

A formulation of kernel matrix

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Correction: I should have removed the symmetric constraints from mapping (1).

Then, everything goes through. In fact, the bilinear mapping (1) is isomorphic to yours in Section 3.

1 Some thoughts about feature mapping Φ

In the last note, the feature mapping Φ is defined as

$$[\Phi(\mathbf{X})]_{ij} \stackrel{\text{def}}{=} \begin{cases} (\phi(\mathbf{X}_{i:}), \phi(\mathbf{X}_{j:})) & \text{if } i \geq j \\ [\Phi(\mathbf{X})]_{ji} & \text{if } i < j. \end{cases} \quad (1)$$

Based on the feature mapping, we define decision function

$$f(\mathbf{X}) \stackrel{\text{def}}{=} \langle \mathbf{B}, \Phi(\mathbf{X}) \rangle, \text{ where } \mathbf{B} \in \text{Sym}_d(\mathcal{H}^2) \text{ and } \text{rank}(\mathbf{B}) \leq r. \quad (2)$$

I have two remarks about this definition.

The number of free parameters in \mathbf{B} is $[\dim(\mathcal{H})]^r + dr$.

Reason: parameterize $\mathbf{B} = \mathbf{P}^\top \mathbf{C} \mathbf{P}$, where the (i,j) -th entry of \mathbf{C} is (c_{-i}, c_{-j})

The number of free parameters = # parameters in $\{c_{-i}\}$ for $i=1, \dots, r$ and # parameters in \mathbf{P}

Remark 1. I cannot find this definition is equivalent to the previous decision function which is defined as

$$f(\mathbf{X}) = \langle \mathbf{B}, (\phi(\mathbf{X}_{1:}), \dots, \phi(\mathbf{X}_{d:})) \rangle. \text{ where } \mathbf{B} \in (\mathcal{H})^d \text{ and } \mathbf{B} = \mathbf{CP} \text{ for some } \mathbf{C} \in (\mathcal{H})^r, \mathbf{P} \in \mathbb{R}^{r \times d}.$$

Not true. also r , not $r(r-1)$. (3)

Many parameters are redundant because of the replication in features. We need to choose a parameterization that minimizes the number of free parameters

Notice that the decision function (3) is what we use for optimization problem. One reason is that the free parameters in (2) is $\dim(\mathcal{H})^{r(r-1)} + dr$ while $\dim(\mathcal{H})^r + rd$ in (3). Even if we consider $\Phi(\mathbf{X})$ has $2d$ same variables for each $\phi(\mathbf{X}_{i:})$ for $i = 1, \dots, d$, it cannot justify this big difference of the free parameters. Therefore, Equation (2) and (3) have similar formula but not exactly equivalent. In addition, I am not sure that newly defined matrix feature mapping $\Phi(\mathbf{X})$ has better representation than $(\phi(\mathbf{X}_{1:}), \dots, \phi(\mathbf{X}_{d:}))$ where there is no replicates of features.

True.
Nevertheless, the advantage of parameterization (1) is its interpretation and generality to non-asymmetric cases.
Suppose that our feature \mathbf{X} represents the social connectivity among n individuals.
The connection between a pair of individuals (i,j) is determined by their latent features (such as age, gender, social).
We use $\phi(\mathbf{X}_{-i})$, $\phi(\mathbf{X}_{-j})$ to denote the latent features for individuals i , and j , respectively.

Remark 2. The decision function (2) is not equivalent to

$$f(\mathbf{X}) = \sum_{k=1}^n \alpha_k \sum_{i,j \in [d]} w_{i,j} K(\mathbf{X}_{i:}^k, \mathbf{X}_{j:}),$$

where $\mathbf{X}^1, \dots, \mathbf{X}^n$ are n -samples, $\alpha_k \in \mathbb{R}$ and $w_{i,j} = [\mathbf{W}]_{ij}$ such that $\text{rank}(\mathbf{W}) \leq r$. To check this, let us consider simple case where $d = 2$ and $r = 1$. We consider $\langle \mathbf{P}\Phi(\mathbf{X}), \mathbf{P}\Phi(\mathbf{X}') \rangle$ where $\mathbf{P} \in \mathbb{R}^{2 \times 1}$ which equals to

We should consider $\langle \mathbf{P} \Phi(\mathbf{X}) \mathbf{P}^\top, \mathbf{P} \Phi(\mathbf{X}') \mathbf{P}^\top \rangle$ with both row- and column-wise projections.

See Proposition 1 in my note.

$$\left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top \begin{pmatrix} (\phi(\mathbf{X}_{1:}), \phi(\mathbf{X}_{1:})) & (\phi(\mathbf{X}_{2:}), \phi(\mathbf{X}_{1:})) \\ (\phi(\mathbf{X}_{2:}), \phi(\mathbf{X}_{1:})) & (\phi(\mathbf{X}_{2:}), \phi(\mathbf{X}_{2:})) \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top \begin{pmatrix} (\phi(\mathbf{X}'_{1:}), \phi(\mathbf{X}'_{1:})) & (\phi(\mathbf{X}'_{2:}), \phi(\mathbf{X}'_{1:})) \\ (\phi(\mathbf{X}'_{2:}), \phi(\mathbf{X}'_{1:})) & (\phi(\mathbf{X}'_{2:}), \phi(\mathbf{X}'_{2:})) \end{pmatrix} \right\rangle$$

$$= (3p_1^2 + 2p_1p_2 + p_2^2)K(\mathbf{X}_{1:}, \mathbf{X}'_{1:}) + 2p_1p_2K(\mathbf{X}_{1:}, \mathbf{X}'_{2:}) + 2p_1p_2K(\mathbf{X}_{2:}, \mathbf{X}'_{1:}) + (p_1^2 + 2p_1p_2 + 3p_2^2)K(\mathbf{X}_{2:}, \mathbf{X}'_{2:})$$

$$= \left\langle \begin{pmatrix} 3p_1^2 + 2p_1p_2 + p_2^2 & 2p_1p_2 \\ 2p_1p_2 & p_1^2 + 2p_1p_2 + 3p_2^2 \end{pmatrix} \mathbf{K}, \right\rangle$$

where $[\mathbf{K}]_{ij} = K(\mathbf{X}_{i:}, \mathbf{X}'_{j:}) = \langle \phi(\mathbf{X}_{i:}), \phi(\mathbf{X}'_{j:}) \rangle$. Notice the matrix $\mathbf{W} = \begin{pmatrix} 3p_1^2 + 2p_1p_2 + p_2^2 & 2p_1p_2 \\ 2p_1p_2 & p_1^2 + 2p_1p_2 + 3p_2^2 \end{pmatrix}$ is not rank 1. The reason for \mathbf{W} not being rank 1 is $\Phi(\mathbf{X})$ has many duplicates of elements of \mathbf{K} .

2 A formulation of feature mapping Φ

We go back to feature mapping with suggested Hilbert space notation. We define feature mapping $\Phi: \mathbb{R}^{d \times d} \rightarrow \mathcal{H}^d$ as

$$\Phi(\mathbf{X}) = (\phi(\mathbf{X}_{1:}), \dots, \phi(\mathbf{X}_{d:})).$$

With this definition we have the same decision function (3) that we can use our current optimization algorithm. In addition, we can address problem mentioned in Remark 2. From Equation (3), we have

$$f(\mathbf{X}) = \langle \mathbf{B}, \Phi(\mathbf{X}) \rangle = \langle \mathbf{C}\mathbf{P}, \Phi(\mathbf{X}) \rangle = \langle \mathbf{C}, \Phi(\mathbf{X})\mathbf{P}^T \rangle \text{ where } \mathbf{C} \in (\mathcal{H})^r, \mathbf{P} \in \mathbb{R}^{r \times d}. \quad (4)$$

Therefore, $f(\mathbf{X})$ is an element of RKHS induced by $\langle \Phi(\mathbf{X})\mathbf{P}^T, \Phi(\mathbf{X}')\mathbf{P}^T \rangle$. Let $\mathbf{P} = (P_1, \dots, P_r)^T$.

$$\begin{aligned} \langle \Phi(\mathbf{X})\mathbf{P}^T, \Phi(\mathbf{X}')\mathbf{P}^T \rangle &= \left\langle (\Phi(\mathbf{X})P_1, \dots, \Phi(\mathbf{X})P_r), (\Phi(\mathbf{X}')P_1, \dots, \Phi(\mathbf{X}')P_r) \right\rangle \\ &= \sum_{i=1}^r \langle \Phi(\mathbf{X})P_i, \Phi(\mathbf{X}')P_i \rangle \\ &= \sum_{i=1}^r \left\langle \sum_{j=1}^d P_{ij}\phi(\mathbf{X}_{j:}), \sum_{j=1}^d P_{ij}\phi(\mathbf{X}'_{j:}) \right\rangle \\ &= \sum_{i=1}^r P_i^T \mathbf{K} P_i \text{ where } [\mathbf{K}]_{ij} = K(\mathbf{X}_{i:}, \mathbf{X}'_{j:}) = \langle \phi(\mathbf{X}_{i:}), \phi(\mathbf{X}'_{j:}) \rangle \\ &= \left\langle \sum_{i=1}^r P_i P_i^T, \mathbf{K} \right\rangle \\ &= \sum_{i,j \in [d]} w_{ij} K(\mathbf{X}_{i:}, \mathbf{X}'_{j:}) \text{ where } \mathbf{W} = \sum_{i=1}^r P_i P_i^T. \quad = \text{(matrix P)}^T \text{ times (matrix P)} \end{aligned}$$

Therefore, we can write considered decision function class as

$$\begin{aligned} \mathcal{F} &= \left\{ f: \mathbf{X} \mapsto \langle \mathbf{C}, \Phi(\mathbf{X})\mathbf{P}^T \rangle \mid \mathbf{P}\mathbf{P}^T = \mathbf{I}, \mathbf{P} \in \mathbb{R}^{r \times d}, \mathbf{C} \in \mathcal{H}^r \right\} \\ &= \left\{ f \in \text{RKHS generated by } \langle \cdot, \cdot \rangle_{\mathbf{P}} \mid \mathbf{P}\mathbf{P}^T = \mathbf{I}, \mathbf{P} \in \mathbb{R}^{r \times d} \right\} \\ &= \left\{ f \in \text{RKHS generated by } \mathcal{K}(\mathbf{W}) \mid \mathbf{W} \succeq 0, \text{rank}(\mathbf{W}) \leq r \right\}. \end{aligned}$$

Remark 3. Connection to our learning algorithm: We consider the optimization over the union of RKHS induced by low rank \mathbf{W} :

$$\max_{f \in \mathcal{F}(r)} L(f) = \max_{\text{rank}(\mathbf{W}) \leq r, \mathbf{W} \succeq 0} \max_{f \in \text{RKHS}(\mathcal{K}(\mathbf{W}))} L(f).$$

Remark 4. In this case, feature matrix \mathbf{X} does not have to be symmetric matrix. However, when $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2}$, we are considering only row-wise extended feature map images. This might be enough when the feature matrix is known to have only one of columnwise or rowwise correlation. In the following section, we propose method for more general feature matrix case where we need to analyze both column and row structures.

3 Case for non symmetric feature matrices

We consider the case where feature matrices are not symmetric $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2}$. Define feature mapping $\Phi_r(\mathbf{X}) \stackrel{\text{def}}{=} (\phi_r(\mathbf{X}_{1:}), \dots, \phi_r(\mathbf{X}_{d_1:}))$ and $\Phi_c(\mathbf{X}) \stackrel{\text{def}}{=} (\phi_c(\mathbf{X}_{:1}), \dots, \phi_c(\mathbf{X}_{:d_2}))$ where ϕ_c, ϕ_r is feature map induced by kernel K . Notice we use the same kernel K in the sense that if we use Gaussian kernel on row vectors, we use the same kernel on column vectors so that the formula of ϕ_r, ϕ_c are the same except the dimension. We define comprehensive feature map $\tilde{\Phi}: \mathbb{R}^{d_1 \times d_2} \rightarrow \mathcal{H}_r^{d_1} \times \mathcal{H}_c^{d_2}$ where \mathcal{H}_r and \mathcal{H}_c are image space of ϕ_c and ϕ_r .

This mapping is isomorphic to the bilinear mapping in Section 1

$$\tilde{\Phi}(\mathbf{X}) = (\Phi_r(\mathbf{X}), \Phi_c(\mathbf{X})).$$

We define decision function,

$$\begin{aligned} f(\mathbf{X}) &= \langle \tilde{\mathbf{B}}, \tilde{\Phi}(\mathbf{X}) \rangle, \text{ where } \tilde{\mathbf{B}} = (\tilde{\mathbf{B}}_r, \tilde{\mathbf{B}}_c) \in \mathcal{H}_r^{d_1} \times \mathcal{H}_c^{d_2} \\ &= \langle \tilde{\mathbf{B}}_r, \Phi_r(\mathbf{X}) \rangle + \langle \tilde{\mathbf{B}}_c, \Phi_c(\mathbf{X}) \rangle \\ &= \sum_{k=1}^n \beta_k \sum_{\substack{i,j \in [d_2] \\ \text{d_1}}} w_{ij}^{row} K(\mathbf{X}_{i:}^k, \mathbf{X}_{j:}) + \sum_{k=1}^n \alpha_k \sum_{i,j \in [d_2]} w_{ij}^{col} K(\mathbf{X}_{:i}^k, \mathbf{X}_{:j}), \end{aligned} \quad (5)$$

where $\mathbf{X}^1, \dots, \mathbf{X}^n$ are sampled matrix features and $\mathbf{W}^{\text{col}}, \mathbf{W}^{\text{row}}$ are some positive semi definite **low-rank** matrices. We estimate $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \dots, \beta_n), \mathbf{W}^{\text{col}}$, and \mathbf{W}^{row} from the training data set. Define new symmetric feature matrix $\tilde{\mathbf{X}} = \begin{pmatrix} 0_{d_1 \times d_2} & \mathbf{X} \\ \mathbf{X}^t & 0_{d_2 \times d_1} \end{pmatrix} \in \mathbb{R}^{(d_1+d_2) \times (d_1+d_2)}$ and feature map $\Phi: \mathbb{R}^{(d_1+d_2) \times (d_1+d_2)} \rightarrow \mathcal{H}^{d_1+d_2}$ as in symmetric case (1),

$$\Phi(\tilde{\mathbf{X}}) = \left(\phi(\tilde{\mathbf{X}}_{1:}), \dots, \phi(\tilde{\mathbf{X}}_{d_1+d_2:}) \right),$$

where ϕ is induced by kernel $K: \mathbb{R}^{(d_1+d_2) \times (d_1+d_2)} \times \mathbb{R}^{(d_1+d_2) \times (d_1+d_2)} \rightarrow \mathbb{R}$. Since all entries of $\Phi_r(\mathbf{X})$ are corresponding to $[\Phi(\tilde{\mathbf{X}})]_{1:d_1}$ and $\Phi_c(\mathbf{X})$ to $[\Phi(\tilde{\mathbf{X}})]_{d_1+1:d_1+d_2}$, we have an equivalent representation of (5)

Explit form of this kernel?

$$\begin{aligned} f(\mathbf{X}) &= \langle \tilde{\mathbf{B}}_r, \Phi_r(\mathbf{X}) \rangle + \langle \tilde{\mathbf{B}}_c, \Phi_c(\mathbf{X}) \rangle \\ &= \langle \mathbf{B}_r, [\Phi(\tilde{\mathbf{X}})]_{1:d_1} \rangle + \langle \mathbf{B}_c, [\Phi(\tilde{\mathbf{X}})]_{d_1+1:d_1+d_2} \rangle, \text{ where } \mathbf{B}_r \in \mathcal{H}^{d_1}, \mathbf{B}_c \in \mathcal{H}^{d_2} \\ &= \langle \mathbf{B}, \Phi(\tilde{\mathbf{X}}) \rangle, \text{ where } \mathbf{B} = (\mathbf{B}_r, \mathbf{B}_c) \in \mathcal{H}^{d_1+d_2}. \end{aligned}$$

Therefore, we can apply low-rank structure on $\mathbf{B} = \mathbf{CP}$ where $\mathbf{C} \in \mathcal{H}^r$ and $\mathbf{P} \in \mathbb{R}^{r \times (d_1+d_2)}$ and use the same optimization procedure with Section 2.