Two polynomial-time estimation algorithms

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Assumption 1. [β-monotone degree] We call a smooth tensor $\Theta \in \mathcal{P}(L)$ is degree-identifiable, if there exists a constant $\beta \in [0,1]$ and small tolerance $\epsilon_d \lesssim d^{-(m-1)/2}$ such that

$$\deg(i) - \deg(j) \gtrsim \left(\frac{i-j}{d}\right)^{1/\beta} - \epsilon_d, \quad \text{for all } i \geq j \in [d].$$

Assumption 2. [β-detectable functions] A function f is called weakly β-detectable, if the function f has at least a local $(1/\beta)$-polynomial fluctuation in each coordinate,

$$\max_{y \in [0,1]} |f(y, \boldsymbol{x}_{-1}) - f(y + d^{-1}, \boldsymbol{x}_{-1})| \geq d^{-1/\beta} \quad \text{for all } \boldsymbol{x}_{-1} \in [0,1]^{m-1}.$$

Which assumptions are more relaxed?: The answer depends on the property of f. We can easily check this by the following two examples.

1. Assumption 1 implies Assumption 2 if f is a constant on all coordinates except the first one. That is, for given y, there exists $c \in [0,1]$ such that

$$f(y, \boldsymbol{x}_{-1}) = c \text{ for all } \boldsymbol{x}_{-1} \in [0,1]^{m-1}. \tag{1}$$

Proof. For $i > j \in [d]$,

$$\begin{aligned}
\deg(i) - \deg(j) &= \int_{\boldsymbol{x}_{-1}} f\left(\frac{i}{d}, \boldsymbol{x}_{-1}\right) d\boldsymbol{x}_{-1} - \int_{\boldsymbol{x}_{-1}} f\left(\frac{j}{d}, \boldsymbol{x}_{-1}\right) d\boldsymbol{x}_{-1} \\
&\leq \int_{\boldsymbol{x}_{-1}} \left| f\left(\frac{i}{d}, \boldsymbol{x}_{-1}\right) - f\left(\frac{j}{d}, \boldsymbol{x}_{-1}\right) \right| d\boldsymbol{x}_{-1} \\
&= \int_{\boldsymbol{x}_{-1}} \left| f\left(y + \frac{i-j}{d}, \boldsymbol{x}_{-1}\right) - f(y, \boldsymbol{x}_{-1}) \right| d\boldsymbol{x}_{-1} \quad (\text{define } y = j/d) \\
&= \left| f\left(y + \frac{i-j}{d}, \boldsymbol{x}_{-1}\right) - f(y, \boldsymbol{x}_{-1}) \right| \quad \text{for all } \boldsymbol{x}_{-1} \in [0,1]^{m-1} \\
&\leq \max_{y \in [0,1]} |f(y, \boldsymbol{x}_{-1}) - f(y + (i-j)d^{-1}, \boldsymbol{x}_{-1})| \quad \text{for all } \boldsymbol{x}_{-1} \in [0,1]^{m-1},
\end{aligned}$$

where the fourth line comes from the condition (1). Therefore, $\deg(i) - \deg(j) \leq \max_{y \in [0,1]} |f(y, \boldsymbol{x}_{-1}) - f(y + (i-j)d^{-1}, \boldsymbol{x}_{-1})|$, which completes the proof. □

2. Assumption 2 implies Assumption 1 if $f(y, \cdot)$ is non-decreasing function.

Proof. For $i > j \in [d]$,

$$\begin{aligned}
\deg(i) - \deg(j) &= \int_{\boldsymbol{x}_{-1}} f\left(\frac{i}{d}, \boldsymbol{x}_{-1}\right) d\boldsymbol{x}_{-1} - \int_{\boldsymbol{x}_{-1}} f\left(\frac{j}{d}, \boldsymbol{x}_{-1}\right) d\boldsymbol{x}_{-1} \\
&= \int_{\boldsymbol{x}_{-1}} \left| f\left(\frac{i}{d}, \boldsymbol{x}_{-1}\right) - f\left(\frac{j}{d}, \boldsymbol{x}_{-1}\right) \right| d\boldsymbol{x}_{-1} \\
&= \int_{\boldsymbol{x}_{-1}} \left| f\left(y + \frac{i-j}{d}, \boldsymbol{x}_{-1}\right) - f(y, \boldsymbol{x}_{-1}) \right| d\boldsymbol{x}_{-1} \quad (\text{define } y = j/d)
\end{aligned}$$

$$\geq \min_{\boldsymbol{x}_{-1} \in [0,1]^{m-1}} \left| f\left(y + \frac{i-j}{d}, \boldsymbol{x}_{-1}\right) - f\left(y, \boldsymbol{x}_{-1}\right) \right|,$$

where the second equality is from the non-decreasing condition. This inequality completes the proof. □