Polynomial-time estimation of permutation equivarant tensors

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1 Permutation equivarant tensor model

Given a symmetric tensor $\Theta \in \mathbb{R}^{d\times\cdots\times d}$ and a permutation $\sigma\colon [d]\to[d]$, we use $\Theta\circ\sigma$ to denote the permuted tensor

$$(\Theta\circ\sigma)(i_1,\ldots,i_m)=\Theta(\sigma(i_1),\ldots,\sigma(i_m)),\quad \text{for all } (i_1,\ldots,i_m)\in[d]^m.$$

Definition 1 (Lipschitz smooth tensor)**.**

$$\mathcal{P}(L)=\left\{\Theta\colon\ |\Theta(\omega)-\Theta(\omega')|\leq \frac{L|\omega-\omega'|_1}{d},\ \text{for all } \omega,\omega'\in[d]^m\right\}. \tag{1}$$

For simplicity, we consider $L=1$ thoughout this note.

Model 1 (Permutated tensor model)**.** Let $\mathcal{Y}\in\mathbb{R}^{d\times\cdots\times d}$ be a data tensor generated from the model

$$\mathcal{Y}=\Theta\circ\sigma+\mathcal{E} \tag{2}$$

where $\Theta\in\mathcal{P}(L)$ is an unknown structured tensor, $\sigma\colon[d]\to[d]$ is an unknown permutation, and $\mathcal{E}\in\mathbb{R}^{d\times\cdots\times d}$ is a noise tensor consisting of zero-mean standard normal entries.

Remark 1 (Random design vs. fixed design)**.** Our results below assume no randomness in the signal tensor Θ. This is the major distinction between our model and the hypergraphon model. In the graphon model, the data tensor $\mathcal{Y}$ has two randomnesses: one from the noise tensor $\mathcal{E}$, and the other from signal tensor Θ,

$$\Theta(i_1,\ldots,i_m)=f(U_{i_1},\ldots,U_{i_m}),\quad \text{with } (U_{i_\ell})_{\ell\in[m]}\sim_{\text{i.i.d}}\text{Unif}[0,1]. \tag{3}$$

We refer to (3) as the random design, and refer to the grid samples

$$\Theta(i_1,\ldots,i_m)=f\left(\frac{i_1}{d},\ldots,\frac{i_m}{d}\right),\quad \text{for all } (i_1,\ldots,i_m)\in[d]^m, \tag{4}$$

as the fixed design. Our permutation equivarant tensor model specified by (1) and (2) is equivalent to classical hypergraphon model with fixed design (4).

Assumption 1 (Degree-identifiable tensors)**.** Define the degree function $\text{deg}(\cdot)$ associated with a tensor Θ,

$$\begin{aligned}\text{deg}\colon [d]&\to\mathbb{R}\\ i&\mapsto \frac{1}{d^{m-1}}\sum_{i_1,\ldots,i_m}\Theta(i_1,\ldots,i_m)\mathbb{1}(i_1=i)\end{aligned}$$

We call a smooth tensor $\Theta\in\mathcal{P}(L)$ is degree-identifiable, if there exists a constant $\beta\in[0,1]$ and a small tolerance $\varepsilon_d\lesssim d^{-(m-1)/2}$ such that

$$\text{deg}(i)-\text{deg}(j)\gtrsim\left(\frac{i-j}{d}\right)^{1/\beta}-\varepsilon_d,\quad \text{for all } i\geq j\in[d]. \tag{5}$$

Remark 2. The condition (5) assumes the polynomial growth of population degree function up to a small error. The tolerance $\mathcal{O}(d^{-(m-1)/2})$ allows for small fluctuations within statistical accuracy. We call β the signal level, because it quantifies the identifiability of permutation from the degree. A lower value of β implies flatness of the function. We make the convention that a constant degree function is 0-monotonic.

Polynomial-time algorithm for estimating Θ: Input: $\mathcal{Y}$, k; Output: $\hat{\sigma}$ and $\hat{\Theta}_{\text{LS}}$.

1. Sorting: Sort the nodes based on the empirical degree of $\mathcal{Y}$. The sorting returns the node permutation $\hat{\sigma}\colon[d]\to[d]$ for which the degree function associated with $\mathcal{Y}\circ\hat{\sigma}^{-1}$ is non-decreasing in $i\in[d]$.

2. Blocking: Estimate Θ based on block tensor approximation

$$\hat{\Theta}_{\text{LS}} = \text{Block}_k(\mathcal{Y} \circ \hat{\sigma}),$$

where the operator $\text{Block}_k(\cdot)$ converts a tensor to a block tensor with k equal-sized blocks; i.e,

$$\hat{\Theta}_{\text{LS}}(\omega') := \text{Block}_k(\mathcal{Y} \circ \hat{\sigma})(\omega') = \text{Average}\left\{\Theta(\omega) \colon \lfloor \omega k/d \rfloor = \lfloor \omega' k/d \rfloor \right\}, \quad \text{for all } \omega' \in [d]^m.$$

We quantify the estimation error using risk

$$\mathcal{R}(\hat{\Theta}, \Theta) = \frac{1}{d^m}\mathbb{E}_{\mathcal{Y}}\|\hat{\Theta} - \Theta\|_F^2.$$

Theorem 1.1 (Sorting-and-blocking under β-monotonicity of degree)**.** Consider model 2 under Assumption 1. Set $k = d^{\frac{m}{2+m}}$ in Algorithm 1. Then, with probability at least $1 - d^{-1}$,

$$\mathcal{R}(\hat{\Theta}_{\text{LS}}, \Theta) \leq \underbrace{d^{-\frac{2m}{2+m}}}_{\text{statistical error}} + \underbrace{d^{-\beta(m-1)}}_{\text{algorithmic error}}.$$

Remark 3. When $\beta \geq \frac{2m}{(m-1)(m+2)}$, the statistical error dominates the algorithmic error. In this regime, we have

$$\mathcal{R}(\hat{\Theta}_{\text{LS}}, \Theta) \leq d^{-\frac{2m}{2+m}}.$$

The rate agrees with the best possible rate known for this problem [2]. However, the estimate proposed in [2] is based on a combinatoric search with exponentially computational complexity. In contrast, our estimate is polynomial-time solveable. We show that, under the degree monotonicity assumption, our estimate achieves both statical accuracy and computational efficiency.

Furthermore, the required β-monotonicity becomes weaker as the tensor order m increases. Recall that a lower value of β implies less constrained degree function. We find that the required lower bound threshold $\beta \geq \frac{2m}{(m-1)(m+2)}$ vanishes to zero as $m \to \infty$.

Proof of theorem 1.1. We decompose the estimation error into three terms,

$$\begin{aligned}
\|\hat{\Theta} - \Theta\|_F^2 &\leq \underbrace{\|\text{Block}_k(\mathcal{Y} \circ \hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y} \circ \sigma^{-1})\|_F^2}_{\text{Permutation error; Lemmas 1-2}} + \underbrace{\|\text{Block}_k(\mathcal{Y} \circ \sigma^{-1}) - \text{Block}_k(\Theta)\|_F^2}_{\text{Nonparametric error; Lemma 3}} + \underbrace{\|\text{Block}_k(\Theta) - \Theta\|_F^2}_{\text{Approximation error; Lemma 4}} \\
&\leq d^m \text{Loss}^2(\sigma, \hat{\sigma}) + k^m + \frac{d^m}{k^2} \\
&\leq d^{-\beta(m-1)+m} + k^m + \frac{d^m}{k^2} \\
&\leq d^{-\beta(m-1)+m} + d^{\frac{m^2}{m+2}}
\end{aligned}$$

□

Lemma 1 (Permutation error)**.** Step 1 in the algorithm yields the permutation error

$$\text{Loss}(\sigma, \hat{\sigma}) := \frac{1}{d}\max_{i \in [d]}|\sigma(i) - \hat{\sigma}(i)| \leq d^{-(m-1)\beta/2},$$

with probability at least $1 - \exp(-d)$.

Lemma 2 (Estimation error due to permutation; Lemma 3 in [3])**.** With probability at least $1 - \exp(-d)$,

$$\|\text{Block}_k(\mathcal{Y} \circ \hat{\sigma}^{-1}) - \text{Block}_k(\mathcal{Y} \circ \sigma^{-1})\|_F^2 \leq d^m \text{Loss}^2(\sigma, \hat{\sigma}).$$

Remark 4. Lemma 2 quantifies the estimation error due to permutation error.

Lemma 3 (Denoising error; Lemma 4 in [3])**.** With probability at least $1 - \exp(-d)$,

$$\|\text{Block}_k(\mathcal{Y} \circ \sigma^{-1}) - \text{Block}_k(\Theta)\|_F^2 \leq k^m.$$

Lemma 4 (Approximation error from Lee's 0225 note; corrected Lemma 1 in [3])**.** Suppose the true parameter Θ is from (1). For every fixed integer $k \leq d$, we have

$$\|\text{Block}_k(\Theta) - \Theta\|_F^2 \leq \frac{d^m}{k^2}.$$

2 Proofs

Proof of Lemma 1. By definition, $\deg(i)$ is the sample average of roughly $d^{(m-1)}$ i.i.d. terms except for at most a few diagonal terms. With high probability, the stochastic deviation satisfies

$$\deg(i) - \widehat{\deg}(i) \lesssim d^{-(m-1)/2}.$$

By definition,

$$\deg(1) \leq \deg(2) \leq \cdots \leq \deg(d). \tag{6}$$

The estimated permutation $\hat{\sigma}$ is obtained based on empirical degree of $\mathcal{Y}$. Since the empirical degree of $\mathcal{Y}$ is $\widehat{\deg} \circ \sigma$, we have

for simplicity, you could plug in sigma = identity.

$$\widehat{\deg} \circ \sigma \circ \hat{\sigma}^{-1}(1) \leq \widehat{\deg} \circ \sigma \circ \hat{\sigma}^{-1}(2) \leq \cdots \leq \widehat{\deg} \circ \sigma \circ \hat{\sigma}^{-1}(d). \tag{7}$$

Now, for any given index i, we examine the error $|i - \hat{\sigma} \circ \sigma^{-1}(i)|$. By (6) and (7), we have

$$i = |\underbrace{\{j \colon \deg(j) \leq \deg(i)\}}_{=:\mathrm{I}}|, \quad \text{and} \quad \hat{\sigma} \circ \sigma^{-1}(i) = |\underbrace{\{j \colon \widehat{\deg}(j) \leq \widehat{\deg}(i)\}}_{=:\mathrm{II}}|,$$

where $|\cdot|$ denotes the cardinality of the set. We claim that the sets I and II differ only in at most $d^{(m-1)\beta/2}$ elements. To prove this, we partition the nodes in $[d]$ in two cases.

1. long-distance nodes in $\{j \colon |i-j| \gg d^{1-(m-1)\beta/2}\}$. In this case, the ordering of (i,j) remains the same in (7) and (6), i.e,

   $$\deg(i) < \deg(j) \quad \Longleftrightarrow \quad \widehat{\deg}(i) < \widehat{\deg}(j). \tag{8}$$

   The $\Longrightarrow$ in (8) is because

   $$\widehat{\deg}(j) - \widehat{\deg}(i) \geq \underbrace{\left\{\widehat{\deg}(j) - \deg(j)\right\}}_{\leq d^{-(m-1)/2}} - \underbrace{\left\{\widehat{\deg}(i) - \deg(i)\right\}}_{\leq d^{-(m-1)/2}} + \underbrace{\{\deg(j) - \deg(i)\}}_{\gg d^{-(m-1)/2}} > 0,$$

   where the third term in the inequality is due to β-smoothness of $\deg(\cdot)$ and the assumption $|j-i| \gg d^{1-\beta(m-1)/2}$. The other direction in (8) can be similarly proved. Therefore, we conclude that none of long-distance nodes belong to $\mathrm{I} \,\Delta\, \mathrm{II}$.

2. short-distance nodes in $\{j \colon |j-i| \leq d^{1-\beta(m-1)/2}\}$. In this case, (7) and (6) may yield different ordering of (i,j).

Combining the above two cases gives that

$$\{j \colon |j-i| \leq d^{1-\beta(m-1)}\} \supset \mathrm{I}\Delta\mathrm{II}.$$

Therefore,

$$\mathrm{Loss}(\sigma, \hat{\sigma}) := \frac{1}{d} \max_i |\sigma(i) - \hat{\sigma}(i)| \leq \frac{1}{d}|\mathrm{I}\Delta\mathrm{II}| \leq d^{-\beta(m-1)/2}.$$

Index here is defined w.r.t. ground truth ranked list

3 Further thoughts

Step 1 is equivalent to

$$\hat{\tau} = \operatorname*{arg\,min}_{\tau \colon [d] \to [d]} \sum_{i \in [d-1]} \mathrm{dist}(\tau(i), \tau(i+1)), \quad \text{where} \quad \mathrm{dist}(x,y) := |\widehat{\deg} \circ \sigma(x) - \widehat{\deg} \circ \sigma(y)|. \tag{9}$$

The optimization (9) has closed form solution under the degree-based distance function. Specifically, the optimizer of (9) is uniquely determined by the sorting

$$\widehat{\deg} \circ \sigma \circ \hat{\tau}(1) \leq \cdots \leq \widehat{\deg} \circ \sigma \circ \hat{\tau}(d).$$

Can the above framework incorporate the neighborhood estimator in [1,4]? List the corresponding Lemmas 1-4 for the estimator in [4]. Which steps make the estimate [4] less optimal?

References

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