

Hypergraphon estimation error 2

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February 25, 2021

1 Notation and problem setting

Let E be a set of possible m -uniform hyperedges from n vertices without diagonal entries,

$$E = \{(i_1, \dots, i_m) \in [n]^m : |\{i_1, \dots, i_m\}| = m\}.$$

We denote an index of m -uniform hyperedges as $\omega = (\omega_1, \dots, \omega_m) \in [n]^m$ and a membership vector of m -vertices as $a = (a_1, \dots, a_m) \in [k]^m$. Let $z: [n] \rightarrow [k]$ be a membership function. For a given membership function z and a membership vector $a \in [k]^m$, define $E_{z^{-1}(a)}$ as a set of m -uniform hyperedges whose clustering group belongs to a i.e.,

$$E_{z^{-1}(a)} = \{\omega \in E : z(\omega_\ell) = a_\ell \text{ for all } \ell \in [m]\}.$$

We define a block average on a set $E_{z^{-1}(a)}$ for a given membership function z , a membership vector a , and a tensor $\Theta \in ([n])^{\otimes m}$ as

$$\bar{\Theta}_a(z) = \frac{1}{|E_{z^{-1}(a)}|} \sum_{\omega \in E_{z^{-1}(a)}} \Theta_\omega.$$

Now we consider an undirected m -uniform hypergraph. The connectivity is encoded by an adjacency tensor $\{\mathcal{A}_\omega\}_{\omega \in E}$ which takes values in $\{0, 1\}$. We assume that $\mathcal{A}_\omega \sim \text{Bernoulli}(\Theta_\omega)$, where

$$\Theta_\omega = f(\xi_{\omega_1}, \dots, \xi_{\omega_m}), \text{ for all } \omega = (\omega_1, \dots, \omega_m) \in E, \quad (0)$$

where $f: [0, 1]^m \rightarrow [0, 1]$ is a symmetric function called graphon such that $f(\xi_{\omega_1}, \dots, \xi_{\omega_m}) = f(\xi_{\sigma(\omega_1)}, \dots, \xi_{\sigma(\omega_m)})$ for all permutation $\sigma: [m] \rightarrow [m]$. Conventionally, we set $\Theta_\omega = 0$ for all $\omega \in [n]^m \setminus E$. In addition, we further assume that a graphon f is α -Hölder smooth with a constant L .

Definition 1. A function $f: [0, 1]^m \rightarrow [0, 1]$ is a α -Hölder smooth with a constant L , denoted as $f \in \mathcal{H}(\alpha, L)$, if there exists a polynomial function $\mathcal{P}_{\lfloor \alpha \rfloor}(x)$ of degree $\lfloor \alpha \rfloor$ such that

$$|f(x) - \mathcal{P}_{\lfloor \alpha \rfloor}(x' - x)| \leq L \|x' - x\|_{\alpha - \lfloor \alpha \rfloor}^{\alpha - \lfloor \alpha \rfloor},$$

where the norm $\|x\|_p^p := \sum_{i=1}^m |x_i|^p$ for $x \in \mathbb{R}^m$.

It follows from the standard embedding theorem for any $f \in \mathcal{H}(\alpha, L)$,

Can we define the function class of interest, $\mathcal{H}(\alpha, L)$, using property (1) with $\alpha \in [0, 1]$?
 $|f(x) - f(y)| \leq M \|x - y\|_{\alpha \wedge 1}^{\alpha \wedge 1}, \quad (1)$

where $M > 0$ is a global constant only depending on α and L . We only use the property (1) over the note for α -Hölder smoothness.

How does M depend on L?

2 Probability matrix estimation

For every integer $k \leq n$, there exists

Lemma 1. There exists $z^*: [n] \rightarrow [k]$, satisfying

add name to Lemma 1

$$\frac{1}{|E|} \sum_{a \in [k]^m} \sum_{\omega \in E_{(z^*)^{-1}(a)}} (\Theta_\omega - \bar{\Theta}_a(z^*))^2 \leq CM^2 \left(\frac{m^2}{k^2} \right)^{\alpha \wedge 1}.$$

Proof. Define $z^*: [n] \rightarrow [k]$ by

$$(z^*)^{-1}(\ell) = \left\{ i \in [n] : \xi_i \in \left[\frac{\ell-1}{k}, \frac{\ell}{k} \right) \right\}, \quad \text{for each } \ell \in [k].$$

By the construction of z^* for $\xi_{\omega_\ell} \in [(a_{\ell-1}-1)/k, a_\ell/k]$,

$$\begin{aligned} |f(\xi_{\omega_1}, \dots, \xi_{\omega_m}) - \bar{\Theta}_a(z^*)| &= \left| f(\xi_{\omega_1}, \dots, \xi_{\omega_m}) - \frac{1}{|E_{z^{-1}(a)}|} \sum_{(\omega'_1, \dots, \omega'_m) \in E_{z^{-1}(a)}} f(\xi_{\omega'_1}, \dots, \xi_{\omega'_m}) \right| \\ &\leq \frac{1}{|E_{z^{-1}(a)}|} \sum_{(\omega'_1, \dots, \omega'_m) \in E_{z^{-1}(a)}} |f(\xi_{\omega_1}, \dots, \xi_{\omega_m}) - f(\xi_{\omega'_1}, \dots, \xi_{\omega'_m})| \\ &\leq \frac{1}{|E_{z^{-1}(a)}|} \sum_{(\omega_1, \dots, \omega_m) \in E_{z^{-1}(a)}} M \|(\xi_{\omega_1}, \dots, \xi_{\omega_m}) - (\xi_{\omega'_1}, \dots, \xi_{\omega'_m})\|_{\alpha \wedge 1}^{\alpha \wedge 1} \\ &\leq CM \left(\frac{m}{k} \right)^{\alpha \wedge 1}. \end{aligned}$$

□

Let $\tilde{\Theta}$ be a minimizer of the least square error from the adjacency tensor \mathcal{A} ,

$$\tilde{\Theta} = \arg \min_{\Theta \in \mathcal{P}_k} \sum_{\omega \in E} (\mathcal{A}_\omega - \Theta_\omega)^2,$$

where

$$\mathcal{P}_k = \{ \Theta \in ([0, 1]^n)^{\otimes m} : \Theta = \mathcal{C} \times_2 \mathbf{M} \times_2 \dots \times_m \mathbf{M}, \text{ with a membership matrix } \mathbf{M} \text{ and a core tensor } \mathcal{C} \in ([0, 1]^k)^{\otimes m} \}.$$

We estimate the probability tensor by $\hat{\Theta} = \text{cut}(\tilde{\Theta})$ such that

$$\text{cut}(\Theta_\omega) = \begin{cases} \Theta_\omega & \text{if } \omega \in E, \\ 0 & \text{if } \omega \in [n]^m \setminus E. \end{cases} \quad (2)$$

Notice $\|\mathcal{A} - \hat{\Theta}\|_F^2 \leq \|\mathcal{A} - \Theta\|_F^2$ for any k block tensor $\Theta \in \text{cut}(\mathcal{P}_k)$.

Theorem 2.1 (hypergraphon model). Let $\hat{\Theta}$ be the estimator from (2). Then, there exist two constants $C_1, C_2 > 0$ such that,

$$\frac{1}{n^m} \|\hat{\Theta} - \Theta\|_F^2 \leq C_1 \left(n^{\frac{-2m(\alpha \wedge 1)}{m-1} + \frac{\log n}{m-1}} \right) \quad \text{with probability at least } 1 - \exp \left(-C_2 \left(n \log n + n^{\frac{m^2}{m+2(\alpha \wedge 1)}} \right) \right) \text{ uniformly over } f \in \mathcal{H}(\alpha, L).$$

Proof. First, we can find a block tensor Θ^* close to true Θ by Lemma 1. By triangular inequality,

$$\|\hat{\Theta} - \Theta\|_F^2 \leq \underbrace{\|\hat{\Theta} - \Theta^*\|_F^2}_{(i)} + \underbrace{\|\Theta^* - \Theta\|_F^2}_{(ii)}. \quad (3)$$

Since we have already shown the error bound of (ii) in Lemma 1, we bound the error from (i). From the definition of $\hat{\Theta}$, we have

$$\|\hat{\Theta} - \mathcal{A}\|_F^2 \leq \|\Theta^* - \mathcal{A}\|_F^2. \quad (4)$$

Combining (4) with the fact

$$\begin{aligned}\|\hat{\Theta} - \mathcal{A}\|_F^2 &= \|\hat{\Theta} - \Theta^* + \Theta^* - \mathcal{A}\|_F^2 \\ &= \|\hat{\Theta} - \Theta^*\|_F^2 + \|\Theta^* - \mathcal{A}\|_F^2 + 2\langle \hat{\Theta} - \Theta^*, \Theta^* - \mathcal{A} \rangle,\end{aligned}$$

yields

$$\begin{aligned}\|\hat{\Theta} - \Theta^*\|_F^2 &\leq 2\langle \hat{\Theta} - \Theta^*, \mathcal{A} - \Theta^* \rangle \\ &= 2\left(\langle \hat{\Theta} - \Theta^*, \mathcal{A} - \Theta \rangle + \langle \hat{\Theta} - \Theta^*, \Theta - \Theta^* \rangle\right) \\ &\leq 2\|\hat{\Theta} - \Theta^*\|_F \left(\left\langle \frac{\hat{\Theta} - \Theta^*}{\|\hat{\Theta} - \Theta^*\|_F}, \mathcal{A} - \Theta \right\rangle + \|\Theta - \Theta^*\|_F \right).\end{aligned}$$

Let $\mathcal{M} = \{\mathbf{M} : \mathbf{M} \text{ is the collection of membership matrices}\}$. Then,

$$\begin{aligned}\left\langle \frac{\hat{\Theta} - \Theta^*}{\|\hat{\Theta} - \Theta^*\|_F}, \mathcal{A} - \Theta \right\rangle &\leq \sup_{\Theta' \in \mathcal{P}_k} \sup_{\Theta'' \in \mathcal{P}_k} \left\langle \frac{\text{cut}(\Theta') - \text{cut}(\Theta'')}{\|\text{cut}(\Theta') - \text{cut}(\Theta'')\|_F}, \mathcal{A} - \Theta \right\rangle \\ &\leq \sup_{\mathbf{M}, \mathbf{M}' \in \mathcal{M}} \sup_{\mathcal{C}, \mathcal{C}' \in ([0,1]^n)^{\otimes m}} \left\langle \frac{\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))}{\|\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))\|_F}, \mathcal{A} - \Theta \right\rangle.\end{aligned}$$

Notice that $\mathcal{A} - \Theta$ is sub-Gaussian with proxy parameter $\sigma^2 = 1/4$. By union bound and the property of sub-Gaussian, we have, for any $t > 0$.

$$\begin{aligned}\mathbb{P} \left(\sup_{\mathbf{M}, \mathbf{M}' \in \mathcal{M}} \sup_{\mathcal{C}, \mathcal{C}' \in ([0,1]^n)^{\otimes m}} \left| \left\langle \frac{\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))}{\|\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))\|_F}, \mathcal{A} - \Theta \right\rangle \right| \geq t \right) \\ \leq \sum_{\mathbf{M}, \mathbf{M}' \in \mathcal{M}} \mathbb{P} \left(\sup_{\mathcal{C}, \mathcal{C}' \in ([0,1]^n)^{\otimes m}} \left| \left\langle \frac{\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))}{\|\text{cut}(\Theta(\mathbf{M}, \mathcal{C})) - \text{cut}(\Theta(\mathbf{M}', \mathcal{C}'))\|_F}, \mathcal{A} - \Theta \right\rangle \right| \geq t \right) \\ \leq |\mathcal{M}|^2 C_1^{k^m} \exp(-C_2 t^2) \\ = \exp(2n \log k + C_1 k^m - C_2 t^2),\end{aligned}$$

where $C_1, C_2 > 0$ are universal constants. The secibd line follows from [Phillippe Rigollet \[2015\]](#) and the fact that $\Theta = \Theta(\mathbf{M}, \cdot)$ lies in a linear space of dimension k^m . Choosing $t = C\sqrt{n \log k + k^m}$ yields

$$(ii) \leq C_1 \left(\left(\frac{m}{k} \right)^{2(\alpha \wedge 1)} + \left(\frac{k}{n} \right)^m + \frac{\log k}{n^{m-1}} \right),$$

with probability at least $1 - \exp(-C_2(n \log k + k^m))$. Combinations of two error bounds in (3) and setting $k = \left\lceil n^{\frac{m}{m+2(\alpha \wedge 1)}} \right\rceil$, completes the theorem. \square

Theorem 2.2 (stochastic block model). Let $\hat{\Theta}$ be the estimator from (2). Suppose true probability tensor $\Theta \in \text{cut}(\mathcal{P}_k)$ for fixed block size k . Then, there exists two constants $C_1, C_2 > 0$, such that

$$\frac{1}{n^m} \|\hat{\Theta} - \Theta\|_F^2 \leq C_1 \left(\frac{k}{n} \right)^m + \frac{\log k}{n^{m-1}},$$

with probability at least $1 - \exp(-C_2(n \log k + k^m))$. In particular, suppose $k \asymp n^\delta$ for some $\delta \in [0, 1]$.

Then, the convergence rate becomes

Perform the similar discussion for
Thm 2.1

$$\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}} \asymp \begin{cases} n^{-m} & k = 1, \\ n^{-m+1} & \delta = 0, k \geq 2, \\ n^{-m+1} \log(n) & \delta \in (0, 1/m], \\ n^{-m(1-\delta)} & \delta \in (1/m, 1]. \end{cases}$$

Proof. By similar way in the proof of Theorem 2.1, we have

$$\begin{aligned} \|\hat{\Theta} - \Theta\|_F^2 &\leq 2\langle \hat{\Theta} - \Theta, \mathcal{A} - \Theta \rangle \\ &= 2\|\hat{\Theta} - \Theta\|_F \left\langle \frac{\hat{\Theta} - \Theta}{\|\hat{\Theta} - \Theta\|_F}, \mathcal{A} - \Theta \right\rangle \\ &\leq \sup_{\Theta' \in \mathcal{P}_k} \sup_{\Theta'' \in \mathcal{P}_k} \left\langle \frac{\text{cut}(\Theta') - \text{cut}(\Theta'')}{\|\text{cut}(\Theta') - \text{cut}(\Theta'')\|_F}, \mathcal{A} - \Theta \right\rangle. \end{aligned}$$

Notice the last inequality holds because $\Theta \in \text{cut}(\mathcal{P}_k)$. Therefore, we have the result following the proof of Theorem 2.1. \square

3 Hypergraphon estimation

For a given probability tensor Θ , define the empirical hypergraphon $f_\Theta: [0, 1]^m \rightarrow [0, 1]$ as the following piecewise constant function:

$$\tilde{f}_\Theta(x_1, \dots, x_m) = \Theta_{\lfloor x_1 \rfloor, \dots, \lfloor x_m \rfloor}.$$

For any hypergraphon estimator \hat{f} , we define the squared error

$$\delta^2(\hat{f}, f) := \inf_{\tau \in \mathcal{T}} \int_{(0,1)^m} |f(\tau(x)) - \hat{f}(x)|^2 dx,$$

where \mathcal{T} is the set of all measure-preserving bijection $\tau: [0, 1] \rightarrow [0, 1]$.

Our goal is to construct the upper bound of error $\mathbb{E}[\delta^2(f_\Theta, f)]$. By triangular inequality, we have

$$\mathbb{E}[\delta^2(f_\Theta, f)] \leq \underbrace{\frac{2}{n^m} \mathbb{E}\|\hat{\Theta} - \Theta\|_F^2}_{(i)} + \underbrace{2\mathbb{E}[\delta^2(f_\Theta, f)]}_{(ii)}.$$

(i) directly follows from Thm 2.1

Currently, I only derived the upper bound of (ii).

Lemma 2. Suppose $f \in \mathcal{H}(\alpha, L)$. Then

Name this Lemma. (Agnostic error)

$$\mathbb{E}[\delta^2(f_\Theta, f)] \leq \frac{C}{n^{\alpha \wedge 1}},$$

for some constant C only depending on constants m and L .

Proof. By triangular inequality, we have

$$\mathbb{E}[\delta^2(f_\Theta, f)] \leq 2\mathbb{E}[\delta^2(f_\Theta, f_{\Theta'})] + 2\mathbb{E}[\delta^2(f_{\Theta'}, f)],$$

where $\Theta' \in ([0, 1]^n)^{\otimes m}$ such that $\Theta'_\omega = f(\xi_{\omega_1}, \dots, \xi_{\omega_m})$ for all $\omega \in [n]^m$. Notice $\Theta'_\omega = \Theta_\omega$ for $\omega \in E$ but

For all future results:
Make the dependence on m explicit
—> Our goal is to investigate the impact
of higher-order (m) compared to
m=2.

$\Theta_\omega = 0$ for $\omega \in [n]^m \setminus E$. By definition of Θ' ,

$$\mathbb{E} [\delta^2(f_\Theta, f_{\Theta'})] = \int_{[0,1]^m} |f_\Theta(x) - f_{\Theta'}(x)| dx < \frac{C}{n},$$

for some $C > 0$ only depending on m . This is because $|f_\Theta(x) - f_{\Theta'}(x)| = 0$ outside of a set of measure $(n^m - \binom{n}{m}n!)/n^m = C/n$. Hence it suffices to prove that

$$\mathbb{E} [\delta^2(f_{\Theta'}, f)] \leq \frac{C}{n^{\alpha \wedge 1}}.$$

We have

$$\delta^2(f_{\Theta'}, f) = \inf_{\tau \in \mathcal{T}} \sum_{i_1, \dots, i_m=1}^n \int_{(i_1-1)/n}^{i_1/n} \cdots \int_{(i_m-1)/n}^{i_m/n} |f(\tau(x_1), \dots, \tau(x_m)) - \Theta'_{i_1, \dots, i_m}|^2 dx_1 \cdots dx_m$$

The infimum over all measure-preserving bijection is smaller than the minimum over the subclass of measure-preserving bijection τ such that

$$\int_{(i_1-1)/n}^{i_1/n} \cdots \int_{(i_m-1)/n}^{i_m/n} f(\tau(x_1), \dots, \tau(x_m)) dx_1 \cdots dx_m = \int_{(\sigma(i_1)-1)/n}^{\sigma(i_1)/n} \cdots \int_{(\sigma(i_m)-1)/n}^{\sigma(i_m)/n} f(x_1, \dots, x_m) dx_1 \cdots dx_m$$

for some permutation σ . For $x \in \prod_{\ell=1}^m [(\sigma(i_\ell) - 1)/n, \sigma(i_\ell)/n]$,

$$\begin{aligned} |f(x_1, \dots, x_m) - f(\xi_1, \dots, \xi_m)|^2 &\leq 2 \left| f(x_1, \dots, x_m) - f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) \right|^2 \\ &\quad + 2 \left| f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) - f(\xi_{(\sigma(i_1))}, \dots, \xi_{(\sigma(i_m))}) \right|^2 \\ &\quad + 2 |f(\xi_{(\sigma(i_1))}, \dots, \xi_{(\sigma(i_m))}) - f(\xi_{i_1}, \dots, \xi_{i_m})|^2, \end{aligned} \quad (5)$$

where $\xi_{(\ell)}$ denotes the ℓ -th largest element of the set $\{\xi_1, \dots, \xi_n\}$. Choose random permutation σ such that $\xi_{\sigma^{-1}(1)} \leq \xi_{\sigma^{-1}(2)} \leq \dots \leq \xi_{\sigma^{-1}(n)}$. Then the third summand in (5) is 0 almost surely.

For the first summand in (5), notice $(\sigma(i_1)/(n+1), \dots, \sigma(i_m)/(n+1)) \in \prod_{\ell=1}^m [(\sigma(i_\ell) - 1)/n, \sigma(i_\ell)/n]$. From (1), we obtain

$$\left| f(x_1, \dots, x_m) - f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) \right|^2 \leq m^2 M^2 \left(\frac{1}{n}\right)^{2(\alpha \wedge 1)}$$

Integrating and taking expectation on the first summand yields,

$$\mathbb{E} \left[\sum_{i_1, \dots, i_m=1}^n \int_{(\sigma(i_1)-1)/n}^{\sigma(i_1)/n} \cdots \int_{(\sigma(i_m)-1)/n}^{\sigma(i_m)/n} \left| f(x_1, \dots, x_m) - f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) \right|^2 dx_1 \cdots dx_m \right] \leq m^2 M^2 \left(\frac{1}{n}\right)^{2(\alpha \wedge 1)}. \quad (6)$$

With (1), the second summand on (5) is bounded,

$$\begin{aligned} \left| f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) - f(\xi_{(\sigma(i_1))}, \dots, \xi_{(\sigma(i_m))}) \right|^2 &\leq \left(M \sum_{\ell=1}^m \left| \frac{\sigma(i_\ell)}{n+1} - \xi_{(\sigma(i_\ell))} \right|^{\alpha \wedge 1} \right)^2 \\ &\leq 2M^2 \sum_{\ell=1}^m \left| \frac{\sigma(i_\ell)}{n+1} - \xi_{(\sigma(i_\ell))} \right|^{2(\alpha \wedge 1)}. \end{aligned}$$

Integrating and taking expectation on the second summand yields,

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i_1, \dots, i_m=1}^n \int_{(\sigma(i_1)-1)/n}^{\sigma(i_1)/n} \cdots \int_{(\sigma(i_m)-1)/n}^{\sigma(i_m)/n} \left| f\left(\frac{\sigma(i_1)}{n+1}, \dots, \frac{\sigma(i_m)}{n+1}\right) - f(\xi_{(\sigma(i_1))}, \dots, \xi_{(\sigma(i_m))}) \right|^2 dx_1 \cdots dx_m \right] \\
& \leq \left[\sum_{i_1, \dots, i_m=1}^n \int_{(\sigma(i_1)-1)/n}^{\sigma(i_1)/n} \cdots \int_{(\sigma(i_m)-1)/n}^{\sigma(i_m)/n} 2M^2 \sum_{\ell=1}^m \left| \frac{\sigma(i_\ell)}{n+1} - \xi_{(\sigma(i_\ell))} \right|^{2(\alpha \wedge 1)} dx_1 \cdots dx_m \right] \\
& = 2mM^2 \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \left| \frac{i}{n+1} - \xi_{(i)} \right|^{2(\alpha \wedge 1)} \right] \\
& \leq 2mM^2 \max_{i=1, \dots, n} \mathbb{E} \left[\left| \frac{i}{n+1} - \xi_{(i)} \right|^{2(\alpha \wedge 1)} \right] \\
& \leq 2mM^2 \max_{i=1, \dots, n} [\text{Var}(\xi_{(i)})]^{\alpha \wedge 1} \leq C \left(\frac{1}{n} \right)^{\alpha \wedge 1}, \tag{7}
\end{aligned}$$

where we have used $\mathbb{E}(\xi_{(\ell)}) = \ell/(n+1)$, $\text{Var}(\xi_{(\ell)}) \leq C/n$ and Jensen's inequality. Combining (6) and (7) proves the lemma. \square

References

Jan-Christian Hitter Phillippe Rigollet. High dimensional statistics. *Lecture notes for course 18S997*, 2015.