Assumption for degree sorting algorithm

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1 Recap

Denote deg and $\widehat{\deg}$ as

$$\deg(i) = \frac{1}{d^{m-1}} \sum_{i_2,\ldots,i_m \in [d]} \Theta(i, i_2, \ldots, i_m), \quad \widehat{\deg}(i) = \frac{1}{d^{m-1}} \sum_{i_2,\ldots,i_m \in [d]} \mathcal{Y}(i, i_2, \ldots, i_m),$$

Without loss of generality, assume σ is identity map so that

$$\deg(1) \leq \cdots \leq \deg(d).$$

Recall that $\hat{\sigma}$ is defined to satisfy

$$\widehat{\deg}(\hat{\sigma}^{-1}(1)) \leq \cdots \leq \widehat{\deg}(\hat{\sigma}^{-1}(d)).$$

Based on the estimated permutation $\hat{\sigma}$ We estimate Θ by

$$\hat{\Theta} = \text{Block}_k(\mathcal{Y} \circ \hat{\sigma}^{-1}).$$

Recall that we are able to bound the error of $\hat{\Theta}$ by

$$\|\hat{\Theta} - \Theta\|_F \leq \underbrace{\|\text{Block}_k(\mathcal{Y} \circ \hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y} \circ \sigma^{-1})\|_F}_{\text{permutation error}} + \underbrace{\|\text{Block}_k(\mathcal{Y} \circ \sigma^{-1}) - \text{Block}_k(\Theta)\|_F}_{\text{nonparametric error}} + \underbrace{\|\text{Block}_k(\Theta) - \Theta\|_F}_{\text{approximation error}}.$$

Since nonparametric error is bounded by $\sqrt{k^m}$ and approximation error by $\sqrt{d^m/k^2}$ regardless of assumption on degree, it suffices to consider permutation error.

2 Assumption for degree sorting algorithm

Assumption 1. For any interval $|I| > d^{(m-1)/2}$, $|\{i \in [d] : \deg(i) \in I\}| \leq \mathcal{O}\left(d/|I|\right)$.

Remark 1. This assumption implies that degrees of signal tensor Θ are not concentrated too much.

Remark 2. Previous β-monotonicity assumption with $\beta \in [0, 1]$,

$$\left|\frac{i-j}{d}\right|^{1/\beta} \leq \deg(i) - \deg(j), \text{ for all } i > j \in [d]$$

implies Assumption 1.

Remark 3. Main role of the assumption is to quantify

$$\left|\{j : \deg(\omega_\ell) - d^{-(m-1)/2} \leq \deg(j) \leq \deg(\omega_\ell)\}\right|, \text{ for } \omega_\ell \in [k_\ell h, (k_\ell + 1)h - 1], \ell \in [m]. \tag{1}$$

Although ω_ℓ has local concept, the number of j satisfying (1) is not restricted to local constraint.

So I am not sure how well and effectively we can relax β-monotonicity assumption with locality concept.

Based on Assumption 1, we bound permutation error.

Theorem 2.1 (Upper bound for permutation error)**.** With probability at least $1-k^m\exp(-h^m\epsilon^2)$,

$$\|\text{Block}_k(\mathcal{Y}\circ\hat{\sigma}^{-1})-\text{Block}_k(\mathcal{Y}\circ\sigma^{-1})\|_F \lesssim d^m(\epsilon^2+d^{-(m-1)/2})$$

Proof. For notational simplicity, denote permuted observed tensors as $\mathcal{A}=\mathcal{Y}\circ\sigma^{-1}$ and $\mathcal{A}=\mathcal{Y}\circ\hat{\sigma}^{-1}$. By definition, permutation error becomes

$$h^m\sum_{k_i\in\{0,\ldots,k-1\},i\in[m]}\left(\frac{1}{h^m}\sum_{h_j\in\{0,\ldots,h-1\},j\in[m]}\hat{\mathcal{A}}(k_1h+h_1,\ldots,k_mh+h_m)-\mathcal{A}(k_1h+h_1,\ldots,k_mh+h_m)\right)^2 \tag{2}$$

Notice for any $\omega\in[k_1h,(k_1+1)h-1]\times\cdots\times[k_mh,(k_m+1)h-1]$, we decompose

$$\hat{\mathcal{A}}(\omega)-\mathcal{A}(\omega)=\underbrace{\hat{\mathcal{A}}(\omega)-\Theta\circ\sigma\circ\hat{\sigma}^{-1}(\omega)}_{\text{Hoeffding's inequality}}+\underbrace{\mathcal{A}(\omega)-\Theta(\omega)}_{\text{Hoeffding's inequality}}+\underbrace{\Theta\circ\sigma\circ\hat{\sigma}^{-1}(\omega)-\Theta(\omega)}_{(*)}.$$

By Hoeffding's inequality, first two terms are bounded by ϵ with probability at least $1-\exp(-h^m\epsilon^2)$. For the $(*)$, Lipchitz assumption on Θ gives us,

$$\begin{aligned}
(*) &\leq \frac{L}{d}\sum_{\ell\in[m]}|\sigma\circ\hat{\sigma}^{-1}(\omega_\ell)-\omega_\ell| \\
&= \frac{L}{d}\sum_{\ell\in[m]}|\hat{\sigma}^{-1}(\omega_\ell)-\omega_\ell| \\
&= \frac{L}{d}\sum_{\ell\in[m]}\left|\,|\underbrace{\{j\colon\widehat{\deg}(j)\leq\widehat{\deg}(\omega_\ell)\}}_{(\text{II}_\ell)}|-|\underbrace{\{j\colon\deg(j)\leq\deg(\omega_\ell)\}}_{(\text{I}_\ell)}|\,\right| \\
&\leq \frac{L}{d}\sum_{\ell\in[m]}|\text{I}_\ell\Delta\text{II}_\ell|,
\end{aligned} \tag{3}$$

where the first equality uses the fact that σ is identity without loss of generality and the second equality uses

$$\sigma^{-1}(i)=i=|\underbrace{\{j\colon\deg(j)\leq\deg(i)\}}_{(\text{I})}|,\quad \hat{\sigma}^{-1}(i)=|\underbrace{\{j\colon\widehat{\deg}(j)\leq\widehat{\deg}(i)\}}_{(\text{II})}|.$$

By the same argument from the note 062521_pemutation.pdf, if $\deg(i)-\deg(j)\gg d^{-(m-2)/2}$, we have

$$\deg(j)<\deg(i)\Longleftrightarrow\widehat{\deg}(j)<\widehat{\deg}(i)$$

because of the following equality,

$$\widehat{\deg}(i) - \widehat{\deg}(j) = \underbrace{\widehat{\deg}(i) - \deg(i)}_{\lesssim d^{(m-1)/2}} + \underbrace{\deg(j) - \widehat{\deg}(j)}_{\lesssim d^{(m-1)/2}} + \deg(i) - \deg(j).$$

Therefore, (3) can be further bounded by

$$\begin{aligned}
\frac{L}{d} \sum_{\ell \in [m]} |\mathrm{I}_\ell \Delta \mathrm{II}_\ell| &\leq \frac{L}{d} \sum_{\ell \in [m]} \left| \{ j \colon \deg(\omega_\ell) - d^{-(m-1)/2} \leq \deg(j) \leq \deg(\omega_\ell) \} \right| \\
&\overset{(*)}{\leq} \frac{L}{d} \sum_{\ell \in [m]} d / d^{(m-1)/2} \\
&\leq L m d^{-(m-1)/2}.
\end{aligned}$$

For inequality $(*)$, we use the assumption that for any interval $|I| > d^{(m-1)/2}$, $|\{i \in [d] \colon \deg(i) \in I\}| \leq \mathcal{O}\left(d/|I|\right)$. Combining all results into the equation (2) completes the proof. $\square$