Blockwise polynomial approximation to permutation-equivalence tensor model

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1 Results

For notational convenience, we make the convention that blockwise constant tensor is of degree 1 (not 0 as in classical conventions). We use $z\colon [d] \to [k]$ to denote the canonical clustering function that partitions $[d]$ into k equal-sized clusters; i.e.,

$$\begin{aligned} z\colon [d] &\to [k] \\ i &\mapsto z(i) = \lceil ki/d \rceil. \end{aligned}$$

By construction, the inverse images $\{z^{-1}(j)\colon j \in [k]\}$ is a collection of disjoint, equal-sized subsets satisfying $\cup_{j\in[k]} z^{-1}(j) = [d]$. We use $\mathcal{E}_k$ to denote the m-way partition that collects k^m disjoint, equal-sized blocks in $[d]^m$; i.e.,

$$\mathcal{E}_k = \{z^{-1}(j_1) \times \cdots \times z^{-1}(j_m)\colon (j_1, \ldots, j_m) \in [k]^m\}.$$

- blockwise degree-1 (constant) tensor:

$$\begin{aligned} \mathscr{B}(k,1) &= \left\{ \mathcal{B} \in (\mathbb{R}^d)^{\otimes m}\colon \mathcal{B}(\omega) = \sum_{\Delta \in \mathcal{E}_k} c_\Delta \mathbb{1}\{\omega \in \Delta\} \right\} \\ &\cong \mathbb{R}^{k^m}, \end{aligned}$$

  where, for each block $\Delta \in \mathcal{E}_k$, the coefficients $c_\Delta \in \mathbb{R}$ represent the block means. Note that there are in total k^m free parameters in $\mathscr{B}(k,1)$, so the parameter space $\mathscr{B}(k,1)$ is isomorphic to the linear space $\mathbb{R}^{k^m}$.

- blockwise degree-2 linear tensor:

$$\begin{aligned} \mathscr{B}(k,2) &= \left\{ \mathcal{B} \in (\mathbb{R}^d)^{\otimes m}\colon \mathcal{B}(\omega) = \sum_{\Delta \in \mathcal{E}_k} \left[c_\Delta + \langle \boldsymbol{\beta}_\Delta, \omega \rangle\right] \mathbb{1}\{\omega \in \Delta\} \text{ for all indices } \omega \in [d]^m \right\} \\ &\cong \mathbb{R}^{(1+m)k^m}, \end{aligned}$$

  where, for each block $\Delta \in \mathcal{E}_k$, the coefficients $(c_\Delta, \boldsymbol{\beta}_\Delta) \in \mathbb{R} \times \mathbb{R}^d$ represent the means and coordinate-wise slopes within blocks. Note that there are in total k^m blocks in $\mathcal{E}_k$, each of which is associated with R^{1+d} free coefficients. By the same argument as before, the parameter space $\mathscr{B}(k,2)$ is isomorphic to the linear space $\mathbb{R}^{(1+m)k^m}$.

- blockwise degree-$(\ell+1)$ polynomial tensor:

$$\mathscr{B}(k,\ell+1)=\left\{\mathcal{B}\in(\mathbb{R}^d)^{\otimes m}\colon \mathcal{B}(\omega)=\sum_{\Delta\in\mathcal{E}_k}\mathrm{Poly}_{\ell,\Delta}(\omega)\mathbb{1}\{\omega\in\Delta\}\text{ for all indices }\omega\in[d]^m\right\}$$
$$\subset\mathbb{R}^{(\ell+m)^\ell k^m},$$

  where, for each block $\Delta\in\mathcal{E}_k$, the polynomial function $\mathrm{Poly}_{\ell,\Delta}(\cdot)$ has at most $(\ell+m)^\ell$ free coefficients. By the same argument as before, the parameter space $\mathscr{B}(k,\ell+1)$ is embedded in the linear space $\mathbb{R}^{(\ell+m)^\ell k^m}$.

Model. Suppose the data tensor $\mathcal{Y}$ is generated from the model

$$\mathcal{Y}=\Theta\circ\pi+\mathcal{E},\quad\text{where}\quad\Theta(i_1,\ldots,i_m)=f\left(\frac{i_1}{d},\ldots,\frac{i_m}{d}\right)\text{ for all }(i_1,\ldots,i_d)\in[d]^m,\tag{1}$$

where $\pi\colon[d]\to[d]$ is an *unknown* permutation, $f\colon\mathbb{R}^m\to\mathbb{R}$ is an *unknown* α-Hölder smooth function with $\alpha\in(0,\infty)$, and $\mathcal{E}$ is a noise tensor with i.i.d. sub-Gaussian entries. We use $\mathcal{P}(\alpha)$ to denote the collection of signal tensors from model (1). The goal is to estimate signal $\Theta\in\mathcal{P}(\alpha)$ from data $\mathcal{Y}$.

The parameters (Θ,π) are not separately identifiable from model (1). However, the tensor $\Theta\circ\pi$ is always identifiable as a composite parameter. We impose the following marginal monotonicity assumption to ensure the separate identifiability.

Theorem 1 (Identifiability)**.** Suppose $f\in\mathcal{M}(\beta)$ with $\beta\in(0,\infty)$. Then, the parameters (Θ,π) are separately identifiable from model (1).

Theorem 2. (Blockwise polynomial tensor approximation) Suppose the function $f\colon[0,1]^m\to\mathbb{R}$ generating the signal tensor Θ is α-Hölder smooth with $\alpha\in(0,\infty)$. Then, for every block size $k\leq d$ and degree $\ell\in\mathbb{N}_+$, we have the approximation error

$$\inf_{\mathcal{B}\in\mathscr{B}(k,\ell)}\frac{1}{d^m}\|\Theta-\mathcal{B}\|_F^2\lesssim\frac{m^2}{k^{2\min(\alpha,\ell)}}.$$

We propose a least-square estimate based on the blockwise polynomial tensor approximation,

$$(\hat{\Theta}^{\mathrm{LSE}},\hat{\pi}^{\mathrm{LSE}})=\mathop{\arg\min}_{\substack{\Theta\in\mathscr{B}(k,\ell)\\ \pi\colon[d]\to[d]}}\|\mathcal{Y}-\Theta\circ\pi\|_F^2.$$

Although not reflected in the notation, the least-square estimate $\hat{\Theta}^{\mathrm{LSE}}$ depends on the tuning parameters (k,ℓ). We provide the optimal choice of (k,ℓ) in the following theorem. We focus on the asymptotic error rates as $d\to\infty$ while treating (m,α) as constants.

Theorem 3 (Least-square estimator)**.** Let $(\hat{\Theta}^{\mathrm{LSE}},\hat{\pi}^{\mathrm{LSE}})$ denote the least-square estimate with degree $\ell^* = \min(\lceil\alpha\rceil, \frac{m(m-1)}{2})$ with block size $k^* = \lceil d^{\frac{m}{m+2\ell^*}}\rceil$. Then, $(\hat{\Theta}^{\text{LSE}}, \hat{\pi}^{\text{LSE}})$ obeys the error bound

$$\begin{aligned}\frac{1}{d^m}\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \Theta \circ \pi\|_F^2 &\lesssim \inf_{(k,\ell)\in[d]\times\mathbb{N}_+} \left\{ \frac{m^2}{k^{2\min(\alpha,\ell)}} + \frac{k^m(\ell+m)^\ell}{d^m} + \frac{\log d}{d^{m-1}} \right\} \\ &\asymp \begin{cases} d^{-\frac{2m\alpha}{m+2\alpha}} & \text{when } \alpha < m(m-1)/2, \\ d^{-(m-1)}\log d & \text{when } \alpha \geq m(m-1)/2. \end{cases}\end{aligned}$$

Remark 1 (Comparison with block tensor approximation)**.** For matrices (i.e., $m = 2$), the optimal polynomial is obtained by block matrix approximation. For order-3 α-smooth tensors the optimal degree and block size are $(\ell^*, k^*) = (3, \lceil d^{1/3}\rceil)$ for all $\alpha \geq 3$. In other words, blockwise quadratic tensors suffice for estimating sufficiently smooth tensors. Further increment of polynomial degree ℓ is of no help for smooth signal estimation.

Theorem 4 (Polynomial-time estimator)**.** Suppose that the signal tensor Θ is generated from model (1) with $f \in \mathcal{H}(\alpha) \cap \mathcal{M}(\beta)$. Let $\hat{\Theta}^{\text{BC}}$ be the estimator in with degree $\ell^* = \min(\lceil\alpha\rceil, \frac{m(m-1)}{2})$ and block size $k^* = \lceil d^{\frac{m}{m+2\ell^*}}\rceil$. Then the estimator $\hat{\Theta}^{\text{BC}}$ satisfies

$$\frac{1}{d^m}\|\hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi\|_F^2 \lesssim d^{-\beta(m-1)} + \begin{cases} d^{-\frac{2m\alpha}{m+2\alpha}} & \text{when } \alpha < m(m-1)/2, \\ d^{-(m-1)}\log d & \text{when } \alpha \geq m(m-1)/2. \end{cases}$$

with very high probability.

Theorem 5 (Minimax lower bound)**.** For any given $\alpha \in (0, \infty)$, the estimation problem based on model (1) obeys the minimax lower bound

$$\inf_{(\hat{\Theta},\hat{\pi})} \sup_{\substack{\Theta\in\mathcal{P}(\alpha) \\ \pi\colon [d]\to[d]}} \mathbb{P}\left(\|\Theta \circ \pi - \hat{\Theta} \circ \hat{\pi}\|_F^2 \geq d^{-\frac{2m\alpha}{m+2\alpha}} + d^{-(m-1)}\log d\right) > 0.8.$$

Remark 2. By comparing Theorems 3 and 5, we find that the constrained least-square estimator achieves the minimax optimal rate.

2 Proofs

Proof of Theorem 3. The proof is similar to theorem 2.1 on note 030721. By Theorem 2, there exists a blockwise polynomial tensor $\mathcal{B} \in \mathscr{B}(k, \ell)$ such that

$$\|\mathcal{B} - \Theta\|_F^2 \lesssim \frac{d^m m^2}{k^{2\min(\alpha,\ell)}}. \tag{2}$$

By the triangle inequality,

$$\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \Theta \circ \pi\|_F^2 \leq 2\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi\|_F^2 + 2\underbrace{\|\mathcal{B} \circ \pi - \Theta \circ \pi\|_F^2}_{\text{Theorem 2}}. \tag{3}$$

Therefore, it suffices to bound $\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi\|_F^2$. By the global optimality of least-square estimator, we have

$$\begin{aligned}
\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi\|_F &\leq \left\langle \frac{\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi}{\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi\|_F},\ \mathcal{E} + (\mathcal{B} \circ \pi - \Theta \circ \pi) \right\rangle \\
&\leq \sup_{\pi, \pi' \colon [d] \to [d]} \sup_{\mathcal{B}, \mathcal{B}' \in \mathscr{B}(k,\ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle + \underbrace{\|\mathcal{B} \circ \pi - \Theta \circ \pi\|_F}_{\text{Theorem 2}}.
\end{aligned}$$

Now, for fixed π, π', the space embedding $\mathscr{B}(k,\ell) \subset \mathbb{R}^{(\ell+m)^\ell k^m}$ implies the space embedding $\{(\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi) \colon \mathcal{B}, \mathcal{B}' \in \mathscr{B}(k,\ell)\} \subset \mathbb{R}^{2(\ell+m)^\ell k^m}$. Therefore, with very high probability,

$$\sup_{\mathcal{B}, \mathcal{B}' \in \mathscr{B}(k,\ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle \lesssim \sup_{\boldsymbol{x} \in \mathbb{R}^{2(\ell+m)^\ell k^m}} \left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2},\ e \right\rangle \lesssim \sqrt{(\ell+m)^\ell k^m},$$

where e is a vector of consistent length that consists of i.i.d. sub-Gaussian entries. By the union bound of Gaussian maxima over countable set $\{\pi, \pi' \colon [d] \to [d]\}$, we obtain

$$\mathbb{E}\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \mathcal{B} \circ \pi\|_F^2 \lesssim (\ell+m)^\ell k^m + d \log d. \tag{4}$$

Combining the inequalities (2), (2) and (2) yields the desired conclusion

$$\mathbb{E}\|\hat{\Theta}^{\mathrm{LSE}} \circ \hat{\pi}^{\mathrm{LSE}} - \Theta \circ \pi\|_F^2 \lesssim \frac{d^m m^2}{k^{2\min(\alpha,\ell)}} + (\ell+m)^\ell k^m + d \log d.$$

□

Proof of Theorem 5. By the definition of the tensor space, we seek the minimax rate ε^2 in the following expression

$$\inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\Theta \in \mathcal{P}(\alpha)} \sup_{\pi \colon [d] \to [d]} \mathbb{P}\left(\|\Theta \circ \pi - \hat{\Theta} \circ \hat{\pi}\|_F^2 \geq \varepsilon^2 \right).$$

On one hand, if we fix permutation $\pi \colon [d] \to [d]$, the problem can be viewed as a classical m-dimensional α-smooth nonparametric regression with d^m sample points. The minimax lower bound is known to be $\varepsilon^2 = d^{-\frac{2m\alpha}{m+2\alpha}}$. On the other hand, if we fix $\Theta \in \mathcal{P}(\alpha)$, the problem become a new type of convergence rate due to the unknown permutation. We refer it to the permutation rate, and will prove that $\varepsilon^2 = d^{-(m-1)} \log d$. Since our target is the sum of the two rate, it suffice to prove the two different rates separately. In the following arguments, we will proceed by this strategy.

Nonparametric rate. The nonparametric rate for α-smooth function is readily available in the literature; see **?**, Example 16 and **?**, Section 2. We state the results here for self-completeness.

Lemma 6 (Minimax rate for α-smooth function estimation)**.** Consider data $(\boldsymbol{x}_1, Y_1), \ldots, (\boldsymbol{x}_N, Y_N)$, where $\boldsymbol{x}_n = (\frac{i_1}{d}, \ldots, \frac{i_m}{d}) \in [0,1]^d$ is the m-dimensional predictor and $Y_n \in \mathbb{R}$ is the scalar response. Consider the observation model

$$Y_n = f(\boldsymbol{x}_n) + \varepsilon_n, \quad \text{with } \varepsilon_n \sim \text{i.i.d. } N(0,1), \quad \text{for all } n \in [N].$$

Assume f is in the α-Holder smooth function class, denoted by $\mathcal{F}(\alpha)$. Then,

$$\inf_{\hat{f}} \sup_{f \in \mathcal{F}(\alpha)} \|f - \hat{f}\|_2 \geq N^{-\frac{2\alpha}{m+2\alpha}}.$$

Our conclusion readily follows from Lemma 6 by taking sample size $N = d^m$ and function norm $\|f - \hat{f}\|_2 = \frac{1}{d^m}\|\Theta - \hat{\Theta}\|_F^2$.

Permutation rate. The permutation rate is obtained by the following two steps. We first show that estimating the unknown permutation for a α-smooth ($\alpha \geq 1$) function is at least as difficult as that for a block tensor ($\alpha = 0$). Then, we prove the permutation rate for the tensor block problem is lower bounded by $d \log d$. For $\alpha \in (0,1)$, the permutation rate is dominated by the nonparametric rate, therefore, we $\square$