SMM dual problem and simulations

Chanwoo Lee, April 02, 2020

1 SMM dual problem

Let us consider the following primal problem for SMM optimization.

$$
\begin{aligned}
(P) \quad & \min_{U,V} \frac{1}{2}\|UV^T\|^2 + C\sum_{i=1}^N \xi_i \\
\text{subject to} \quad & y_i(\langle UV^T, X_i\rangle + b) \geq 1 - \xi_i, \\
& \xi_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

The Laglange function with multiplier $\boldsymbol{\alpha}$ and $\boldsymbol{\mu}$ is

$$
L_p = \frac{1}{2}\|UV^T\|^2 + C\sum_{i=1}^N \xi_i - \sum_{i=1}^N \alpha_i \left(y_i(\langle UV^T, X_i\rangle + b) - (1 - \xi_i)\right) - \sum_{i=1}^N \mu_i \xi_i, \tag{1}
$$

which we minimize w.r.t. U, V, b, and ξ_i. Setting the respective derivatives to zero, we get

$$
\begin{aligned}
U(V^TV) &= \sum_{i=1}^N \alpha_i y_i X_i V, \qquad (2) \\
(U^TU)V^T &= \sum_{i=1}^N \alpha_i y_i U^T X_i. \\
0 &= \sum_{i=1}^N \alpha_i y_i, \\
\alpha_i &= C - \mu_i, \forall i.
\end{aligned}
$$

From this, we have

$$
\begin{aligned}
UV^T &= \left(\sum_{i=1}^N \alpha_i y_i X_i\right) H_V = H_U \left(\sum_{i=1}^N \alpha_i y_i X_i\right) \\
&= \sum_{i=1}^N \alpha_i y_i H_U X_i H_V, \\
&\text{where } H_U = U(U^TU)^{-1}U^T \text{ and } H_V = V(V^TV)^{-1}V^T.
\end{aligned}
$$

Using this UV^T expression, we have

$$
\begin{aligned}
\frac{1}{2}\|UV^T\|^2 = \langle UV^T, UV^T\rangle &= \frac{1}{2}\langle \sum_{i=1}^N \alpha_i y_i X_i H_V, \sum_{j=1}^N \alpha_j y_j H_U X_j\rangle \qquad (3) \\
&= \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \langle X_i H_V, H_U X_i\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{N}\alpha_i y_i\langle UV^T, X_i\rangle = \langle UV^T, \sum_{i=1}^{N}\alpha_i y_i X_i\rangle &= \langle(\sum_{i=1}^{N}\alpha_i y_i X_i)H_V, \sum_{j=1}^{N}\alpha_j y_j X_j\rangle \\
&= \langle(\sum_{i=1}^{N}\alpha_i y_i X_i)H_V, (\sum_{i=1}^{N}\alpha_i y_i X_i)H_V\rangle \\
&= \langle(\sum_{i=1}^{N}\alpha_i y_i X_i)H_V, H_U(\sum_{i=1}^{N}\alpha_i y_i X_i)\rangle \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j y_i y_j\langle X_i H_V, H_U X_i\rangle.
\end{aligned}
$$

By substituting (3) and (2) into (1), we obtain the Lagrangian dual objective function

$$
\begin{aligned}
L_d &= \sum_{i=1}^{N}\alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j y_i y_j\langle X_i H_V, H_U X_j\rangle \\
&= \sum_{i=1}^{N}\alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j y_i y_j\langle H_U X_i H_V, H_U X_j H_V\rangle
\end{aligned} \tag{4}
$$

We maximize L_d subject to $0 \le \alpha_i \le C$ and $\sum_{i=1}^{N}$. The dual problem (4) shows the intuition about how SMM model captures feature of predictor data X_i: SMM model projects X_i into row space and column space then use projected features as predictors. However, this dual problem is hard to find the optimizer $\boldsymbol{\alpha}$ because U and V are unknown in L_d. Therefore, alternating updates for U and V fixing the other is reasonable.

2 Simulations

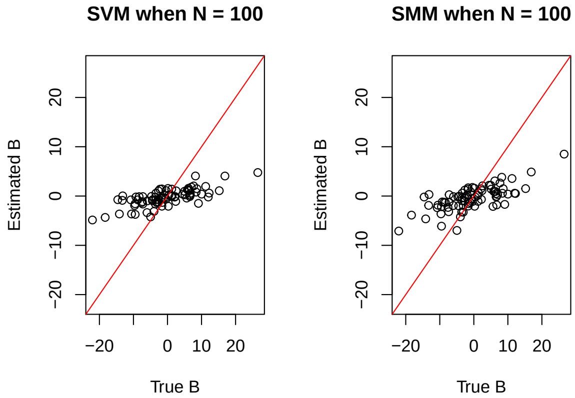
Our training data consists of N pairs $(X_1, y_1), \ldots, (X_N, y_N)$, with $X_i \in \mathbb{R}^{10\times 8}$ and $y_i \in \{-1, 1\}$. We define a hyper plane by $\{X : f(X) = \langle X, B\rangle + 0.1 = 0\}$ where the rank of B is five. A classification rule induced by $f(X)$ is $y_i = \text{sign}\left(f(X_i)\right)$.

I perform three main simulations. In the first simulation, I check the consistency of SMM and SVM estimations. Figure 1 shows both SVM and SMM are consistent estimation because both estimations have small errors as N increases. In addition, we can check SMM outperforms SVM under B being low rank.

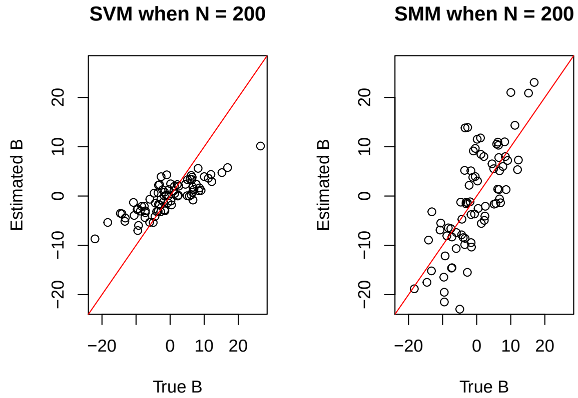
In the second simulation, I check whether SMM method match with SVM when when we assume B as a full rank matrix. The Figure 2 shows that SVM estimator and SMM estimator perfectly match each other under the full rank assumption.

In the last simulation, I do 5 folded cross validation to check prediction performance. Simulation 1 generates a low-rank B and matrix ensembles $\{X_i\}$. I assign $y_i \in \{-1, 1\}$ based the rule, $y_i = \text{sign}(\langle X_i, B\rangle + b)$. Simulation 2 generates data set $(X_1, y_1), \ldots, (X_{200}, y_{200})$ based on the following rule

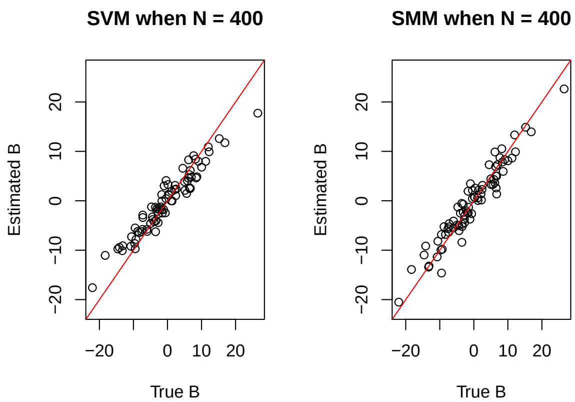
$$
\{(X_i, 1) : X_i = u_1 v_1^T + E_i \quad i = 1, \ldots, 100\} \text{ and } \{(X_j, -1) : X_j = u_2 v_2^T + E_i \quad j = 101, \ldots, 200\},
$$



(a) When the number of data set is $N = 100$



(b) When the number of data set is $N = 200$



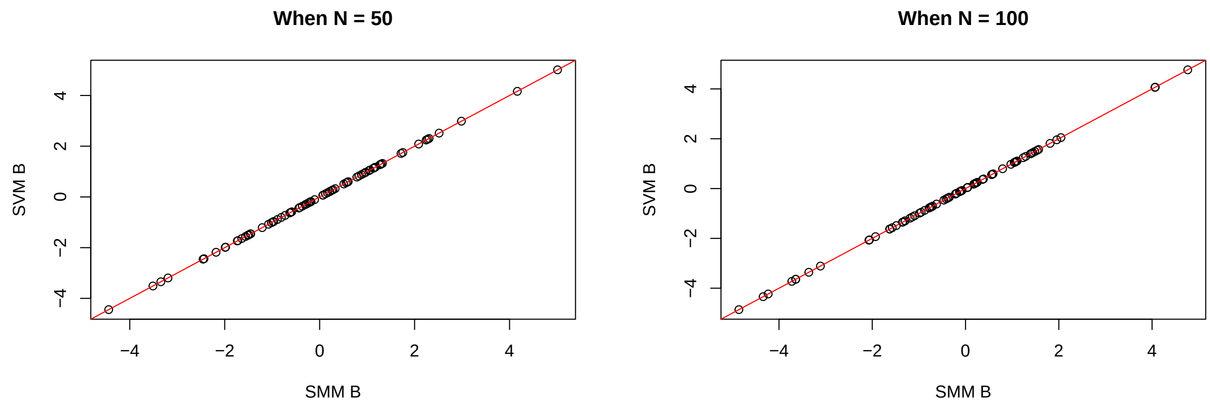
(c) When the number of data set is $N = 400$

Figure 1: I compare true parameter B with estimated parameter $\hat{B}$ under the several number of data sets $N \in \{100, 200, 400\}$. The horizontal axis is entries of B and the vertical axis is entries of $\hat{B}$. For each sub figure, the left figure is for SVM method and the right figure for SMM method.

whose entries of E_i i.i.d from $N(0, 4^2)$. This setting makes the data set inseparable space The following table shows the cross validation results in Simulation 1 and 2. We can check SMM outperforms SVM when the data set is inseparable.

3 An issue for stability

One issue of the current algorithm is that the output depends on initial points quite a lot. Whenever I run the algorithm under the same settings, I get different output values. Figure 3 shows this issue well. Therefore, setting good initial points is needed.



(a) When the number of data set is $N = 50$

(b) When the number of data set is $N = 100$

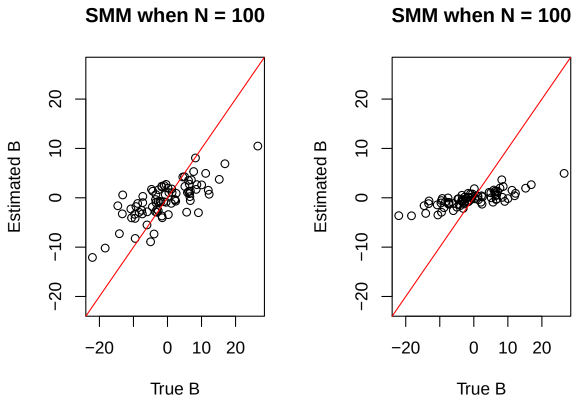
Figure 2: I compare SVM estimator $\hat{B}_{\text{svm}}$ with SMM estimator $\hat{B}_{\text{smm}}$ under the several number of data sets $N \in \{50, 100\}$. The horizontal axis is entries of $\hat{B}_{\text{smm}}$ and the vertical axis is entries of $\hat{B}_{\text{svm}}$.

		CV1	CV2	CV3	CV4	CV5	Mean
Simulation 1	SVM	0.875	0.725	0.675	0.8	0.875	0.790
	SMM	0.725	0.800	0.725	0.8	0.875	0.785
Simulation 2	SVM	0.75	0.700	0.800	0.725	0.625	0.720
	SMM	0.75	0.725	0.775	0.725	0.700	0.735

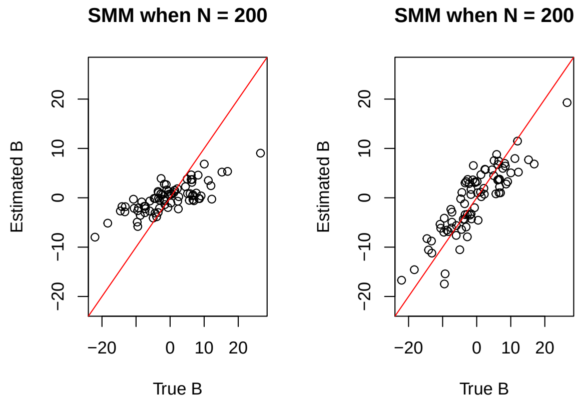
4 R-codes

4.1 Updated R functions

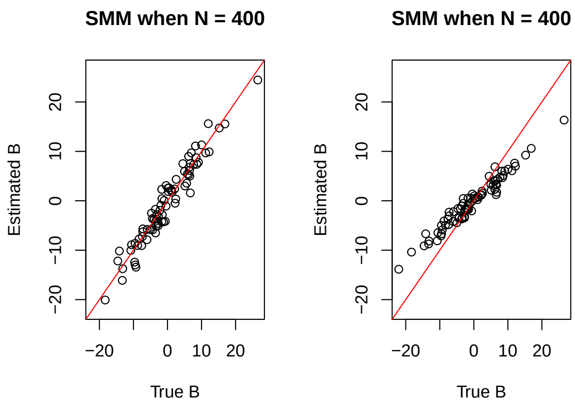
```
library(pracma)
library(quadprog)

eps = 10^-5

sqrtH = function(Us,U){
  h = eigen(Us%*%t(U))
  return(h$vectors%*%diag(sqrt(pmax(h$values,eps))))
}

objv = function(B,b0,X,y,cost = 10){
  return(sum(B*B)/2+cost*sum(pmax(1-y*unlist(lapply(X,function(x) sum(B*x)+b0)),0)
    ))
}

# Generating dataset
gendat = function(m,n,r,N,b0){
  result = list()
  # simulation
  # Weight
  rU = matrix(runif(m*r,-1,1),nrow = m)
  rV = matrix(runif(n*r,-1,1),nrow = n)
  B = rU%*%t(rV)

```



(a) When the number of data set is $N = 100$



(b) When the number of data set is $N = 200$



(c) When the number of data set is $N = 400$

Figure 3: I compare true parameter B with the alternating algorithm for SMM result $\hat{B}$ under the several number of data sets $N \in \{100, 200, 400\}$. The horizontal axis is entries of B and the vertical axis is entries of $\hat{B}$. For each sub figure, We can check that the output differs at each run.

```
# predictor matrix
X = list()
for (i in 1:N) {
  X[[i]] <- matrix(runif(m*n,-1,1),nrow = m,ncol=n)
}

# classification
y = list()
for (i in 1:N) {
  y[[i]] = sign(sum(B*X[[i]])+b0)
}
y = unlist(y)

# predictor vector
x = matrix(nrow =N,ncol = m*n)
for(i in 1:N){
  x[i,] = as.vector(X[[i]])
}
```

```r
  dat = data.frame(y = factor(y), x)

  result$B = B
  result$X = X; result$y = y; result$dat = dat
  return(result)
}


kernelm = function(X,H,y,type = c("u","v")){
  n = length(X)
  x = matrix(unlist(X),nrow = length(X),byrow = T)
  if (type == "u") {
    hx = matrix(unlist(lapply(X,function(x) x%*%H)),nrow = length(X),byrow = T)
  } else {
    hx = matrix(unlist(lapply(X,function(x) H%*%x)),nrow = length(X),byrow = T)
  }
  Q = matrix(nrow = n,ncol = n)
  for (i in 1:n) {
    for(j in i:n){
      Q[i,j] = sum(x[i,]*hx[j,])*y[i]*y[j]
      Q[j,i] = Q[i,j]
    }
  }
  h = eigen(Q)
  Q = (h$vectors)%*%diag(pmax(h$values,eps))%*%t(h$vectors)
  return(Q)
}



smm = function(X,y,r,cost = 10){
  result = list()
  error = 10
  iter = 0
  # SMM
  m= nrow(X[[1]]); n = ncol(X[[1]]); N = length(X)

  #initialization
  U = randortho(m)[,1:r]
  # U = matrix(runif(m*r,-1,1),nrow = m)
  V = randortho(n)[,1:r]
  # V = matrix(runif(n*r,-1,1),nrow = n)
  obj = objv(U%*%t(V),0,X,y,cost);obj

  while((iter <20)&(error>10^-4)){
    # update U fixing V
    Vs = V%*%solve(t(V)%*%V)
    H = Vs%*%t(V)
    dvec = rep(1,length(X))
    Dmat = kernelm(X,H,y,"u")
    Amat = cbind(y,diag(1,N),-diag(1,N))
    bvec = c(rep(0,1+N),rep(-cost,N))
    alpha = solve.QP(Dmat,dvec,Amat,bvec,meq =1)
    Bpart=matrix(t(y*alpha$solution)%*%matrix(unlist(X),nrow = length(X),byrow = T
    ),nrow = m)
    U = Bpart%*%Vs;U


    # update V fixing U
```

```
    Us = U%*%solve(t(U)%*%U)
    H = Us%*%t(U)
    Dmat = kernelm(X,H,y,"v")
    alpha = solve.QP(Dmat,dvec,Amat,bvec,meq = 1)
    Bpart=matrix(t(y*alpha$solution)%*%matrix(unlist(X),nrow = length(X),byrow = T),nrow = m)
    V = t(Bpart)%*%Us;V


    ## intercept estimation
    Bhat = U%*%t(V);Bhat
    b0hat = -(min(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==1)])+
               max(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==-1)]))/2
    obj = c(obj,objv(Bhat,b0hat,X,y,cost));obj
    iter = iter+1
    error = abs(-obj[iter+1]+obj[iter])/obj[iter];error

  }
  predictor = function(x) sign(sum(Bhat*x)+b0hat)
  result$B = Bhat; result$b0 = b0hat; result$obj = obj; result$iter = iter
  result$error = error; result$predict = predictor
  return(result)
}


svm = function(X,y,cost = 10){
  result = list()
  error = 10
  iter = 0
  # SVM
  m= nrow(X[[1]]); n = ncol(X[[1]]); N = length(X)

  H = diag(1,n)
  dvec = rep(1,length(X))
  Dmat = kernelm(X,H,y,"u")
  Amat = cbind(y,diag(1,N),-diag(1,N))
  bvec = c(rep(0,1+N),rep(-cost,N))
  alpha = solve.QP(Dmat,dvec,Amat,bvec,meq =1)
  Bhat=matrix(t(y*alpha$solution)%*%matrix(unlist(X),nrow = length(X),byrow = T),nrow = m)
  b0hat = -(min(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==1)])+
             max(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==-1)]))/2
  obj = objv(Bhat,b0hat,X,y,cost)

  predictor = function(x) sign(sum(Bhat*x)+b0hat)
  result$B = Bhat; result$b0 = b0hat; result$obj = obj;
  result$predict = predictor
  return(result)
}
```

4.2 Simulation codes

```
source("SMMfunctions.R")
set.seed(1818)
m = 10; n = 8; r = 5; N = 800; b0 = 0.1
result = gendat(m,n,r,N,b0)
X = result$X; y = result$y; dat = result$dat
B = result$B
k = 400
```

```r
svmres = svm(X[1:k],y[1:k])
smmres = smm(X[1:k],y[1:k],5);smmres

lim = c(min(B/b0),max(B/b0))

par(mfrow = c(1,2))
plot(B/b0,svmres$B/abs(svmres$b0),xlab = "True B",ylab = "Estimated B",main =
    paste("SVM when N =", k),
      xlim = lim, ylim = lim)
abline(0,1,col = "red")
plot(B/b0,smmres$B/abs(smmres$b0),xlab = "True B",ylab = "Estimated B",main =
    paste("SMM when N =", k),
      xlim = lim, ylim = lim)
abline(0,1,col = "red")


#### consistency test
k = 50
par(mfrow = c(1,1))
svmres = svm(X[1:k],y[1:k])
fsmmres = smm(X[1:k],y[1:k],8)
plot(fsmmres$B/abs(fsmmres$b0),svmres$B/abs(svmres$b0),xlab = "SMM B",ylab = "SVM
    B",main = paste("When N =", k),
      xlim = c(min(fsmmres$B/abs(fsmmres$b0)),max(fsmmres$B/abs(fsmmres$b0))),
      ylim = c(min(svmres$B/abs(svmres$b0)),max(svmres$B/abs(svmres$b0))))
abline(0,1,col = "red")




#### Stability test
k = 100
smmres = smm(X[1:k],y[1:k],5);smmres

lim = c(min(B/b0),max(B/b0))

par(mfrow = c(1,2))
plot(B/b0,smmres$B/abs(smmres$b0),xlab = "True B",ylab = "Estimated B",main =
    paste("SMM when N =", k),
      xlim = lim, ylim = lim)
abline(0,1,col = "red")


smmres = smm(X[1:k],y[1:k],5);smmres
plot(B/b0,smmres$B/abs(smmres$b0),xlab = "True B",ylab = "Estimated B",main =
    paste("SMM when N =", k),
      xlim = lim, ylim = lim)
abline(0,1,col = "red")
```