α-Hölder smoothness

Chanwoo Lee, August 13, 2021

For one variate function, many papers follow definition of α-Hölder class in [2]. The following properties focuse on $\alpha \in (0, \infty)$.

Definition 1 (Univariate case [2])**.** Let $\alpha \in (0, \infty)$. A function $f\colon \mathcal{D} \to \mathbb{R}$ is in α-Hölder class if

1. $\sup_{x\in\mathcal{D}} |f^{(j)}(x)| < \infty$, for all $j = 0, 1, \ldots, \lfloor\alpha\rfloor$.
2. $\sup_{x\neq x'\in\mathcal{D}} \frac{|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(x')|}{|x-x'|^{\alpha-\lfloor\alpha\rfloor}} < \infty$.

Define α-Hölder norm

$$\|f\|_{\mathcal{H}_\alpha} = \max_{j\leq\lfloor\alpha\rfloor} \sup_{x\in\mathcal{D}} |f^{(j)}(x)| + \sup_{x\neq x'\in\mathcal{D}} \frac{|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(x')|}{|x-x'|^{\alpha-\lfloor\alpha\rfloor}}.$$

Remark 1. When $\mathcal{D}$ is a compact set (e.g. closed interval), the first condition in Definition (1) can be dropped. The existence of $f^{(\lfloor\alpha\rfloor)}$ directly implies the boundness of lower-order derivatives.

Notice we have equivalent definition of Hölder class with Definition 1 using α-Hölder norm.

Definition 1* (Variation for univariate case)**.** A function $f\colon \mathcal{D} \to \mathbb{R}$ is in Hölder class if

$$\|f\|_{\mathcal{H}_\alpha} \leq M,$$

for some $M = M(\alpha) > 0$.

Now consider multivariate function $f\colon \mathcal{X} \to \mathbb{R}$, where $\mathcal{X} = \mathcal{D}^d$. For multi-index κ, we denote partial derivatives,

$$\nabla_\kappa f(x) = \frac{\partial^{|\kappa|} f(x)}{(\partial x)^\kappa}.$$

Then we define multivariate version of α-Hölder class as in [3],

Definition 2 (Multivariate case [3])**.** Define α-Hölder norm with respect to norm $\|\cdot\|$,

$$\|f\|_{\mathcal{H}_\alpha} = \max_{\boldsymbol{j}\colon |\boldsymbol{j}|\leq\lfloor\alpha\rfloor} \sup_{\boldsymbol{x}\in\mathcal{X}} |\nabla_{\boldsymbol{j}} f(\boldsymbol{x})| + \max_{\kappa=\lfloor\alpha\rfloor} \sup_{\boldsymbol{x}\neq\boldsymbol{x}'\in\mathcal{D}} \frac{|\nabla_\kappa f(\boldsymbol{x}) - \nabla_\kappa f(\boldsymbol{x}')|}{\|\boldsymbol{x}-\boldsymbol{x}'\|^{\alpha-\lfloor\alpha\rfloor}}.$$

Then f is in α-Hölder class if $\|f\|_{\mathcal{H}_\alpha} \leq M$, for some $M > 0$.

Remark 2. In [3], they choose to use $\|\cdot\|$ as zero norm when they define α-Hölder norm.

In a relation to α-Hölder class, we define α-Hölder smooth function. This smoothness definition is equivalent to α-Hölder class [1] (see Lemma 1 for one way) and is used in [4].

Definition 2* (α-Hölder smooth [4])**.** A function $f\colon \mathcal{X} \to \mathbb{R}$ is α-Hölder smooth with respect to $\|\cdot\|$ if there exists a polynomial $P_k(\cdot - \boldsymbol{x}_0)$ of degree $k = \lfloor\alpha\rfloor$, such that

$$|f(\boldsymbol{x}) - P_k(\boldsymbol{x}-\boldsymbol{x}_0)| \leq c\|\boldsymbol{x}-\boldsymbol{x}_0\|^\alpha, \text{ for all } \boldsymbol{x}, \boldsymbol{x}_0 \in \mathcal{X}.$$

Lemma 1. A function f is in α-Hölder class, if and only if f is a α-Hölder smooth.

Proof. Here we prove only one way. Let $P_k(\cdot - \boldsymbol{x}_0)$ be the Taylor polynomial of degree $\lfloor\alpha\rfloor$,

$$P_k(\cdot - \boldsymbol{x}_0) = \sum_{\kappa\colon |\kappa|\le\lfloor\alpha\rfloor} \frac{\nabla_\kappa f(\boldsymbol{x}_0)}{\kappa}(\boldsymbol{x}-\boldsymbol{x}_0)^\kappa.$$

Then, $P_k(\cdot - \boldsymbol{x}_0)$ satisfies,

$$\begin{aligned}
|f(\boldsymbol{x}) - P_k(\boldsymbol{x}-\boldsymbol{x}_0)| &= \sum_{\kappa\colon |\kappa|=\lfloor\alpha\rfloor} \frac{|\nabla_\kappa f(\boldsymbol{z}) - \nabla_\kappa f(\boldsymbol{x}_0)|}{\kappa!}(\boldsymbol{x}-\boldsymbol{x}_0)^\kappa,\ \text{where } \boldsymbol{z} = \boldsymbol{x}_0 + c(\boldsymbol{x}-\boldsymbol{x}_0), c\in(0,1),\\
&\lesssim \sum_{\kappa\colon |\kappa|=\lfloor\alpha\rfloor} \frac{|\nabla_\kappa f(\boldsymbol{z}) - \nabla_\kappa f(\boldsymbol{x}_0)|}{\kappa!}\|\boldsymbol{x}-\boldsymbol{x}_0\|^{\lfloor\alpha\rfloor}\\
&\le \sum_{\kappa\colon |\kappa|=\lfloor\alpha\rfloor} \frac{\|\boldsymbol{x}-\boldsymbol{x}_0\|^{\alpha-\lfloor\alpha\rfloor}}{\kappa!}\|\boldsymbol{x}-\boldsymbol{x}_0\|^{\lfloor\alpha\rfloor}\\
&\le M_\alpha\|\boldsymbol{x}-\boldsymbol{x}_0\|^\alpha.
\end{aligned}$$

□

Important fact:

1. α-Hölder smoothness is the same as α-Lip smoothness only if $\alpha \in (0,1)$. These two smoothness notions are different when $\alpha \ge 1$.
2. The smoothness index depends on the $\mathcal{D}$. E.g. function $f(x) = x^2$ is ∞-Hölder smooth at $x \in [0,1]$, but it is 0-Hölder smooth at $x \in \mathbb{R}$.
3. For a given function $f(x)$, an easy way to determine $\alpha \in (0,\infty)$ is to take derivative until it blows up. E.g. $f(x) = \log x$ is 0-Hölder at $x = 0$ and ∞-Hölder everywhere else.

References

[1] Patrik Andersson. Characterization of pointwise hölder regularity. *Applied and Computational Harmonic Analysis*, 4(4):429–443, 1997.

[2] Lawrence C Evans. Partial differential equations and monge-kantorovich mass transfer. *Current developments in mathematics*, 1997(1):65–126, 1997.

[3] Chao Gao, Yu Lu, and Harrison H Zhou. Rate-optimal graphon estimation. *The Annals of Statistics*, 43(6):2624–2652, 2015.

[4] Olga Klopp, Alexandre B Tsybakov, and Nicolas Verzelen. Oracle inequalities for network models and sparse graphon estimation. *The Annals of Statistics*, 45(1):316–354, 2017.