SMM kernel method and posterior distribution

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1 Strong duality condition

Consider a convex optimization problem

$$\begin{aligned} p^* = \min_{x\in\mathcal{D}} f_0(x) : & f_i(x) \geq 0, \quad i = 1,\ldots,m, \\ & h_i(x) = 0, \quad i = 1,\ldots,p, \end{aligned}$$

where the functions $f_0, f_1, \ldots, f_m$ are convex and $h_1, \ldots, h_p$ are affine. We can have strong duality condition through Slater's condition.

Definition 1 (Slater's condition)**.** We say that the problem satisfies Slater's condition if it is strictly feasible, that is:

$$\exists x_0 \in \mathcal{D} : f_i(x_0) < 0, i = 1,\ldots,m \quad h_i(x_0) = 0, i = 1,\ldots,p$$

We can replace the above by a weak form of Slater's condition, where strict feasibility is not required whenever the function f_i is affine. We then have the

Theorem 1.1 (Slater condition Duality)**.** *If the primal problem is convex, and satisfies the weak Slater's condition, the the strong duality holds.*

There are another sufficient condition for strong duality.

Theorem 1.2 (Quadratic convex optimization duality)**.** *If f_0 is quadratic convex, and the functions $f_1, \ldots, f_m, h_1, \ldots, h_p$ are all affine, then the strong duality holds, provided one of the primal or dual problems is feasible.*

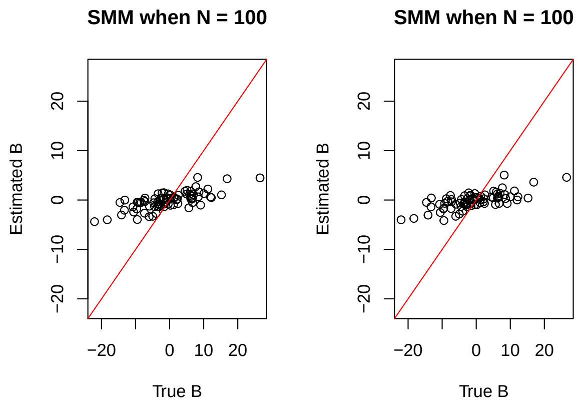
We can apply the second theorem to ensure the strong duality holds in our case.

2 Instability issue of the SMM algorithm

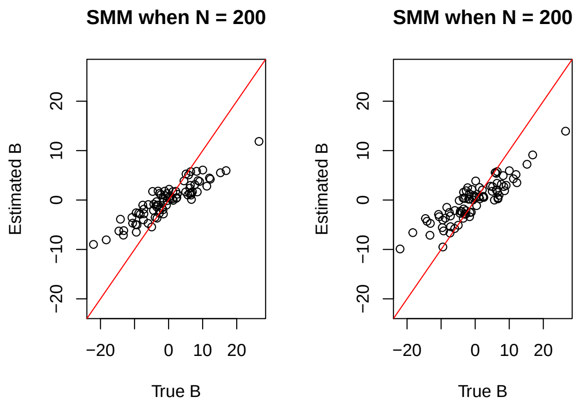
Multiple initialization can solve unstable issue of the SMM algorithm in the last meeting note. In the modified algorithm, we can set multiple initialization method. In this option, the algorithm choose the best output among multiple outputs in respect to loss function value. Figure 1 shows consistent outputs from repetitions.

3 Comparison SVM and SMM

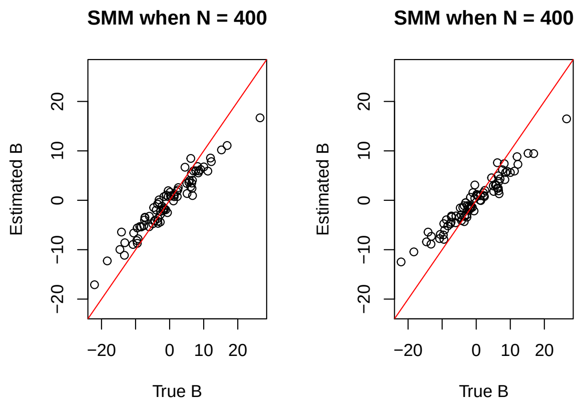
Since there are multiple solutions for the best hyperplane, we have to compare B and $\hat{B}$ with the same scale. I realized I did not compare B and $\hat{B}$ in the right way in the sense that I standardize intercept as 1 not the slope B. If I standardize slope B, Figure 1 changes. Figure 2 is the right figure to compare the parameters. In addition, I compare SVM and SMM method result with $\left\| B/\|B\| - \hat{B}/\|\hat{B}\| \right\|_F$ and Table 1 and Figure 2 shows the result.



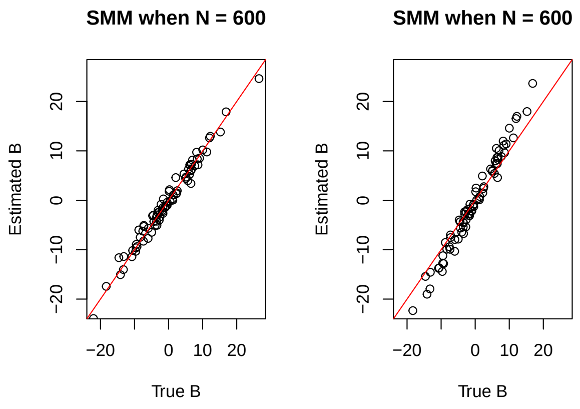
(a) When the number of data set is $N = 100$



(b) When the number of data set is $N = 200$



(c) When the number of data set is $N = 400$



(d) When the number of data set is $N = 600$

Figure 1: True parameter B is compared with multiple initialized SMM result $\hat{B}$ under the several number of data sets $N \in \{100, 200, 400, 600\}$. The horizontal axis is entries of B and the vertical axis is entries of $\hat{B}$. The number of initialization is 10. For each sub figure, we can check that the outputs are pretty much the same.

4 Kernel functions for matrices

We fit the SM classifier using input feature $h(X_i), i = 1, \dots, N$. From this feature, we have the Lagrange dual problem

$$L_D = \sum_{i=1}^{N} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \langle h(X_i), h(X_j) \rangle. \tag{1}$$

By solving (1), we obtain the nonlinear function $\hat{f}(X) = \sum_{i=1}^{N} \alpha_i y_i \langle h(X), h(X_i) \rangle$. Since all related equations require only knowledge of the kernel function,

$$K(X, X') = \langle h(X), h(X') \rangle.$$

	N = 100	N = 200	N = 300	N = 400
Error (SMM)	0.4827625	0.1903077	0.1020411	0.0663940
Error (SVM)	0.5405979	0.3149391	0.1361625	0.0734705

Table 1: The columns are the number of sample size and rows are the Frobenius norm error according to the two methods.

Our goal is to define the kernel function which catches matrix structure well. In SVM case, three popular choices for K are

$$\begin{aligned} \text{dth-Degree polynomial}: \ & K(\boldsymbol{x},\boldsymbol{x}') = (1+\langle \boldsymbol{x},\boldsymbol{x}'\rangle)^d, \\ \text{Radial basis}: \ & K(\boldsymbol{x},\boldsymbol{x}') = \exp(-\gamma\|\boldsymbol{x}-\boldsymbol{x}'\|^2), \\ \text{Sigmoid}: \ & K(\boldsymbol{x},\boldsymbol{x}') = \tanh(\gamma_1\langle \boldsymbol{x},\boldsymbol{x}'\rangle+\gamma_2). \end{aligned}$$

I define two measures to generalize $\langle \boldsymbol{x},\boldsymbol{x}'\rangle$ and $\|\boldsymbol{x}-\boldsymbol{x}'\|^2$ into matrices case not vectorizing matrices. For two matrices $X, X' \in \mathbb{R}^{m\times n}$, we have singular value decomposition of two matrices.

$$X = \sum_{k=1}^{m\vee n}\sigma_k u_k v_k^T \quad \text{and} \quad X' = \sum_{k=1}^{m\vee n}\sigma_k' u_k' (v_k')^T.$$

From this notation, I define weighted inner product between two matrices.

$$\langle X, X'\rangle_M = \sum_{k=1}^{m\vee n}\sigma_k\sigma_k'\langle u_k, u_k'\rangle\langle v_k, v_k'\rangle. \tag{2}$$

In (2), $\sigma\sigma'$ works as weight on principal inner products of subspace and $\langle u,u'\rangle, \langle v,v'\rangle$ represent principal inner product in column space and row space respectively. From this new definition, we can generalize d-th degree polynomial kernel and sigmoid kernel into matrices case.

$$\begin{aligned} \text{dth-Degree polynomial}: \ & K(X,X') = (1+\langle X,X'\rangle_M)^d, \\ \text{Sigmoid}: & K(X,X') = \tanh(\gamma_1\langle X,X'\rangle_M+\gamma_2). \end{aligned}$$

In addition to inner product, we can define weighted matrices distance as

$$\|X-X'\|_M^2 = \sum_{k=1}^{m\vee n}\sigma_k\sigma_k'(\|u_k-u_k'\|^2+\|v_k-v_k'\|^2). \tag{3}$$

In (3), $\sigma\sigma'$ works as weight on principal row and column distances. $\|u_k-u_k'\|^2$ and $\|v_k-v_k'\|^2$ are column-wise and row-wise distances between principal vectors. With this definition we define generalized Radial basis kernel as

$$\text{Radial basis}: \ K(X,X') = \exp(-\gamma\|X-X'\|^2).$$

If two vectors $\boldsymbol{x},\boldsymbol{x}'$ are expressed as

$$\boldsymbol{x} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\|\boldsymbol{x}\|\cdot 1 \quad \text{and} \quad \boldsymbol{x}' = \frac{\boldsymbol{x}'}{\|\boldsymbol{x}'\|}\|\boldsymbol{x}'\|\cdot 1$$

We can check those definitions are consistent to vector case as follows

$$\begin{aligned} \langle \boldsymbol{x},\boldsymbol{x}'\rangle_M &= \|\boldsymbol{x}\|\|\boldsymbol{x}'\|\langle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|},\frac{\boldsymbol{x}'}{\|\boldsymbol{x}'\|}\rangle\langle 1,1\rangle = \langle \boldsymbol{x},\boldsymbol{x}'\rangle, \\ \|\boldsymbol{x}-\boldsymbol{x}'\|_M &= \|\boldsymbol{x}\|\|\boldsymbol{x}'\|\left(\left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}-\frac{\boldsymbol{x}'}{\|\boldsymbol{x}'\|}\right\|_2+\|1-1\|\right) = \|\boldsymbol{x}-\boldsymbol{x}'\| \end{aligned}$$

5 Weighted binary classification

To obtain posterior distribution given feature data, we solve the regularization problem based on the weighted hinge loss.

$$\min_{\boldsymbol{\beta},\boldsymbol{\xi}} \frac{1}{2}\|\boldsymbol{\beta}\|^2 + C\left[(1-\pi)\sum_{y_i=1}\xi_i + \pi\sum_{y_i=-1}\xi_i\right] \tag{4}$$
$$\text{subject to } y_i(\langle x_i,\boldsymbol{\beta}\rangle + b) \geq 1-\xi_i,$$
$$\xi_i \geq 0.$$

The related dual problem for (4) is

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^N \alpha_i - \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle, \tag{5}$$
$$\text{subject to } 0 \leq \alpha_i \leq C(1-\pi) \text{ for } y_i = 1,$$
$$0 \leq \alpha_i \leq C\pi \text{ for } y_i = -1,$$
$$\sum_{i=1}^N \alpha_i y_i = 0.$$

From the solution of (5), we can find the primal solution as $\boldsymbol{\beta} = \sum_{i=1}^N y_i\alpha_i\boldsymbol{x}_i$. With this relation, `smm` in R-codes section solves the equation (4). Figure 4 shows the weighted hinge loss SVM classifier with $\pi \in \{0.001, 0.5, 0.999\}$. It is known that the minimizer $\text{sign}(f_\pi(x))$ to Equation (4) is a consistent estimate of $\text{sign}(\mathbb{P}(y=1|x)-\pi)$. Therefore solving Equation (4) using different π values such that $\pi_1 < \cdots < \pi_m$, we can estimate

$$\hat{\mathbb{P}}(y=1|x) = \frac{1}{2}\left(\arg\max_{\pi_j}\{\text{sign}(f_{\pi_j}(x)) = 1\} + \arg\max_{\pi_j}\{\text{sign}(f_{\pi_j}(x)) = -1\}\right). \tag{6}$$

Figure 3 shows the posterior probability estimation with the rule of (6).

6 One issue for posterior estimation

I found one issue to estimate posterior probability $\mathbb{P}(y=1|\boldsymbol{x})$. There are some points $\boldsymbol{x}_i$'s such that $\text{sign}(\mathbb{P}(y|\boldsymbol{x}_i)-\pi)$ is not decreasing in respect to π. We can check that the red point in Figure 4 has $\text{sign}(\mathbb{P}(y=1|\boldsymbol{x})-0.0001) = -1$ but $\text{sign}(\mathbb{P}(y=1|\boldsymbol{x})-0.9999) = 1$ which does not make sense. This phenomenon happens in all points located below classification boundary when $\pi = 0.0001$ and above the boundary when $\pi = 0.9999$ at the same time. In addition, this area is inevitable unless two classification boundaries are parallel which is hard to be satisfied. If I stick to the rule in (6), all points in the area has 0.5 as posterior probability.

7 R-codes

7.1 Updated functions

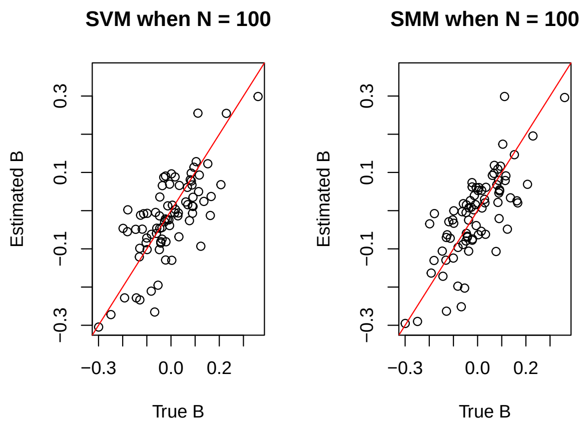
```r
library(pracma)
library(quadprog)

eps = 10^-5



objv = function(B,b0,X,y,cost = 10,prob = F){
  if (prob == F) {
    value = sum(B*B)/2+cost*sum(pmax(1-y*unlist(lapply(X,function(x) sum(B*x)+b0))
   ,0))
  }else{
    ind = which(y==1)
    value = sum(B*B)/2 +
      (1-prob)*cost*sum(pmax(1-y[ind]*unlist(lapply(X[ind],function(x) sum(B*x)+b0
   )),0)) +
      prob*cost*sum(pmax(1-y[-ind]*unlist(lapply(X[-ind],function(x) sum(B*x)+b0))
   ,0))

  }
  return(value)
}

# Generating dataset
gendat = function(m,n,r,N,b0){
  result = list()
  # simulation
  # Weight
  rU = matrix(runif(m*r,-1,1),nrow = m)
  rV = matrix(runif(n*r,-1,1),nrow = n)
  B = rU%*%t(rV)

  # predictor matrix
  X = list()
  for (i in 1:N) {
    X[[i]] <- matrix(runif(m*n,-1,1),nrow = m,ncol=n)
  }

  # classification
  y = list()
  for (i in 1:N) {
    y[[i]] = sign(sum(B*X[[i]])+b0)
  }
  y = unlist(y)

  # predictor vector
  x = matrix(nrow =N,ncol = m*n)
  for(i in 1:N){
    x[i,] = as.vector(X[[i]])
  }
  dat = data.frame(y = factor(y), x)

  result$B = B
  result$X = X; result$y = y; result$dat = dat
  return(result)
}


kernelm = function(X,H,y,type = c("u","v")){
  n = length(X)
  x = matrix(unlist(X),nrow = length(X),byrow = T)
  if (type == "u") {
    hx = matrix(unlist(lapply(X,function(x) x%*%H)),nrow = length(X),byrow = T)
  } else {
    hx = matrix(unlist(lapply(X,function(x) H%*%x)),nrow = length(X),byrow = T)
  }
  Q = matrix(nrow = n,ncol = n)
  for (i in 1:n) {
    for(j in i:n){
      Q[i,j] = sum(x[i,]*hx[j,])*y[i]*y[j]
      Q[j,i] = Q[i,j]
    }
  }
  h = eigen(Q)
  Q = (h$vectors)%*%diag(pmax(h$values,eps))%*%t(h$vectors)
  return(Q)
}




## SMM with multiple initialization
smm = function(X,y,r,cost = 10,rep = 10){
  result = list()

  # SMM
  m= nrow(X[[1]]); n = ncol(X[[1]]); N = length(X)

  compareobj = 10^100
  for (i in 1:rep) {
    error = 10
    iter = 0
    #initialization
    U = randortho(m)[,1:r]
    # U = matrix(runif(m*r,-1,1),nrow = m)
    V = randortho(n)[,1:r]
    # V = matrix(runif(n*r,-1,1),nrow = n)
    obj = objv(U%*%t(V),0,X,y,cost);obj

    while((iter <20)&(error>10^-3)){
      # update U fixing V
      Vs = V%*%solve(t(V)%*%V)
      H = Vs%*%t(V)
      dvec = rep(1,length(X))
      Dmat = kernelm(X,H,y,"u")
      Amat = cbind(y,diag(1,N),-diag(1,N))
      bvec = c(rep(0,1+N),rep(-cost,N))
      alpha = solve.QP(Dmat,dvec,Amat,bvec,meq =1)
      Bpart=matrix(t(y*alpha$solution)%*%matrix(unlist(X),nrow = length(X),byrow =
     T),nrow = m)
      U = Bpart%*%Vs


      # update V fixing U
      Us = U%*%solve(t(U)%*%U)
      H = Us%*%t(U)
      Dmat = kernelm(X,H,y,"v")
      alpha = solve.QP(Dmat,dvec,Amat,bvec,meq = 1)
```

```r
      Bpart=matrix(t(y*alpha$solution)%*%matrix(unlist(X),nrow = length(X),byrow =
    T),nrow = m)
      V = t(Bpart)%*%Us


      ## intercept estimation
      Bhat = U%*%t(V);Bhat
      positiv = min(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==1)])
      negativ = max(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==-1)])
      if ((1-positiv)<(-1-negativ)) {
        b0hat = -(positiv+negativ)/2
      }else{
        gridb0 = seq(from = -1-negativ,to = 1-positiv,length = 100)
        b0hat = gridb0[which.min(sapply(gridb0,function(b) objv(Bhat,b,X,y)))]
      }
      obj = c(obj,objv(Bhat,b0hat,X,y,cost));obj
      iter = iter+1
      error = abs(-obj[iter+1]+obj[iter])/obj[iter];error

    }
    if (compareobj>obj[iter+1]) {
      compareobj = obj[iter+1]
      predictor = function(x) sign(sum(Bhat*x)+b0hat)
      result$B = Bhat; result$b0 = b0hat; result$obj = obj; result$iter = iter
      result$error = error; result$predict = predictor
    }

  }
  return(result)
}


kernelmat = function(x,y,kernels = function(x1,x2) sum(x1*x2)){
  N = length(y)
  Q = matrix(nrow = N,ncol = N)
  for (i in 1:N) {
    for(j in i:N){
      Q[i,j] =kernels(x[i,],x[j,])*y[i]*y[j]
      Q[j,i] = Q[i,j]
    }
  }
  h = eigen(Q)
  Q = (h$vectors)%*%diag(pmax(h$values,eps))%*%t(h$vectors)
  return(Q)
}


# SVM with kernel functions and weighted cost function
svm = function(X,y,cost = 10, kernels = function(x1,x2) sum(x1*x2), p = .5){
  if (p==.5) {
    cost = 2*cost
  }
  result = list()
  error = 10
  iter = 0
  # SVM
  m= nrow(X[[1]]); n = ncol(X[[1]]); N = length(X)

  x = matrix(unlist(X),nrow = N,byrow = T)
  dvec = rep(1,length(X))
  Dmat = kernelmat(x,y,kernels)
  Amat = cbind(y,diag(1,N),-diag(1,N))
  bvec = c(rep(0,1+N),ifelse(y==1,-cost*(1-p),-cost*p))
  alpha = solve.QP(Dmat,dvec,Amat,bvec,meq =1)

  Bhat=matrix(t(y*alpha$solution)%*%x,nrow = m)
  b0hat = -(min(unlist(lapply(X,function(x) sum(Bhat*x))))[which(y==1)])+
              max(unlist(lapply(X,function(x) sum(Bhat*x)))[which(y==-1)]))/2
  obj = objv(Bhat,b0hat,X,y,cost,prob = p)

  predictor = function(x) sign(sum(Bhat*x)+b0hat)
  result$B = Bhat; result$b0 = b0hat; result$obj = obj;
  result$predict = predictor
  return(result)
}




posterior = function(X,y,cost = 10,test,kernels = function(x1,x2) sum(x1*x2)){
  a = 1:99
  for(i in 1:99){
    fit = svm(X,y,cost, kernels, p = i*0.01)$predict
    a[i] = fit(test)
  }
  if (all(a==1)) {
    return(1)
  }else if(all(a==-1)){
    return(0)
  }else{
    return((max(which(a==1))+min(which(a==-1)))/200)
  }
}
```

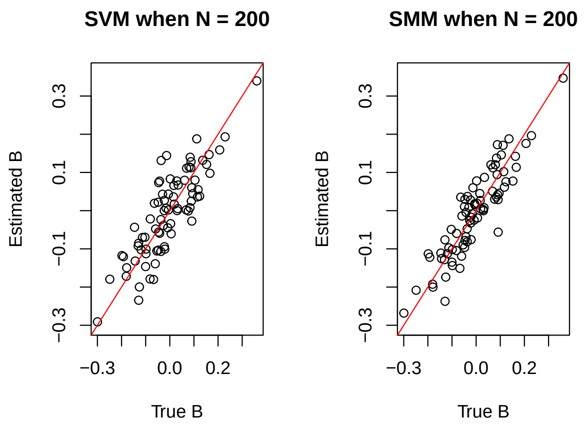
7.2 Simulations

```r
load(file = "ESL.mixture.rda")
names(ESL.mixture)
rm(x,y)
attach(ESL.mixture)
y = ifelse(y==1,1,-1)
X = lapply(seq_len(nrow(x)),function(i) x[i,,drop = F])
par(mfrow = c(1,1))
plot(x, col = y + 3)
dat = data.frame(y = factor(y), x)



a1 = matrix(nrow = 2, ncol = 99)
a1[1,] = (1:99)*0.01
for(i in 1:99){
  fit = svm(X,y,cost, kernels, p = i*0.01)$predict
  a1[2,i] = fit(test)
}

a2 = matrix(nrow = 2, ncol = 99)
a2[1,] = (1:99)*0.01
for(i in 1:99){
  fit = svm(X,y,cost, kernels, p = i*0.01)$predict
  a2[2,i] = fit(test2)
}

a1



### Changing svm according to weight
par(mfrow = c(1,3))

fit = svm(X,y,cost = 10,p=0.0001)$predict
xgrid = expand.grid(X1 = px1, X2 = px2)
ygrid = apply(xgrid,1,fit)
plot(xgrid, col = as.numeric(ygrid+2), pch = 20, cex = .2,main = "(a) : pi =
    0.0001")
points(x, col = y + 2, pch = 19)
points(test,col = 'red',pch = 19)
points(test2,col = 'blue',pch = 19)

fit = svm(X,y,cost = 10,p=0.5)$predict
xgrid = expand.grid(X1 = px1, X2 = px2)
ygrid = apply(xgrid,1,fit)
plot(xgrid, col = as.numeric(ygrid+2), pch = 20, cex = .2,main = "(b) : pi = 0.5")
points(x, col = y + 2, pch = 19)
points(test,col = 'red',pch = 19)
points(test2,col = 'blue',pch = 19)


fit = svm(X,y,cost = 10,p=0.9999)$predict
xgrid = expand.grid(X1 = px1, X2 = px2)
ygrid = apply(xgrid,1,fit)
plot(xgrid, col = as.numeric(ygrid+2), pch = 20, cex = .2,main = "(c) : pi =
    0.9999")
points(x, col = y + 2, pch = 19);
points(test,col = "red",pch = 19)
points(test2,col = "blue",pch = 19)



### posterior
posterior(X,y,cost = 10,test)
posterior(X,y,cost = 10,test2)
yposterior = vector(length = 200)
for(i in 1:200){
  yposterior[i] = posterior(X,y,cost = 10,x[i,])
  print(paste(i,"th point is done lol"))
}
ypost = ifelse(yposterior==-1,0,yposterior)


par(mfrow = c(1,1))
xgrid = expand.grid(X1 = px1, X2 = px2)
ygrid = apply(xgrid,1,fit)
datgrid = cbind(ygrid,xgrid)

colnames(x) = c("X1","X2 ")
realdat =as.data.frame(cbind(ypost,x))
breaks <- c(0.5,1,3.2)
```

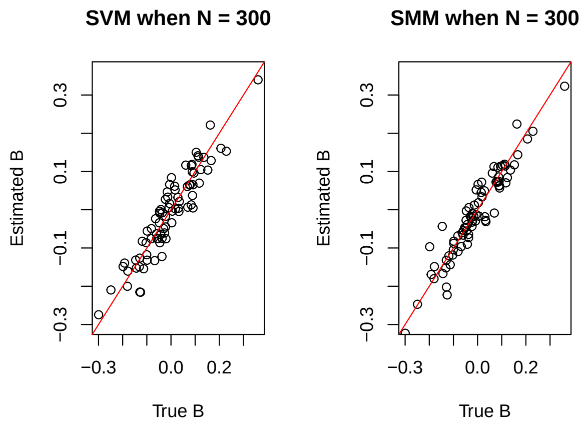
```r
ggplot(data = datgrid,aes(x = X1,y= X2,colour = ifelse(ygrid==-1,0,ygrid)))+geom_point(size = 0.001)+
  geom_point(data = realdat,aes(x = realdat[,2],y = realdat[,3],colour = ypost))+
  labs(colour = "Posterior pb")+
  scale_colour_gradientn(colours = c("darkblue","orange","darkgreen"),
                                                          breaks = breaks, labels = format(breaks))
```



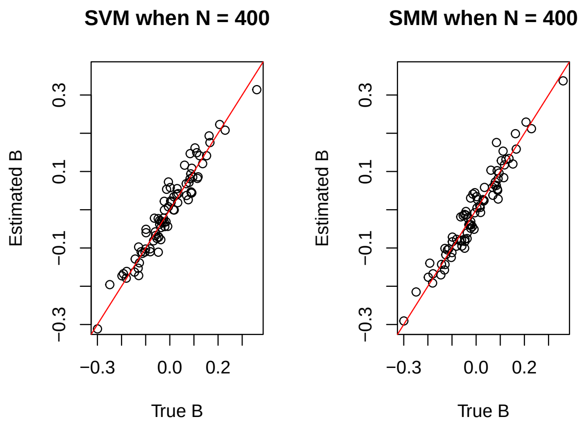
(a) When the number of data set is $N = 100$



(b) When the number of data set is $N = 200$



(c) When the number of data set is $N = 400$



(d) When the number of data set is $N = 600$

Figure 2: True parameter $B/\|B\|_F$ is compared with multiple initialized SMM result $\hat{B}/\|\hat{B}\|_F$ under the several number of data sets $N \in \{100, 200, 300, 400\}$. The left and right figures are SVM and SMM method result respectively. The horizontal axis is entries of B and the vertical axis is entries of $\hat{B}$.

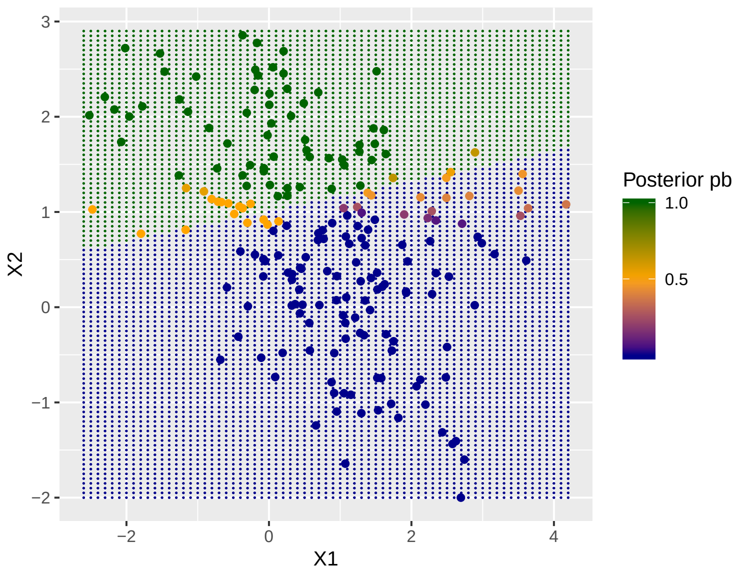


Figure 3: The green area is labeled as 1 with the SVM and blue area is -1. We can obtain non trivial posterior probability around the classification boundary.

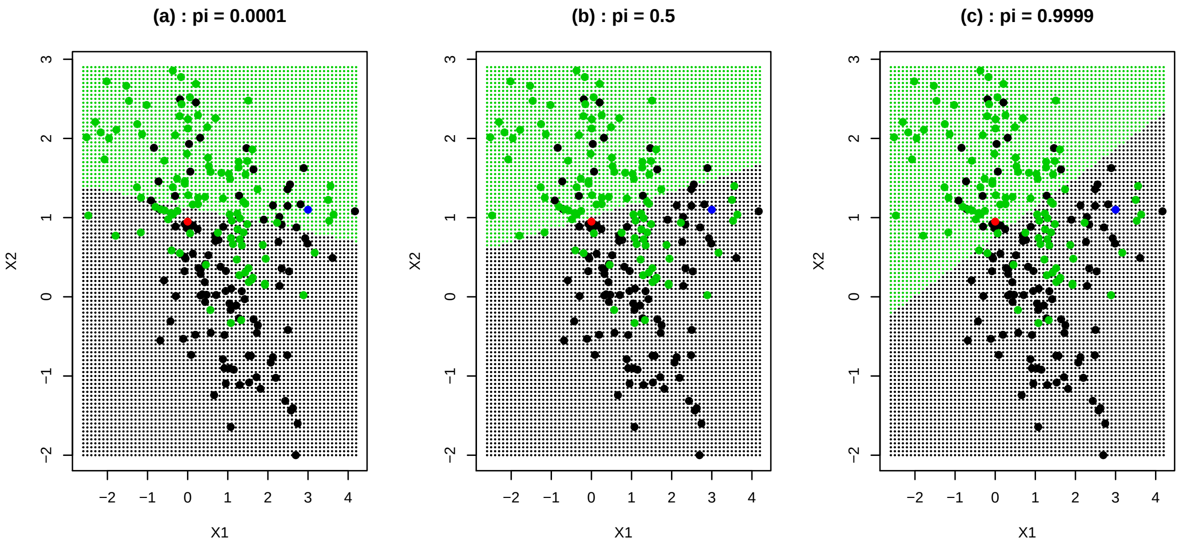


Figure 4: The sub figures show weighted hinge loss SVM classifier when $\pi = 0.001, 0.5, 0.999$. Green points represent for $y = 1$ and black for $y = -1$. The middle sub figure ($\pi = 0.5$) is the regular SVM. The red point changes its label from -1 to 1 as π increases. The blue point shows vice versa.