Improved bounds for random projection

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Notation. We say that an event A occurs "with high probability" if $\mathbb{P}(A)$ tends to 1 as the dimension $d_{\min} = \min\{d_1, d_2\}$ tends to infinity. We say that A occurs "with very high probability" if $\mathbb{P}(A)$ tends to 1 faster than any polynomial of $d_{\min}$.

Let $\boldsymbol{E} \in \mathbb{R}^{d_1 \times d_2}$ be a real matrix. The matrix spectral norm is defined as

$$\|\boldsymbol{E}\|_\sigma = \max_{(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{S}^{d_1-1} \times \mathbb{S}^{d_2-1}} \boldsymbol{a}^T \boldsymbol{E} \boldsymbol{b},$$

where $\mathbb{S}^{d-1} = \{\boldsymbol{a} \in \mathbb{R}^d : \|\boldsymbol{a}\|_2 = 1\}$ denotes the unit sphere in $\mathbb{R}^d$.

Theorem 0.1. *Consider a noisy rank-1 matrix model $\boldsymbol{D} = \lambda \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{E}$, where $\lambda \in \mathbb{R}_+$ is a scalar, $\boldsymbol{a} \in \mathbb{R}^{d_1}, \boldsymbol{b} \in \mathbb{R}^{d_2}$ are unit-1 vectors, and $\boldsymbol{E} \in \mathbb{R}^{d_1 \times d_2}$ is a Gaussian matrix with i.i.d. $N(0, \sigma^2)$ entries. Define a random projection*

$$\hat{\boldsymbol{a}} = \boldsymbol{D}\boldsymbol{\Omega}, \ \text{ where } \boldsymbol{\Omega} = (z_1, \ldots, z_{d_2})^T \overset{i.i.d.}{\sim} N(0,1).$$

Suppose $\lambda \gg \sigma\sqrt{d_2 \max(d_1, d_2)}$ as $d_1, d_2 \to \infty$. Then, $\cos\Theta(\boldsymbol{a}, \hat{\boldsymbol{a}}) \to 1$ in probability.

Proof. The perturbed rank-1 model $\boldsymbol{D} = \lambda \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{E}$ implies that

$$\hat{\boldsymbol{a}} = \boldsymbol{D}\boldsymbol{\Omega} = \lambda Z \boldsymbol{a} + \boldsymbol{E}\boldsymbol{\Omega}, \tag{1}$$

where $Z \overset{\text{def}}{=} \langle \boldsymbol{b}, \boldsymbol{\Omega} \rangle$ is an $N(0,1)$ random variable. Note that $\boldsymbol{E}\boldsymbol{\Omega} \in \mathbb{R}^{d_1}$ is a random vector with length $\|\boldsymbol{E}\boldsymbol{\Omega}\|_2 \leq \|\boldsymbol{E}\|_\sigma \|\boldsymbol{\Omega}\|_2$.

To show $\cos(\boldsymbol{a}, \hat{\boldsymbol{a}}) \to 1$, it suffices to show that $\cot(\boldsymbol{a}, \hat{\boldsymbol{a}}) \to \infty$. Based on (1), we have

$$\cot\Theta(\boldsymbol{a}, \hat{\boldsymbol{a}}) \geq \frac{\|\lambda Z \boldsymbol{a}\|_2}{\|\boldsymbol{E}\boldsymbol{\Omega}\|_2} \geq \frac{\lambda |Z|}{\|\boldsymbol{E}\|_\sigma \|\boldsymbol{\Omega}\|_2}. \tag{2}$$

Now consider the asymptotical property of (2) as $d_1, d_2 \to \infty$. The fact $\|\boldsymbol{\Omega}\|_2^2 \sim \chi^2(d_2)$ implies that $\|\boldsymbol{\Omega}\|_2 \asymp (1 + o(1))\sqrt{d_2}$ with very high probability, for some constant $C > 0$. Furthermore, we have that $\|\boldsymbol{E}\|_\sigma \asymp (2 + o(1))\sigma\sqrt{\max(d_1, d_2)}$ by Lemma 1. Therefore, for any fixed $L > 0$,

$$\begin{aligned} \mathbb{P}(\cot(\boldsymbol{a}, \hat{\boldsymbol{a}}) \geq L) &\geq \mathbb{P}\left(\frac{\lambda |Z|}{\|\boldsymbol{E}\|_\sigma \|\boldsymbol{\Omega}\|_2} \geq L \right) \\ &\geq \mathbb{P}\left(|Z| \geq \frac{2L\sigma\sqrt{d_2 \max(d_1, d_2)}}{\lambda} \right) \\ &\geq 1 - \frac{4\lambda_*}{\sqrt{2\pi}}, \end{aligned} \tag{3}$$

where $\lambda_* \overset{\text{def}}{=} L\sigma\sqrt{d_2 \max(d_1, d_2)}/\lambda$. Here the last line of (3) uses the fact that $\mathbb{P}(|N(0,1)| \geq t) \geq$

$1-2t\phi(0)$ for any $t \geq 0$, where $\phi(\cdot)$ is the pdf for standard normal. Based on assumption $\lambda_* \to 0$ we conclude that $\cot(\boldsymbol{a}, \hat{\boldsymbol{a}}) \geq L$ with high probability. Sending $L \to \infty$ gives the desired result. $\square$

Lemma 1 (Spectral norm of Gaussian matrix [1]). *Let $\boldsymbol{E} \in \mathbb{R}^{d_1 \times d_2}$ be a random matrix with i.i.d. $N(0,1)$ entries. Then, we have, with very high probability,*

$$\|\boldsymbol{E}\|_\sigma \asymp (2 + o(1))\sqrt{\max(d_1, d_2)}.$$

References

[1] Mark Rudelson and Roman Vershynin. Non-asymptotic theory of random matrices: extreme singular values. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 1576–1602. World Scientific, 2010.