π_1 on $[r_1]$ such that $(z_1^{(0)})_j = \pi_1(z_1)$ for all $j \in S^c$. Thus, for any $a \in [r_1]$

$$
\begin{aligned}
\left\|\theta^*_a - \hat{\theta}_{\pi_1(a)}\right\|_2^2 &= \frac{\sum_{j\in\mathcal{C}_a}\left\|\theta^*_{(z_1)_j} - \hat{\theta}_{(z_1^{(0)})_j}\right\|_2^2}{|\mathcal{C}_a|} \\
&\overset{(38)}{\le} C\frac{\sum_{j=1}^{p_1}\left\|\theta^*_{(z_1)_j} - \hat{\theta}_{(z_1^{(0)})_j}\right\|^2}{p_1/r_1} \\
&\overset{(34)}{\le} CM\frac{r_1}{p_1}\left(r_* + \bar{p}\bar{r}^2 + p_*^{1/2}\bar{r}\right).
\end{aligned} \tag{39}
$$

Then,

$$
\begin{aligned}
l_1^{(0)} &= \frac{1}{p_1}\sum_{j\in[p_1]}\left\|(\mathcal{M}_1(\mathcal{S}))_{(z_1)_j:} - (\mathcal{M}_1(\mathcal{S}))_{\pi_1^{-1}\left((z_1^{(0)})_j\right):}\right\|_2^2 \\
&= \frac{1}{p_1}\sum_{j\in[p_1]}\left\|(\mathcal{M}_1(\mathcal{S}))_{(z_1)_j:} - (\mathcal{M}_1(\mathcal{S}))_{\pi_1^{-1}\left((z_1^{(0)})_j\right):}\right\|_2^2 \cdot \mathbb{I}\{(z_1^{(0)})_j \neq \pi_1(z_1)_j\} \\
&\le C\frac{1}{p_1}\cdot\prod_{k=2}^{d}\lambda_{r_k}^{-2}(\mathbf{M}_k)\sum_{j\in[p_1]}\left\|\theta^*_{(z_1)_j} - \theta^*_{\pi_1^{-1}\left((z_1^{(0)})_j\right)}\right\|_2^2 \cdot \mathbb{I}\{(z_1^{(0)})_j \neq \pi_1(z_1)_j\}\,(\mathcal{M}_1(\mathcal{S}))_{(z_1)_j:} \\
&\overset{(35)}{\le} C\frac{r_{-1}}{p_*}\sum_{j\in[p_1]}\left\|\theta^*_{(z_1)_j} - \theta^*_{\pi_1^{-1}\left((z_1^{(0)})_j\right)}\right\|_2^2 \cdot \mathbb{I}\{(z_1^{(0)})_j \neq \pi_1(z_1)_j\} \\
&\le 2C\frac{r_{-1}}{p_*}\sum_{j\in[p_1]}\left(\left\|\theta^*_{(z_1)_j} - \hat{\theta}_{(z_1^{(0)})_j}\right\|_2^2 + \left\|\hat{\theta}_{(z_1^{(0)})_j} - \theta^*_{\pi_1^{-1}\left((z_1^{(0)})_j\right)}\right\|_2^2\right)\mathbb{I}\{(z_1^{(0)})_j \neq \pi_1(z_1)_j\}. \\
&\le 2C\frac{r_{-1}}{p_*}\left(\sum_{j\in[p_1]}\left\|\theta^*_{(z_1)_j} - \hat{\theta}_{(z_1^{(0)})_j}\right\|_2^2 + \max_{a\in[r_1]}\left\|\hat{\theta}_a - \theta^*_{\pi_1^{-1}(a)}\right\|_2^2\sum_{j\in[p_1]}\mathbb{I}\{(z_1^{(0)})_j \neq \pi_1(z_1)_j\}\right). \\
&\le 2C\frac{r_{-1}}{p_*}\left(\sum_{j\in[p_1]}\left\|\theta^*_{(z_1)_j} - \hat{\theta}_{(z_1^{(0)})_j}\right\|_2^2 + |S|\max_{a\in[r_1]}\left\|\hat{\theta}_a - \theta^*_{\pi_1^{-1}(a)}\right\|_2^2\right) \\
&\overset{(34),(39)}{\lesssim} M\frac{r_{-1}}{p_*}\left(r_* + \bar{p}\bar{r}^2 + p_*^{1/2}\bar{r}\right).
\end{aligned}
$$

Now Theorem 2 follows by applying Lemma 1. □

A.2 Proof of Proposition 1

Without loss of generality, we assume $\sigma = 1$. We start by introducing several notations and assumptions. For each $k = 1, \ldots, d$, denote

new z = z+x-rank_r(x)

$$
\mathbf{X}_k = \mathcal{M}_k(\mathcal{X}),\ \mathbf{Z}_k = \mathcal{M}_k(\mathcal{Z}),\ \mathbf{Y}_k = \mathcal{M}_k(\mathcal{Y}).
$$

Recall that $\mathrm{rank}(\mathbf{X}_k) \le r_k$. We further denote $\mathbf{U}_k = \mathrm{SVD}_{r_k}(\mathbf{X}_k)$ and $\tilde{\mathbf{U}}_k = \mathrm{SVD}_{r_k}(\mathbf{Y}_k)$. For some constant C_0 which will be specified later, define

$$
r'_k = \max\left\{r' \in \{0, \ldots, r_k\} : \sigma_{r'}(\mathbf{X}_k) \ge C_0(p_*^{1/4} \vee \bar{p}^{1/2})\right\}.
$$

We set $r'_k = 0$ if $\sigma_1(\mathbf{X}_k) < C_0(p_*^{1/4} \vee \bar{p}^{1/2})$. We use $\mathbf{U}'_k$ to denote the leading r'_k singular vectors of $\mathbf{U}_k$ and use $\mathbf{V}'_k$ to denote the rest $r_k - r'_k$ singular vectors and thus $\mathbf{U}_k$ can be written as $[\mathbf{U}'_k \ \mathbf{V}'_k]$. We next define

$$\mathbf{X}'_k = \mathbf{X}_k \left(\mathbb{P}_{\mathbf{U}'_{k+1}} \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}'_d} \otimes \mathbb{P}_{\mathbf{U}'_1} \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}'_{k-1}} \right)$$

We also denote

$$\begin{aligned}
\bar{\mathbf{Y}}_k &= \mathbf{Y}_k(\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}), \\
\bar{\mathbf{X}}_k &= \mathbf{X}_k(\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}), \\
\bar{\mathbf{Z}}_k &= \mathbf{Z}_k(\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}).
\end{aligned}$$

Now we define the following events under which we conduct the subsequent analysis.

$$A_1 = \left\{ \left\| \tilde{\mathbf{U}}_{k\perp}^\top \mathbf{U}'_k \right\| \le \frac{1}{\sqrt{2}}, \quad k = 1, \ldots, d. \right\} \tag{40}$$

steps needs to verified.
Should be easy using triangular inequality of spectral norm.

$$A_2 = \left\{ \left\| \bar{\mathbf{Z}}_k \right\| \le C(\sqrt{p_k} + \sqrt{r_{-k}} + \sum_{l \neq k} \sqrt{p_l r_l}), \quad k = 1, \ldots, d. \right\} \tag{41}$$

$$A_3 = \left\{ \left\| \mathcal{Z} \times_1 \hat{\mathbf{U}}_1 \times \cdots \times_d \hat{\mathbf{U}}_d \right\|_{\mathrm{F}} \le C(\sqrt{r_*} + \sum_{k=1}^{d} \sqrt{p_k r_k}) \right\} \tag{42}$$

By Lemma 2, with probability at least $1 - C\exp(-\underline{p})$, for each $k \in [d]$,

$$\left\| \tilde{\mathbf{U}}_{k\perp}^\top \mathbf{U}'_k \right\| \le \frac{C\sqrt{p_k}(\sigma_{r'_k}(\mathbf{X}_k) + \sqrt{p_{-k}})}{\sigma^2_{r'_k}(\mathbf{X}_k)} \le \frac{C}{C_0} \left(\frac{\sqrt{p_k}}{\sqrt{\bar{p}}} + \frac{\sqrt{p_*}}{\sqrt{p_*}} \right) \le \frac{1}{\sqrt{2}},$$

where the last inequality is obtained by specifying $C_0 = 2\sqrt{2}C$. Meanwhile, By Lemma 8, $\mathbb{P}(A_2 \cap A_3) \ge 1 - \exp(-c\underline{p})$. Therefore, $\mathbb{P}(A_1 \cap A_2 \cap A_3) \ge 1 - \exp(-c\underline{p})$. Now we prove the Theorem under $A_1 \cap A_2 \cap A_3$.

We provide an upper bound for $\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k \right\|_{\mathrm{F}}$. First of all,

$$\begin{aligned}
\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k \right\|_{\mathrm{F}} &= \left\| \hat{\mathbf{U}}_{k\perp}^\top (\mathbf{X}'_k + \mathbf{X}_k - \mathbf{X}'_k) \right\|_{\mathrm{F}} \\
&\le \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}'_k \right\|_{\mathrm{F}} + \left\| \hat{\mathbf{U}}_{k\perp}^\top (\mathbf{X}_k - \mathbf{X}'_k) \right\|_{\mathrm{F}} \\
&\le \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}'_k \right\|_{\mathrm{F}} + \left\| \mathbf{X}_k - \mathbf{X}'_k \right\|_{\mathrm{F}}.
\end{aligned} \tag{43}$$

To bound $\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}'_k \right\|_{\mathrm{F}}$, we notice that

$$\begin{aligned}
&\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}'_k (\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}) \right\|_{\mathrm{F}} \\
\le &\left\| \hat{\mathbf{U}}_{k\perp}^\top \bar{\mathbf{X}}_k \right\|_{\mathrm{F}} + \left\| \hat{\mathbf{U}}_{k\perp}^\top (\mathbf{X}_k - \mathbf{X}'_k)(\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}) \right\|_{\mathrm{F}} \\
\le &\left\| \hat{\mathbf{U}}_{k\perp}^\top \bar{\mathbf{X}}_k \right\|_{\mathrm{F}} + \left\| \mathbf{X}_k - \mathbf{X}'_k \right\|_{\mathrm{F}}.
\end{aligned} \tag{44}$$

Also, since the right singular space of $\mathbf{X}_k'$ is $\mathbf{U}_{k+1}' \otimes \cdots \otimes \mathbf{U}_d' \otimes \mathbf{U}_1' \otimes \cdots \otimes \mathbf{U}_{k-1}'$, we have

$$\begin{aligned}
&\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' (\tilde{\mathbf{U}}_{k+1} \otimes \cdots \otimes \tilde{\mathbf{U}}_d \otimes \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \tilde{\mathbf{U}}_{k-1}) \right\|_{\mathrm{F}} \\
=& \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' (\mathbb{P}_{\mathbf{U}_k'} \tilde{\mathbf{U}}_k \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}_d'} \tilde{\mathbf{U}}_d \otimes \mathbb{P}_{\mathbf{U}_1'} \tilde{\mathbf{U}}_1 \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}_{k-1}'} \tilde{\mathbf{U}}_{k-1}) \right\|_{\mathrm{F}} \\
\geq& \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' \right\|_{\mathrm{F}} \cdot \prod_{l \neq k} \sigma_{r_k'} (\mathbf{U}_k'^\top \tilde{\mathbf{U}}_k) \qquad (45) \\
=& \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' \right\|_{\mathrm{F}} \cdot \prod_{l \neq k} \sqrt{1 - \left\| \tilde{\mathbf{U}}_{k\perp}^\top \mathbf{U}_k' \right\|^2} \overset{(40)}{\geq} \frac{1}{\sqrt{2}^{d-1}} \left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' \right\|_{\mathrm{F}}.
\end{aligned}$$

Combining (43), (44) and (45), we obtain

$$\begin{aligned}
\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k' \right\|_{\mathrm{F}} &\leq 2^{(d-1)/2} \left(\left\| \hat{\mathbf{U}}_{k\perp}^\top \bar{\mathbf{X}}_k \right\|_{\mathrm{F}} + \left\| \mathbf{X}_k - \mathbf{X}_k' \right\|_{\mathrm{F}} \right) \\
\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k \right\|_{\mathrm{F}} &\leq 2^{(d-1)/2} \left\| \hat{\mathbf{U}}_{k\perp}^\top \bar{\mathbf{X}}_k \right\|_{\mathrm{F}} + (2^{(d-1)/2} + 1) \left\| \mathbf{X}_k - \mathbf{X}_k' \right\|_{\mathrm{F}}.
\end{aligned} \qquad (46)$$

min(r, l), where l = # {singular value of bar Y > threshold}

By Lemma 7, since $\bar{\mathbf{Y}}_k = \bar{\mathbf{X}}_k + \bar{\mathbf{Z}}_k$, $\mathrm{SVD}_{r_k}(\bar{\mathbf{Y}}_k) = \hat{\mathbf{U}}_k$, we have

step needs to modified

$$\left\| \hat{\mathbf{U}}_{k\perp}^\top \bar{\mathbf{X}}_k \right\|_{\mathrm{F}} \leq 2\sqrt{r_k} \left\| \bar{\mathbf{Z}}_k \right\| \overset{(41)}{\lesssim} \sqrt{r_*} + \sum_{l=1}^d \sqrt{p_l r_l \bar{r}}. \qquad (47)$$

Now it suffices to bound $\|\mathbf{X}_k - \mathbf{X}_k'\|_{\mathrm{F}}$. For notation similicity, we focus on $k = 1$, while the analysis for other modes can be similarly carried on.

$$\begin{aligned}
\left\| \mathbf{X}_1 - \mathbf{X}_1' \right\|_{\mathrm{F}} &= \left\| \mathbf{X}_1 \left((\mathbb{P}_{\mathbf{U}_2'} + \mathbb{P}_{\mathbf{V}_2'}) \otimes \cdots \otimes (\mathbb{P}_{\mathbf{U}_d'} + \mathbb{P}_{\mathbf{V}_d'}) - \mathbb{P}_{\mathbf{U}_2'} \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}_d'} \right) \right\|_{\mathrm{F}} \\
&= \left\| \mathbf{X}_1 \left(\mathbb{P}_{\mathbf{V}_2'} \otimes \mathbf{I}_{p_3} \otimes \cdots \otimes \mathbf{I}_{p_d} + \mathbb{P}_{\mathbf{U}_2'} \otimes \mathbb{P}_{\mathbf{V}_3'} \otimes \cdots \otimes \mathbf{I}_{p_d} + \cdots + \mathbb{P}_{\mathbf{U}_2'} \otimes \cdots \otimes \mathbb{P}_{\mathbf{U}_{d-1}'} \otimes \mathbb{P}_{\mathbf{V}_d'} \right) \right\|_{\mathrm{F}} \\
&\leq \sum_{k=2}^d \left\| \mathbf{V}_k'^\top \mathcal{M}_k(\mathcal{X}) \right\|_{\mathrm{F}} \\
&\leq \sum_{k=2}^d \sqrt{r_k - r_k'} \sigma_{r_k'+1}(\mathbf{X}_k) \leq \sum_{k=2}^d C_0 (p_*^{1/4} + \bar{p}^{1/2}) \sqrt{r_k}.
\end{aligned}$$

Here, the last inequality comes from the definition of r_k', i.e., the $r_k' + 1$th singular value of $\mathbf{X}_k$ is smaller than $C_0(p_*^{1/4} \vee \bar{p}^{1/2})$. In general, for any $k \in [d]$, we have

$$\left\| \mathbf{X}_k - \mathbf{X}_k' \right\|_{\mathrm{F}} \leq C_0 d (p_*^{1/4} \vee \bar{p}^{1/2}) \bar{r}^{1/2}. \qquad (48)$$

Combining (46), (47) and (48), it follows that

$$\left\| \hat{\mathbf{U}}_{k\perp}^\top \mathbf{X}_k \right\|_{\mathrm{F}} \leq C_d \left(\sqrt{r_*} + p_*^{1/4} \bar{r}^{1/2} + \bar{p}^{1/2} \bar{r} \right). \qquad (49)$$

Now we are ready to bound $\left\| \hat{\mathcal{X}} - \mathcal{X} \right\|$. Recall that $\hat{\mathcal{X}} = \mathcal{Y} \times_1 \mathbb{P}_{\hat{\mathbf{U}}_1} \times \cdots \times_d \mathbb{P}_{\hat{\mathbf{U}}_d}$. Then,

$$\begin{aligned}
&\left\| \mathcal{Y} \times_1 \hat{\mathbf{U}}_1 \hat{\mathbf{U}}_1^\top \times \cdots \times_d \hat{\mathbf{U}}_d \hat{\mathbf{U}}_d^\top - \mathcal{X} \right\|_{\mathrm{F}} \\
\leq& \left\| \mathcal{X} \times_1 \hat{\mathbf{U}}_1 \hat{\mathbf{U}}_1^\top \times \cdots \times_d \hat{\mathbf{U}}_d \hat{\mathbf{U}}_d^\top - \mathcal{X} \right\|_{\mathrm{F}} + \left\| \mathcal{Z} \times_1 \hat{\mathbf{U}}_1^\top \times \cdots \times_d \hat{\mathbf{U}}_d^\top \right\|_{\mathrm{F}}.
\end{aligned} \qquad (50)$$