Probability bound of Lemma 2 and 3

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Lemma 2 (Estimation error due to permutation (Expectation version))**.**

$$\mathbb{E}\|\text{Block}_k(\mathcal{Y}\circ\hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y}\circ\sigma^{-1})\|_F^2 \leq d^m \text{Loss}^2(\sigma,\hat{\sigma}).$$

Lemma 2* (Estimation error due to permutation (Probability version))**.** With probability $1 - k^m\exp(-h^m\epsilon^2)$,

$$\|\text{Block}_k(\mathcal{Y}\circ\hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y}\circ\sigma^{-1})\|_F^2 \leq d^m(\epsilon^2 + \text{Loss}^2(\sigma,\hat{\sigma})).$$

Proof. Define $\mathcal{A} := \mathcal{Y}\circ\sigma^{-1}$ and $\hat{\mathcal{A}} := \mathcal{Y}\circ\hat{\sigma}^{-1}$. Notice that

$$\|\text{Block}_k(\mathcal{Y}\circ\hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y}\circ\sigma^{-1})\|_F^2 = \\ h^m \sum_{\substack{k_i\in\{0,\ldots,k-1\}\\ i=1,\ldots,m}} \underbrace{\left(\frac{1}{h^m}\sum_{\substack{h_j\in\{0,\ldots,h-1\}\\ j=1,\ldots,m}} \hat{\mathcal{A}}_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)} - \mathcal{A}_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)}\right)^2}_{(*)}, \tag{1}$$

where $\omega(k_1,\ldots,k_m,h_1,\ldots h_m) = (k_1h+h_1,\ldots,k_mh+h_m)$.

Notice that $(*)$ in (1) is divided into three terms by

$$\hat{\mathcal{A}}_\omega - \mathcal{A}_\omega = \underbrace{\hat{\mathcal{A}}_\omega - [\Theta\circ\sigma\circ\hat{\sigma}^{-1}]_\omega}_{(a)} + \underbrace{\mathcal{A}_\omega - \Theta_\omega}_{(b)} + \underbrace{[\Theta\circ\sigma\circ\hat{\sigma}^{-1}]_\omega - \Theta_\omega}_{(c)}$$

Therefore, we bound (a),(b), and (c) in $(*)$ respectively. Since $\mathbb{E}(\hat{\mathcal{A}}_\omega) = [\Theta\circ\sigma\circ\hat{\sigma}^{-1}]_\omega$ and $\mathbb{E}(\mathcal{A}_\omega) = \Theta_\omega$, Hoeffding's inequality, we bound (a) and (b) by

$$\mathbb{P}\left(\frac{1}{h^m}\left|\sum_{\substack{h_j\in\{0,\ldots,h-1\}\\ j=1,\ldots,m}} \hat{\mathcal{A}}_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)} - [\Theta\circ\sigma\circ\hat{\sigma}^{-1}]_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)}\right| \geq \epsilon\right) \leq 2\exp\left(-2h^m\epsilon^2\right),$$

$$\mathbb{P}\left(\frac{1}{h^m}\left|\sum_{\substack{h_j\in\{0,\ldots,h-1\}\\ j=1,\ldots,m}} \mathcal{A}_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)} - \Theta_{\omega(k_1,\ldots,k_m,h_1,\ldots h_m)}\right| \geq \epsilon\right) \leq 2\exp\left(-2h^m\epsilon^2\right), \tag{2}$$

for any $\epsilon > 0$. For (c) term, notice that for any $\omega\in[d]^m$,

$$\begin{aligned} |[\Theta\circ\sigma\circ\hat{\sigma}^{-1}]_\omega - \Theta_\omega| &= |[\Theta\circ\sigma]_{\omega'} - [\Theta\circ\hat{\sigma}]_{\omega'})|, \quad \text{for some } \omega'\in[d]^m \\ &\leq \frac{L|\sigma(\omega') - \hat{\sigma}(\omega'|_1}{d} \\ &\lesssim \text{Loss}(\sigma,\hat{\sigma}). \end{aligned} \tag{3}$$

Combining (2), (3) and triangular inequality, for probability at least $1-\exp(-h^m\epsilon^2)$,

$$(*) \lesssim \epsilon^2 + \text{Loss}^2(\sigma, \hat{\sigma}).$$

Going back to (1), we show that

$$h^m \sum_{\substack{k_i\in\{0,\dots,k-1\}\\ i=1,\dots,m}} \left(\frac{1}{h^m} \sum_{\substack{h_j\in\{0,\dots,h-1\}\\ j=1,\dots,m}} \hat{\mathcal{A}}_{\omega(k_1,\dots,k_m,h_1,\dots h_m)} - \mathcal{A}_{\omega(k_1,\dots,k_m,h_1,\dots h_m)} \right)^2 \lesssim d^m(\epsilon^2 + \text{Loss}^2(\sigma,\hat{\sigma})),$$

with probability at least, $1-k^m\exp(-h^m\epsilon^2)$. □

Lemma 3 (Denoising error (Expectation version))**.**

$$\mathbb{E}\|\text{Block}_k(\mathcal{Y}\circ\sigma^{-1}) - \text{Block}_k(\Theta)\|_F^2 \le k^m.$$

Lemma 3* (Denoising error (Probability version))**.** With probability $1-k^m\exp(-h^m\epsilon^2)$,

$$\|\text{Block}_k(\mathcal{Y}\circ\sigma^{-1}) - \text{Block}_k(\Theta)\|_F^2 \le d^m\epsilon^2.$$

Proof. Remember that we define $\mathcal{A} = \mathcal{Y}\circ\sigma^{-1}$. By definition, we have

$$\begin{aligned}&\|\text{Block}_k(\mathcal{Y}\circ\sigma^{-1}) - \text{Block}_k(\Theta\|_F^2 = \\ &\qquad h^m \sum_{\substack{k_i\in\{0,\dots,k-1\}\\ i=1,\dots,m}} \left(\frac{1}{h^m} \sum_{\substack{h_j\in\{0,\dots,h-1\}\\ j=1,\dots,m}} \mathcal{A}_{\omega(k_1,\dots,k_m,h_1,\dots h_m)} - \Theta_{\omega(k_1,\dots,k_m,h_1,\dots h_m)} \right)^2,\end{aligned} \tag{4}$$

where $\omega(k_1,\dots,k_m,h_1,\dots h_m) = (k_1h+h_1,\dots,k_mh+h_m)$. Combining (2) and (4) yields,

$$\|\text{Block}_k(\mathcal{Y}\circ\sigma^{-1}) - \text{Block}_k(\Theta\|_F^2 \lesssim d^m\epsilon^2,$$

with probability at least $1-k^m\exp(-2h^m\epsilon^2)$. □

Based on the changed lemmas, our main theorem becomes

Theorem 0.1 (Sorting-and-blocking under β-monotonicity of degree)**.** With probability at least $1-k^m\exp(-h^m\epsilon^2)$,

$$\begin{aligned}\|\hat{\Theta}-\Theta\|_F^2 &\lesssim d^m(\epsilon^2+\text{Loss}^2(\sigma,\hat{\sigma})) + d^m\epsilon^2 + \frac{d^m}{k^2}\\ &\lesssim d^{-\beta(m-1)+m} + d^m\epsilon^2 + \frac{d^m}{k^2}.\end{aligned}$$

Furthermore, setting $\epsilon^2 = \frac{m\log d}{h^m}$ yields,

$$\|\hat{\Theta}-\Theta\|_F^2 \lesssim d^{-\beta(m-1)+m} + k^m(m\log d) + \frac{d^m}{k^2}$$

$$\lesssim d^{-\beta(m-1)+m} + d^{\frac{m^2}{m+2}}(m \log d),$$

with probability at least $1 - (k/d)^m$. The last line comes from balancing the block size as $k = \mathcal{O}(d^{\frac{m}{m+2}})$.