Extension to sparse regime and algorithm performance in smooth settings

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1 Algorithm performance in smooth settings

We assume that $\mathcal{A}_{\boldsymbol{\omega}} \sim$ Bernoulli$(\Theta_{\boldsymbol{\omega}})$, where

$$\Theta_{\boldsymbol{\omega}} = f(\xi_{\omega_1}, \ldots, \xi_{\omega_m}), \text{ for all } \boldsymbol{\omega} = (\omega_1, \ldots, \omega_m) \in E,$$

[m] to [m]?

where $f\colon [0,1]^m \to [0,1]$ is a symmetric function called hypergraphon such that $f(\xi_{\omega_1}, \ldots, \xi_{\omega_m}) = f(\xi_{\sigma(\omega_1)}, \ldots, \xi_{\sigma(\omega_m)})$ for all permutation $\sigma\colon [n] \to [n]$. I checked our algorithm performance under this hypergraphon model with different symmetric function $f\colon [0,1]^3 \to [0,1]$.

- **Smooth 1:** $f(x_1, x_2, x_3) = 1/(1 + \exp(-(x_1^2 + x_2^2 + x_3^2))$.
- **Smooth 2:** $f(x_1, x_2, x_3) = x_1 x_2 x_3$.
- **Smooth 3:** $f(x_1, x_2, x_3) = \log(1 + \max(x_1, x_2, x_3)$.
- **Smooth 4:** $f(x_1, x_2, x_3) = \exp(-\min(x_1, x_2, x_3))$.

Figure 1 shows the distributions of Θ from each model when $n = 100$.

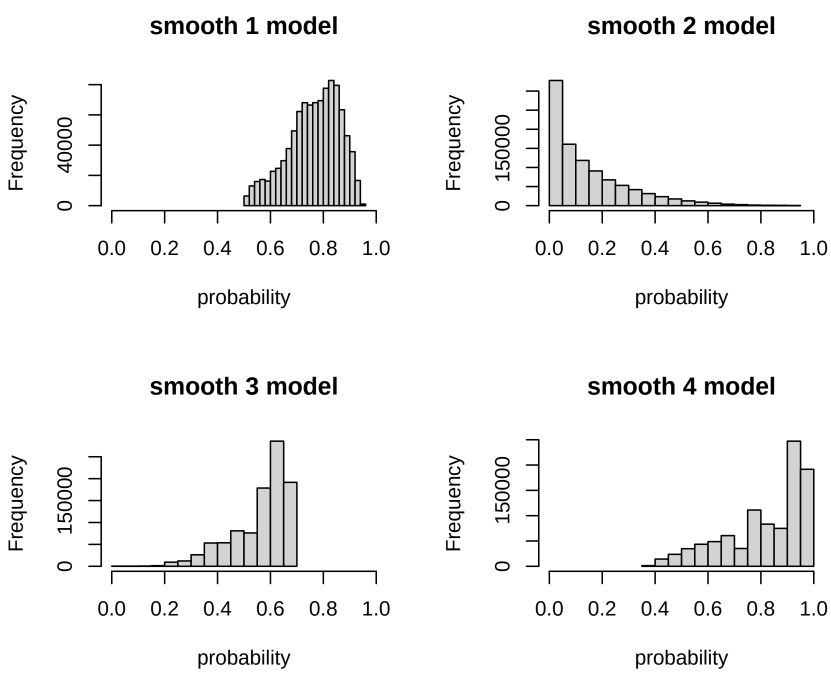


Figure 1: Empirical density of the probability tensor Θ for each smooth model (Smooth 1-4).

In hypergraphon model, there is no clusters. So I choose to use $k \in \{5, 10, 15, 20\}$ for each $n \in \{50, 100, 150, 200, 250\}$. I use `functions_sbm` for the updates. Figure 2 shows the MSE result according to different smooth models. It turns out that our algorithm works great in the smooth settings except small group size case in Smooth 3 and Smooth 4. I found that `functions_sbm` is sometimes trapped in local minimums from which

Add a plot for MSE vs. K for n= 50 under smooth model 1.

Our theory suggests a variance-bias trade off determined by k —> want to verify this from simulation as well.

Current simulation shows under-parameterized region where higher k is better. would be interesting to show the overparameterized region where higher k is worse as well.

functions_sbm2 escape. Figure 3 shows that functions_sbm2 succeeded to find the good optimal points where functions_sbm failed.

Could you merge *sbm.R and *sbm2.R into a single file so we have a cleaner folder?

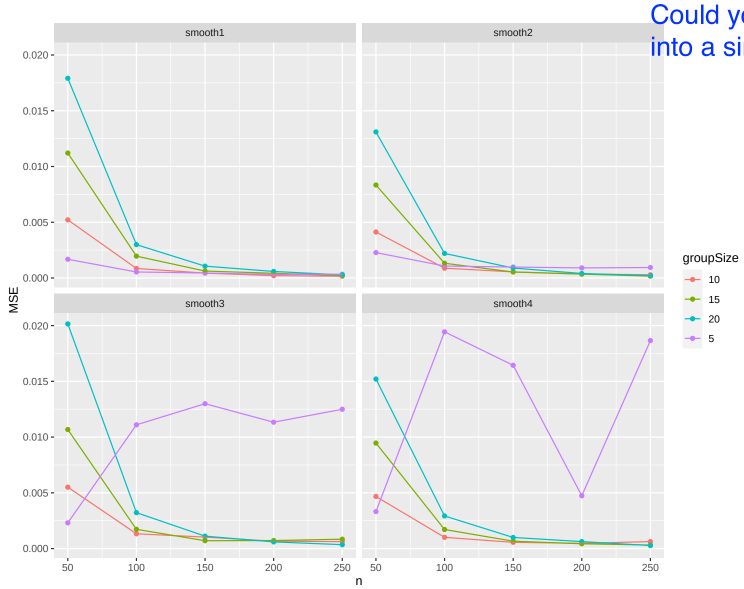
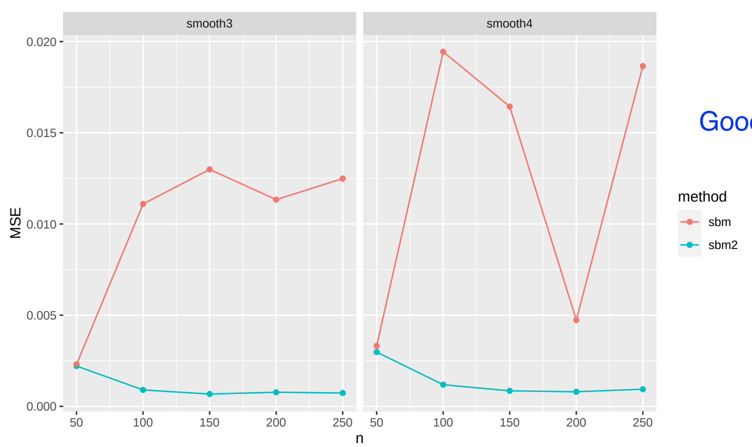


Figure 2: MSE error depending on smooth model, the number of node $n \in \{50, 100, 150, 200, 250\}$ and the number of clusters $k \in \{5, 10, 15, 20\}$. functions_sbm is used for the algorithm. functions_sbm is used on red lines (sbm) while functions_sbm2 on blue lines (sbm2).



Good. Let's stick with sbm2 for future use.

Figure 3: MSE errors in Smooth 3 and 4 depending on the number of nodes. The cluster size is set to be 5 where previous functions_sbm failed to escape local minimums (Figure 2).

2 Extension to sparse regime

We denote $\rho \in [0,1]$ as the sampling probability or the sparsity parameter. When the observed tensor $\mathcal{A} \in \{0,1\}^{d_1 \times \cdots \times d_K}$ is complete, we interpret ρ as the sparsity parameter. On the other hand, we interpret ρ as the sampling probability when the data is incomplete, i.e.,

$$\mathbb{P}\left[\mathcal{A}_\omega \text{ is observed } |\Theta_\omega^{\text{true}}\right] = \rho.$$

If we assign missing entries as 0 in $\mathcal{A}$, the marginal probability of observed network being connected has

$$\mathbb{P}(\mathcal{A}_\omega = 1) = \rho\Theta_\omega^{\text{true}},$$

for all $\rho \in E$ in both complete and incomplete cases. Here we choose to call the parameter ρ the sparsity parameter. Notice that when $\rho = 1$, the problem reduces to previous settings.

From known ρ, we estimate Θ^{true} by

$$\hat{\Theta} = \text{cut}(\tilde{\Theta}), \qquad \text{where } \tilde{\Theta} = \underset{\Theta \in \mathcal{P}_k}{\arg\min} \sum_{\omega \in E} |\mathcal{A}_\omega - \rho\Theta_\omega|^2. \tag{1}$$

With adaptation of the new parameter ρ, we modify previous theorems incorporating ρ.

2.1 Probability tensor estimation

Under k-piecewise constant hypergraphon model, we construct theoretical guarantees for estimated probability tensor from observed network tensor $\mathcal{A} \in \{0,1\}^{d_1 \times \cdots \times d_m}$.

Theorem 2.1 (Stochastic block model with the sparsity parameter ρ)**.** Let $\hat{\Theta}$ be the estimator from (1). Suppose true probability tensor $\Theta^{\text{true}} \in \text{cut}(\mathcal{P}_k)$ for fixed block size k Then, there exists two constants $C_1, C_2 > 0$, such that

$$\frac{1}{n^m}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq \frac{C_1}{\rho}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}}\right), \tag{2}$$

with probability at least $1 - \exp\left(-C_2(n\log k + k^m)\right)$. Furthermore, expected mean square error is bounded by

$$\frac{1}{n^m}\mathbb{E}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq \frac{C}{\rho}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}}\right),$$

for some constant $C > 0$.

Remark 1. The sparsity parameter ρ makes both nonparametric and clustering rates worse by the same rate. For the nonparametric rate, the number of observation becomes $\mathcal{O}(\rho n^m)$ while the number of parameters remains the same as $\mathcal{O}(k^m)$. This reduced observation is reflected on the rate. Similarly, the number of possible k-clusters of n-vertices remains $\mathcal{O}(k^n)$ while the number of observation changes to $\mathcal{O}(\rho n^m)$. Therefore, the clustering rate is also reduced by ρ.

Now we assume that a hypergraphon f is α-Hölder continuous with a constant L. We define a block average on a set $E_{z^{-1}(a)}$ for a given membership function z, a membership vector a, and a tensor $\Theta \in ([n])^{\otimes m}$ as

$$\bar{\Theta}_a(z) = \frac{1}{|E_{z^{-1}(a)}|}\sum_{\boldsymbol{\omega} \in E_{z^{-1}(a)}} \Theta_\omega.$$

Notice that we define $\Theta_\omega = f(\xi_{\omega_1}, \ldots, \xi_{\omega_m})$ for all $\omega = (\omega_1, \ldots, \omega_m) \in E$ where there is no sparsity parameter ρ. Therefore the following block approximation lemma remains the same.

Lemma 2.1 (Block approximation)**.** Suppose the true parameter Θ^{true} admits the hypergraphon model with $f \in \mathcal{H}(\alpha, L)$. For every integer $k \leq n$, there exists $z^*\colon [n] \to [k]$, satisfying

$$\frac{1}{|E|}\sum_{a \in [k]^m}\sum_{\omega \in E_{(z^*)^{-1}(a)}} (\Theta_\omega^{\text{true}} - \bar{\Theta}_a(z^*))^2 \leq m^2 L^2 \left(\frac{1}{k^2}\right)^\alpha.$$

Theorem 2.2 (Hölder continuous hypergraphon model with the sparsity parameter ρ)**.** Suppose the true parameter Θ^{true} admits the hypergraphon model with $f \in \mathcal{H}(\alpha, L)$ Let $\hat{\Theta}$ be the estimator from (1). Then, there exist two constants $C_1, C_2 > 0$ such that,

$$\frac{1}{n^m}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq C_1\left((\rho n^m)^{\frac{-2\alpha}{m+2\alpha}} + \frac{\log n}{\rho n^{m-1}}\right),$$

with probability at least $1 - \exp\left(-C_2\left(n\log(\rho n^m) + (\rho n^m)^{\frac{m}{m+2\alpha}}\right)\right)$ uniformly over $f \in \mathcal{H}(\alpha, L)$. Furthermore, the expected mean square error is bounded by

$$\frac{1}{n^m}\mathbb{E}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq C\left((\rho n^m)^{\frac{-2\alpha}{m+2\alpha}} + \frac{\log n}{\rho n^{m-1}}\right),$$

consistency is achieved provided that rho > n^(-(m-1))

for some constant $C > 0$.

the tolerance for sparsity improves as order increases. Agree with intuition?

Remark 2. Notice that the recovery of the true probability tensor Θ^{true} is influenced by the expected number of observations, ρn^m. The probability $1 - \exp\left(-C_2\left(n\log(\rho n^m) + (\rho n^m)^{\frac{m}{m+2\alpha}}\right)\right)$ tells us that the sparsity parameter should be greater than $1/n^m$ to expect the meaningful MSE. Depending on constants m, α and the sparsity parameter ρ, convergence rate becomes

$$(\rho n^m)^{\frac{-2\alpha}{m+2\alpha}} + \frac{\log n}{\rho n^{m-1}} \asymp \begin{cases} (\rho n^m)^{\frac{-2\alpha}{m+2\alpha}} & \text{if } \rho \geq n^{1-m+2\alpha/m}, \\ \frac{\log n}{\rho n^{m-1}} & \text{if } \rho < n^{1-m+2\alpha/m}, \end{cases}$$

upto $\log n$ factors. The nonparametric rate tends to dominate the error when dense observation while the clustering rate dominates the error under the sparse regime.

3 Proof

3.1 Proof of Theorem 2.1

Proof. We consider two exclusive cases

1. Case 1: when $\|\hat{\Theta} - \Theta^{\text{true}}\|_F \leq \sqrt{C(k^m + n\log k)/\rho}$, then we have directly (2).
2. Case 2: when $\|\hat{\Theta} - \Theta^{\text{true}}\|_F > \sqrt{C(k^m + n\log k)/\rho}$.

   By the definition of $\hat{\Theta}$ in (1), we have

$$\begin{aligned}\|\rho\hat{\Theta} - \rho\Theta^{\text{true}}\|_F^2 &\leq 2\langle \rho\hat{\Theta} - \rho\Theta^{\text{true}}, \mathcal{A} - \rho\Theta^{\text{true}}\rangle \\ &= 2\|\rho\hat{\Theta} - \rho\Theta^{\text{true}}\|_F\left\langle \frac{\hat{\Theta} - \Theta^{\text{true}}}{\|\hat{\Theta} - \Theta^{\text{true}}\|_F}, \mathcal{A} - \rho\Theta^{\text{true}}\right\rangle.\end{aligned} \tag{3}$$

   Then,

$$\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq 2\left|\left\langle \frac{\hat{\Theta} - \Theta^{\text{true}}}{\|\hat{\Theta} - \Theta^{\text{true}}\|_F}, \frac{\mathcal{A} - \rho\Theta^{\text{true}}}{\rho}\right\rangle\right| \leq 2\sup_{\boldsymbol{M},\boldsymbol{M}'\in\mathcal{M}}\sup_{\mathcal{C},\mathcal{C}'\in([0,1]^k)^{\otimes m}}\left|\left\langle \mathcal{T}(\boldsymbol{M},\boldsymbol{M}',\mathcal{C},\mathcal{C}'), \frac{\mathcal{A} - \rho\Theta^{\text{true}}}{\rho}\right\rangle\right|,$$

   where $\mathcal{T} = \frac{\text{cut}(\Theta(\boldsymbol{M},\mathcal{C}) - \text{cut}(\Theta(\boldsymbol{M}',\mathcal{C}')}{\|\text{cut}(\Theta(\boldsymbol{M},\mathcal{C}) - \text{cut}(\Theta(\boldsymbol{M}',\mathcal{C}')\|_F}$. Under the event $\|\hat{\Theta} - \Theta^{\text{true}}\|_F > \sqrt{C(k^m + n\log k)/\rho}$ for some constant C, we have

$$|\mathcal{T}_\omega| \leq \frac{1}{\|\hat{\Theta} - \Theta^{\text{true}}\|_F} \leq \sqrt{\frac{\rho}{C(k^m + n\log k)}}, \text{ for all } \omega \in [n]^{\otimes m}.$$

   Therefore, combination of Lemma 3.1 and 3.2 yields

$$\mathbb{P}\left(\sup_{\boldsymbol{M},\boldsymbol{M}'\in\mathcal{M}}\sup_{\mathcal{C},\mathcal{C}'\in([0,1]^k)^{\otimes m}}\left|\left\langle \mathcal{T}(\boldsymbol{M},\boldsymbol{M}',\mathcal{C},\mathcal{C}'),\frac{\mathcal{A}-\rho\Theta^{\text{true}}}{\rho}\right\rangle\right|\geq t\right)$$
$$\leq \exp\left(C'\left(k^m+n\log k\right)\right)\cdot\exp\left(-\min\left(\frac{\rho t^2}{24},\frac{t\sqrt{C\rho(k^m+n\log k)}}{4}\right)\right),$$

for some constant $C'>0$. Setting $t=\sqrt{C_1(k^m+n\log k)/\rho}$ for sufficiently large C_1 depending $C>0$ completes the proof.

Expectation bound follows from the probability tail bound. □

Lemma 3.1. [Gao et al., 2016, Lemma 13] Let $\{\mathcal{A}_\omega\}_{\omega\in E}$ be independent sub-Gaussian random variables with mean $\rho\Theta_\omega$ and proxy variance σ^2, where $\Theta_\omega\in[-M,M]$, $\rho\in[0,1]$, and E is an index set. Then, for $|\lambda|\leq\rho/(M\vee\sigma)$, we have

$$\mathbb{E}e^{\lambda\left(\frac{\mathcal{A}_\omega-\rho\Theta_\omega}{\rho}\right)}\leq 2e^{(M^2+2\sigma^2)\lambda^2/\rho}.$$

Moreover, for $\sum_{\omega\in}c_\omega^2=1$,

$$\mathbb{P}\left\{\left|\sum_{\omega\in E}c_\omega\left(\frac{\mathcal{A}_\omega-\rho\Theta_\omega}{\rho}\right)\right|\geq t\right\}\leq 4\exp\left\{-\min\left(\frac{\rho t^2}{4(M^2+2\sigma^2)},\frac{\rho t}{2(M\vee\sigma)\|c\|_\infty}\right)\right\},$$

for any $t>0$.

Lemma 3.2. Let $\mathcal{X}\in(\mathbb{R}^n)^{\otimes m}$ be a random tensor drawn from any distributions. Denote $\mathcal{B}_2$ as a unit ball in $(\mathbb{R}^n)^{\otimes m}$ with $\|\cdot\|_2$ distance. Suppose that

$$\mathbb{P}(|\langle\mathcal{T},\mathcal{X}\rangle|\geq t)\leq\phi(t),\ \text{for all } t>0 \text{ and } \mathcal{T}\in\mathcal{B}_2, \tag{4}$$

for some function $\phi\colon\mathbb{R}_+\to[0,1]$. Define

$$\mathcal{B}_c=\left\{\frac{\text{cut}(\Theta(\boldsymbol{M},\mathcal{C}))-\text{cut}(\Theta(\boldsymbol{M}',\mathcal{C}'))}{\|\text{cut}(\Theta(\boldsymbol{M},\mathcal{C}))-\text{cut}(\Theta(\boldsymbol{M}',\mathcal{C}'))\|_F}\colon \boldsymbol{M},\boldsymbol{M}'\in\mathcal{M}\text{ and }\mathcal{C},\mathcal{C}'\in([0,1]^k)^{\otimes m}\right\}$$

Then,

$$\mathbb{P}\left(\sup_{\mathcal{T}\in\mathcal{B}_c}|\langle\mathcal{T},\mathcal{X}\rangle|\geq t\right)\leq\exp(Ck^m+2n\log k)\phi(t/2),$$

for some constant $C>0$.

Proof. Notice that

$$\text{vec}(\Theta(\boldsymbol{M}_1,\mathcal{C}_1)-\Theta(\boldsymbol{M}_2,\mathcal{C}_2))=\underbrace{\begin{bmatrix}\boldsymbol{M}_1^{\otimes m} & -\boldsymbol{M}_2^{\otimes m}\end{bmatrix}}_{=:\boldsymbol{A}\in\{0,1\}^{n^m\times 2k^m}}\underbrace{\begin{bmatrix}\text{vec}(\mathcal{C}_1)\\ \text{vec}(\mathcal{C}_2)\end{bmatrix}}_{=:\boldsymbol{c}\in[0,1]^{2k^m\times 1}}.$$

With careful allocation, we always find a matrix $\tilde{\boldsymbol{A}}\in\{0,1\}^{n^m\times 2k^m}$ such that,

$$\text{vec}\left(\text{cut}(\Theta(M_1,\mathcal{C}_1))-\text{cut}(\Theta(M_2,\mathcal{C}_2))\right)=\tilde{\boldsymbol{A}}\boldsymbol{c}.$$

Notice

$$\begin{aligned}
\mathbb{P}\left(\sup_{\mathcal{T}\in\mathcal{S}}|\langle\mathcal{T},\mathcal{X}\rangle|\geq t\right) &= \mathbb{P}\left(\sup_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\sup_{\mathcal{C}_1,\mathcal{C}_2\in([0,1]^k)^{\otimes m}}\left\langle\frac{\mathrm{cut}(\Theta(\boldsymbol{M}_1,\mathcal{C}_1))-\mathrm{cut}(\Theta(\boldsymbol{M}_2,\mathcal{C}_2))}{\|\mathrm{cut}(\Theta(\boldsymbol{M}_1,\mathcal{C}_1))-\mathrm{cut}(\Theta(\boldsymbol{M}_2,\mathcal{C}_2))\|_F},\mathcal{X}\right\rangle\geq t\right)\\
&\leq \mathbb{P}\left(\sup_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\sup_{\boldsymbol{c}\in\mathbb{R}^{2k^m}}\left\langle\frac{\tilde{\boldsymbol{A}}\boldsymbol{c}}{\|\tilde{\boldsymbol{A}}\boldsymbol{c}\|_2},\mathrm{vec}(\mathcal{X})\right\rangle\geq t\right)\\
&\leq \sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\left\langle\frac{\tilde{\boldsymbol{A}}\boldsymbol{c}}{\|\tilde{\boldsymbol{A}}\boldsymbol{c}\|_2},\mathrm{vec}(\mathcal{X})\right\rangle\geq t\right).
\end{aligned}$$

Let $r:=\mathrm{rank}(\tilde{\boldsymbol{A}})\leq 2k^m$ be the rank of $\tilde{\boldsymbol{A}}$. Then we express $\tilde{\boldsymbol{A}}\boldsymbol{c}=\sum_{i=1}^r\lambda_i\boldsymbol{u}_i\boldsymbol{v}_i^T\boldsymbol{c}=\sum_{i=1}^r(\lambda_i\boldsymbol{v}_i^T\boldsymbol{c})\boldsymbol{u}_i$, where $\boldsymbol{u}_i,\boldsymbol{v}_i$ and λ_i are i-th singular vectors and singular value respectively. Defining $\boldsymbol{\alpha}=(\lambda_1\boldsymbol{v}_1^T\boldsymbol{c},\ldots,\lambda_r\boldsymbol{v}_r^T\boldsymbol{c})\in\mathbb{R}^r$, we have,

$$\begin{aligned}
\sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\left\langle\frac{\tilde{\boldsymbol{A}}\boldsymbol{c}}{\|\tilde{\boldsymbol{A}}\boldsymbol{c}\|_2},\mathrm{vec}(\mathcal{X})\right\rangle\geq t\right) &= \sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\left\langle\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|_2},(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_r)^T\mathrm{vec}(\mathcal{X})\right\rangle\geq t\right)\\
&\leq \sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\max_{\boldsymbol{c}\in\mathbb{R}^r}\left\langle\frac{\boldsymbol{c}}{\|\boldsymbol{c}\|_2},\boldsymbol{x}\right\rangle\geq t\right)\\
&\leq \sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\max_{\boldsymbol{c}\in\mathcal{S}}\langle\boldsymbol{c},\boldsymbol{x}\rangle\geq t\right), \qquad (5)
\end{aligned}$$

where we define $\boldsymbol{x}:=(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_r)^T\mathrm{vec}(\mathcal{X})\in\mathbb{R}^r$ and $\mathcal{S}$ is a unit sphere in $\mathbb{R}^r$. Notice the upper bound (4) still holds for $\boldsymbol{x}$ by the orthonormality of $(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_r)$. Let $\mathcal{S}'$ be a 1/2-net of $\mathcal{S}$ with respect to the Euclidean norm that satisfies $|\mathcal{S}'|\leq 6^r$. Observed that for every $\boldsymbol{c}\in\mathbb{R}^r$, there exists $\boldsymbol{c}'\in\mathcal{S}'$ such that $\|\boldsymbol{c}-\boldsymbol{c}'\|_2\leq 1/2$. Then, we have

$$\begin{aligned}
|\langle\boldsymbol{c},\boldsymbol{x}\rangle| &\leq |\langle\boldsymbol{c}-\boldsymbol{c}',\boldsymbol{x}\rangle|+|\langle\boldsymbol{c}',\boldsymbol{x}\rangle|\\
&= \|\boldsymbol{c}-\boldsymbol{c}'\|_2\left|\left\langle\frac{\boldsymbol{c}-\boldsymbol{c}'}{\|\boldsymbol{c}-\boldsymbol{c}'\|_2},\boldsymbol{x}\right\rangle\right|+|\langle\boldsymbol{c}',\boldsymbol{x}\rangle|\\
&\leq \frac{1}{2}\sup_{\boldsymbol{c}\in\mathcal{S}}|\langle\boldsymbol{c},\boldsymbol{x}\rangle|+|\langle\boldsymbol{c}',\boldsymbol{x}\rangle|.
\end{aligned}$$

Taking sup and max on both side yields,

$$\sup_{\boldsymbol{c}\in\mathcal{S}}|\langle\boldsymbol{c},\boldsymbol{x}\rangle|\leq 2\max_{\boldsymbol{c}'\in\mathcal{S}'}|\langle\boldsymbol{c}',\boldsymbol{x}\rangle|.$$

Finally applying this result to (5) yields,

$$\begin{aligned}
\mathbb{P}\left(\sup_{\mathcal{T}\in\mathcal{S}}|\langle\mathcal{T},\mathcal{X}\rangle|\geq t\right) &\leq \sum_{\boldsymbol{M}_1,\boldsymbol{M}_2\in\mathcal{M}}\mathbb{P}\left(\max_{\boldsymbol{c}'\in\mathcal{S}'}\langle\boldsymbol{c}',\boldsymbol{x}\rangle\geq t/2\right)\\
&\leq (k^{2n})6^r\phi(t/2)\\
&\leq \exp(Ck^m+2n\log k)\phi(t/2),
\end{aligned}$$

where the last inequality used (4) and the fact $r\leq k^m$.

□

3.2 Proof of Thoerm 2.2

Proof. First, we prove the probability tail bound. By Lemma 2.1, we can always find a block tensor Θ^* close to the true probability tensor Θ^{true} such that

$$\frac{1}{n^m}\|\Theta^* - \Theta^{\text{true}}\|_F^2 \leq m^2 L^2 \left(\frac{1}{k^2}\right)^{\alpha}. \tag{6}$$

By triangular inequality,

$$\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \leq 2\underbrace{\|\hat{\Theta} - \Theta^*\|_F^2}_{\text{(i)}} + 2\underbrace{\|\Theta^* - \Theta^{\text{true}}\|_F^2}_{\text{(ii)}}. \tag{7}$$

Since we have the error bound (ii) as in (6), we find the upper bound of the error (i). Based on definition of $\hat{\Theta}$, we have the following inequality similar to (3).

$$\begin{aligned}
\|\hat{\Theta} - \Theta^*\|_F^2 &\leq 2\left\langle \hat{\Theta} - \Theta^*, \frac{\mathcal{A} - \rho\Theta^*}{\rho} \right\rangle \\
&= 2\left(\left\langle \hat{\Theta} - \Theta^*, \frac{\mathcal{A} - \rho\Theta^{\text{true}}}{\rho} \right\rangle + \langle \hat{\Theta} - \Theta^*, \Theta^{\text{true}} - \Theta^* \rangle\right) \\
&\leq 2\|\hat{\Theta} - \Theta^*\|_F \left(\left\langle \frac{\hat{\Theta} - \Theta^*}{\|\hat{\Theta} - \Theta^*\|_F}, \frac{\mathcal{A} - \rho\Theta^{\text{true}}}{\rho} \right\rangle + \|\Theta^{\text{true}} - \Theta^*\|_F\right).
\end{aligned}$$

It suffices to bound the inner product term because of (6). Notice Θ^* is a block tensor such that $\Theta^* \in \text{cut}(\mathcal{P}(k))$. Therefore, by the same way in the proof of Theorem 2.1, we obtain

$$\mathbb{P}\left(\left\langle \frac{\hat{\Theta} - \Theta^*}{\|\hat{\Theta} - \Theta^*\|_F}, \frac{\mathcal{A} - \rho\Theta}{\rho} \right\rangle \geq t\right) \leq \exp\left(C'\left(k^m + n\log k\right)\right) \cdot \exp\left(-\min\left(\frac{\rho t^2}{24}, \frac{t\sqrt{C\rho(k^m + n\log k)}}{4}\right)\right),$$

for some universal constants $C, C' > 0$. Setting $t = \sqrt{C''(k^m + n\log k)/\rho}$ for sufficiently large C'' depending $C > 0$ yields

$$\text{(i)} \lesssim m^2 L^2 \left(\frac{1}{k}\right)^{2\alpha} + \frac{1}{\rho}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}}\right)$$

with probability at least $1 - \exp\left(-C_2(n\log k + k^m)\right)$. Combinations of two error bounds in (7) and setting $k = \left\lceil (\rho n^m)^{\frac{1}{m+2\alpha}} \right\rceil$, completes the theorem.

The moment bound follows by the probability tail bound. □

References

Chao Gao, Yu Lu, Zongming Ma, and Harrison H Zhou. Optimal estimation and completion of matrices with biclustering structures. *The Journal of Machine Learning Research*, 17(1):5602–5630, 2016.