

Another polynomial-time estimation algorithm

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1 Rank- \sqrt{d} approximatable

Definition 1 (Rank- \sqrt{d} approximatable tensor). Let Θ be an order-3 tensor. We use $f: [d] \rightarrow \mathbb{R}$ to denote the distance function, in the sense of matrix spectral norm $\|\mathcal{M}(\cdot)\|_{\text{sp}}$, between Θ and its rank- r projection,

$$f(r) = \inf\{\|\mathcal{M}(\Theta - \mathcal{A})\|_{\text{sp}} : \text{Rank}(\mathcal{A}) \leq (r, r, r)\}.$$

The tensor Θ is called rank- \sqrt{d} approximatable, if $f(\sqrt{d}) \leq \sqrt{d}$. Geometrically, the intersection point between two curves $f(r)$ and $g(r) = r$ is smaller than \sqrt{d} .

Equivalently, Θ admits the decomposition

$$\Theta = \mathcal{A} + \mathcal{A}^\perp, \quad \text{s.t.} \quad \text{Rank}(\mathcal{A}) \leq (\sqrt{d}, \sqrt{d}, \sqrt{d}), \quad \text{and} \quad \|\text{Unfold}(\mathcal{A}^\perp)\|_{\text{sp}} \leq \sqrt{d}. \quad (1)$$

Proposition 1 (Smooth matrix). Every Lipschitz smooth matrix is rank- \sqrt{d} approximatable.

Proof of Proposition 1. Let Θ be a Lipschitz smooth matrix. Set $\mathcal{A} = \text{Block}(\Theta, \sqrt{d})$ and $\mathcal{A}^\perp = \Theta - \text{Block}(\Theta, \sqrt{d})$. Then, by approximation theorem,

$$\|\text{Unfold}(\mathcal{A}^\perp)\|_{\text{sp}} \leq \|\mathcal{A}^\perp\|_F \leq \sqrt{\frac{d^2}{d}} = \sqrt{d}.$$

Since \mathcal{A} is of rank at most \sqrt{d} , the decomposition satisfies the condition (1). \square

Conjecture 1 (Higher-order spectral algorithm). Suppose Θ is an order-3, rank- \sqrt{d} approximatable tensor. Then, the rank- \sqrt{d} higher-order spectral algorithm [1] yields the estimate $\hat{\Theta}$ with error bound

$$\mathcal{R}(\hat{\Theta}, \Theta) \lesssim d^{-1}.$$

Intuition: We decompose the error into estimation error and approximation bias

$$\begin{aligned} \|\hat{\Theta} - \Theta\|_F^2 &\leq \|\hat{\Theta} - \mathcal{A}\|_F^2 + \|\mathcal{A}^\perp\|_F^2 \\ &\lesssim \underbrace{(d^{3/2}r + dr^2 + r^3)}_{\text{by Proposition 1 in [1]}} + \underbrace{d[f(r)]^2}_{\leq d^2 \text{ by Assumption 1}} \\ &\lesssim d^2 \text{ if } r \asymp \sqrt{d}. \end{aligned}$$

More careful analysis is needed though, e.g. additive Gaussian vs. Bernoulli models, non-uniqueness of \mathcal{A} and its singular space, etc. Also, the rank choice $\asymp \sqrt{d}$ is meaningful only in asymptotical sense. In practice, we should choose rank $C\sqrt{d}$ where the constant C may depend on actual Θ , noise, etc.

SBM (HOS+iteration)	sort-and-smoothing	square spectral	higher-order spectral (HOS)	NN
$d^{-6/5}$	$d^{-6/5}$ (restricted model)	$d^{-2/3}$	d^{-1} (restricted model)	?

Table 1: Convergence rate for order-3 smooth tensors.

Questions 1. Unlike matrices, not every order-3 smooth tensor is \sqrt{d} -approximatable. How large is the order-3 tensor family that satisfy (1)? Does the signal tensor in our simulations satisfy (1)? How about general order- m tensors? Fill in the rate for NN.

2 Block approximatable

Based on the proof of [1, Proposition 1], Conjecture 1 also applies to the block approximatable tensor. More generally, we aim to carve out the regimes for which HOS algorithm works.

Definition 2 (Block- d^β approximatable tensor). An order- m tensor Θ is called block approximatable with index $\beta \in [0, 1]$ if it admits the decomposition $\Theta = \mathcal{A} + \mathcal{A}^\perp$ satisfying the following two constraints:

1. \mathcal{A} is a d^β -block tensor;
2. \mathcal{A}^\perp has controlled spectral complexity in that

$$\|\mathcal{A}^\perp\|_{\text{sp}} \leq \sqrt{d}, \quad \text{and} \quad \|\text{Unfold}(\mathcal{A}^\perp)\|_{\text{sp}} \leq d^{\frac{m}{4}}. \quad (2)$$

By definition, every tensor is block approximatable with trivial $\beta = 1$. We make the convention that β denotes the minimal block complexity in the decomposition for which the residual tensor satisfy (2).

Proposition 2 (Examples).

- Every Lipschitz smooth matrix is block approximatable with $\beta = 1/2$;
- Every low-rank tensor with bounded factors has $\beta = 0$ (conjecture).
- Gaussian random tensor has $\beta \rightarrow 1$ for every $m \geq 2$ (conjecture).

Remark 2. Not sure which of (2) and (1) has better intuitive interpretation. On one hand, the block assumption on \mathcal{A} is more restricted than the rank assumption. On the other hand, the spectral constraint on \mathcal{A}^\perp in (2) is more relaxed than (1), because $\|\mathcal{A}^\perp\|_{\text{sp}} \leq \|\text{Unfold}(\mathcal{A}^\perp)\|_{\text{sp}}$ [2]. In both cases, we need the $\|\text{Unfold}(\mathcal{A}^\perp)\|_{\text{sp}} \leq d^{m/4}$ for convergence guarantee of HOS algorithm [1, Proposition 1].

Conjecture 2. Suppose Θ is a block approximatable tensor with $\beta \leq \frac{m}{m+2}$. Then the HOS algorithm in [1] with rank specification

$$r_* = \begin{cases} d^{\frac{1}{3}}, & \text{when } m = 2; \\ d^{\frac{1}{2}}, & \text{when } m = 3; \\ d^{\frac{m}{m+2}}, & \text{when } m \geq 4, \end{cases} \quad (3)$$

gives the estimator $\hat{\Theta}$ with error rate

$$\begin{aligned} \mathcal{R}(\Theta, \hat{\Theta}) &\leq d^{-m} \{d^{\frac{m}{2}+\beta} + d^{\beta m} + \min(d^{m-2\beta}, d^{\frac{m}{2}+1})\} \\ &\leq \begin{cases} d^{-\frac{2}{3}}, & \text{when } m = 2; \\ d^{-1}, & \text{when } m = 3; \\ d^{-\frac{2m}{m+2}}, & \text{when } m \geq 4. \end{cases} \end{aligned}$$

Questions 2. What is the rate when $\beta \geq \frac{m}{m+2}$? Implication in the matrix case. Compare with other methods in theory and in simulation. How large is the order-3 tensor family that satisfy (2)? Give two examples of smooth tensors that satisfy and violate this constraint, respectively. How about non-smooth tensors, e.g., single index tensors, glm tensors?

3 Intuition

- Oracle risk:

$$\underbrace{r^m}_{\text{block mean}} + \underbrace{d \log r}_{\text{block position}} \asymp \underbrace{\frac{d^m}{r^2}}_{m\text{-way approximation}}$$

Therefore, the best $r \asymp d^{\frac{m}{m+2}}$. When $m = 2$ (matrix), $r = \sqrt{d}$.

- Oracle Spectral risk:

$$\underbrace{dr + r^m}_{\text{d.f. in spectral method}} \asymp \underbrace{\frac{d^m}{r^2}}_{m\text{-way approximation}}.$$

When $m = 2$, the left hand side is computable by matrix SVD. The best $r = d^{1/3}$.

When $m \geq 3$, no polynomial time algorithm is able to solve exact SVD. The best-so-far polynomial algorithm increases the risk to

$$\underbrace{d^{m/2}r + r^m}_{\text{d.f. in spectral method}} \asymp \underbrace{\frac{d^m}{r^2}}_{m\text{-way approximation}}.$$

Notice the extra cost one has to pay on d when $m \geq 3$. The best r is solved in (3).

- NN risk for $m = 2$:

$$\underbrace{dr}_{\text{d.f. in row-based NN}} \asymp \underbrace{\frac{d^2}{r}}_{\text{row-based approximation}}.$$

The best $r = \sqrt{d}$, which yields the risk $d^{-1/2}$. Why $\frac{d^2}{r}$ on right hand side? Because row-based NN partitions the rows into r groups, but the keep the d columns as they are. The accuracy is suboptimal even when the true two-way clustering patten is known a prior (check...).

References

- [1] Rungang Han, Yuetian Luo, Miaoyan Wang, and Anru R Zhang, *Exact clustering in tensor block model: Statistical optimality and computational limit*, arXiv preprint arXiv:2012.09996 (2020).
- [2] Miaoyan Wang, Khanh Dao Duc, Jonathan Fischer, and Yun S Song, *Operator norm inequalities between tensor unfoldings on the partition lattice*, Linear Algebra and Its Applications **520** (2017), 44–66.