α-Hölder smoothness

Chanwoo Lee, August 12, 2021

For one variate function, many papers follow definition of α-Hölder class in [1].

Definition 1 (Univariate case [1])**.** A function $f\colon \mathcal{D} \to \mathbb{R}$ is in Hölder class if

1. $\sup_{x\in\mathcal{D}} |f^{(j)}(x)| \leq \infty$, for all $j = 0, 1, \ldots, \lfloor \alpha \rfloor$.
2. $\sup_{x\neq x'\in\mathcal{D}} \frac{|f^{(\lfloor\alpha\rfloor)}(x)-f^{(\lfloor\alpha\rfloor)}(x')|}{|x-x'|^{\alpha-\lfloor\alpha\rfloor}} \leq \infty$.

Define α-Hölder norm

$$\|f\|_{\mathcal{H}_\alpha} = \max_{j\leq\lfloor\alpha\rfloor} \sup_{x\in\mathcal{D}} |f^{(j)}(x)| + \sup_{x\neq x'\in\mathcal{D}} \frac{|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(x')|}{|x-x'|^{\alpha-\lfloor\alpha\rfloor}}.$$

Notice we have equivalent definition of Hölder class with Definition 1 using α-Hölder norm.

Definition 1* (Variation for univariate case)**.** A function $f\colon \mathcal{D} \to \mathbb{R}$ is in Hölder class if

$$\|f\|_{\mathcal{H}_\alpha} \leq M,$$

for some $M > 0$.

Now consider multivariate function $f\colon \mathcal{X} \to \mathbb{R}$, where $\mathcal{X} = \mathcal{D}^d$. For multi-index κ, we denote partial derivatives,

$$\nabla_\kappa f(x) = \frac{\partial^{|\kappa|} f(x)}{(\partial x)^\kappa}.$$

Then we define multivariate version of α-Hölder class as in [2],

Definition 2 (Multivariate case [2])**.** Define α-Hölder norm with respect to norm $\|\cdot\|$,

$$\|f\|_{\mathcal{H}_\alpha} = \max_{\boldsymbol{j}\colon |\boldsymbol{j}|\leq\lfloor\alpha\rfloor} \sup_{\boldsymbol{x}\in\mathcal{X}} |\nabla_{\boldsymbol{j}} f(\boldsymbol{x})| + \max_{\kappa=\lfloor\alpha\rfloor} \sup_{\boldsymbol{x}\neq\boldsymbol{x}'\in\mathcal{D}} \frac{|\nabla_\kappa f(\boldsymbol{x}) - \nabla_\kappa f(\boldsymbol{x}')|}{\|\boldsymbol{x}-\boldsymbol{x}'\|^{\alpha-\lfloor\alpha\rfloor}}.$$

Then f is in α-Hölder class if $\|f\|_{\mathcal{H}_\alpha} \leq M$, for some $M > 0$.

Remark 1. In [2], they choose to use $\|\cdot\|$ as zero norm when they define α-Hölder norm.

Remark 2. In a relation to α-Hölder class, we define α-Hölder smooth function. We will show that α-Hölder class is included in a collection of α-Hölder smooth functions in Lemma 1.

Definition 3 (α-Hölder smooth)**.** A function $f\colon \mathcal{X} \to \mathbb{R}$ is α-Hölder smooth with respect to $\|\cdot\|$ if there exists a polynomial $P_k(\cdot - \boldsymbol{x}_0)$ of degree $k = \lfloor\alpha\rfloor$, such that

$$|f(\boldsymbol{x}) - P_k(\boldsymbol{x}-\boldsymbol{x}_0)| \leq c\|\boldsymbol{x}-\boldsymbol{x}_0\|^\alpha, \text{ for all } \boldsymbol{x}, \boldsymbol{x}_0 \in \mathcal{X}.$$

Lemma 1. If a function f is in α-Hölder class, then f is a α-Hölder smooth.

Proof. Let $P_k(\cdot - \boldsymbol{x}_0)$ be the Taylor polynomial of degree $\lfloor \alpha \rfloor$,

$$P_k(\cdot - \boldsymbol{x}_0) = \sum_{\kappa:\, |\kappa| \leq \lfloor \alpha \rfloor} \frac{\nabla_\kappa f(\boldsymbol{x}_0)}{\kappa} (\boldsymbol{x} - \boldsymbol{x}_0)^\kappa.$$

Then, $P_k(\cdot - \boldsymbol{x}_0)$ satisfies,

$$\begin{aligned}
|f(\boldsymbol{x}) - P_k(\boldsymbol{x} - \boldsymbol{x}_0)| &= \sum_{\kappa:\, |\kappa| = \lfloor \alpha \rfloor} \frac{|\nabla_\kappa f(\boldsymbol{z}) - \nabla_\kappa f(\boldsymbol{x}_0)|}{\kappa!} (\boldsymbol{x} - \boldsymbol{x}_0)^\kappa, \text{ where } \boldsymbol{z} = \boldsymbol{x}_0 + c(\boldsymbol{x} - \boldsymbol{x}_0), c \in (0,1), \\
&\lesssim \sum_{\kappa:\, |\kappa| = \lfloor \alpha \rfloor} \frac{|\nabla_\kappa f(\boldsymbol{z}) - \nabla_\kappa f(\boldsymbol{x}_0)|}{\kappa!} \|\boldsymbol{x} - \boldsymbol{x}_0\|^{\lfloor \alpha \rfloor} \\
&\leq \sum_{\kappa:\, |\kappa| = \lfloor \alpha \rfloor} \frac{\|\boldsymbol{x} - \boldsymbol{x}_0\|^{\alpha - \lfloor \alpha \rfloor}}{\kappa!} \|\boldsymbol{x} - \boldsymbol{x}_0\|^{\lfloor \alpha \rfloor} \\
&\leq M_\alpha \|\boldsymbol{x} - \boldsymbol{x}_0\|^\alpha.
\end{aligned}$$

□

References

[1] Lawrence C Evans. Partial differential equations and monge-kantorovich mass transfer. *Current developments in mathematics*, 1997(1):65–126, 1997.

[2] Chao Gao, Yu Lu, and Harrison H Zhou. Rate-optimal graphon estimation. *The Annals of Statistics*, 43(6):2624–2652, 2015.