Correction of Theorem 0.1

Miaoyan Wang, March 1, 2020

We now extend Theorem 4.1 to the case of unknown cut-off points $\boldsymbol{b}$. Assume that the true parameters $(\Theta^{\text{true}}, \boldsymbol{b}^{\text{true}}) \in \mathcal{P} \times \mathcal{B}$, where the feasible sets are defined as

$$\mathcal{P} = \{\Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K} \colon \text{rank}(\mathcal{P}) \leq \boldsymbol{r},\ \langle \Theta, \mathcal{J} \rangle = 0,\ \|\Theta\|_\infty \leq \alpha\},$$
$$\mathcal{B} = \{\boldsymbol{b} \in \mathbb{R}^{L-1} \colon \|\boldsymbol{b}\|_\infty \leq \beta,\ \min_\ell (b_\ell - b_{\ell-1}) \geq \Delta\}.$$

Here, $\mathcal{J} = [\![1]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ denotes a tensor of all ones. The constraint $\langle \Theta, \mathcal{J} \rangle = 0$ is imposed to ensure the identifiability of Θ and $\boldsymbol{b}$. We propose the constrained M-estimator

$$(\hat{\Theta}, \hat{\boldsymbol{b}}) = \underset{(\Theta, \boldsymbol{b}) \in \mathcal{P} \times \mathcal{B}}{\arg\max}\ \mathcal{L}_{\mathcal{Y}}(\Theta, \boldsymbol{b}). \tag{1}$$

The estimation accuracy is assessed using the mean squared error (MSE):

$$\text{MSE}\left(\hat{\Theta}, \Theta^{\text{true}}\right) = \frac{1}{\prod_k d_k} \|\hat{\Theta} - \Theta^{\text{true}}\|_F, \quad \text{MSE}\left(\hat{\boldsymbol{b}}, \boldsymbol{b}^{\text{true}}\right) = \frac{1}{L-1} \|\hat{\boldsymbol{b}} - \boldsymbol{b}^{\text{true}}\|_F.$$

To facilitate the examination of MSE, we define an order-$(K+1)$ tensor, $\mathcal{Z} = [\![z_{\omega,\ell}]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K \times (L-1)}$, by stacking the parameters $\Theta = [\![\theta_\omega]\!]$ and $\boldsymbol{b} = [\![b_\ell]\!]$ together. Specifically, let $z_{\omega,\ell} = -\theta_\omega + b_\ell$ for all $\omega \in [d_1] \times \cdots \times [d_K]$ and $\ell \in [L-1]$; that is,

$$\mathcal{Z} = -\Theta \otimes \mathbf{1} + \mathcal{J} \otimes \boldsymbol{b},$$

where $\mathbf{1}$ denotes a length-$(L-1)$ vector of all ones. Under the identifiability constraint $\langle \Theta, \mathcal{J} \rangle = 0$, there is an one-to-one mapping between $\mathcal{Z}$ and $(\Theta, \boldsymbol{b})$, with $\text{rank}(\mathcal{Z}) \leq (\text{rank}(\Theta), 2)^T$. Furthermore,

$$\|\hat{\mathcal{Z}} - \mathcal{Z}^{\text{true}}\|_F^2 = \|\hat{\Theta} - \Theta^{\text{true}}\|_F^2 (L-1) + \|\hat{\boldsymbol{b}} - \boldsymbol{b}^{\text{true}}\|_F^2 \left(\prod_k d_k\right), \tag{2}$$

where $\mathcal{Z}^{\text{true}} = \Theta^{\text{true}} \otimes \mathbf{1} + \mathcal{J} \otimes \boldsymbol{b}^{\text{true}}$ and $\hat{\mathcal{Z}} = \hat{\Theta} \otimes \mathbf{1} + \mathcal{J} \otimes \hat{\boldsymbol{b}}$.

We make the following assumptions about the link function.

Assumption 1. *The link function $f \colon \mathbb{R} \mapsto [0,1]$ satisfies the following properties:*

1. *$f(z)$ is twice-differentiable and strictly increasing in z.*
2. *$\dot{f}(z)$ is strictly log-concave and symmetric with respect to $z = 0$.*

We define the following constants that will be used in the theory:

$$C_{\alpha,\beta,\Delta} = \max_{|z| \leq \alpha+\beta} \max_{\substack{z' \leq z - \Delta \\ z'' \geq z + \Delta}} \max \left\{ \frac{\dot{f}(z)}{f(z) - f(z')}, \frac{\dot{f}(z)}{f(z'') - f(z)} \right\}, \tag{3}$$

$$D_{\alpha,\beta,\Delta} = \max_{|z| \leq \alpha+\beta} \max_{\substack{z' \leq z - \Delta \\ z'' \geq z + \Delta}} \max \left\{ -\frac{\partial}{\partial z} \left(\frac{\dot{f}(z)}{f(z) - f(z')} \right), \frac{\partial}{\partial z} \left(\frac{\dot{f}(z)}{f(z'') - f(z)} \right) \right\},$$

$$A_{\alpha,\beta,\Delta} = \min_{|z| \leq \alpha+\beta} \min_{z' \leq z - \Delta} \left(f(z) - f(z') \right).$$

Remark 1. The condition $\Delta = \min_\ell(b_\ell - b_{\ell-1}) > 0$ on the feasible set $\mathcal{B}$ guarantees the strict positiveness of $f(z) - f(z')$ and $f(z'') - f(z)$. Therefore, the denominators in the above quantities $C_{\alpha,\beta,\Delta}, D_{\alpha,\beta,\Delta}$ are well-defined. Furthermore, by Theorem 8.1, $f(z) - f(z')$ is strictly log-concave in (z, z') for $z \leq z' - \Delta, z, z' \in [-\alpha - \beta,\ \alpha + \beta]$. Based on Assumption 1 and closeness of the feasible set, we have $C_{\alpha,\beta,\Delta} > 0$, $D_{\alpha,\beta,\Delta} > 0$, $A_{\alpha,\beta,\Delta} > 0$.

Remark 2. Add the specific bound for logistic link.

Theorem 0.1 (Statistical convergence with unknown $\boldsymbol{b}$)**.** *Consider an ordinal tensor $\mathcal{Y} \in [L]^{d_1 \times \cdots \times d_K}$ generated from model (1) with the link function f and parameters $(\Theta^{\text{true}}, \boldsymbol{b}^{\text{true}}) \in \mathcal{P} \times \mathcal{B}$. Suppose the link function f satisfies Assumption 1. Define $r_{\max} = \max_k r_k$, and assume $r_{\max} = \mathcal{O}(1)$.*

Then with very high probability, the estimator in (1) satisfies

$$\|\hat{\mathcal{Z}} - \mathcal{Z}^{\text{true}}\|_F^2 \leq \frac{c_1 r_{\max}^{K-1} C_{\alpha,\beta,\Delta}^2}{A_{\alpha,\beta,\Delta}^2 D_{\alpha,\beta,\Delta}^2} \left(L - 1 + \sum_k d_k \right), \tag{4}$$

In particular,

$$\text{MSE}\left(\hat{\Theta}, \Theta^{\text{true}}\right) \leq \min\left(4\alpha^2,\ \frac{c_1 r_{\max}^{K-1} C_{\alpha,\beta,\Delta}^2}{A_{\alpha,\beta,\Delta}^2 D_{\alpha,\beta,\Delta}^2} \frac{L - 1 + \sum_k d_k}{(L-1)\prod_k d_k}\right),$$

and

$$\text{MSE}\left(\hat{\boldsymbol{b}}, \boldsymbol{b}^{\text{true}}\right) \leq \min\left(4\beta^2,\ \frac{c_1 r_{\max}^{K-1} C_{\alpha,\beta,\Delta}^2}{A_{\alpha,\beta,\Delta}^2 D_{\alpha,\beta,\Delta}^2} \frac{L - 1 + \sum_k d_k}{(L-1)\prod_k d_K}\right),$$

where $c_1, C_{\alpha,\beta,\Delta}, D_{\alpha,\beta,\Delta}$ are positive constants independent of the tensor dimension, rank, and number of ordinal levels.

Proof. (sketch)

Let $\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{Y}} = [\![\frac{\partial \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell}}]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K \times [L-1]}$ denote the score function, and $\boldsymbol{H} = \nabla_{\mathcal{Z}}^2 \mathcal{L}_{\mathcal{Y}}$ the Hession matrix. Following the same argument in the previous version (Taylor expansion, $r_{\max}(\mathcal{Z}) = r_{\max}(\Theta)$, etc), we have

$$\|\hat{\mathcal{Z}} - \mathcal{Z}^{\text{true}}\|_F^2 \leq c_1 r_{\max}^{K-1} \frac{\|\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{Y}}(\mathcal{Z}^{\text{true}})\|_\sigma^2}{\lambda_1^2\left(\boldsymbol{H}(\check{\mathcal{Z}})\right)}, \tag{5}$$

where $\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{Y}}(\mathcal{Z}^{\text{true}})$ is the score evaluated at $\mathcal{Z}^{\text{true}}$, $\boldsymbol{H}(\check{\mathcal{Z}})$ is the Hession evaluated at $\check{\mathcal{Z}}$, for some $\check{\mathcal{Z}}$ between $\hat{\mathcal{Z}}$ and $\mathcal{Z}^{\text{true}}$, and $\lambda_1(\cdot)$ is the largest matrix eigenvalue.

Hence, it suffices to bound the score and the Hession.

1. (Score.) The (ω, ℓ)-th entry in $\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{Y}}$ is

$$\frac{\partial \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell}} = \mathbb{1}_{\{y_\omega = \ell\}} \frac{\dot{f}(z)}{f(z) - f(z')}\Bigg|_{(z,\ z') = (z_{\omega,\ell},\ z_{\omega,\ell-1})} - \mathbb{1}_{\{y_\omega = \ell+1\}} \frac{\dot{f}(z)}{f(z'') - f(z)}\Bigg|_{(z'',\ z) = (z_{\omega,\ell+1},\ z_{\omega,\ell})},$$

which is upper bounded in magnitude by $C_{\alpha,\beta,\Delta} > 0$. Therefore, with very high probability,

$$\|\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{Y}}(\mathcal{Z}^{\text{true}})\|_\sigma \leq C_{\alpha,\beta,\Delta} \sqrt{L - 1 + \sum_k d_k}.$$

2. (Hession.) The entries in the Hession matrix are

$$\text{Diagonal: } \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell}^2} = \mathbb{1}_{\{y_\omega=\ell\}} \frac{\ddot{f}(z)\left(f(z)-f(z')\right)-\dot{f}^2(z)}{\left(f(z)-f(z')\right)^2}\Bigg|_{(z,\ z')=(z_{\omega,\ell},\ z_{\omega,\ell-1})} -$$

$$\mathbb{1}_{\{y_\omega=\ell+1\}} \frac{\ddot{f}(z)\left(f(z'')-f(z)\right)+\dot{f}^2(z)}{\left(f(z'')-f(z)\right)^2}\Bigg|_{(z'',\ z)=(z_{\omega,\ell+1},\ z_{\omega,\ell})},$$

$$\text{Off-diagonal: } \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell} z_{\omega,\ell+1}} = \mathbb{1}_{\{y_\omega=\ell+1\}} \frac{\dot{f}(z_{\omega,\ell})\dot{f}(z_{\omega,\ell+1})}{\left(f(z_{\omega,\ell+1})-f(z_{\omega,\ell})\right)^2} \quad \text{and} \quad \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell} z_{\omega',\ell'}} = 0 \text{ otherwise.}$$

Based on Assumption 1, the Hession matrix $\boldsymbol{H}$ has the following three properties:

(a) The Hession matrix is a block matrix, $\boldsymbol{H} = \operatorname{diag}\{\boldsymbol{H}_\omega : \omega \in [d_1] \times \cdots \times [d_K]\}$, and each block $\boldsymbol{H}_\omega \in \mathbb{R}^{(L-1)\times(L-1)}$ is a tridiagonal matrix.

(b) The off-diagonal entries are either zero or strictly positive.

(c) The diagonal entries are either zero or strictly negative. Furthermore,

$$\begin{aligned}
&\boldsymbol{H}_\omega(\ell,\ell) + \boldsymbol{H}_\omega(\ell,\ell-1) + \boldsymbol{H}_\omega(\ell,\ell+1) \\
=&\frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell}^2} + \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell} z_{\omega,\ell+1}} + \frac{\partial^2 \mathcal{L}_{\mathcal{Y}}}{\partial z_{\omega,\ell-1} z_{\omega,\ell}} \\
=&\mathbb{1}_{\{y_\omega=\ell\}} \frac{\partial}{\partial z}\left(\frac{\dot{f}(z)}{f(z)-f(z')}\right)\Bigg|_{(z,\ z')=(z_{\omega,\ell},\ z_{\omega,\ell-1})} - \mathbb{1}_{\{y_\omega=\ell+1\}} \frac{\partial}{\partial z}\left(\frac{\dot{f}(z)}{f(z)-f(z')}\right)\Bigg|_{(z'',\ z)=(z_{\omega,\ell+1},\ z_{\omega,\ell})} \\
\leq& - D_{\alpha,\beta,\Delta} < 0.
\end{aligned}$$

We will show that, with very high probability over $\mathcal{Y}$, $\boldsymbol{H}$ is negative definite in that

$$\lambda_1(\boldsymbol{H}) = \max_{\boldsymbol{z}} \frac{\boldsymbol{z}^T \boldsymbol{H} \boldsymbol{z}}{\|\boldsymbol{z}\|_F^2} \leq -c_2 A_{\alpha,\beta,\Delta} D_{\alpha,\beta,\Delta}, \tag{6}$$

where $A_{\alpha,\beta,\Delta}, D_{\alpha,\beta,\Delta} > 0$ are constants defined in (3), and $c_1 > 0$ is a constant.

Let $\boldsymbol{z}_\omega = (z_{\omega,1}, \ldots, z_{\omega,L-1})^T \in \mathbb{R}^{L-1}$ and $\boldsymbol{z} = (\boldsymbol{z}_{1,\ldots,1}, \ldots, \boldsymbol{z}_{d_1,\ldots,d_K})^T \in \mathbb{R}^{(L-1)\prod_k d_k}$. It follows from property (a) that

$$\boldsymbol{z}^T \boldsymbol{H} \boldsymbol{z} = \sum_\omega \boldsymbol{z}_\omega^T \boldsymbol{H}_\omega \boldsymbol{z}_\omega.$$

Furthermore, properties (b) and (c) (or similar arguments as in page 29, arXiv preprint) imply that

$$\boldsymbol{z}_\omega^T \boldsymbol{H}_\omega \boldsymbol{z}_\omega \leq -D_{\alpha,\beta,\Delta} \sum_\ell z_{\omega,\ell}^2 \underbrace{\mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}}}_{\text{Bernoulli r.v. with probability bounded by } A_{\alpha,\beta,\Delta}}.$$

Therefore,

$$\boldsymbol{z}^T \boldsymbol{H} \boldsymbol{z} = \sum_\omega \boldsymbol{z}_\omega^T \boldsymbol{H}_\omega \boldsymbol{z}_\omega \leq -D_{\alpha,\beta,\Delta} \sum_\omega \sum_\ell z_{\omega,\ell}^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}}. \tag{7}$$

Based on central limit theorem (and concentration properties of Bernoulli r.v.'s), as the tensor dimension goes to infinity,

$$\sum_{\omega}\sum_{\ell} z_{\omega,\ell}^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} \to \sum_{\omega}\sum_{\ell} z_{\omega,\ell}^2 \mathbb{P}(y_\omega=\ell \text{ or } \ell+1) \geq c_2 A_{\alpha,\beta,\Delta}\|\boldsymbol{z}\|_F^2 \tag{8}$$

holds with very high probability.

By (7) and (8), we have

$$\boldsymbol{z}^T\boldsymbol{H}\boldsymbol{z} \leq -c_2 A_{\alpha,\beta,\Delta} D_{\alpha,\beta,\Delta}\|\boldsymbol{z}\|_F^2,$$

and therefore (6) is proved. Plugging (4) and (6) into (5) yields

$$\|\hat{\mathcal{Z}}-\mathcal{Z}^{\text{true}}\|_F^2 \leq \frac{c_1 r_{\max}^{K-1} C_{\alpha,\beta,\Delta}^2}{A_{\alpha,\beta,\Delta}^2 D_{\alpha,\beta,\Delta}^2}\left(L-1+\sum_k d_k\right).$$

The MSEs for $\hat{\Theta}$ and $\hat{\boldsymbol{b}}$ readily follow from (2).

Correction of (8). Define the subspace:

$$\mathcal{S} = \{\text{Vec}(\mathcal{Z}) : \mathcal{Z} = -\Theta \otimes \mathbf{1} + \mathcal{J} \otimes \boldsymbol{b},\ (\Theta, \boldsymbol{b}) \in (\mathcal{P}, \mathcal{B})\}.$$

We show that Hession is definite negative restricted in the subspace $\mathcal{S}$. Specifically, for any vector $\boldsymbol{z} = [\![z_{\omega,\ell}]\!] \in \mathcal{S}$,

$$\begin{aligned}
\sum_{\omega,\ell} z_{\omega,\ell}^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} &= \sum_{\omega,\ell}(-\theta_\omega+b_\ell)^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} \\
&= \sum_{\omega,\ell}(\theta_\omega^2 - 2\theta_\omega b_\ell + b_\ell^2)\mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} \\
&= \sum_{\omega,\ell}\theta_\omega^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} - 2\sum_{\omega,\ell}\theta_\omega b_\ell \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} + \sum_{\omega,\ell} b_\ell^2 \mathbb{1}_{\{y_\omega=\ell \text{ or } \ell+1\}} \\
&\geq \sum_{\omega}\theta_\omega^2 - 2\sum_{\omega,\ell}\theta_\omega b_\ell + \sum_{\ell} b_\ell^2\,(n_\ell+n_{\ell+1}) \\
&= \sum_{\omega}\theta_\omega^2 + \min_{\ell}\,(n_\ell+n_{\ell+1})\sum_{\ell} b_\ell^2
\end{aligned}$$

On the other hand,

$$\|\boldsymbol{z}\|_F^2 = \sum_{\omega,\ell} z_{\omega,\ell}^2 = \sum_{\omega,\ell}(-\theta_\omega+b_\ell)^2 = (L-1)\sum_{\omega}\theta_\omega^2 + \left(\prod_k d_k\right)\sum_{\ell} b_\ell^2.$$

Therefore, there exists a positive constant $c_1 > 0$ (? perhaps depending on α, β, Δ etc...needs verification..) such that

$$\frac{\boldsymbol{z}^T\boldsymbol{H}\boldsymbol{z}}{\|\boldsymbol{z}\|_F^2} \geq c_1 \frac{\min_\ell(n_\ell+n_{\ell_1})}{(L-1)\prod_k d_k}.$$

□