=1}^n$, which we will use Equation 1 to solve in a least squares fasion for $\mathbf{V}$ and $\mathbf{\Omega}$. We will discuss several special cases. The general case will not be reviewed here but you can refer to the slides or the 1992 paper "Subspace Methods for Recovering Rigid Motion I: Algorithm and Implementation" by Heeger and Jepson.

3.1 Rotation only

In this case, where $\mathbf{V} = \mathbf{0}$, Equation 1 reduces to:

$$\dot{\mathbf{p}} = B(\mathbf{p})\mathbf{\Omega}$$

We can solve this in a straightforward fashion, by solving a least squares problem:

$$\Omega^* = \arg\min_{\Omega} \sum_{i=1}^{n} \|B(\mathbf{p}_i)\mathbf{\Omega} - \dot{\mathbf{p}}_i\|^2$$

3.2 Translation only

In the case where we either know the translation and can subtract it off or only have translational motion ($\mathbf{\Omega} = \mathbf{0}$), we can estimate the velocity $\mathbf{V}$. However, due to the depth variables Z_i (which vary with each point), we cannot quite use a straightforward least squares approach. Here we will present a slightly different method than that shown in the slides for how to solve it. Recalling Equation 2 and Equation 3, with pure translational motion we get:

$$\dot{\mathbf{p}} = \frac{1}{Z_i} A(\mathbf{p})\mathbf{V}$$

To make this a homogenous equation (thus getting rid of Z_i), we can take the cross product of $\dot{\mathbf{p}}$ on both sides (using $[\dot{\mathbf{p}}]_\times$ to denote the cross product matrix):

$$\mathbf{0} = [\dot{\mathbf{p}}]_\times A(\mathbf{p})\mathbf{V}$$

As $\mathbf{V} = \mathbf{0}$ satisfies this, eq require $\|\mathbf{V}\| = 1$. Using this we can write a optimization problem and simpify:

$$
\begin{aligned}
&\underset{\mathbf{V}:\|\mathbf{V}\|=1}{\arg\min} \sum_{i=1}^{n} \|[\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)\mathbf{V}\|^2 \\
= &\underset{\mathbf{V}:\|\mathbf{V}\|=1}{\arg\min} \sum_{i=1}^{n} \left([\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)\mathbf{V}\right)^T \left([\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)\mathbf{V}\right) \\
= &\underset{\mathbf{V}:\|\mathbf{V}\|=1}{\arg\min} \sum_{i=1}^{n} \mathbf{V}^T A(\mathbf{p}_i)^T [\dot{\mathbf{p}}_i]_\times^T [\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)\mathbf{V} \\
= &\underset{\mathbf{V}:\|\mathbf{V}\|=1}{\arg\min} \mathbf{V}^T \left(\sum_{i=1}^{n} A(\mathbf{p}_i)^T [\dot{\mathbf{p}}_i]_\times^T [\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i) \right) \mathbf{V}
\end{aligned}
$$

This is just finding the minimum eigenvector of $\sum_{i=1}^{n} A(\mathbf{p}_i)^T [\dot{\mathbf{p}}_i]_\times^T [\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)$, which we know how to do. (Note this is equivalent to finding the null space of the matrix made by stacking the $[\dot{\mathbf{p}}_i]_\times A(\mathbf{p}_i)$ matrices all together)

3.3 Known Depth

In the case where $\mathbf{V}$ and $\mathbf{\Omega}$ are both unknown but we have depth estimates (from a Lidar or some other means), we can also solve this as a least squares problem.

$$
\dot{\mathbf{p}} = \frac{1}{Z} A(\mathbf{p})\mathbf{V} + B(\mathbf{p})\mathbf{\Omega} = \begin{pmatrix} \frac{1}{Z} A(\mathbf{p}) & B(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{\Omega} \end{pmatrix}
$$

Thus this becomes once again a straightforward least squares problem:

$$
\mathbf{V}^*, \mathbf{\Omega}^* = \underset{\mathbf{V}, \mathbf{\Omega}}{\arg\min} \sum_{i=1}^{n} \left\| \begin{pmatrix} \frac{1}{Z_i} A(\mathbf{p}_i) & B(\mathbf{p}_i) \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{\Omega} \end{pmatrix} - \dot{\mathbf{p}}_i \right\|^2
$$