Supplements for "Tensor denoising and completion based on ordinal observations"

1 Proofs

1.1 Estimation error for tensor denoising

Proof of Theorem **??**. We suppress the subscript Ω in the proof, because the tensor denoising assumes complete observation $\Omega = [d_1] \times \cdots \times [d_K]$. It follows from the expression of $\mathcal{L}_{\mathcal{Y}}(\Theta)$ that

$$\frac{\partial \mathcal{L}_{\mathcal{Y}}}{\partial \theta_\omega} = \sum_{\ell \in [L]} \mathbb{1}_{\{y_\omega = \ell\}} \frac{\dot{g}_\ell(\theta_\omega)}{g_\ell(\theta_\omega)},$$
$$\frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial \theta_\omega^2} = \sum_{\ell \in [L]} \mathbb{1}_{\{y_\omega = \ell\}} \frac{\ddot{g}_\ell(\theta_\omega) g_\ell(\theta_\omega) - \dot{g}_\ell^2(\theta_\omega)}{g_\ell^2(\theta_\omega)} \text{ and } \quad \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial \theta_\omega \theta_{\omega'}} = 0 \text{ if } \omega \neq \omega', \tag{1}$$

for all $\omega \in [d_1] \times \cdots \times [d_K]$. Define $d_{\text{total}} = \prod_k d_k$. Let $\nabla_\Theta \mathcal{L}_{\mathcal{Y}} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ denote the tensor of gradient with respect to $\Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K}$, and $\nabla^2_\Theta \mathcal{L}_{\mathcal{Y}}$ the corresponding Hession matrix of size d_{total}-by-d_{total}. Here, $\text{Vec}(\cdot)$ denotes the operation that turns a tensor into a vector. By (1), $\nabla^2_\Theta \mathcal{L}_{\mathcal{Y}}$ is a diagonal matrix. Recall that

$$U_\alpha = \max_{\ell \in [L], |\theta| \leq \alpha} \frac{\dot{g}_\ell(\theta)}{g_\ell(\theta)} > 0 \quad \text{and} \quad L_\alpha = \min_{\ell \in [L], |\theta| \leq \alpha} \frac{\dot{g}_\ell^2(\theta) - \ddot{g}_\ell(\theta) g_\ell(\theta)}{g_\ell^2(\theta)} > 0.$$

Therefore, all entries in $\nabla_\Theta \mathcal{L}_{\mathcal{Y}}$ are upper bounded $U_\alpha > 0$, and all diagonal entries in $\nabla^2_\Theta \mathcal{L}_{\mathcal{Y}}$ are upper bounded by $-L_\alpha < 0$.

By the second-order Taylor's expansion of $\mathcal{L}_{\mathcal{Y}}(\Theta)$ around Θ^{true}, we obtain

$$\mathcal{L}_{\mathcal{Y}}(\Theta) = \mathcal{L}_{\mathcal{Y}}(\Theta^{\text{true}}) + \langle \text{Vec}(\nabla_\Theta \mathcal{L}_{\mathcal{Y}}),\ \text{Vec}(\Theta - \Theta^{\text{true}}) \rangle + \frac{1}{2} \text{Vec}(\Theta - \Theta^{\text{true}})^T \nabla^2_\Theta \mathcal{L}_{\mathcal{Y}}(\check{\Theta}) \text{Vec}(\Theta - \Theta^{\text{true}}), \tag{2}$$

$\check{\Theta} = \gamma \Theta^{\text{true}} + (1 - \gamma)\Theta$ for some $\gamma \in [0, 1]$, and $\nabla^2_\Theta \mathcal{L}_{\mathcal{Y}}(\check{\Theta})$ denotes the $\prod_k d_k$-by-$\prod_k d_k$ Hession matrix evaluated at $\check{\Theta}$.

We first bound the linear term in (2). Note that, by Lemma 3,

$$|\langle \text{Vec}(\nabla_\Theta \mathcal{L}_{\mathcal{Y}}(\Theta^{\text{true}}), \text{Vec}(\Theta - \Theta^{\text{true}}) \rangle| \leq \|\nabla_\Theta \mathcal{L}_{\mathcal{Y}}(\Theta^{\text{true}})\|_\sigma \|\Theta - \Theta^{\text{true}}\|_*, \tag{3}$$

where $\|\cdot\|_\sigma$ denotes the tensor spectral norm and $\|\cdot\|_*$ denotes the tensor nuclear norm. Define

$$s_\omega = \frac{\partial \mathcal{L}_{\mathcal{Y}}}{\partial \theta_\omega}\Big|_{\Theta = \Theta^{\text{true}}} \quad \text{for all } \ \omega \in [d_1] \times \cdots \times [d_K].$$

Based on (1) and the definition of U_α, $\nabla_\Theta \mathcal{L}_{\mathcal{Y}}(\Theta^{\text{true}}) = [\![s_\omega]\!]$ is a random tensor whose entries are independently distributed satisfying

$$\mathbb{E}(s_\omega) = 0, \quad |s_\omega| \leq U_\alpha, \quad \text{for all } \omega \in [d_1] \times \cdots \times [d_K]. \tag{4}$$

By lemma 5, with probability at least $1 - \exp(-C_1 \sum_k d_k)$, we have

$$\|\nabla_\Theta \mathcal{L}_{\mathcal{Y}}(\Theta^{\text{true}})\|_\sigma \leq C_2 U_\alpha \sqrt{\sum_k d_k}, \tag{5}$$

where C_1, C_2 are two positive constants that depend only on K. Furthermore, note that $\text{rank}(\Theta) \leq \boldsymbol{r}$, $\text{rank}(\Theta^{\text{true}}) \leq \boldsymbol{r}$, so $\text{rank}(\Theta - \Theta^{\text{true}}) \leq 2\boldsymbol{r}$. By lemma 2, $\|\Theta - \Theta^{\text{true}}\|_* \leq (2r_{\max})^{\frac{K-1}{2}} \|\Theta - \Theta^{\text{true}}\|_F$. Combining (3), (4) and (5), we have that, with probability at least $1 - \exp(-C_1 \sum_k d_k)$,

$$|\langle \text{Vec}(\nabla_\Theta \mathcal{L}_\mathcal{Y}(\Theta^{\text{true}})), \text{Vec}(\Theta - \Theta^{\text{true}}) \rangle| \leq C_2 U_\alpha \sqrt{r_{\max}^{K-1} \sum_k d_k} \|\Theta - \Theta^{\text{true}}\|_F. \tag{6}$$

We next bound the quadratic term in (2). Note that

$$\begin{aligned} \text{Vec}(\Theta - \Theta^{\text{true}})^T \nabla^2_\Theta \mathcal{L}_\mathcal{Y}(\check{\Theta}) \, \text{Vec}(\Theta - \Theta^{\text{true}}) &= \sum_\omega \left(\frac{\partial^2 \mathcal{L}_\mathcal{Y}}{\partial \theta_\omega^2} \Big|_{\Theta = \check{\Theta}} \right) (\theta_\omega - \theta_{\text{true},\omega})^2 \\ &\leq -L_\alpha \sum_\omega (\Theta_\omega - \Theta_{\text{true},\omega})^2 \\ &= -L_\alpha \|\Theta - \Theta^{\text{true}}\|_F^2, \end{aligned} \tag{7}$$

where the second line comes from the fact that $\|\check{\Theta}\|_\infty \leq \alpha$ and the definition of L_α.

Combining (2), (6) and (7), we have that, for all $\Theta \in \mathcal{P}$, with probability at least $1 - \exp(-C_1 \sum_k d_k)$,

$$\mathcal{L}_\mathcal{Y}(\Theta) \leq \mathcal{L}_\mathcal{Y}(\Theta^{\text{true}}) + C_2 U_\alpha \left(r_{\max}^{K-1} \sum_k d_k \right)^{1/2} \|\Theta - \Theta^{\text{true}}\|_F - \frac{L_\alpha}{2} \|\Theta - \Theta^{\text{true}}\|_F^2.$$

In particular, the above inequality also holds for $\hat{\Theta} \in \mathcal{P}$. Therefore,

$$\mathcal{L}_\mathcal{Y}(\hat{\Theta}) \leq \mathcal{L}_\mathcal{Y}(\Theta^{\text{true}}) + C_2 U_\alpha \left(r_{\max}^{K-1} \sum_k d_k \right)^{1/2} \|\hat{\Theta} - \Theta^{\text{true}}\|_F - \frac{L_\alpha}{2} \|\hat{\Theta} - \Theta^{\text{true}}\|_F^2.$$

Since $\hat{\Theta} = \arg\max_{\Theta \in \mathcal{P}} \mathcal{L}_\mathcal{Y}(\Theta)$, $\mathcal{L}_\mathcal{Y}(\hat{\Theta}) - \mathcal{L}_\mathcal{Y}(\Theta^{\text{true}}) \geq 0$, which gives

$$C_2 U_\alpha \left(r_{\max}^{K-1} \sum_k d_k \right)^{1/2} \|\hat{\Theta} - \Theta^{\text{true}}\|_F - \frac{L_\alpha}{2} \|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 \geq 0.$$

Henceforth,

$$\frac{1}{\sqrt{\prod_k d_k}} \|\hat{\Theta} - \Theta^{\text{true}}\|_F \leq \frac{2 C_2 U_\alpha \sqrt{r_{\max}^{K-1} \sum_k d_k}}{L_\alpha \sqrt{\prod_k d_k}} = \frac{2 C_2 U_\alpha r_{\max}^{(K-1)/2}}{L_\alpha} \sqrt{\frac{\sum_k d_k}{\prod_k d_k}}.$$

This completes the proof. □

Proof of Corollary **??**. The result follows immediately from Theorem **??** and Lemma 7. □

Proof of Theorem **??**. Let $d_{\text{total}} = \prod_{k \in [K]} d_k$, and $\gamma \in [0, 1]$ be a constant to be specified later. Our strategy is to construct a finite set of tensors $\mathcal{X} = \{\Theta_i : i = 1, \ldots\} \subset \mathcal{P}$ satisfying the properties of (i)-(iv) in Lemma 8. By Lemma 8, such a subset of tensors exist. For any tensor $\Theta \in \mathcal{X}$, let $\mathbb{P}_\Theta$ denote the distribution of $\mathcal{Y}|\Theta$, where $\mathcal{Y}$ is the ordinal tensor. In particular, $\mathbb{P}_\mathbf{0}$ is the distribution of $\mathcal{Y}$ induced by the zero parameter tensor $\mathbf{0}$, i.e., the distribution of $\mathcal{Y}$ conditional on the parameter tensor $\Theta = \mathbf{0}$. Based on the Remark for Lemma 7, we have

$$KL(\mathbb{P}_\Theta || \mathbb{P}_\mathbf{0}) \leq C \|\Theta\|_F^2, \tag{8}$$

where $C = \frac{(4L-6)\dot{f}^2(0)}{A_\alpha} > 0$ is a constant independent of the tensor dimension and rank. Combining the inequality (8) with property (iii) of $\mathcal{X}$, we have

$$\mathrm{KL}(\mathbb{P}_\Theta || \mathbb{P}_{\mathbf{0}}) \leq \gamma^2 R_{\max} d_{\max}. \tag{9}$$

From (9) and the property (i), we deduce that the condition

$$\frac{1}{\mathrm{Card}(\mathcal{X}) - 1} \sum_{\Theta \in \mathcal{X}} \mathrm{KL}(\mathbb{P}_\Theta, \mathbb{P}_{\mathbf{0}}) \leq \varepsilon \log_2 \{\mathrm{Card}(\mathcal{X}) - 1\} \tag{10}$$

holds for any $\varepsilon \geq 0$ when $\gamma \in [0, 1]$ is chosen to be sufficiently small depending on ε, e.g., $\gamma \leq \sqrt{3\varepsilon}$. By applying Lemma 10 to (10), and in view of the property (iv), we obtain that

$$\inf_{\hat{\Theta}} \sup_{\Theta^{\mathrm{true}} \in \mathcal{X}} \mathbb{P}\left(\|\hat{\Theta} - \Theta^{\mathrm{true}}\|_F \geq \frac{\gamma}{8} \min\left\{ \alpha \sqrt{d_{\mathrm{total}}}, C^{-1/2} \sqrt{R_{\max} d_{\max}} \right\} \right) \geq \frac{1}{2} \left(1 - 2\varepsilon - \sqrt{\frac{16\varepsilon}{R_{\max} d_{\max}}} \right). \tag{11}$$

Note that $\mathrm{Loss}(\hat{\Theta}, \Theta^{\mathrm{true}}) = \|\hat{\Theta} - \Theta^{\mathrm{true}}\|_F^2 / d_{\mathrm{total}}$ and $\mathcal{X} \subset \mathcal{P}$. By taking $\varepsilon = 1/8$ and $\gamma = 1/2$, we conclude from (11) that

$$\inf_{\hat{\Theta}} \sup_{\Theta^{\mathrm{true}} \in \mathcal{P}} \mathbb{P}\left(\mathrm{Loss}(\hat{\Theta}, \Theta^{\mathrm{true}}) \geq \frac{1}{256} \min\left\{ \alpha^2, \frac{C^{-1} R_{\max} d_{\max}}{d_{\mathrm{total}}} \right\} \right) \geq \frac{1}{2} \left(\frac{3}{4} - \frac{2}{R_{\max} d_{\max}} \right) \geq \frac{1}{8}.$$

This completes the proof. □

1.2 Sample complexity for tensor completion

Proof of Theorem **??**. For notational convenience, we use $\|\Theta\|_{F,\Omega} = \sum_{\omega \in \Omega} \Theta_\omega^2$ to denote the sum of squared entries over the observed set Ω, for a tensor $\Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K}$.

Following a similar argument as in the proof of Theorem **??**, we have

$$\mathcal{L}_{\mathcal{Y},\Omega}(\Theta) = \mathcal{L}_{\mathcal{Y},\Omega}(\Theta^{\mathrm{true}}) + \langle \mathrm{Vec}(\nabla_\Theta \mathcal{L}_{\mathcal{Y},\Omega}),\ \mathrm{Vec}(\Theta - \Theta^{\mathrm{true}}) \rangle + \frac{1}{2} \mathrm{Vec}(\Theta - \Theta^{\mathrm{true}})^T \nabla_\Theta^2 \mathcal{L}_{\mathcal{Y},\Omega}(\check{\Theta}) \mathrm{Vec}(\Theta - \Theta^{\mathrm{true}}), \tag{12}$$

where

1. $\nabla_\Theta \mathcal{L}_{\mathcal{Y},\Omega}$ is a $d_1 \times \cdots \times d_K$ tensor with $|\Omega|$ nonzero entries, and each entry is upper bounded by $U_\alpha > 0$.
2. $\nabla_\Theta^2 \mathcal{L}_{\mathcal{Y},\Omega}$ is a diagonal matrix of size d_{total}-by-d_{total} with $|\Omega|$ nonzero entries, and each entry is upper bounded by $-L_\alpha < 0$.

Similar to (3) and (7), we have

$$|\langle \mathrm{Vec}(\nabla_\Theta \mathcal{L}_{\mathcal{Y},\Omega}),\ \mathrm{Vec}(\Theta - \Theta^{\mathrm{true}}) \rangle| \leq C_2 U_\alpha \sqrt{r_{\max}^{K-1} \sum_k d_k} \|\Theta - \Theta^{\mathrm{true}}\|_{F,\Omega}$$

and

$$\mathrm{Vec}(\Theta - \Theta^{\mathrm{true}})^T \nabla_\Theta^2 \mathcal{L}_{\mathcal{Y}}(\check{\Theta}) \mathrm{Vec}(\Theta - \Theta^{\mathrm{true}}) \leq -L_\alpha \|\Theta - \Theta^{\mathrm{true}}\|_{F,\Omega}^2. \tag{13}$$

Combining (12)-(13) with the fact that $\mathcal{L}_{\mathcal{Y},\Omega}(\hat{\Theta}) \geq \mathcal{L}_{\mathcal{Y},\Omega}(\Theta^{\mathrm{true}})$, we have

$$\|\hat{\Theta} - \Theta^{\mathrm{true}}\|_{F,\Omega} \leq \frac{2 C_2 U_\alpha r_{\max}^{(K-1)/2}}{L_\alpha} \sqrt{\sum_k d_k}. \tag{14}$$

Lastly, we invoke the result regarding the closeness of Θ to its sampled version Θ_Ω, under the entrywise bound condition. Note that $\|\hat{\Theta}-\Theta^{\text{true}}\|_\infty \le 2\alpha$ and $\text{rank}(\hat{\Theta}-\Theta^{\text{true}}) \le 2\boldsymbol{r}$. By Lemma 1, $\|\hat{\Theta}-\Theta^{\text{true}}\|_M \le 2^{(3K-1)/2}\alpha\left(\frac{\prod r_k}{r_{\max}}\right)^{3/2}$. Therefore, the condition in Lemma 11 holds with $\beta = 2^{(3K-1)/2}\alpha\left(\frac{\prod r_k}{r_{\max}}\right)^{3/2}$. Applying Lemma 11 to (14) gives

$$\begin{aligned}\|\hat{\Theta}-\Theta^{\text{true}}\|_{F,\Pi}^2 &\le \frac{1}{m}\|\hat{\Theta}-\Theta^{\text{true}}\|_{F,\Omega}^2 + c\beta\sqrt{\frac{\sum_k d_k}{|\Omega|}} \\ &\le C_2 r_{\max}^{K-1}\frac{\sum_k d_k}{|\Omega|} + C_1\alpha r_{\max}^{3(K-1)/2}\sqrt{\frac{\sum_k d_k}{|\Omega|}},\end{aligned}$$

with probability at least $1-\exp(-\frac{\sum_k d_k}{\sum_k \log d_k})$ over the sampled set Ω. Here $C_1, C_2 > 0$ are two constants independent of the tensor dimension and rank. Therefore,

$$\|\hat{\Theta}-\Theta^{\text{true}}\|_{F,\Pi}^2 \to 0, \quad \text{as} \quad \frac{|\Omega|}{\sum_k d_k} \to \infty,$$

provided that $r_{\max} = O(1)$. □

1.3 Auxiliary lemmas

We begin with various notion of tensor norms that are useful for the proofs of the main theorems.

Definition 1 (Atomic M-norm [Ghadermarzy et al., 2019]). Define $T_\pm = \{\mathcal{T} \in \{\pm 1\}^{d_1\times\cdots\times d_K} : \text{rank}(\mathcal{T}) = 1\}$. The atomic M-norm of a tensor $\Theta \in \mathbb{R}^{d_1\times\cdots\times d_K}$ is defined as

$$\begin{aligned}\|\Theta\|_M &= \inf\{t>0 : \Theta \in t\,\text{conv}(T_\pm)\} \\ &= \inf\left\{\sum_{\mathcal{X}\in T_\pm} c_x : \Theta = \sum_{\mathcal{X}\in T_\pm} c_x\mathcal{X},\ c_x > 0\right\}.\end{aligned}$$

Definition 2 (Spectral norm [Lim, 2005]). The spectral norm of a tensor $\Theta \in \mathbb{R}^{d_1\times\cdots\times d_K}$ is defined as

$$\|\Theta\|_\sigma = \sup\left\{\langle\Theta, \boldsymbol{x}_1\otimes\cdots\otimes\boldsymbol{x}_K\rangle : \|\boldsymbol{x}_k\|_2 = 1,\ \boldsymbol{x}_k\in\mathbb{R}^{d_k},\ \text{for all } k\in[K]\right\}.$$

Definition 3 (Nuclear norm [Friedland and Lim, 2018]). The nuclear norm of a tensor $\Theta \in \mathbb{R}^{d_1\times\cdots\times d_K}$ is defined as

$$\|\Theta\|_* = \inf\left\{\sum_{i\in[r]}|\lambda_i| : \Theta = \sum_{i=1}^r \lambda_i \boldsymbol{x}_1^{(i)}\otimes\cdots\otimes\boldsymbol{x}_K^{(i)},\ \|\boldsymbol{x}_k^{(i)}\|_2 = 1,\ \boldsymbol{x}_k^{(i)}\in\mathbb{R}^{d_k},\ \text{for all}k\in[K],\ i\in[r]\right\},$$

where the infimum is taken over all $r\in\mathbb{N}$ and $\left\|\boldsymbol{x}_k^{(i)}\right\|_2 = 1$ for all $i\in[r]$ and $k\in[K]$.

Lemma 1 (M-norm and infinity norm [Ghadermarzy et al., 2019]). *Let $\Theta \in \mathbb{R}^{d_1\times\cdots\times d_K}$ be an order-K, rank-$(r_1,\ldots,r_K)$ tensor. Then*

$$\|\Theta\|_\infty \le \|\Theta\|_M \le \left(\frac{\prod_k r_k}{r_{\max}}\right)^{\frac{3}{2}}\|\Theta\|_\infty.$$

Lemma 2 (Nuclear norm and F-norm). *Let $\mathcal{A} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ be an order-K tensor with Tucker rank$(\mathcal{A}) = (r_1, \ldots, r_K)$. Then*

$$\|\mathcal{A}\|_* \leq \sqrt{\frac{\prod_k r_k}{\max_k r_k}} \|\mathcal{A}\|_F,$$

where $\|\cdot\|_*$ denotes the nuclear norm of the tensor.

Proof. Without loss of generality, suppose $r_1 = \min_k r_k$. Let $\mathcal{A}_{(k)}$ denote the mode-k matricization of $\mathcal{A}$ for all $k \in [K]$. By Wang et al. [2017, Corollary 4.11], and the invariance relationship between a tensor and its Tucker core [Jiang et al., 2017, Section 6], we have

$$\|\mathcal{A}\|_* \leq \sqrt{\frac{\prod_{k\geq 2} r_k}{\max_{k\geq 2} r_k}} \|\mathcal{A}_{(1)}\|_*, \tag{15}$$

where $\mathcal{A}_{(1)}$ is a d_1-by-$\prod_{k\geq 2} d_k$ matrix with matrix rank r_1. Furthermore, the relationship between the matrix norms implies that $\|\mathcal{A}_{(1)}\|_* \leq \sqrt{r_1}\|\mathcal{A}_{(1)}\|_F = \sqrt{r_1}\|\mathcal{A}\|_F$. Combining this fact with the inequality (15) yields the final claim. □

Lemma 3. *Let $\mathcal{A}$, $\mathcal{B}$ be two order-K tensors of the same dimension. Then*

$$|\langle \mathcal{A}, \mathcal{B} \rangle| \leq \|\mathcal{A}\|_\sigma \|\mathcal{B}\|_*.$$

Proof. By Friedland and Lim [2018, Proposition 3.1], there exists a nuclear norm decomposition of $\mathcal{B}$, such that

$$\mathcal{B} = \sum_r \lambda_r \boldsymbol{a}_r^{(1)} \otimes \cdots \otimes \boldsymbol{a}_r^{(K)}, \quad \boldsymbol{a}_r^{(k)} \in \mathbf{S}^{d_k - 1}(\mathbb{R}), \quad \text{for all } k \in [K],$$

and $\|\mathcal{B}\|_* = \sum_r |\lambda_r|$. Henceforth we have

$$\begin{aligned} |\langle \mathcal{A}, \mathcal{B} \rangle| &= |\langle \mathcal{A}, \sum_r \lambda_r \boldsymbol{a}_r^{(1)} \otimes \cdots \otimes \boldsymbol{a}_r^{(K)} \rangle| \leq \sum_r |\lambda_r| |\langle \mathcal{A}, \boldsymbol{a}_r^{(1)} \otimes \cdots \otimes \boldsymbol{a}_r^{(K)} \rangle| \\ &\leq \sum_r |\lambda_r| \|\mathcal{A}\|_\sigma = \|\mathcal{A}\|_\sigma \|\mathcal{B}\|_*, \end{aligned}$$

which completes the proof. □

The following lemma provides the bound on the spectral norm of random tensors. The result was firstly presented in Nguyen et al. [2015], and we adopt the version from Tomioka and Suzuki [2014].

Lemma 4 (Tomioka and Suzuki [2014]). *Suppose that $\mathcal{S} = [\![s_\omega]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ is an order-K tensor whose entries are independent random variables that satisfy*

$$\mathbb{E}(s_\omega) = 0, \quad \text{and} \quad \mathbb{E}(e^{t s_\omega}) \leq e^{t^2 L^2 / 2}.$$

Then the spectral norm $\|\mathcal{S}\|_\sigma$ satisfies that,

$$\|\mathcal{S}\|_\sigma \leq \sqrt{8L^2 \log(12K) \sum_k d_k + \log(2/\delta)},$$

with probability at least $1 - \delta$.

Lemma 5. *Suppose that $\mathcal{S} = [\![s_\omega]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ is an order-K tensor whose entries are independent random variables that satisfy*

$$\mathbb{E}(s_\omega) = 0, \quad \text{and} \quad |s_\omega| \leq U.$$

Then we have

$$\mathbb{P}\left(\|\mathcal{S}\|_\sigma \geq C_2 U \sqrt{\sum_k d_k} \right) \leq \exp\left(-C_1 \log K \sum_k d_k \right)$$

where $C_1 > 0$ is an absolute constant, and $C_2 > 0$ is a constant that depends only on K.

Proof. Note that the random variable $U^{-1} s_\omega$ is zero-mean and supported on $[-1, 1]$. Therefore, $U^{-1} s_\omega$ is sub-Gaussian with parameter $\frac{1-(-1)}{2} = 1$, i.e.

$$\mathbb{E}(U^{-1} s_\omega) = 0, \quad \text{and} \quad \mathbb{E}(e^{tU^{-1}s_\omega}) \leq e^{t^2/2}.$$

It follows from Lemma 4 that, with probability at least $1 - \delta$,

$$\|U^{-1}\mathcal{S}\|_\sigma \leq \sqrt{(c_0 \log K + c_1) \sum_k d_k + \log(2/\delta)},$$

where $c_0, c_1 > 0$ are two absolute constants. Taking $\delta = \exp(-C_1 \log K \sum_k d_k)$ yields the final claim, where $C_2 = c_0 \log K + c_1 + 1 > 0$ is another constant. □

Lemma 6. *Let X, Y be two discrete random variables taking values on L possible categories, with category probabilities $\{p_\ell\}_{\ell \in [L]}$ and $\{q_\ell\}_{\ell \in [L]}$, respectively. Suppose p_ℓ, $q_\ell > 0$ for all $i \in [L]$. Then, the Kullback-Leibler (KL) divergence satisfies that*

$$KL(X||Y) \stackrel{def}{=} -\sum_{\ell \in [L]} \mathbb{P}_X(\ell) \log\left\{ \frac{\mathbb{P}_Y(\ell)}{\mathbb{P}_X(\ell)} \right\} \leq \sum_{\ell \in [L]} \frac{(p_\ell - q_\ell)^2}{q_\ell}.$$

Proof. Using the fact $\log x \leq x - 1$ for $x > 0$, we have that

$$\begin{aligned}
\mathrm{KL}(X||Y) &= \sum_{\ell \in [L]} p_\ell \log \frac{p_\ell}{q_\ell} \\
&\leq \sum_{\ell \in [L]} \frac{p_\ell}{q_\ell}(p_\ell - q_\ell) \\
&= \sum_{\ell \in [L]} \left(\frac{p_\ell}{q_\ell} - 1 \right)(p_\ell - q_\ell) + \sum_{\ell \in [L]} (p_\ell - q_\ell).
\end{aligned}$$

Note that $\sum_{\ell \in [L]} (p_\ell - q_\ell) = 0$. Therefore,

$$\mathrm{KL}(X||Y) \leq \sum_{\ell \in [L]} \left(\frac{p_\ell}{q_\ell} - 1 \right)(p_\ell - q_\ell) = \sum_{\ell \in [L]} \frac{(p_\ell - q_\ell)^2}{q_\ell}.$$

□

Lemma 7 (KL divergence and F-norm)**.** *Let $\mathcal{Y} \in [L]^{d_1 \times \cdots \times d_K}$ be an ordinal tensor generated from the model (??) with the link function f and parameter tensor Θ. Let $\mathbb{P}_\Theta$ denote the joint categorical distribution of $\mathcal{Y}|\Theta$ induced by the parameter tensor Θ, where $\|\Theta\|_\infty \leq \alpha$. Define*

$$A_\alpha = \min_{\ell \in [L], |\theta| \leq \alpha} [f(\theta + b_\ell) - f(\theta + b_{\ell-1})]. \tag{16}$$

Then, for any two tensors Θ, Θ^* in the parameter spaces, we have

$$KL(\mathbb{P}_\Theta || \mathbb{P}_{\Theta^*}) \leq \frac{2(2L-3)}{A_\alpha} \dot{f}^2(0) \|\Theta - \Theta^*\|_F^2.$$

Proof. Suppose that the distribution over the ordinal tensor $\mathcal{Y} = [\![y_\omega]\!]$ is induced by $\Theta = [\![\theta_\omega]\!]$. Then, based on the generative model (**??**),

$$\mathbb{P}(y_\omega = \ell | \theta_\omega) = f(\theta_\omega + b_\ell) - f(\theta_\omega + b_{\ell-1}),$$

for all $\ell \in [L]$ and $\omega \in [d_1] \times \cdots \times [d_K]$. For notational convenience, we suppress the subscribe in θ_ω and simply write θ (and respectively, θ^*). Based on Lemma 6 and Taylor expansion,

$$\begin{aligned}
\mathrm{KL}(\theta || \theta^*) &\leq \sum_{\ell \in [L]} \frac{[f(\theta + b_\ell) - f(\theta + b_{\ell-1}) - f(\theta^* + b_\ell) + f(\theta^* + b_{\ell-1})]^2}{f(\theta^* + b_\ell) - f(\theta^* + b_{\ell-1})} \\
&\leq \sum_{\ell=2}^{L-1} \frac{\left[\dot{f}(\eta_\ell + b_\ell) - \dot{f}(\eta_{\ell-1} + b_{\ell-1})\right]^2}{f(\theta^* + b_\ell) - f(\theta^* + b_{\ell-1})} (\theta - \theta^*)^2 + \frac{\dot{f}^2(\eta_1 + b_1)}{f(\theta^* + b_1)} (\theta - \theta^*)^2 \\
&\qquad + \frac{\dot{f}^2(\eta_{L-1} + b_{L-1})}{1 - f(\theta^* + b_{L-1})} (\theta - \theta^*)^2,
\end{aligned}$$

where η_ℓ and $\eta_{\ell-1}$ fall between θ and θ^*. Therefore,

$$\mathrm{KL}(\theta || \theta^*) \leq \left(\frac{4(L-2)}{A_\alpha} + \frac{2}{A_\alpha} \right) \dot{f}^2(0) (\theta - \theta^*)^2 = \frac{2(2L-3)}{A_\alpha} \dot{f}^2(0) (\theta - \theta^*)^2, \tag{17}$$

where we have used Taylor expansion, the bound (16), and the fact that $\dot{f}(\cdot)$ peaks at zero for an unimodal and symmetric function. Now summing (17) over the index set $\omega \in [d_1] \times \cdots \times [d_K]$ gives

$$\mathrm{KL}(\mathbb{P}_\Theta || \mathbb{P}_{\Theta^*}) = \sum_{\omega \in [d_1] \times \cdots \times [d_K]} \mathrm{KL}(\theta_\omega || \theta^*_\omega) \leq \frac{2(2L-3)}{A_\alpha} \dot{f}^2(0) \|\Theta - \Theta^*\|_F^2.$$

□

Remark 1. In particular, let $\mathbb{P}_{\mathbf{0}}$ denote the distribution of $\mathcal{Y}|\mathbf{0}$ induced by the zero parameter tensor. Then we have

$$\mathrm{KL}(\mathbb{P}_\Theta || \mathbb{P}_{\mathbf{0}}) \leq \frac{2(2L-3)}{A_\alpha} \dot{f}^2(0) \|\Theta\|_F^2.$$

Lemma 8. *Assume the same setup as in Theorem **??**. Without loss of generality, suppose $d_1 = \max_k d_k$. Define $R = \max_k r_k$ and $d_{total} = \prod_{k \in [K]} d_k$. For any constant $0 \leq \gamma \leq 1$, there exist a finite set of tensors $\mathcal{X} = \{\Theta_i : i = 1, \ldots\} \subset \mathcal{P}$ satisfying the following four properties:*

(i) Card$(\mathcal{X}) \geq 2^{Rd_1/8} + 1$, where Card denotes the cardinality;

(ii) $\mathcal{X}$ *contains the zero tensor* $\mathbf{0} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$;

(iii) $\|\Theta\|_\infty \le \gamma \min\left\{\alpha,\ C^{-1/2}\sqrt{\frac{Rd_1}{d_{total}}}\right\}$ *for any element* $\Theta \in \mathcal{X}$;

(iv) $\|\Theta_i - \Theta_j\|_F \ge \frac{\gamma}{4}\min\left\{\alpha\sqrt{d_{total}},\ C^{-1/2}\sqrt{Rd_1}\right\}$ *for any two distinct elements* $\Theta_i,\ \Theta_j \in \mathcal{X}$,

Here $C = C(\alpha, L, f, \boldsymbol{b}) = \frac{(4L-6)\dot{f}^2(0)}{A_\alpha} > 0$ *is a constant independent of the tensor dimension and rank.*

Proof. Given a constant $0 \le \gamma \le 1$, we define a set of matrices:

$$\mathcal{C} = \left\{\boldsymbol{M} = (m_{ij}) \in \mathbb{R}^{d_1 \times R} : a_{ij} \in \left\{0, \gamma \min\left\{\alpha, C^{-1/2}\sqrt{\frac{Rd_1}{d_{\text{total}}}}\right\}\right\},\ \forall (i,j) \in [d_1] \times [R]\right\}.$$

We then consider the associated set of block tensors:

$$\mathcal{B} = \mathcal{B}(\mathcal{C}) = \{\Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K} : \Theta = \boldsymbol{A} \otimes \mathbf{1}_{d_3} \otimes \cdots \otimes \mathbf{1}_{d_K},\\ \text{where } \boldsymbol{A} = (\boldsymbol{M}|\cdots|\boldsymbol{M}|\boldsymbol{O}) \in \mathbb{R}^{d_1 \times d_2},\ \boldsymbol{M} \in \mathcal{C}\},$$

where $\mathbf{1}_d$ denotes a length-d vector with all entries 1, $\boldsymbol{O}$ denotes the $d_1 \times (d_2 - R\lfloor d_2/R \rfloor)$ zero matrix, and $\lfloor d_2/R \rfloor$ is the integer part of d_2/R. In other words, the subtensor $\Theta(\boldsymbol{I}, \boldsymbol{I}, i_3, \ldots, i_K) \in \mathbb{R}^{d_1 \times d_2}$ are the same for all fixed $(i_3, \ldots, i_K) \in [d_3] \times \cdots \times [d_K]$, and furthermore, each subtensor $\Theta(\boldsymbol{I}, \boldsymbol{I}, i_3, \ldots, i_K)$ itself is filled by copying the matrix $\boldsymbol{M} \in \mathbb{R}^{d_1 \times R}$ as many times as would fit.

By construction, any element of $\mathcal{B}$, as well as the difference of any two elements of $\mathcal{B}$, has Tucker rank at most $\max_k r_k \le R$, and the entries of any tensor in $\mathcal{B}$ take values in $[0, \alpha]$. Thus, $\mathcal{B} \subset \mathcal{P}$. By Lemma 9, there exists a subset $\mathcal{X} \subset \mathcal{B}$ with cardinality $\text{Card}(\mathcal{X}) \ge 2^{Rd_1/8} + 1$ containing the zero $d_1 \times \cdots \times d_K$ tensor, such that, for any two distinct elements Θ_i and Θ_j in $\mathcal{X}$,

$$\|\Theta_i - \Theta_j\|_F^2 \ge \frac{Rd_1}{8}\gamma^2 \min\left\{\alpha^2, \frac{C^{-1}Rd_1}{d_{\text{total}}}\right\}\lfloor\frac{d_2}{R}\rfloor\prod_{k\ge 3} d_k \ge \frac{\gamma^2 \min\left\{\alpha^2 d_{\text{total}}, C^{-1}Rd_1\right\}}{16}.$$

In addition, each entry of $\Theta \in \mathcal{X}$ is bounded by $\gamma \min\left\{\alpha, C^{-1/2}\sqrt{\frac{Rd_1}{d_{\text{total}}}}\right\}$. Therefore the Properties (i) to (iv) are satisfied. □

Lemma 9 (Varshamov-Gilbert bound)**.** *Let* $\Omega = \{(w_1, \ldots, w_m) : w_i \in \{0, 1\}\}$. *Suppose* $m > 8$. *Then there exists a subset* $\{w^{(0)}, \ldots, w^{(M)}\}$ *of* Ω *such that* $w^{(0)} = (0, \ldots, 0)$ *and*

$$\|w^{(j)} - w^{(k)}\|_0 \ge \frac{m}{8}, \quad \text{for } 0 \le j < k \le M,$$

where $\|\cdot\|_0$ *denotes the Hamming distance, and* $M \ge 2^{m/8}$.

Lemma 10 (Theorem 2.5 in Tsybakov [2009])**.** *Assume that a set* $\mathcal{X}$ *contains element* $\Theta_0, \Theta_1, \ldots, \Theta_M$ *(*$M \ge 2$*) such that*

- $d(\Theta_j,\ \Theta_k) \ge 2s > 0$, $\forall 0 \le j \le k \le M$;
- $\mathbb{P}_j \ll \mathbb{P}_0$, $\forall j = 1, \ldots, M$, *and*

$$\frac{1}{M}\sum_{j=1}^{M} KL(\mathbb{P}_j || \mathbb{P}_0) \le \alpha \log M$$

where $d\colon \mathcal{X} \times \mathcal{X} \mapsto [0, +\infty]$ *is a semi-distance function,* $0 < \alpha < 1/8$ *and* $P_j = P_{\Theta_j}$, $j = 0, 1 \ldots, M$.

Then

$$\inf_{\hat{\Theta}} \sup_{\Theta \in \mathcal{X}} \mathbb{P}_\Theta(d(\hat{\Theta}, \Theta) \geq s) \geq \frac{\sqrt{M}}{1+\sqrt{M}} \left(1 - 2\alpha - \sqrt{\frac{2\alpha}{\log M}}\right) > 0.$$

Lemma 11 (Lemma 28 in Ghadermarzy et al. [2019]). *Define* $\mathbb{B}_M(\beta) = \{\Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K} : \|\Theta\|_M \leq \beta\}$. *Let* $\Omega \subset [d_1] \times \cdots \times [d_K]$ *be a random set with* $m = |\Omega|$, *and assume that each entry in* Ω *is drawn with replacement from* $[d_1] \times \cdots \times [d_K]$ *using probability* Π. *Define*

$$\|\Theta\|_{F,\Pi}^2 = \frac{1}{m} \mathbb{E}_{\Omega \in \Pi} \|\Theta\|_{F,\Omega}^2.$$

Then, there exists a universal constant $c > 0$, *such that, with probability at least* $1 - \exp\left(-\frac{\sum_k d_k}{\sum_k \log d_k}\right)$ *over the sampled set* Ω,

$$\frac{1}{m}\|\Theta\|_{F,\Omega}^2 \geq \|\Theta\|_{F,\Pi}^2 - c\beta\sqrt{\frac{\sum_k d_k}{m}}$$

holds uniformly for all $\Theta \in \mathbb{B}_M(\beta)$.

2 Convexity of the likelihood function

Theorem 2.1.

$$\mathcal{L}_{\mathcal{Y},\Omega}(\Theta, \boldsymbol{b}) = \sum_{\omega \in \Omega} \sum_{\ell \in [L]} \left\{ \mathbb{1}_{\{y_\omega = \ell\}} \log\left[f(\theta_\omega + b_\ell) - f(\theta_\omega + b_{\ell-1})\right] \right\}, \text{ where } f(x) = \frac{e^x}{1+e^x}.$$

is a concave function to $(\Theta, \boldsymbol{b})$

Proof. We use $(\Theta, \boldsymbol{b})$ abusively as $(Vec(\Theta)^T, \boldsymbol{b}^T)^T$ in this proof. Let us denote $\lambda(u, v) = \log\left[f(u) - f(v)\right]$. It is enough to show that $\lambda(u, v)$ is a concave function to (u, v) where $u > v$. Because if $\lambda(u, v)$ is a concave function and u, v are both linear functions of $(\Theta, \boldsymbol{b})$ such that

$$u = \boldsymbol{a}_1^T(\Theta, \boldsymbol{b}), v = \boldsymbol{a}_2^T(\Theta, \boldsymbol{b}) \text{ for some } \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^{d_{\text{total}}+L-1},$$

then $\lambda(u, v) = \lambda(\boldsymbol{a}_1^T(\Theta, \boldsymbol{b}), \boldsymbol{a}_2^T(\Theta, \boldsymbol{b})))$ is a concave function to $(\Theta, \boldsymbol{b})$ by the definition of the convexity. Furthermore, we can conclude that $\mathcal{L}_{\mathcal{Y},\Omega}(\Theta, \boldsymbol{b})$ is a concave function to $(\Theta, \boldsymbol{b})$ because $\mathcal{L}_{\mathcal{Y},\Omega}(\Theta, \boldsymbol{b})$ can be written as the form of summations of $\lambda(u, v)$. To prove the convexity of $\lambda(u, v)$, we have the following.

$$\lambda(u, v) = \log\left[f(u) - f(v)\right] = \log\left[\int \mathbb{1}_{(u,v)}(x) f'(x) dx\right].$$

Notice that $\mathbb{1}_{(u,v)}(x)$ is log-concave to (u, v, x) and $f'(x)$ is log-concave to x because $f'(x) = \frac{e^x}{(1+e^x)^2}$. Those convexities imply that $\mathbb{1}_{(u,v)}(x) f'(x)$ is a log-concave function to (u, v, x). Therefore, we can conclude that the $\lambda(u, v)$ is a concave function from the lemma 12. □

Lemma 12 (Corollary 3.5 in Brascamp and Lieb [2002]). *Let* $F(x, y) : \mathbb{R}^{m+n} \to \mathbb{R}$ *be an integrable function where* $x \in \mathbb{R}^m, y \in \mathbb{R}^n$. *Let*

$$G(x) = \int_{\mathbb{R}^n} F(x, y) dy.$$

If $F(x, y)$ *is a log concave function to (x,y), then* $G(x)$ *is a log concave function.*

3 Brain clustering method and result

3.1 Clustering method

We perform clustering based on the estimated latent parameter $\hat{\Theta}$. Our clustering method is based on multidimensional version of Principal Component Analysis (PCA) and Singular Value Decomposition (SVD). In matrices case, we perform clustering on a $m \times n$ data matrix X based on the following procedure. First, we factorize X into

$$X = U\Sigma V^T,$$

where Σ is a diagonal matrix and U, V are factor matrices with orthogonal columns. We interpret each column of V as a principal direction or axis and $U\Sigma$ as principal components. From this interpretation, we perform K-means method to m rows of $U\Sigma$. We suggest to apply this procedure to the estimated latent parameter tensor $\hat{\Theta}$. For the notational convenience, we use Θ as $\hat{\Theta}$ in this section. From the Tucker decomposition which is multidimensional version of SVD, we have

$$\Theta = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K,$$

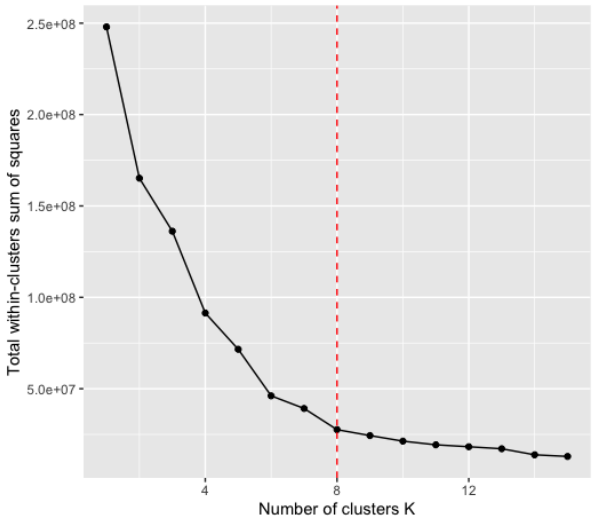
where $\mathcal{C} \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ is a core tensor, $\boldsymbol{M}_k \in \mathbb{R}^{d_k \times r_k}$ are factor matrices with orthogonal columns, and $\times_k$ denotes the tensor-by-matrix multiplication Kolda and Bader [2009]. We perform mode-k matricization of the tensor Θ to cluster data from mode-k variable as follows.

$$\Theta_{(k)} = \boldsymbol{M}_k \mathcal{C}_{(k)} \left(\boldsymbol{M}_K \otimes \cdots \otimes \boldsymbol{M}_1\right)$$

We interpret $\left(\boldsymbol{M}_K \otimes \cdots \otimes \boldsymbol{M}_1\right)$ as principal axes and $\boldsymbol{M}_k \mathcal{C}_{(k)}$ as principal components. By putting other mode variables as principal axes, we can exclude effects from those variables and can see k-th mode effect from checking principal components, $\boldsymbol{M}_k \mathcal{C}_{(k)}$. Therefore, we perform K-means method to d_k rows of the principal components like in the matrices case.

3.2 Clustering result on HCP

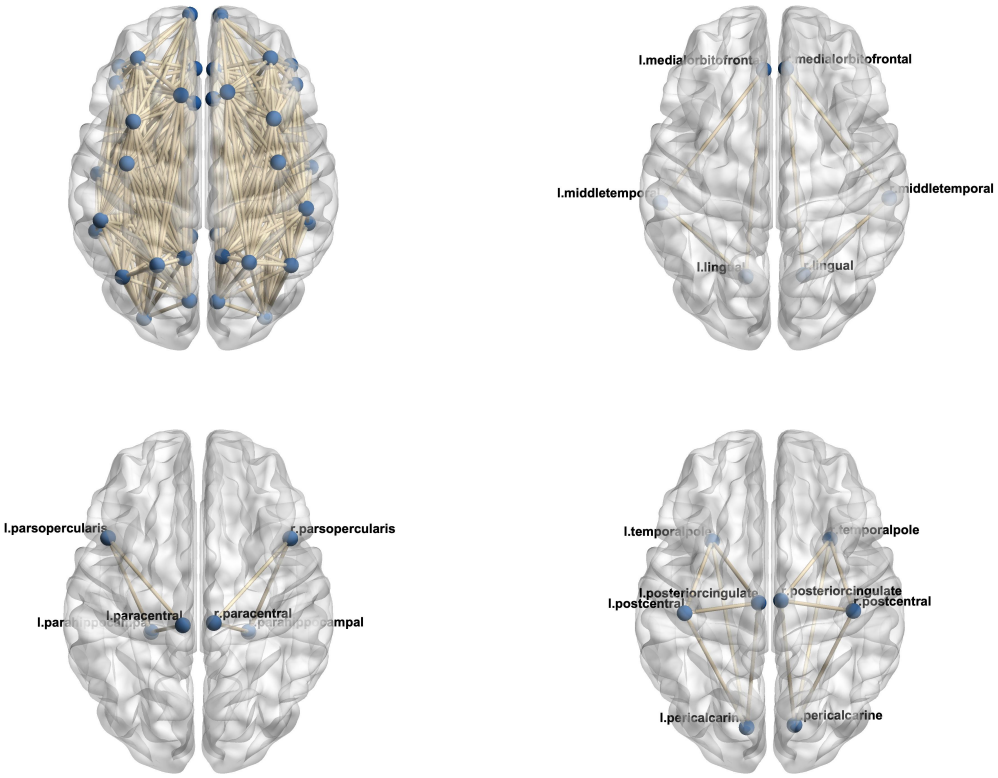
We perform K-means method to cluster brain nodes based on the suggested above method. We have the estimated latent parameter $\hat{\Theta}$ with the rank $(23, 23, 8)$ which minimizes the BIC value. The number of clusters is chosen to be eight from the elbow method in figure 3.2. The table in 3.2 shows eight cluster groups and their entries. Most of brain nodes fall into the cluster #1 and the cluster #2 which can be represented as the left side brain and the right side brain. From the cluster #3 to the cluster #8 , entries of each cluster share the same name encoded in the data and play a similar role in the brain. For example, the cluster #3 and the cluster #4 are the left and the right side of the regions called Supramarginal gyrus. Those regions are known to be in charge of interpreting tactile sensory data and involved in perception of space and limbs location Carlson [2012], Reed and Caselli [1994]. The figure 3.2 shows the nodes in the dataset on the brain image. Each node is connected by edges within the same cluster and the groups can be paired as the right part and the left part of the brain. The results show that the clustering based on our model successfully group the brain nodes without knowledge of the brain region.



Supplementary Figure S1: The elbow plot for the number of clusters

Group	#1			#2		
Brain nodes	"RMF_L","'Fpole_L","Insula_L", "SupF_L","CaudMF_L", "parsTriangularis_L","parsOpercularis_L", "preCentral_L","Tpole_L","SupT_L", "SupT_L","postCentral_L", "SupP_L", "IP_L","LO_L","MOF_L","SupF_L", "IsthmusC_L","Precuneus_L", "Cuneus_L","paraHippo_L", "Lingual_L","SupT_L","LO_L"			"RMF_R","Fpole_R","Insula_R", "SupF_R", "CaudMF_R", "parsTriangularis_R" ,"parsOpercularis_R", "preCentral_R", "Tpole_R","SupT_R", "SupT_R","postCentral_R","SupP_R", "IP_R","LO_R","MOF_R", "SupF_R", "IsthmusC_R","Precuneus_R", "Cuneus_R","paraHippo_R", "Lingual_R","SupT_R","LO_R"		
Group	#3	#4	#5	#6	#7	#8
Brain nodes	"SupraM_L" "SupraM_L" "SupraM_L" "SupraM_L"	"SupraM_R" "SupraM_R" "SupraM_R" "SupraM_R"	"IT_L" "IT_L" "IT_L"	"IT_R" "IT_R" "IT_R"	"MT_L" "MT_L" "MT_L"	"MT_R" "MT_R" "MT_R"

Supplementary Table S1: Result of the brain nodes clustering: the last alphabet in each node name means the left side for 'L' and the right side for 'R' in the brain



Supplementary Figure S2: Brain image of eight clusters: entries of each cluster are connected by edges

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