

# SMM conditional probability

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## 1 Simulation of SMM Conditional Probability

I generate feature data matrix with the following rule.

$$\begin{aligned}\mathbb{P}(X_{\cdot 1}|y=1) &\sim N((1, 1)^T, I_2), \quad \mathbb{P}(X_{\cdot 2}|y=1) \sim N((-1, -1)^T, I_2), \\ \mathbb{P}(X_{\cdot 1}|y=-1) &\sim N((0, 0)^T, I_2), \quad \mathbb{P}(X_{\cdot 2}|y=-1) \sim N((0, 0)^T, I_2), \\ X = (X_{\cdot 1}, X_{\cdot 2}) &\in \mathbb{R}^{2 \times 2}.\end{aligned}$$

With this rule, we have a test data set  $(X^1, 1), \dots, (X^{20}, 1), (X^{21}, -1), \dots, (X^{40}, -1)$ . Define

$$Z_1 = X_{11} + X_{21} \text{ and } Z_2 = X_{12} + X_{22}.$$

We can plot those feature matrices into two dimension transforming the matrices as  $((Z_1^1, Z_2^1)^T, 1), \dots, ((Z_1^{20}, Z_2^{20})^T, 1), ((Z_1^{21}, Z_2^{21})^T, -1), \dots, ((Z_1^{40}, Z_2^{40})^T, -1)$ . Figure 1 shows the data distribution.

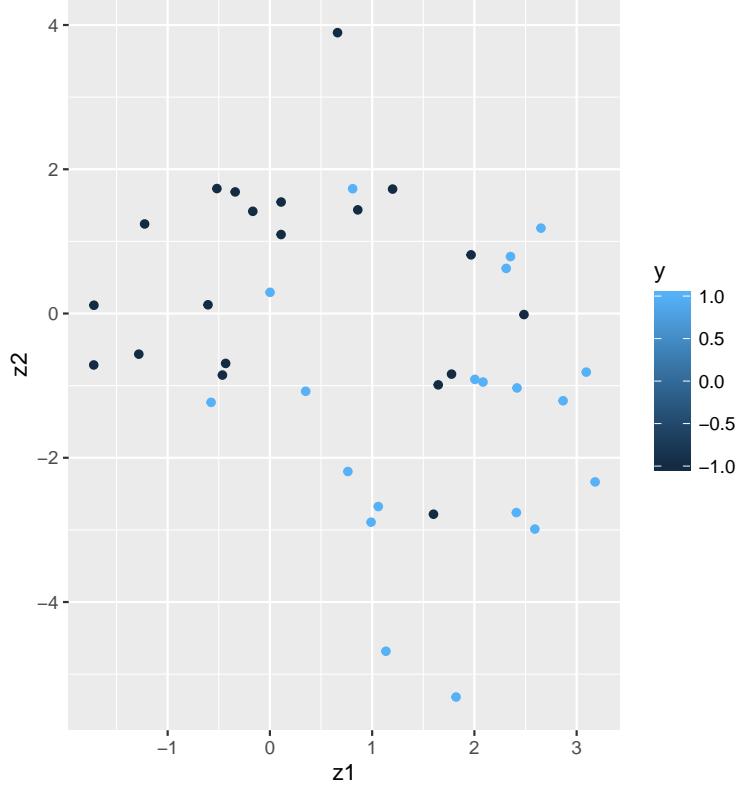


Figure 1: Visualization of the feature matrices.  $Z_1$  is the sum of the first column and  $Z_2$  is of the second one.

Notice that we can calculate the distribution of  $(Z_1, Z_2)^T$  as,

$$\mathbb{P}((Z_1, Z_2)^T | y=1) \sim N((2, -2), 2I_2), \tag{1}$$

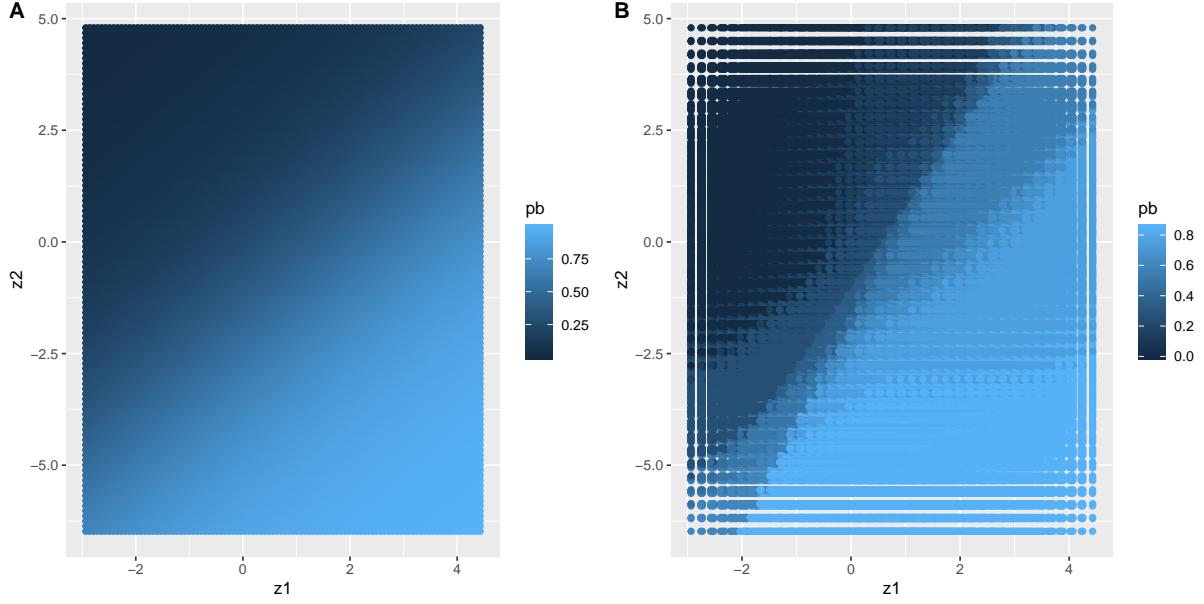


Figure 2: Figure A shows the true conditional probability of  $y$  given  $(Z_1, Z_2)$ . Figure B shows the estimated probability using SMM weighted hinge loss approach.

$$\mathbb{P}((Z_1, Z_2)^T | y = -1) \sim N((0, 0), 2I_2).$$

From the Bayes rule, we calculate the conditional probability  $\mathbb{P}(y = 1 | (Z_1, Z_2))$  with the assumption  $\mathbb{P}(y = 1) = \mathbb{P}(y = -1)$  as

$$\begin{aligned} \mathbb{P}(y = 1 | (Z_1, Z_2)^T) &= \frac{\mathbb{P}((Z_1, Z_2)^T | y = 1) \mathbb{P}(y = 1)}{\mathbb{P}((Z_1, Z_2)^T | y = 1) \mathbb{P}(y = 1) + \mathbb{P}((Z_1, Z_2)^T | y = -1) \mathbb{P}(y = -1)} \\ &= \frac{1}{1 + \exp\left(-(2, -2)((Z_1, Z_2) - (1, -1))^T / 4\right)}. \end{aligned} \quad (2)$$

In the last equality in (2), we use the distribution in (1).

Using the weighted hinge loss approach, I calculate the conditional probability using SMM method (with modified algorithm for conditional probability). I set the rank as one when fitting SMM model to the data set. Figure 2 shows the true conditional probability  $\mathbb{P}(y = 1 | (Z_1, Z_2))$  and the estimated one. We can see that SMM conditional probability estimation works well since the true distribution is linear.

## 2 Kernel method

I am suggesting new kernel method which makes optimization easier. Define feature mapping  $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m' \times n}$  where  $m < m'$ . Kernels for matrix case can be define as

$$K(X, X') = h(X)^T h(X') \in \mathbb{R}^{n \times n}.$$

Our objective primal problem is

$$\min_{U \in \mathbb{R}^{m' \times r}, V \in \mathbb{R}^{n \times r}, \xi} \frac{1}{2} \|UV^T\|^2 + c \sum_{i=1}^N \xi_i \quad (3)$$

$$\begin{aligned} & \text{subject to } y_i(\langle UV^T, h(X_i) \rangle + b) \leq 1 - \xi_i \\ & \quad \xi_i \geq 0, \quad i = 1, \dots, N. \end{aligned}$$

Then fixing  $V$  we have the dual problem of (3).

$$\begin{aligned} & \min_{\alpha} - \sum_{i=1}^N \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \langle h(X_i) H_V, h(X_j) H_V \rangle \\ & \text{subject to } \sum_{i=1}^N y_i \alpha_i = 0 \\ & \quad 0 \leq \alpha_i \leq C, \quad i = 1, \dots, N. \end{aligned}$$

Notice  $\langle h(X_i) H_V, h(X_j) H_V \rangle = \text{tr}(H_V h(X_i)^T h(X_j)) = \text{tr}(H_V K(X_i, X_j))$ . Therefore, we can update  $U$  and  $V$  without the information of the feature map  $h$ . We can define the following kernels.

$$\begin{aligned} \text{Linear: } & K(X, X') = X^T X' \\ \text{Polynomial: } & K(X, X') = (X^T X' + I_n)^d \\ \text{Radial: } & K(X, X') = \exp((X - X')^T (X - X')/\sigma), \end{aligned}$$

where  $\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ . Notice that when  $X \in \mathbb{R}^{m \times 1}$  i.e.  $X$  is a vector, all those definitions are reduced to SVM case. From this way, we can generalize linear SMM method to Kernel SMM with tractable algorithm.