Necessary condition for matrix-valued kernels

Miaoyan Wang, April 26, 2020

Theorem 0.1 (Necessary condition)**.** *Suppose* $\boldsymbol{K}\colon \mathbb{R}^{d'\times d} \times \mathbb{R}^{d'\times d} \mapsto \mathbb{R}^{d\times d}$ *is a function that takes as input a pair of matrices and produces a matrix. Let* $\{\boldsymbol{X}_i \in \mathbb{R}^{d'\times d}\colon i \in [n]\}$ *denote a set of input matrices, and let* $\mathcal{K}$ *denote an order-4* (n, n, d, d)*-dimensional array,*

$$\mathcal{K} = [\![\mathcal{K}(i, i', p, p')]\!], \quad \textit{where } \mathcal{K}(i, i', p, p') \textit{ is the } (p, p')\textit{-th entry of the matrix } \boldsymbol{K}(\boldsymbol{X}_i, \boldsymbol{X}_{i'}).$$

Then, the factorization $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}') = \boldsymbol{h}(\boldsymbol{X})^T \boldsymbol{h}(\boldsymbol{X}')$ *exists for some mapping* $\boldsymbol{h}$*, only if both of the following conditions hold:*

(1) For every index $i \in [n]$*, the slice* $\mathcal{K}(i, i, \cdot, \cdot) \in \mathbb{R}^{d\times d}$ *is a symmetric, positive definite matrix.*

(2) For every index $p \in [d]$*, the slice* $\mathcal{K}(\cdot, \cdot, p, p) \in \mathbb{R}^{n\times n}$ *is a symmetric, positive definite matrix.*