Assumption modification

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Assumption 1 (Boundary noise)**.** There exist constants $\alpha \in (0,1]$ and $C > 0$, such that

$$\max_{\pi \in \Pi'} \mathbb{P}_{\boldsymbol{X}} \left(|p(\boldsymbol{X}) - \pi| \leq t/H \right) \leq C \left(\frac{t}{H} \right)^{\frac{\alpha}{1-\alpha}}, \quad \text{for all } t \in [0,1], \tag{1}$$

where $\Pi' \subset \Pi$ with $|\Pi| - |\Pi'| \leq m$ and m is a finite number independent on H. When $\alpha = 1$, the inequality (1) reads $\mathbb{P}(|p(\boldsymbol{X}) - \pi| \leq t/H) = 0$.

Theorem 0.1. Suppose (1) is satisfied with $\alpha \in (0,1)$. Then, with the condition that $H \leq \frac{C_0}{\max_{\pi \in \Pi'} [R_\pi(S) - R_\pi(S_{\text{bayes}})]^{1-\alpha}}$,

$$\mathbb{P}_{\boldsymbol{X}}[S \Delta S_{\text{bayes}}] \leq C_1 [R_\pi(S) - R_\pi(S_{\text{bayes}}(\pi))]^\alpha,$$

for all sets $S \in \mathbb{R}^{d_1 \times d_2}$, levels $\pi \in \Pi'$. If (1) is satisfied with $\alpha = 1$, Then, with no condition on H,

$$\mathbb{P}_{\boldsymbol{X}}[S \Delta S_{\text{bayes}}] \leq C_2 H [R_\pi(S) - R_\pi(S_{\text{bayes}}(\pi)],$$

for all sets $S \in \mathbb{R}^{d_1 \times d_2}$, levels $\pi \in \Pi'$.

Theorem 0.2 (Nonparametric regression via weighted classifications)**.** Let $p(\boldsymbol{X})$ be a regression function satisfying (1), and $\bar{p}(\boldsymbol{X})$ the linear combination of weighted classifiers based on our method. Then, there exists a constant $C_3 > 0$ such that

$$R_{\text{reg}}(\bar{p}) - R_{\text{reg}}(p) \leq 4\mathbb{E}_{\boldsymbol{X}} |\bar{p}(\boldsymbol{X}) - p(\boldsymbol{X})| \leq \frac{2}{H} + C_3 H^{\mathbb{1}\{\alpha=1\}} \max_{\pi \in \Pi} \left[R_\pi(\bar{S}(\pi)) - R_\pi(S_{\text{bayes}}(\pi)) \right]^\alpha,$$

for all resolution parameter $H = |\Pi| \in \mathbb{N}_+$.

Remark 1. We have two cases of the bound

1. when $\alpha \in (0,1)$, We have

$$\mathbb{E}_{\boldsymbol{X}} |\bar{p}(\boldsymbol{X}) - p(\boldsymbol{X})| \leq C_3 \max_{\pi \in \Pi} \left[R_\pi(\bar{S}(\pi)) - R_\pi(S_{\text{bayes}}(\pi)) \right]^\alpha + C_4 \max_{\pi \in \Pi} \left[R_\pi(\bar{S}(\pi)) - R_\pi(S_{\text{bayes}}(\pi)) \right]^{1-\alpha}$$

$$\leq \underbrace{C_3 \left(\frac{r(s_1+s_2)\log d}{n} \right)^{(1-\alpha)/(2-\alpha)}}_{\text{reduction error}} + \underbrace{C_4 \left(\frac{r(s_1+s_2)\log d}{n} \right)^{\alpha/(2-\alpha)}}_{\text{statistical error}} + \underbrace{C_5 a_n^\alpha}_{\text{approximation error}}$$

2. when $\alpha = 1$,

$$\mathbb{E}_{\boldsymbol{X}} |\bar{p}(\boldsymbol{X}) - p(\boldsymbol{X})| \leq \underbrace{C_3 \frac{1}{H}}_{\text{reduction error}} + C_4 \underbrace{H \left(\frac{r(s_1+s_2)\log d}{n} \right)}_{\text{statistical error}} + \underbrace{C_5 a_n^\alpha}_{\text{approximation error}}$$

We can bound the convergence rate setting $H = \mathcal{O}(\sqrt(n))$.

Remark 2. One way to make our theorem organized without distinguishing $alpha \in (0,1)$ and $\alpha = 1$ is to change Theorem 0.1 as

Theorem 0.3. Suppose (1) is satisfied with $\alpha \in (0,1]$, then (1) implies

$$\mathbb{P}_{\boldsymbol{X}}[S \Delta S_{\text{bayes}}] \leq C_1 H [R_\pi(S) - R_\pi(S_{\text{bayes}}(\pi))]^\alpha,$$

for all sets $S \in \mathbb{R}^{d_1 \times d_2}$, levels $\pi \in \Pi'$.

Then our main bound becomes

$$\mathbb{E}_{\boldsymbol{X}} \left|\bar{p}(\boldsymbol{X}) - p(\boldsymbol{X})\right| \leq \underbrace{C_3 \frac{1}{H}}_{\text{reduction error}} + C_4 \underbrace{H \left(\frac{r(s_1+s_2)\log d}{n}\right)^{\alpha/(2-\alpha)}}_{\text{statistical error}} + \underbrace{C_5 a_n^{\alpha}}_{\text{approximation error}}$$