Necessary condition for matrix-valued kernels

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Theorem 0.1 (Necessary condition). *Suppose $\boldsymbol{K} \colon \mathbb{R}^{d' \times d} \times \mathbb{R}^{d' \times d} \mapsto \mathbb{R}^{d \times d}$ is a function that takes as input a pair of matrices and produces a matrix. Let $\{\boldsymbol{X}_i \in \mathbb{R}^{d' \times d} \colon i \in [n]\}$ denote a set of input matrices, and let $\mathcal{K}$ denote an order-4 (d, d, n, n)-dimensional array,*

$$\mathcal{K} = [\![\mathcal{K}(i, i', p, p')]\!], \quad \text{where } \mathcal{K}(i, i', p, p') \text{ is the } (p, p')\text{-th entry of the matrix } \boldsymbol{K}(\boldsymbol{X}_i, \boldsymbol{X}_{i'}).$$

Then, the factorization $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}') = \boldsymbol{h}(\boldsymbol{X})^T \boldsymbol{h}(\boldsymbol{X}')$ exists for some mapping $\boldsymbol{h}$, only if both of the following conditions hold:

(1) For every index $i \in [n]$, the slice $\mathcal{K}(i, i, \cdot, \cdot) \in \mathbb{R}^{d \times d}$ is a symmetric, positive definite matrix.

(2) For every index $p \in [d]$, the slice $\mathcal{K}(\cdot, \cdot, p, p) \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix.