Algorithm modification and group adaptation

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1 Algorithm modification

Previous algorithm is to find optimal $\Theta \in \mathcal{P}_k$ such that

$$\tilde{\Theta}' = \arg\min_{\Theta \in \mathcal{P}_k} \sum_{\omega \in [n]^m} |\mathcal{A}_\omega - \Theta_\omega|^2,$$

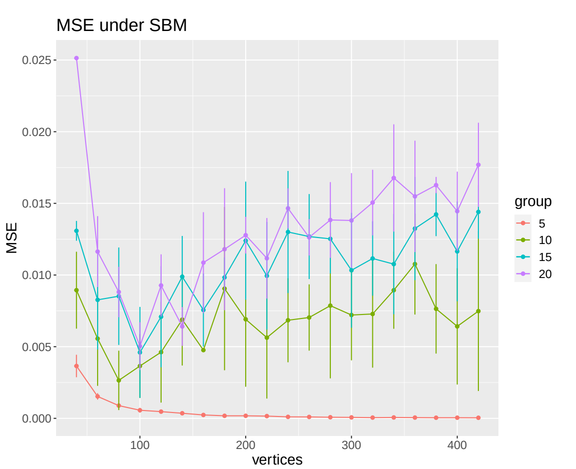
where

$$\mathcal{P}_k = \{\Theta \in ([0,1]^n)^{\otimes m} \colon \Theta = \mathcal{C} \times_2 \boldsymbol{M} \times_2 \cdots \times_m \boldsymbol{M}, \text{ with a membership matrix } \boldsymbol{M} \text{ and a core tensor } \mathcal{C} \in ([0,1]^k)^{\otimes m}\}.$$

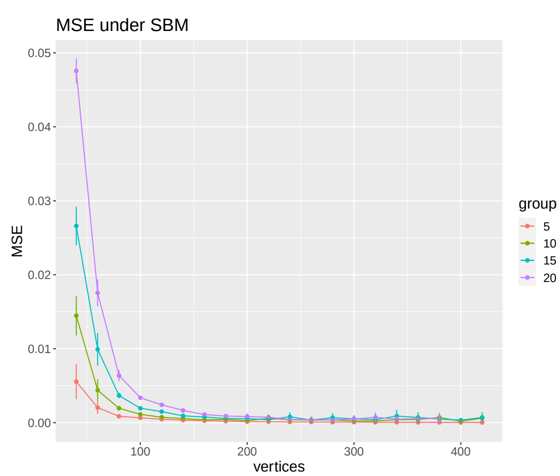
This does not incur any issue when $\Theta^{\text{true}} \in \mathcal{P}_k$, for example, when we allow the self loop in the graph. However, if we consider $\Theta \in \text{cut}(\mathcal{P}_k)$, whose diagonal entries are all 0, $\text{MSE}(\Theta, \tilde{\Theta}')$ does not behave well as the number of vertices n increases. This is because error term from diagonal entries dominates as n increases. Notice that our theorem is based on the estimator

$$\tilde{\Theta} = \arg\min_{\Theta \in \mathcal{P}_k} \sum_{\omega \in E} |\mathcal{A}_\omega - \Theta_\omega|^2,$$

where E is an index set. Notice $E = [n]^m$ when we allow self loop while E is a set without diagonal indices when there is no self loop. To make sure that our estimator behaves as in theorems, we need to make diagonal entries 0 when $\Theta^{\text{true}} \in \text{cut}(\mathcal{P}_k)$. Therefore, I modified the algorithm to give us the output Θ is $\hat{\Theta} = \text{cut}(\tilde{\Theta})$. To be specific, I added an option `diagP`. `diagP==T` means that we allow self loop while `diagP==F` means that all diagonal entries of the probability tensor is 0. Figure 1 shows the improvement of the algorithm when $\Theta^{\text{true}} \in \text{cut}(\mathcal{P}_k)$.



(a) MSE result before the algorithm modification.



(b) MSE result after the algorithm modification.

Figure 1: MSE depending on the group number k and the number of vertices n under stochastic block model when there is no self loop.

2 Case when there are missing values

There are two ways of handling missing values: nonparametric and parametric approach. Denote Ω as an index set of observed entries.

2.1 Nonparametric approach

In this approach, we use loss function based on only the observed entries

$$\tilde{\Theta} = \arg\min_{\Theta\in\mathcal{P}_k} \sum_{\omega\in\Omega} |\mathcal{A}_\omega - \Theta_\omega|^2.$$

We update the core tensor as

$$\mathcal{C}_a^{(t+1)} = \text{Average}\left(\{\mathcal{A}_\omega \colon \boldsymbol{z}^{(t)}(\omega) = a, \omega\in\Omega\}\right) \text{ for } a = (a_1,\ldots,a_m)\in[k]^{\otimes m},$$

where $\boldsymbol{z} = (z_1,\ldots,z_m)\colon [n]^{\otimes m} \to [k]^{\otimes m}$. Here, we assume that membership functions of each mode $z_1,\ldots,z_m$ might differ. For the update for membership functions, we calculate $\mathcal{A}_\ell^{(t)} \in \mathbb{R}^{r\times\cdots\times r\times n\times r\times\cdots\times r}$ as

$$(\mathcal{A}_\ell^{(t)})_{i_1,\ldots,i_{\ell-1},j,i_{\ell+1},\ldots,i_m} = \text{Average}\left(\left\{\mathcal{A}_{j_1,\ldots,j_{\ell-1},j,j_{\ell+1},\ldots,j_d}\colon (z_o)_{j_o}^{(t)} = i_o, \forall o\in[m]\setminus\ell, (j_1,\ldots,j_{\ell-1},j,j_{\ell+1},\ldots,j_m)\in\Omega\right\}\right).$$

Then, we perform the nearest neighbor search to update the membership fucntions.

$$(z_\ell^{(t+1)})(j) = \arg\min_{a_\ell\in[k]} \|(\mathcal{M}_\ell(\mathcal{A}_\ell)_{j:} - \mathcal{M}_\ell(\mathcal{C}^{(t)}))_{a:}\|_F^2.$$

When we calculate the frobenius norm, we ignore the missing entries.

One shortcoming of this method is that some entries of the core tensor $\mathcal{C}$ might not be able to be estimated (Consider the case when all entries of $\mathcal{A}$ whose membership is a are not observed). In addition, we need to think about how to assign initial points with missing entries for example, K-means with missing entries.

2.2 Parametric approach

Another approach is to set sampling probability $p\in[0,1]$. We assume that

$$\mathbb{P}\left(\mathcal{A}_\omega \text{ is observed }\right) = p,$$

and the probability is independent of other entries. If we decode missing entries as 0 in $\mathcal{A}$, we can check that

$$\mathbb{P}(\mathcal{A}_\omega = 1) = p\Theta_\omega^{\text{true}} \tag{1}$$

From (1), we can also view the sampling probability p as sparsity parameter when we have complete observation. Notice the parameter p is interpreted in different ways, this parameter works same in the context of estimation. From known p, we estimate Θ^{true} by

$$\hat{\Theta} = \text{cut}(\tilde{\Theta}), \text{ where } \tilde{\Theta} = \arg\min_{\Theta\in\mathcal{P}_k} \sum_{\omega\in E} |\mathcal{A}_\omega - p\Theta_\omega|^2. \tag{2}$$

With adaptation of the sampling probability p, we have slightly different theorems from previous note. Modified theorem with known sampling probability p are as follows

Theorem 2.1 (Stochastic block model with the sampling probability p)**.** Let $\hat{\Theta}$ be the estimator from

(2). Suppose true probability tensor $\Theta \in \text{cut}(\mathcal{P}_k)$ for fixed block size k Then, there exists two constants $C_1, C_2 > 0$, such that

$$\frac{1}{n^m}\|\hat{\Theta} - \Theta\|_F^2 \leq \frac{C_1}{p}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}}\right),$$

with probability at least $1 - \exp\left(-C_2(n\log k + k^m)\right)$. Furthermore, expected mean square error is bounded by

$$\frac{1}{n^m}\mathbb{E}\|\hat{\Theta} - \Theta\|_F^2 \leq \frac{C}{p}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}}\right),$$

for some constant $C > 0$.

Theorem 2.2 (Hölder continuous hypergraphon model with the sampling probability p)**.** Suppose the true parameter Θ admits the hypergraphon model with $f \in \mathcal{H}(\alpha, L)$ Let $\hat{\Theta}$ be the estimator from (**??**). Then, there exist two constants $C_1, C_2 > 0$ such that,

$$\frac{1}{n^m}\|\hat{\Theta} - \Theta\|_F^2 \leq C_1\left(m^2L^2p^{\frac{-2\alpha}{m+2\alpha}}n^{\frac{-2m\alpha}{m+2\alpha}} + \frac{\log n}{pn^{m-1}}\right),$$

with probability at least $1 - \exp\left(-C_2\left(n\log n + n^{\frac{m^2}{m+2\alpha}}\right)\right)$ uniformly over $f \in \mathcal{H}(\alpha, L)$. Furthermore, the expected mean square error is bounded by

$$\frac{1}{n^m}\mathbb{E}\|\hat{\Theta} - \Theta\|_F^2 \leq C\left(m^2L^2p^{\frac{-2\alpha}{m+2\alpha}}n^{\frac{-2m\alpha}{m+2\alpha}} + \frac{\log n}{pn^{m-1}}\right),$$

for some constant $C > 0$.

Theorem 2.3 (Mean square error of k-piecewise constant hypergraphon with the sampling probability p)**.** Suppose the true parameter Θ admits the form the hypergraphon model with $f \in \mathcal{F}_k$. Let $\hat{\Theta}$ be the estimator from (2). Then, there exists a positive constant $C > 0$ only depending on m and L such that

$$\mathbb{E}\left[\delta^2(f_{\hat{\Theta}}, f)\right] \leq C\left(\frac{1}{p}\left(\frac{k^m}{n^m} + \frac{\log k}{n^{m-1}}\right) + \frac{m}{p^2}\sqrt{\frac{k}{n}}\right).$$

Theorem 2.4 (Mean square error of Hölder continuous hypergraphon with the sampling probability p)**.** Suppose the true parameter Θ admits the form the hypergraphon model with $f \in \mathcal{H}(\alpha, L)$. Let $\hat{\Theta}$ be the estimator from (**??**). Then, there exists a positive constant $C > 0$ only depending on m and L such that

$$\mathbb{E}\left[\delta^2(f_{\hat{\Theta}}, f)\right] \leq C\left(m^2L^2p^{\frac{-2\alpha}{m+2\alpha}}n^{\frac{-2m\alpha}{m+2\alpha}} + \frac{\log n}{pn^{m-1}} + \frac{m^2L^2}{p^2n^\alpha}\right).$$

3 The number of group adaptation

When the generating model is from α-smooth hypergraphon model, we do not have the true number of group k. In this case, we can easily pick $k = \lfloor n^{\frac{m}{m+2\alpha}} \rfloor$ which guarantees the convergence rate $\mathcal{O}(n^{\frac{-2m\alpha}{m+2\alpha}} + \log n/n)$. However, for the stochastic block model with unknown group, we need to set k for the estimation. We have two ways of choosing the parameter k according to different approach for missing value imputation.

3.1 When we take nonparametric approach for missing value

In this case, we can use cross validation and pick the parameter k that minimizes the test MSE error.

3.2 When we take parametric approach for missing value

This approach is similar to 2 folded cross validation except the part that we modify the sampling probability multiplying a half. To be specific, we split the observed entries into two half with probability 1/2 and use one for the training data set and the other for the test dataset. Let Ω_1 be the training set and Ω_2 be the test set from Bernoulli(1/2) sampling. For many different $k \in [n]$, we calculate

$$\hat{\Theta}_k^{\text{test}} = \operatorname*{arg\,min}_{\Theta \in \text{cut}(\mathcal{P}_k)} \sum_{\omega \in \Omega_1} |\mathcal{A}_\omega - \Theta_\omega|^2.$$

We select the parameter which minimizes the MSE error on the test dataset

$$\hat{k} = \operatorname*{arg\,min}_{k \in [n]} \sum_{\omega \in \Omega_2} |\mathcal{A}_\omega - (\hat{\Theta}_k^{\text{test}})_\omega|^2.$$

The final estimation is given by

$$\hat{\Theta} = \operatorname*{arg\,min}_{\Theta \in \text{cut}(\mathcal{P}_{\hat{k}})} \sum_{\omega \in \omega} |\mathcal{A}_\omega - \Theta_\omega|^2. \tag{3}$$

We can show that the convergence rate of the estimator (3) as follows.

Theorem 3.1 (Stochastic block model with the sampling probability p and unknown k)**.** Let $\hat{\Theta}$ be the estimator from (3). Suppose true probability tensor $\Theta \in \text{cut}(\mathcal{P}_k)$ for fixed block size k Then, there exists two constants $C_1, C_2 > 0$, such that

$$\frac{1}{n^m}\|\hat{\Theta} - \Theta\|_F^2 \leq \frac{C_1}{p}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}} + \frac{\log n}{p}\right),$$

with probability at least $1 - \exp\left(-C_2(n\log k + k^m)\right)$. Furthermore, expected mean square error is bounded by

$$\frac{1}{n^m}\mathbb{E}\|\hat{\Theta} - \Theta\|_F^2 \leq \frac{C}{p}\left(\left(\frac{k}{n}\right)^m + \frac{\log k}{n^{m-1}} + \frac{\log n}{p}\right),$$

for some constant $C > 0$.

We have an additional error term $\log n$ and this stems from picking the right number of groups.

4 To do list

1. Write down the proof of the above theorems.
2. Check if we can adapt smoothly to unknown observation rate.
3. Perform more simulations.