Modified proof, new algorithm simulation and ordinal tensor simulation

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1 Accuracy proof for approximated SVD

Our goal is to estimate a signal tensor from one with a noise. Let $\mathcal{A} \in \mathbb{R}^{d_1\times d_2\times d_3}$ be the signal tensor to estimate. We assume this signal tensor has Tucker decomposition,

$$\mathcal{A} = \mathcal{C} \times_1 M^{(1)} \times_2 M^{(2)} \times_3 M^{(3)}$$

where $\mathcal{C} \in \mathbb{R}^{r_1\times r_2\times r_3}$ and $M^{(1)}, M^{(2)}, M^{(3)}$ are orthonormal matrices in $\mathbb{R}^{d_1\times r_1}, \mathbb{R}^{d_2\times r_2}, \mathbb{R}^{d_3\times r_3}$ respectively. We will suggest various methods to recover $\mathcal{A}$ from a tensor

$$\mathcal{D} = \mathcal{A} + \mathcal{E}$$

where $\mathcal{E}$ is a noise tensor from $N(0, \sigma^2)$.

We suggest some theorems to guarantee convergence of estimators to true signal tensor in each method. Theorem 1 and Theorem 2 shows that convergence of a tensor is followed by convergence of orthonormal matrices. Only difference between the two theorems is types of error used for orthonormal matrices: Theorem 1 uses norm while Theorem 2 uses angle to measure error. Theorem 3 suggests sufficient conditions to recover a signal tensor from a noise. From those theorems, proposed algorithms have asymptotic consistency. In section 1.1, we present some linear algebraic tools we need. Section 1.2 uses these methods to derive a generic error bound. Afterward, we guarantee consistency of estimation under some conditions.

1.1 Some background from linear algebra

We go through linear algebraic background in this section.

Proposition 1 (Kronecker Product Norms)**.** *Let* $\|\cdot\|_F$ *be Frobenious norm and* $\|\cdot\|_\sigma$ *be spectral norm. For any* $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times k}$

$$\|A\otimes B\|_F = \|A\|_F\|B\|_F, \quad \|A\otimes B\|_\sigma = \|A\|_\sigma\|B\|_\sigma$$

Proof. From the definition of Frobenius norm and Kronecker product, we can get

$$\|A\otimes B\|_F = \| \begin{bmatrix} a_{11}B & \cdots & a_{mn}B \\ \vdots & \cdots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \|_F = \|A\|_F\|B\|_F$$

For spectral norm case, Let $A = \sum_i \sigma_i u_i v_i^T$ and $B = \sum_j \lambda_j x_j y_j^T$ be the singular value decomposition of two matrices. Then,

$$\begin{aligned} A\otimes B &= \sum_{i,j} \sigma_i\lambda_j (u_i v_i^T)\otimes(x_j y_j^T) \\ &= \sum_{i,j} \sigma_i\lambda_j (u_i\otimes x_j)(v_i^T\otimes y_j^T) \\ &= \sum_{i,j} \sigma_i\lambda_j (u_i\otimes x_j)(v_i\otimes y_j)^T \end{aligned}$$

From this form, we can see $\{\sigma_i\lambda_j : i,j\}$ are singular values of $A\otimes B$. Therefore,

$$\|A\otimes B\|_\sigma = \max_{i,j}\sigma_i\lambda_j = (\max_i \sigma_i)(\max_j \lambda_j) = \|A\|_\sigma\|B\|_\sigma.$$

□

Proposition 2 (Norm of Projection matrices)**.** *Let* $P \in \mathbb{R}^{m\times n}$ *be a projection matrix and* $A \in \mathbb{R}^{n\times k}$ *be an arbitrary matrix. Then,*

$$\|PA\|_F \le \|A\|_F, \quad \|PA\|_\sigma \le \|A\|_\sigma$$

Proof. Notice for $x \in \mathbb{R}^n, \|x\|_2^2 = \|Px\|_2^2 + \|(I-P)x\|_2^2$. Therefore, we have following inequality

$$\|Px\|_2 \leq \|x\|_2$$

This proves the main inequalities of proposition 2. □

Proposition 3 (Submultiplicativity of Frobenius Norm). *Let* $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times k}$ *be arbitrary matrices. Then*

$$\|AB\|_F \leq \|A\|_F\|B\|_F, \quad \|AB\|_F \leq \|A\|_F\|B\|_\sigma$$

Proof. For Frobenius norm's submultiplicativity case, it is an application of Cauchy-Schwarz inequality. So we focus on proving $\|AB\|_F \leq \|A\|_F\|B\|_\sigma$ case here. Let A_i denote i-th row of A.

$$\|AB\|_F^2 = \sum_i \|(AB)_i\|_\sigma^2 = \sum_i \|A_iB\|_\sigma^2 \leq \sum_i \|A_i\|_F^2\|B\|_\sigma^2 = \|A\|_F^2\|B\|_\sigma^2.$$

□

An angle between two subspaces is defined as follows

Definition 1. *For nonzero subspaces* $\mathcal{R}, \mathcal{N} \subset \mathbb{R}^n$ *, a principle angle between* $\mathcal{R}$ *and* $\mathcal{N}$ *is defined to be the number* $0 \leq \theta \leq \pi/2$ *that satisfies*

$$\cos\theta(\mathcal{R},\mathcal{N}) = \max_{u\in\mathcal{R}, v\in\mathcal{N} \|u\|=\|v\|=1} v^t u.$$

A principle angle between two matrices is defined as

$$\cos\theta(A,B) = \cos\theta(\mathrm{span}(A), \mathrm{span}(B))$$

where $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{n\times k}$ *are matrices*

Following 2 propositions offer a way to relate angle and spectral norm.

Proposition 4. *If* P_R *and* P_N *are the orthogonal projectors onto* R *and* N *, respectively, then*

$$\cos\theta = \|P_N P_R\|_\sigma = \|P_R P_N\|_\sigma.$$

Proof. For vectors x and y such that $\|x\| = \|y\| = 1$, we have $P_R x \in \mathcal{R}$ and $P_N y \in \mathcal{N}$ Then

$$\cos\theta = \max_{u\in\mathcal{R}, v\in\mathcal{N}\|u\|=\|v\|=1} v^t u = \max_{u\in\mathcal{R}, v\in\mathcal{N}, \|u\|\le 1\|v\|\le 1} v^t u = \max_{\|x\|\le 1\|y\|\le 1} y^t P_{\mathcal{N}} P_{\mathcal{R}} x = \|P_R P_N\|_\sigma$$

.

$\square$

Proposition 5. *Under the same condition with proposition 4,*

$$\|P_N(I - P_R)\|_\sigma \le \sin(\theta)$$

Proof.

$$\begin{aligned}\|P_N(I - P_R)\|_\sigma^2 &= \max_{u\in\mathcal{R},\|u\|=1} u^t P_N(I - P_R)u = \max_{u\in\mathcal{R},\|u\|=1} u^t P_N u - u^t P_N P_R u \\ &\le 1 - \|P_N P_R\|_\sigma^2 = \sin^2(\theta)\end{aligned}$$

$\square$

1.2 Theoretical results

We are prepared to develop general error bounds for suggested algorithmic methods. First two results show that accuracy of tensor estimation is determined by accuracy of estimations for orthonormal matrices. Last result guarantees consistency of estimation under some conditions.

Theorem 1. *Let $\mathcal{A} = C \times_1 M^{(1)} \times_2 M^{(2)} \times_3 M^{(3)} \in \mathbb{R}^{d_1\times d_2\times d_3}$ where $\mathcal{C} \in \mathbb{R}^{r_1\times r_2\times r_3}$ and $M^{(1)}, M^{(2)}, M^{(3)}$ are orthonormal matrices in $\mathbb{R}^{d_1\times r_1}, \mathbb{R}^{d_2\times r_2}, \mathbb{R}^{d_3\times r_3}$ respectively.*
Suppose that an estimator $\hat{M}^{(i)}$ for each $i \in \{1, 2, 3\}$ has Frobenious norm bound such that $\|M^{(i)} - \hat{M}^{(i)}\|_F \le \epsilon$.
Then, $\hat{\mathcal{A}} = \hat{\mathcal{C}} \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \times_3 \hat{M}^{(3)}$ where $\hat{C} = \mathcal{A} \times_1 (\hat{M}^{(1)})^T \times_2 (\hat{M}^{(2)})^T \times_3 (\hat{M}^{(3)})^T$ has an upper error bound,

$$\|\hat{\mathcal{A}} - \mathcal{A}\|_F \le \|\mathcal{A}\|_F (2\|M^{(1)}\|_F + 2\|M^{(2)}\|_F + 2\|M^{(3)}\|_F + 3\epsilon)\epsilon$$

Furthermore, if we assume that an estimator $\hat{M}^{(i)}$ for each $i \in \{1, 2, 3\}$ has spectral norm

bound such that $\|M^{(i)} - \hat{M}^{(i)}\|_\sigma \le \epsilon$ *instead of Frobenius norm,*
$\hat{\mathcal{A}}$ *has an upper error bound,*

$$\|\hat{\mathcal{A}} - \mathcal{A}\|_F \le \|\mathcal{A}\|_F(2\|M^{(1)}\|_\sigma + 2\|M^{(2)}\|_\sigma + 2\|M^{(3)}\|_\sigma + 3\epsilon)\epsilon$$

Proof. First, assume Frobenius norm bound. Then, for each $i \in \{1, 2, 3\}$, we have,

$$\begin{aligned}
\|M^{(i)}(M^{(i)})^T - \hat{M}^{(i)}(\hat{M}^{(i)})^T\|_F &= \|M^{(i)}(M^{(i)})^T - M^{(i)}(\hat{M}^{(i)})^T + M^{(i)}(\hat{M}^{(i)})^T - \hat{M}^{(i)}(\hat{M}^{(i)})^T\|_F \\
&= \|M^{(i)}((M^{(i)})^T - (\hat{M}^{(i)})^T) + (M^{(i)} - \hat{M}^{(i)})(\hat{M}^{(i)})^T)\|_F \\
&\le (2\|M^{(i)}\|_F + \epsilon)\epsilon
\end{aligned}$$

Let us define $P_A \coloneqq AA^T$. Then, the above inequality can be denoted as,

$$\|P_{M^{(i)}} - P_{\hat{M}^{(i)}}\|_F \le (2\|M^{(i)}\|_F + \epsilon)\epsilon \tag{1}$$

The theorem's main inequality can be gotten as follows

$$\begin{aligned}
\|\mathcal{A} - \hat{\mathcal{A}}\|_F &= \|\mathcal{A} - \hat{\mathcal{C}} \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \times_3 \hat{M}^{(3)}\|_F \\
&= \|\mathcal{A} - \mathcal{A} \times_1 (\hat{M}^{(1)})^T \times_2 (\hat{M}^{(2)})^T \times_3 (\hat{M}^{(3)})^T \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \times_3 \hat{M}^{(3)}\|_F \\
&= \|\mathcal{A} - \mathcal{A} \times_1 P_{\hat{M}^{(1)}} \times_2 P_{\hat{M}^{(1)}} \times_3 P_{\hat{M}^{(1)}}\|_F \\
&= \|A_{(1)} - P_{\hat{M}^{(1)}} A_{(1)}(P_{\hat{M}^{(2)}} \otimes P_{\hat{M}^{(3)}})\|_F \\
&= \|P_{M^{(1)}} A_{(1)}(P_{M^{(2)}} \otimes P_{M^{(3)}}) - P_{\hat{M}^{(1)}} A_{(1)}(P_{\hat{M}^{(2)}} \otimes P_{\hat{M}^{(3)}})\|_F \\
&= \|(P_{M^{(1)}} - P_{\hat{M}^{(1)}})A_{(1)}(P_{M^{(2)}} \otimes P_{M^{(3)}}) + P_{\hat{M}^{(1)}} A_{(1)}(P_{M^{(2)}} \otimes P_{M^{(3)}} - P_{\hat{M}^{(2)}} \otimes P_{\hat{M}^{(3)}})\|_F \\
&\le \|(P_{M^{(1)}} - P_{\hat{M}^{(1)}})A_{(1)}(P_{M^{(2)}} \otimes P_{M^{(3)}})\|_F + \|P_{\hat{M}^{(1)}} A_{(1)}(P_{M^{(2)}} \otimes P_{M^{(3)}} - P_{\hat{M}^{(2)}} \otimes P_{\hat{M}^{(3)}})\|_F
\end{aligned} \tag{2}$$

Using Frobenius versions of the Proposition 1, Proposition 2 and Proposition 3 on (2), following inequality holds true. We proceed the inequality,

$$
\begin{aligned}
\|\mathcal{A}-\hat{\mathcal{A}}\|_F &\le \|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_F\|A_{(1)}\|_F+\|A_{(1)}\|_F\|P_{M^{(2)}}\otimes P_{M^{(3)}}-P_{\hat{M}^{(2)}}\otimes P_{\hat{M}^{(3)}}\|_F \\
&= \big(\|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_F+\|P_{M^{(2)}}\otimes P_{M^{(3)}}-P_{\hat{M}^{(2)}}\otimes P_{\hat{M}^{(3)}}\|_F\big)\|\mathcal{A}\|_F \\
&\le (\|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_F+\|(P_{M^{(2)}}-P_{\hat{M}^{(2)}})\otimes P_{M^{(3)}}\|_F+\|P_{\hat{M}^{(2)}}\otimes(P_{M^{(3)}}-P_{\hat{M}^{(3)}})\|_F)\|\mathcal{A}\|_F \\
&\le (\|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_F+\|P_{M^{(2)}}-P_{\hat{M}^{(2)}}\|_F+\|P_{M^{(3)}}-P_{\hat{M}^{(3)}}\|_F)\|\mathcal{A}\|_F
\end{aligned}
$$

Finally, we get the inequality of the theorem from (1).
Likewise, we can get the same result under the assumption of spectral norm bound. By the same approach for Frobenius norm case, we get,

$$\|M^{(i)}(M^{(i)})^T-\hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma \le (2\|M^{(i)}\|_\sigma+\epsilon)\epsilon \tag{3}$$

Also, From the inequality (2)

$$
\begin{aligned}
\|\mathcal{A}-\hat{\mathcal{A}}\|_F &\le \|(P_{M^{(1)}}-P_{\hat{M}^{(1)}})A_{(1)}(P_{M^{(2)}}\otimes P_{M^{(3)}})\|_F+\|P_{\hat{M}^{(1)}}A_{(1)}(P_{M^{(2)}}\otimes P_{M^{(3)}}-P_{\hat{M}^{(2)}}\otimes P_{\hat{M}^{(3)}})\|_F \\
&\le \|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_\sigma\|A_{(1)}(P_{M^{(2)}}\otimes P_{M^{(3)}})\|_F+\|P_{\hat{M}^{(1)}}A_{(1)}\|_F\|P_{M^{(2)}}\otimes P_{M^{(3)}}-P_{\hat{M}^{(2)}}\otimes P_{\hat{M}^{(3)}}\|_\sigma \qquad (4) \\
&\le \big(\|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_\sigma+\|P_{M^{(2)}}\otimes P_{M^{(3)}}-P_{\hat{M}^{(2)}}\otimes P_{\hat{M}^{(3)}}\|_\sigma\big)\|A_{(1)}\|_F \qquad (5) \\
&\le (\|P_{M^{(1)}}-P_{\hat{M}^{(1)}}\|_\sigma+(\|P_{M^{(2)}}-P_{\hat{M}^{(2)}}\|_\sigma+(\|P_{M^{(3)}}-P_{\hat{M}^{(3)}}\|_\sigma\big)\|A_{(1)}\|_F \qquad (6)
\end{aligned}
$$

Proposition 3 is used in (4). Proposition 2 and 3 are used in (5) and (6). Finally, The main inequality of the theorem is derived from (3) and (6). □

If we use principle angle to measure error, following Theorem 2 holds.

Theorem 2. *Under the same condition in Theorem 1 but instead of Frobenius norm convergence, we assume*

$$\sin(\theta(M^{(i)},\hat{M}^{(i)}))=\sqrt{1-\cos^2(\theta(M^{(i)},\hat{M}^{(i)}))}<\epsilon$$

Then, Following bound holds

$$\|\mathcal{A}-\hat{\mathcal{A}}\|_F \le 6\epsilon\|\mathcal{A}\|_F$$

Proof. If an inequality $\|P_{M^{(i)}} - P_{\hat{M}^{(i)}}\|_\sigma \le 2\epsilon$ is true, the main inequality holds from the last inequality of the proof in the theorem 1.

$$\|\mathcal{A} - \hat{\mathcal{A}}\|_F \le (\|P_{M^{(1)}} - P_{\hat{M}^{(1)}}\|_\sigma + \|P_{M^{(2)}} - P_{\hat{M}^{(2)}}\|_\sigma + \|P_{M^{(3)}} - P_{\hat{M}^{(3)}}\|_\sigma)\|\mathcal{A}\|_F \le 6\epsilon\|\mathcal{A}\|_F$$

So it suffices to show $\|P_{M^{(i)}} - P_{\hat{M}^{(i)}}\|_\sigma \le 2\epsilon$ under the given condition.

$$\begin{aligned}
&\|P_{M^{(i)}} - P_{\hat{M}^{(i)}}\|_\sigma \\
&= \|M^{(i)}(M^{(i)})^T - \hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma \\
&= \|M^{(i)}(M^{(i)})^T - M^{(i)}(M^{(i)})^T\hat{M}^{(i)}(\hat{M}^{(i)})^T + M^{(i)}(M^{(i)})^T\hat{M}^{(i)}(\hat{M}^{(i)})^T - \hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma \\
&\le \|M^{(i)}(M^{(i)})^T - M^{(i)}(M^{(i)})^T\hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma + \|M^{(i)}(M^{(i)})^T\hat{M}^{(i)}(\hat{M}^{(i)})^T - \hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma \\
&= \|M^{(i)}(M^{(i)})^T(I - \hat{M}^{(i)}(\hat{M}^{(i)})^T)\|_\sigma + \|(M^{(i)}(M^{(i)})^T - I)\hat{M}^{(i)}(\hat{M}^{(i)})^T\|_\sigma \\
&\le \sin(\theta) + \sin(\theta) \le 2\epsilon
\end{aligned}$$

Last inequality follows from Property 4. □

Next theorem makes us guarantee angle-wise convergence of $\hat{M}^{(i)}$ to $M^{(i)}$ under certain condition. Thereby, we have a consistent estimator $\hat{\mathcal{A}}$ for the signal tensor $\mathcal{A}$.

Theorem 3. *Let $\mathcal{A} = \mathcal{C} \times_1 M^{(1)} \times_2 M^{(2)} \times_3 M^{(3)}$ be a signal tensor where $\mathcal{C} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ and $M^{(1)}, M^{(2)}, M^{(3)}$ are orthonormal matrices in $\mathbb{R}^{d_1 \times r_1}, \mathbb{R}^{d_2 \times r_2}, \mathbb{R}^{d_3 \times r_3}$ respectively.*
Let $\mathcal{D} = \mathcal{A} + \mathcal{E}$ be a given tensor with a noise tensor $\mathcal{E} \sim N(0, \sigma^2)$
Suppose, $s_{min}(C_{(i)}) >> \sigma\sqrt{\max(d_i, \frac{d_1 d_2 d_3}{d_i})\frac{d_1 d_2 d_3}{d_i r_i}}$ as $d_1, d_2, d_3 \to \infty$, where $s_{min}(C_{(i)})$ is the smallest singular value of $C_{(i)}$.
Then, the following holds true.

$$\cos\theta(M^{(i)}, \hat{M}^{(i)}) \to 1 \textit{ in probability for } i \in [3]$$

$$\|\mathcal{A} - \hat{\mathcal{A}}\|_F \to 0 \textit{ in probability}$$

where $(\hat{\mathcal{C}}, \hat{M}^{(1)}, \hat{M}^{(2)}, \hat{M}^{(3)})$ is an output of Algorithm 1 from input $\mathcal{D}$ and $\mathcal{A}$ is an estimator such that $\hat{\mathcal{A}} = \hat{\mathcal{C}} \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \times_3 \hat{M}^{(3)}$

Proof. It suffices to show when $i = 1$. Notice,

$$\begin{aligned} A_{(1)} &= M^{(1)}(\mathcal{C} \times_2 M^{(2)} \times_3 M^{(3)})_{(1)} \\ &= M^{(1)} C_{(1)} (M^{(3)} \otimes M^{(2)})^T \end{aligned}$$

Define $B = (M^{(3)} \otimes M^{(2)})^T = \begin{pmatrix} M_1^{(3)} \otimes M_1^{(2)}, & M_1^{(3)} \otimes M_2^{(2)}, & \cdots, & M_{r_3}^{(3)} \otimes M_{r_2}^{(2)} \end{pmatrix}$ where $M_j^{(i)}$ is the j-th column of $M^{(i)}$. Step A of algorithm 1 generates test random Gaussian matrix Ω and capture image space of unfolded matrix $A_{(1)}$. Having this procedure in mind, we get,

$$\begin{aligned} A_{(1)}\Omega &= M^{(1)} C_{(1)} (M^{(3)} \otimes M^{(2)})^T \Omega \\ &= M^{(1)} C_{(1)} B \Omega \quad \text{where } \Omega \in R^{d_2 d_3 \times r_1} \text{ of which entries from } i.i.d.N(0,1) \\ &= M^{(1)} C_{(1)} \begin{pmatrix} Z_1, & Z_2, & \cdots & , Z_{r_1} \end{pmatrix} \quad \text{where } Z_i^T = \begin{pmatrix} \sum_{k=1}^{d_2 d_3} B_{1,k} \Omega_{k,i}, & \cdots, & \sum_{k=1}^{d_2 d_3} B_{r_2 r_3, k} \Omega_{k,i} \end{pmatrix} \\ &= M^{(1)} C_{(1)} \mathbf{Z} \quad \text{where } \mathbf{Z} = \begin{pmatrix} Z_1, & Z_2, & \cdots & , Z_{r_1} \end{pmatrix} \end{aligned} \tag{7}$$

However, since our input is $\mathcal{D} = \mathcal{A} + \mathcal{E}$, we have image space of $A_{(1)} + E_{(1)}$ instead of $A_{(1)}$. Therefore, estimator $\hat{M}^{(1)}$ is from the following equality.

$$\begin{aligned} (A_{(1)} + E_{(1)})\Omega &= M^{(1)} C_{(1)} \mathbf{Z} + E_{(1)} \Omega \\ &= \hat{M}^{(1)} R \quad \text{(QR decomposition)} \end{aligned} \tag{8}$$

From the relationship that $\text{span}(A_{(1)}\Omega) \subset \text{span}(M^{(1)})$ and $\text{span}(A_{(1)}\Omega + E_{(1)}\Omega) = \text{span}(\hat{M^{(1)}})$, we have the following.

$$\begin{aligned} \cos\theta(M^{(1)}, \hat{M^{(1)}}) &= \max_{u \in \text{span}(M^{(1)}), v \in \text{span}(\hat{M}^{(1)})} \cos(u, v) \\ &\geq \max_{u \in \text{span}(A_{(1)}\Omega), v \in \text{span}((A_{(1)} + E_{(1)})\Omega)} \cos(u, v) \\ &= \max_{x \in R^{r_1}, y \in R^{r_1}, \|x\|_2 = \|y\|_2 = 1} \cos(A_{(1)}\Omega x, (A_{(1)} + E_{(1)})\Omega y) \end{aligned} \tag{9}$$

The first argument in the theorem holds true by (9) if

$$\max_{x \in R^{r_1}, y \in R^{r_1}, \|x\|_2 = \|y\|_2 = 1} \cos(A_{(1)}\Omega x, (A_{(1)} + E_{(1)})\Omega y) \to 1 \tag{10}$$

Also (10) holds true, if

$$\cot(A_{(1)}\Omega x, (A_{(1)} + E_{(1)})\Omega y) \to \infty \text{ for any fixed } x, y \text{ such that } \|x\| = \|y\| = 1 \tag{11}$$

So main proof of this theorem is to show (11). We prove this by the following inequality.

$$\cot(A_{(1)}\Omega x, (A_{(1)} + E_{(1)})\Omega y) \geq \frac{\|A_{(1)}\Omega x\|_2}{\|E_{(1)}\Omega y\|_2} \geq \frac{s_{min}(C_{(1)})\sqrt{\chi^2_{r_1}}}{\|E\|_2\|\Omega y\|_2} \tag{12}$$

To get numerator part in (12),

$$\|A_{(1)}\Omega x\|_2 = \|M^{(1)}C_{(1)}\mathbf{Z}x\|_2 = \|C_{(1)}\sum_{i=1}^{r_1} Z_i x_i\|_2 \tag{13}$$

Last equality in (13) is from orthonormality of $M^{(1)}$. Lemma 1 shows that all entries of $\mathbf{Z}$ in (7) are $i.i.d.N(0,1)$. So with Lemma 1 and the fact that $\|x\|_2 = 1$, we can get

$$\sum_{i=1}^{r_1} Z_i x_i \sim N_{r_2 r_3}(0, I_{r_2 r_3})$$

Therefore, the equation (13) becomes,

$$\begin{aligned}
\|A_{(1)}\Omega x\|_2 = \|C_{(1)}\sum_{i=1}^{r_1} Z_i x_i\|_2 &\sim \|N_{r_1}(0, C_{(1)}C_{(1)}^T)\|_2 \overset{(*)}{\equiv} \|N_{r_1}(0, Q\Lambda Q^T)\|_2 \overset{(**)}{\equiv} \|N_{r_1}(0,\Lambda)\|_2 \\
&= \sqrt{\lambda_1^2 V_1 + \cdots \lambda_{r_1}^2 V_{r_1}} \text{ where } V_i \sim \chi^2_1 \quad \text{for } i \in [r_1] \\
&\geq |\lambda_1|\sqrt{V_1 + \cdots + V_{r_1}} \\
&= |\lambda_1|\sqrt{\chi^2_{r_1}} \\
&= s_{min}(C_{(1)})\sqrt{\chi^2_{r_1}}
\end{aligned}$$

$(*)$ is from Eigen-Value Decomposition,

$$C_{(1)}C_{(1)}^T = Q\Lambda Q^T$$

where Q is an orthonormal matrix and $\Lambda = diag(\lambda_{r_1} \cdots \lambda_1)$ such that $\lambda_{r_1} \geq \cdots \geq \lambda_1 \geq 0$. $(**)$ uses invariance of the norm under orthornormal transformation.

The denominator part in (11) is from Cauchy Schwarz inequality. We proved that (11) holds true. Lemma 2 shows

$$\|E\|_F \asymp (2+o(1))\sigma\sqrt{\max(d_1, d_2d_3)}$$

Also, $\|\Omega y\|_2^2 \sim \chi^2(d_2d_3)$ since $\|y\|_2 = 1$. Equivalently,

$$\|\Omega y\|_2 \asymp (1+o(1))\sqrt{d_2d_3}$$

Using (11) and above two approximations, we have following inequality with fixed $L > 0$

$$\begin{aligned} P(\cot(A_{(1)}\Omega x, (A_{(1)}+E_{(1)})\Omega y) \geq L) &\geq P(\frac{s_{min}(C_{(1)})\sqrt{\chi^2_{r_1}}}{\|E\|_2\|\Omega y\|_2} \geq L) \\ &\geq P(\sqrt{\chi^2_{r_1}} \geq \frac{2L\sigma\sqrt{d_2d_3\max(d1, d_2d_3)}}{s_{min}(C_{(1)})}) \\ &= P(\chi^2_{r_1} \geq \frac{4L^2\sigma^2 d_2d_3\max(d1, d_2d_3)}{s_{min}(C_{(1)})^2}) \\ &\overset{(***)}{\geq} 1 - \left(4\lambda e^{1-4\lambda}\right)^{\frac{r_1}{2}} \end{aligned} \tag{14}$$

In $(***)$, we defined $\lambda \overset{def}{=} \frac{L^2\sigma^2 d_2d_3\max(d1,d_2d_3)}{r_1 s_{min}(C_{(1)})^2}$ and used Chernoff bounds,

$$P(\chi^2_r \geq t) \geq 1 - (\frac{t}{r}e^{1-\frac{t}{r}})^{\frac{r}{2}} \text{ for any } t \geq 0$$

The main assumption of the theorem implies $\lambda \to 0$ for fixed L. The equation (14) shows that $\cot(A_{(1)}\Omega x, (A_{(1)}+E_{(1)})\Omega y) \geq L$ with high probability. By letting $L \to \infty$, we get desired result.

The last argument that $\|\mathcal{A} - \hat{\mathcal{A}}\| \to 0$ is derived directly from Theorem 2 and Theorem 3 $\square$

Lemma 1. *In the proof of the Theorem 1, all entries of* $\mathbf{Z} = \begin{pmatrix} Z_1, & Z_2, & \cdots & , Z_{r_1}\end{pmatrix}$ *is from* $i.i.dN(0,1)$

Proof. To remind definitions,

$$Z_i^T = \begin{pmatrix}\sum_{k=1}^{d_2d_3} B_{1,k}\Omega_{k,i}, & \cdots, & \sum_{k=1}^{d_2d_3} B_{r_2r_3,k}\Omega_{k,i}\end{pmatrix}^T = (z_{1,i}, \cdots, z_{r_1r_2,i})^T$$

$$B = (M^{(3)} \otimes M^{(2)})^T = \begin{pmatrix} M_1^{(3)} \otimes M_1^{(2)}, & M_1^{(3)} \otimes M_2^{(2)}, & \cdots, & M_{r_3}^{(3)} \otimes M_{r_2}^{(2)}\end{pmatrix}$$

Notice that Z_i and Z_j are independent when $i \neq j$ because entries of Z_i only consist of i-th column of Ω. Therefore, it suffices to show entries of Z_1 are independent and from $N(0,1)$.

1. $z_{1,1} \sim N(0,1)$
   Notice $z_{1,1} = \sum_{k=1}^{d_2 d_3} B_{1,k}\Omega_{k,1} = [B^1]^T \Omega_1$ where $B^1 \overset{def}{=}$ 1st row of B
   Therefore, $z_{1,1}$ is from $N(0,1)$ because $\|B^1\| = 1$ and $\Omega_1 \overset{i.i.d}{\sim} N(0,1)$

2. $z_{1,1}, \cdots, z_{r_2 r_3,1}$ are independent.
   Define a function $(ind_1, ind_2) : N \to N \times N$ which satisfies $B_i = M^{(3)}_{ind_1(i)} \otimes M^{(2)}_{ind_2(i)}$.
   To give a simple example, $(ind_1(1), ind_2(1)) = (1,1)$ because $B_1 = M^{(3)}_1 \otimes M^{(2)}_1$.
   For $i \neq j$,

$$\begin{aligned} Cov(z_{i,1}, z_{j,1}) &= Cov(\sum_{k=1}^{d_2 d_3} B_{i,k}\Omega_{k,1}, \sum_{k=1}^{d_2 d_3} B_{j,k}\Omega_{k,1}) \\ &= (B_i^T)^T (B_j^T) \\ &= 0 \end{aligned}$$

Therefore, all elements of $\mathbf{Z}$ is from $i.i.d.N(0,1)$ by 1,2 $\square$

Lemma 2 (Spectral norm of Gaussin matrix)**.** *Let $E \in R^{m \times n}$ be a random matrix with i.i.d. $N(0,1)$ entries. Then, we have, with very high probability,*

$$\|E\|_\sigma \asymp (2 + o(1))\sqrt{\max(m,n)}$$

We can apply Theorem 3 to the lower dimension case.

Corollary 1. *Consider a rank 1 matrix model, $D = \lambda a \otimes b + E$, where $\lambda \in R_+$, $a \in R^{d_1}, b \in R^{d_2}$ and $E \in R^{d_1 \times d_2}$ is a Gaussian matrix from $i.i.d.N(0, \sigma^2)$. Define a random projection*

$$\hat{a} = D\Omega, \text{ where } \Omega = (z_1, \cdots, z_{d_2})^T \overset{i.i.d.}{\sim} N(0,1)$$

If $\lambda >> \sigma\sqrt{d_2 \max(d_1, d_2)}$ as $d_1, d_2 \to \infty$, then we get

$$\cos\theta(a, \hat{a}) \to 1 \text{ in probability.}$$

Proof. Take $\mathcal{C} = \lambda$, $M^{(1)} = a$, $M^{(2)} = b$ and $M^{(3)} = 1$ in Theorem 3. Then, you get the result of Corollary 1. $\square$

2 New randomized tucker decomposition algorithm and simulations

2.1 Algorithm 2.

The Algorithm 1 has expensive computation cost to generate random test matrices. The algorithm 2 suggests another way to overcome this weak point. Only difference between two algorithms is how to generate random test matrices in step A. For each unfolded matrix $A_{(n)}$, the algorithm 1 generates a $\prod_{i \neq n} d_i \times r_n$ random Gaussian matrix. However, the algorithm 2 draws $N-1$ Gaussian matrices such that $P_i \in \mathbb{R}^{d_i \times r_n}$ for $i \in [N] - \{n\}$. Taking Khatri Rao product on all drawn matrices, the algorithm 2 gets a same sized random matrix with the algorithm 1. Detailed algorithm 2 is as follows.

```
Algorithm 1 Approx tensor SVD 3
  procedure SVD($\mathcal{A}$)
      Step A: Approximate SVD of $\mathcal{A}$
      for $n \leftarrow 1 : N$ do
          Unfold $\mathcal{A}$ as $A_{(n)}$
          Generate Gaussian test matrices $P_1 \in \mathbb{R}^{d_1 \times r_n} \cdots P_{n-1} \in \mathbb{R}^{d_{n-1} \times r_n} P_{n+1} \in \mathbb{R}^{d_{n+1} \times r_n} \cdots P_N \in \mathbb{R}^{d_N \times r_n}$ from $N(0,1)$
          Get $\Omega^{(n)} = P_1 \odot \cdots \odot P_{n-1} \odot P_{n+1} \odot \cdot \odot P_N$
          For $\mathbf{Y^{(n)}} = \mathbf{A_{(n)}} \Omega^{(n)}$
          Construct a matrix $\hat{M}^{(n)}$ whose columns form an orthonormal basis for the range of $\mathbf{Y^{(n)}}$
      get $\hat{\mathcal{C}} = \mathcal{A} \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \cdots \times_N \hat{M}^{(N)}$
      Step B: Get approximated SVD
      return $(\hat{\mathcal{C}}, \hat{M}^{(1)}, \cdots, \hat{M}^{(N)})$
```

Algorithm 3 modifies the algorithm 2 to reduce variability. In this algorithm, we get averaged projected space in terms of angles. Thereby, we can get reduced variability of the image space from repeated projections. The following describes detailed procedure of the algorithm

Algorithm 2 Approx tensor SVD 4

```
procedure SVD($\mathcal{A}$)
    Step A: Approximate SVD of $\mathcal{A}$
    for $n \leftarrow 1 : N$ do
        Unfold $\mathcal{A}$ as $A_{(n)}$
        for $r \leftarrow 1 : R$ do
            Generate Gaussian test matrices $P_1^r \in \mathbb{R}^{d_1 \times r_n} \cdots P_{n-1}^r \in \mathbb{R}^{d_{n-1} \times r_n} P_{n+1}^r \in \mathbb{R}^{d_{n+1} \times r_n} \cdots P_N^r \in \mathbb{R}^{d_N \times r_n}$ from $N(0,1)$
            Get $\Omega^{(n)} = P_1^r \odot \cdots \odot P_{n-1}^r \odot P_{n+1}^r \odot \cdot \odot P_N^r$
            For $\mathbf{Y}_\mathbf{r}^{(\mathbf{n})} = \mathbf{A}_{(\mathbf{n})} \Omega^{(n)}$
        Construct a matrix $\hat{M}^{(n)}$ whose columns form $r_n$ largest left singular vectors for the range of $(\mathbf{Y}_\mathbf{1}^{(\mathbf{n})}, \cdots, \mathbf{Y}_\mathbf{R}^{(\mathbf{n})})$
    get $\hat{\mathcal{C}} = \mathcal{A} \times_1 \hat{M}^{(1)} \times_2 \hat{M}^{(2)} \cdots \times_N \hat{M}^{(N)}$
    Step B: Get approximated SVD
    return $(\hat{\mathcal{C}}, \hat{M}^{(1)}, \cdots, \hat{M}^{(N)})$
```

2.2 Simulation.

I compared 4 various randomized tucker decomposition algorithms:

1. First method: Unstructured Gaussian test matrix + Single Random Projection.
2. Second method: Unstructured Gaussian test matrix + Direct Random Projection to Tensor.
3. Third method: Khatri-Rao Gaussian test matrix + Single Random Projection.
4. Fourth method: Khatri-Rao Gaussian test matrix + Multiple Random Projections.

First simulation investigate the accuracy of each method. We consider an order-3 dimension $(100, 100, 100)$ signal tensor B. All entries of B are i.i.d. drawn from $\mathrm{N}(0, 1)$. The tensor D is generated by adding a noise tensor $\mathcal{E}$ from the signal tensor B, where $\mathcal{E}$ consists of i.i.d $\mathrm{N}(0, \sigma^2)$. To be specific, $\mathcal{D} = \mathcal{B} + \mathcal{E}$. We varied the noise level $\sigma \in \{0.01, 0.02, \cdots 0.49, 0.5\}$ and estimated B using various methods with target rank 20. Figure 1 shows how far estimators are from true signal tensor. While from 1 to 3 methods have the similar performances, the method 4 outperforms other methods across all noise sizes.

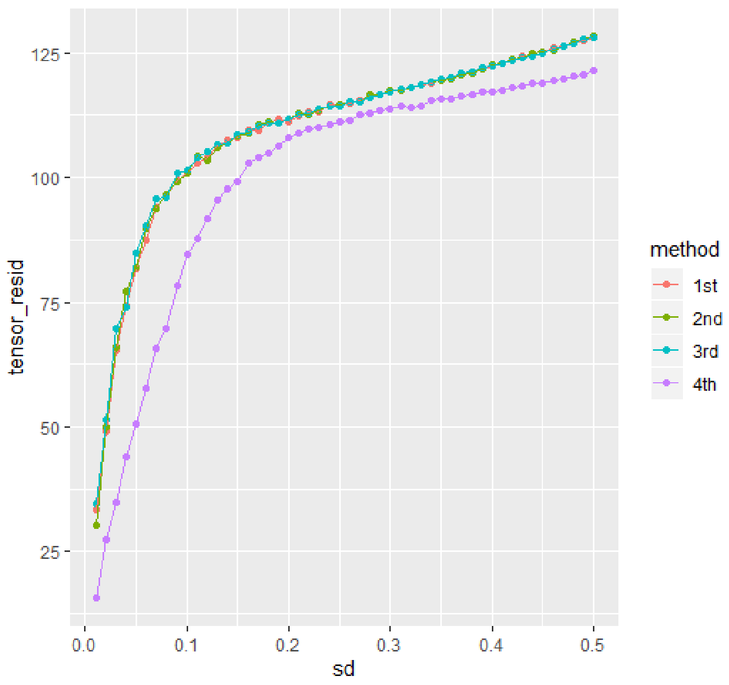


Figure 1: Y axis means Frobenius norm between the true signal and estimator. X axis means the noise size.

Second simulation investigates accuracy of estimators in terms of angles and MSE. We consider an order-3 dimension $(100, 100, 100)$ signal tensor B. We assume B has Tucker decomposition as $B = a \otimes b \otimes c$, where $a, b, c \in \mathbb{R}^{100}$ are the signal vectors. All entries of a, b, c are i.i.d. drawn from $\mathrm{N}(0, 1)$. We vary the noise level $\sigma \in \{0.01, 0.02, \cdots 0.49, 0.5\}$. With target rank 1, we estimate the signal vectors according to each algorithms. We compare angles between true signal vectors and estimators. Figure 2 shows 4-th method outperforms any other methods.

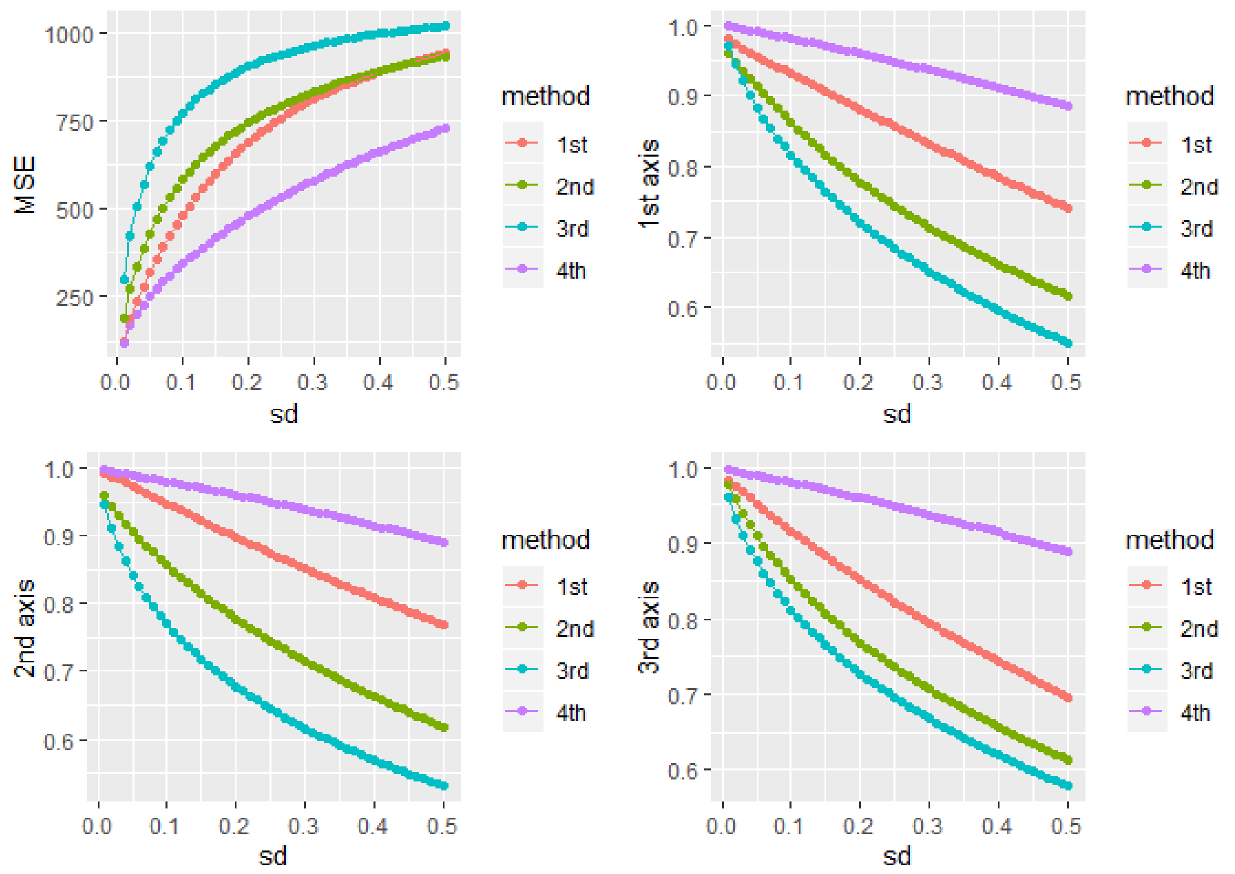


Figure 2: Each y axis means $\cos\theta(a,\hat{a})$, $\cos\theta(b,\hat{b})$ and $\cos\theta(c,\hat{c})$. We have consistent simulation results: The method 4 has the best performance in all respects. First method, second method and third method are followed in order.

3 Ordinal tensor model simulation

We propose to estimate the unknown parameter tensor Θ using a constrained likelihood. The loss function from log-likelihood is in (15).

$$\mathcal{L}_{\mathcal{Y}}(\Theta,\boldsymbol{\omega}) = -\sum_{i_1,\cdots,i_N}\left[\sum_{l=1}^{K}\mathbb{1}(y_{i_1,\cdots,i_N}=l)\log(\pi_l(\theta_{i_1,\cdots,i_N},\boldsymbol{\omega}))\right] \tag{15}$$

where, $\pi_l(\theta_{i_1,\cdots,i_N}, \boldsymbol{\omega}) = \text{logit}(\omega_l + \theta_{i_1,\cdots,i_N}) - \text{logit}(\omega_{l-1} + \theta_{i_1,\cdots,i_N})$ for $l \in [K]$. This loss minimization problem becomes constrained optimization from Tucker decomposition assumption:

$$\arg\min_{\Theta \subset \mathcal{D}} \mathcal{L}_{\mathcal{Y}}(\Theta, \omega) \text{ where } \mathcal{D} \subset \mathcal{S} = \{\Theta = \mathcal{C} \times_1 A_1 \cdots \times_N A_N : \mathcal{C} \in \mathbb{R}^{\otimes_{i=1}^N r_i}, A_i \in \mathbb{R}^{d_i \times r_i}, \|\Theta\|_\infty \leq \alpha\} \tag{16}$$

We assume that the search space $\mathcal{D}$ is a compact set containing the true parameter Θ_{true}. In this section, we describe how to solve (16) optmization problem. Afterward, we simulate our method to check performance.

3.1 Algorithm construction

We introduce an algorithm to solve (16). The objective function $L_{\mathcal{Y}}(\Theta, \boldsymbol{\omega})$ is convex in $(\Theta, \boldsymbol{\omega})$. However, the feasible set $\mathcal{D}$ is non-convex. We utilized a formulation of tucker decomposition, and turn the optimization into a block wise convex problem. We divide cases into two: First, when we know bin boundary $\omega_1, \cdots, \omega_K$. Second, when we have no information about bin boundary.

3.1.1 Known Bin Boundary

From previous data or experience, we may have knowledge about bin boundary parameter ω. In this section we only consider this case. To get a minimizer for (16), we split Θ into several blocks using Tucker decomposition. Specifically, write Θ as a function of $N - mode$ matrices and a core tensor

$$\Theta = g(\mathcal{C}, A_1, \cdots, A_N) = \mathcal{C} \times_1 A_1 \cdots \times_N A_N \tag{17}$$

Then the problem (16) can be rephrased as

$$\arg\min_{(\mathcal{C}, A_1, \cdots, A_N)} \mathcal{L}_{\mathcal{Y}}(g(\mathcal{C}, A_1, \cdots, A_N), \boldsymbol{\omega}) \text{ subject to } g(\mathcal{C}, A_1, \cdots, A_N) \leq \alpha \tag{18}$$

The objective function in (18) is convex in each block individually with all other blocks fixed. This feature enables us to update each block alternatively at a time while others being fixed. To see what happens in each iteration, let us denote $A_i^{(t)}$ as the i-th mode matrix in Tucker decomposition at t-th iteration and vec$(\cdot)$ as the operator that turns a tensor or a matrix into a vector. Then, we have an equivalent problem with (18).

$$\text{vec}(A_i^{(t+1)}) = \underset{A_i\in\mathbb{R}^{d_i\times r_i}}{\arg\min}\, \mathcal{L}_{\mathcal{Y}}\Big(\big((A_N^{(t)}\otimes\cdots\otimes A_{i+1}^{(t)}\otimes A_{i-1}^{(t+1)}\otimes\cdots\otimes A_1^{(t+1)})(C_{(i)}^{(t)})^T\otimes I_{d_i}\big)\,\text{vec}(A_i^{(t)}), \boldsymbol{\omega}\Big), \tag{19}$$

$$\text{subject to } g(\mathcal{C}^{(t)}, A_1^{(t+1)},\cdots, A_{i-1}^{(t+1)}, A_i, A_{i+1}^{(t)},\cdots, A_N^{(t)}) \le \alpha$$

After updating all matrices for each iteration from (19), we update a core tensor as follows.

$$\text{vec}(\mathcal{C}^{(t+1)}) = \underset{\mathcal{C}\in\mathbb{R}^{\otimes_{i=1}^N r_i}}{\arg\min}\, \mathcal{L}_{\mathcal{Y}}\Big(\big(A_N^{(t+1)}\otimes\cdots\otimes A_1^{(t+1)}\big)\,\text{vec}(\mathcal{C}^{(t)})\Big), \tag{20}$$

$$\text{subject to } g(\mathcal{C}, A_1^{(t+1)},\cdots, A_N^{(t+1)}) \le \alpha$$

We repeat this procedure until it converges. "BFGS", quasi-Newton method, is used to update in each iteration. The full algorithm is described in algorithm 3.

Algorithm 3 Ordinal tensor optimization with known boundary $\boldsymbol{\omega}$

```
    Input: $\mathcal{C}^0 \in \mathbb{R}^{r_1\times\cdots\times r_N}, A_1^0\in\mathbb{R}^{d_1\times r_1},\cdots, A_N^0\in\mathbb{R}^{d_N\times r_N}$
    Output: Optimizor of $\mathcal{L}_Z(\boldsymbol{\omega},\Theta)$ given $\boldsymbol{\omega}$
  for t = 1,2,$\cdots$, do until convergence,
     Update $A_n$
     for n = 1,2,$\cdots$,N do
        $\Theta_{(n)} = A_n^t\mathcal{C}_{(n)}^t\big(A_{n+1}^t\otimes\cdots\otimes A_N^t\otimes A_1^{t+1}\otimes\cdots\otimes A_{n-1}^{t+1}\big)^T$
        $\text{vec}(\Theta_{(n)}) = \Big(\big(A_{n+1}^t\otimes\cdots\otimes A_N^t\otimes A_1^{t+1}\otimes\cdots\otimes A_{n-1}^{t+1}\big)(C_{(n)}^t)^T\otimes I_{d_n}\Big)\,\text{vec}(A_n^t)$
        $\text{vec}(A_n^{t+1}) = \arg\max(\mathcal{L}_{\mathcal{Y}}(\alpha, \text{vec}(\Theta_{(n)})))$
        Get $A_n^{t+1}$
     Update $\mathcal{C}$
     $\Theta_{(1)} = A_1^{t+1}\mathcal{C}_{(1)}^t\big(A_N^{t+1}\otimes\cdots\otimes A_2^{t+1}\big)^T$
     $\text{vec}(\Theta_{(1)}) = \big(A_N^{t+1}\otimes\cdots\otimes A_1^{t+1}\big)\,\text{vec}(C_{(1)}^t)$
     $\text{vec}(C_{(1)}^t) = \arg\max(\mathcal{L}_{\mathcal{Y}}(\boldsymbol{\omega}, \text{vec}(\Theta_{(1)})))$
     Get $\mathcal{C}^{t+1}$
  return $\Theta$
```

3.1.2 Unknown Bin Boundary

Having information about bin boundary rarely happens. If we have no knowledge on bin boundary, $\boldsymbol{\omega}$ becomes unknown parameter to estimate. Therefore, we modify to add one more line on algorithm 3 to update $\boldsymbol{\omega}$. Specifically, we update $\boldsymbol{\omega}$ in each iteration as follows

$$\boldsymbol{\omega}^{t+1} = \arg\min_{\boldsymbol{\omega}\in R^K} \mathcal{L}_{\mathcal{Y}}(\mathcal{C}^{t+1} \times_1 A_1^{(t+1)} \cdots \times A_N^{t+1}, \boldsymbol{\omega}) \text{ subject to } \omega_1 < \cdots < \omega_K = \infty \tag{21}$$

Detailed algorithm is in algorithm 4.

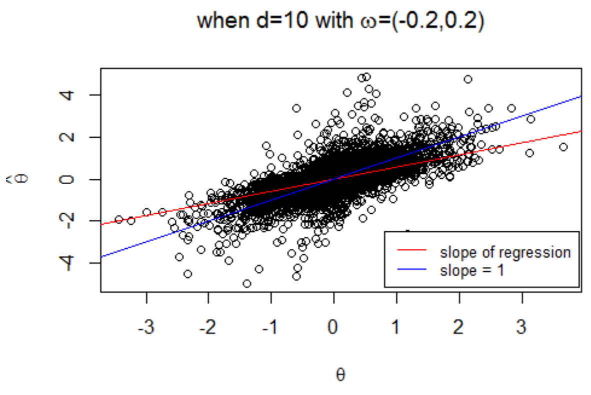
Algorithm 4 Ordinal tensor optimization with unknown boundary $\boldsymbol{\omega}$

```
    Input: $\mathcal{C}^0 \in \mathbb{R}^{r_1\times\cdots\times r_N}, A_1^0 \in \mathbb{R}^{d_1\times r_1}, \cdots, A_N^0 \in \mathbb{R}^{d_N\times r_N}$
    Output: Optimizor of $\mathcal{L}_Z(\boldsymbol{\omega}, \Theta)$ given $\boldsymbol{\omega}$
for t = 1,2,$\cdots$, do until convergence,
    Update $A_n$
    for n = 1,2,$\cdots$,N do
        $\Theta_{(n)} = A_n^t \mathcal{C}_{(n)}^t \left(A_{n+1}^t \otimes \cdots \otimes A_N^t \otimes A_1^{t+1} \otimes \cdots \otimes A_{n-1}^{t+1}\right)^T$
        $\text{vec}(\Theta_{(n)}) = \left(\left(A_{n+1}^t \otimes \cdots \otimes A_N^t \otimes A_1^{t+1} \otimes \cdots \otimes A_{n-1}^{t+1}\right)(C_{(n)}^t)^T \otimes I_{d_n}\right) \text{vec}(A_n^t)$
        $\text{vec}(A_n^{t+1}) = \arg\max(\mathcal{L}_{\mathcal{Y}}(\boldsymbol{\omega}^t, \text{vec}(\Theta_{(n)})))$
        Get $A_n^{t+1}$
    Update $\mathcal{C}$
    $\Theta_{(1)} = A_1^{t+1} \mathcal{C}_{(1)}^t \left(A_N^{t+1} \otimes \cdots \otimes A_2^{t+1}\right)^T$
    $\text{vec}(\Theta_{(1)}) = \left(A_N^{t+1} \otimes \cdots \otimes A_1^{t+1}\right) \text{vec}(C_{(1)}^t)$
    $\text{vec}(C_{(1)}^t) = \arg\max(\mathcal{L}_{\mathcal{Y}}(\boldsymbol{\omega}^t, \text{vec}(\Theta_{(1)})))$
    Get $\mathcal{C}^{t+1}$
    Update $\boldsymbol{\omega}$
    $\boldsymbol{\omega}^{t+1} = \arg\max(\mathcal{L}_{\mathcal{Y}}(\boldsymbol{\omega}, \Theta^{t+1}))$
return $\boldsymbol{\omega}, \Theta$
```

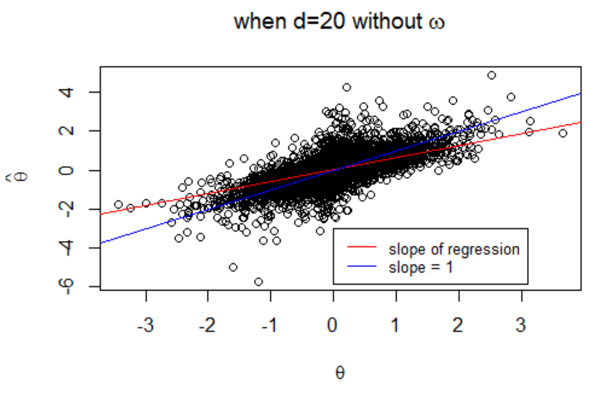
3.2 Simulations.

In this section, we investigate the performance of our method when the data indeed follows the Tucker decomposition tensor model. We set an order-3 dimension (d, d, d) ordinal tensor

$\mathcal{Y}$ with $K = 3$ from the model, where $\boldsymbol{\omega}_{true} = (-0.2, 0.2)$ and $\Theta_{true} = \mathcal{C} \times_1 A_1 \times_2 A_2 \times_3 A_3$ with mode of $\mathcal{C}$ being $(3, 3, 3)$. The entries of $\mathcal{C}$ and A_i are i.i.d. drawn from Uniform$[-1, 1]$. We get $\|\Theta\|_\infty = 2.507$ when $d = 20$ and $\|\Theta\|_\infty = 3.049$ when $d = 30$ for this simulation. We varied the tensor dimension d $\in \{20, 30\}$ and implemented algorithm 3 and algorithm 4. Figure 3 is scatter plot between Θ_{true} and $\hat{\Theta}_{MLE}$ when $d = 20$. Figure 3 is scatter plot between Θ_{true} and $\hat{\Theta}_{MLE}$ when $d = 30$. A larger tensor size has smaller deviation from true parameter. Also algorithm 4 successfully estimated true bin boundary $\boldsymbol{\omega}_{true} = (-0.2, 0.2)$ regardless of tensor size. We had $\hat{\boldsymbol{\omega}}$ = (-0.198,0.201) when $d = 20$ and $\hat{\boldsymbol{\omega}}$ = (-0.211,0.203) when $d = 30$.

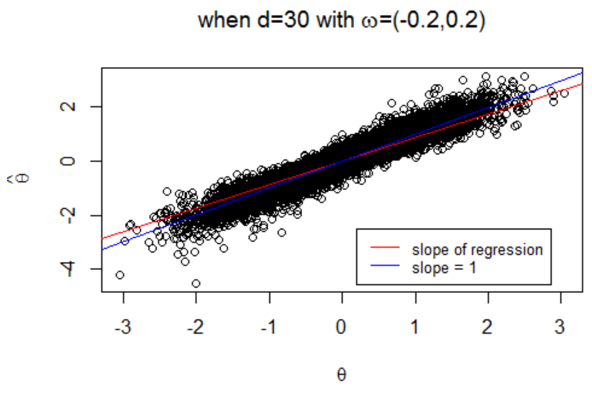


(a) When we know bin boundary ω

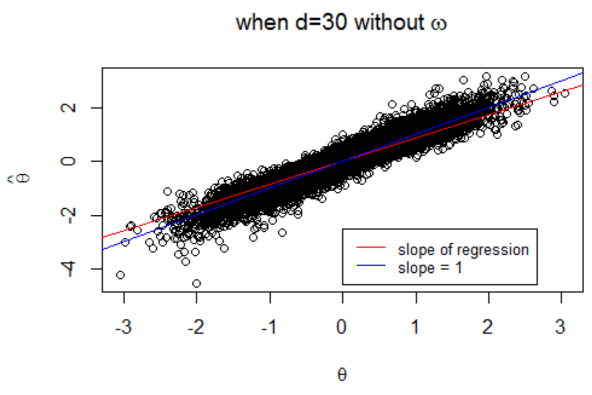


(b) When we don't know bin boundary ω

Figure 3: When d = 20. Red lines are slopes of ordinary least square estimators. Blue lines are line of $y = x$. Each slope from left to right figure is 0.68 and 0.691.



(a) When we know bin boundary ω



(b) When we don't know bin boundary ω

Figure 4: When d = 30. Red lines are slopes of ordinary least square estimators. Blue lines are line of $y = x$. Left figure has slope = 0.867366 and right figure has slope = 0.86410

4 Moderate Sample Size for Simuation 3.2

Empirically, it is known that

$$\|\Theta_{true} - \hat{\Theta}\|_F \sim \sqrt{\frac{df_{parameter}}{n_{sample}}}, \tag{22}$$

where $df_{parameter}$ is a degree of freedom of parameter space and n_{sample} is the sample size. (22) implies that It is helpful to error analysis to check degree of freedom of parameters and the number of samples. Suppose the parameter space is

$$\mathcal{D} = \{\Theta : \mathcal{C} \times_1 A_1 \times_2 A_2 \times_3 A_3, \text{ where } \mathcal{C} \in \mathbb{R}^{r\times r\times r}, \text{ and } A_i \in \mathbb{R}^{d\times r} \text{ for } i \in [3]\}$$

The paper [1] shows that Tucker decomposition with following theorem has uniqueness.

Theorem 4 (N-th order SVD)**.** *Every complex* $(I_1 \times \cdots \times I_N)$ *tensor* $\mathcal{A}$ *can be written as the product*

$$\mathcal{A} = S \times_1 U^{(1)} \times_2 U^{(2)} \cdots \times_N U^{(N)}$$

in which

1. $U^{(n)} = (U_1^{(n)} U_2^{(n)} \cdots U_{I_n}^{(n)})$ *is a unitary* $I_n \times I_n$ *matrix*

2. *S is an $(I_1 \times \cdots \times I_N)$ tensor of which the subtensors $S_{i_n=\alpha}$ obtained by fixing the nth index to α, have properties of*
   (a) *all-orthogonality: two subtensors $S_{i_n=\alpha}$ and $S_{i_n=\beta}$ are orthogonal for all possible values of n, α and β such that for $\alpha \neq \beta$, $\langle S_{i_n=\alpha}, S_{i_n=\beta}\rangle = 0$*
   (b) *Ordering $\|S_{i_n=1}\|_F \geq \|S_{i_n=2}\|_F \geq \cdots \|S_{i_n=I_N}\|_F \geq 0$*

Based on this theorem, we can rewrite $\mathcal{D}$ as,

$$\mathcal{D} = \{\mathcal{S}\times_1 U_1 \times_2 U_2 \times_3 U_3 : \mathcal{S} \in \mathbb{R}^{r\times r\times r}, U_i \in \mathbb{R}^{d\times r} \text{ s.t. } \langle \mathcal{S}_{i=\alpha}, \mathcal{S}_{i=\beta}\rangle = 0 \text{ for } \alpha \neq \beta \text{ and } U_i^t U_i = I \text{ for } i \in [3], \}$$

The condition for $\mathcal{S}$ reduces the number independent parameters in $\mathcal{S}$ from r^3 to r^3-$r\frac{r(r-1)}{2}$. The orthonormal condition U_i reduces the number of independent parameters in U_i from dr to $dr - \frac{r(r-1)}{2} - r$ for each i. Therefore, we get,

$$df_{parameter} = \frac{r^2(r+1)}{2} + 3(dr - \frac{r(r+1)}{2})$$

We can calculate approximated simulation error by combining this degree of freedom and (22). Following is the result.

1. When d=20,

$$\|\Theta_{true} - \hat{\Theta}\|_F \sim \sqrt{\frac{df_{parameter}}{n_{sample}}} = \sqrt{\frac{180}{20^3}} \approx 0.15$$

2. When d=30,

$$\|\Theta_{true} - \hat{\Theta}\|_F \sim \sqrt{\frac{df_{parameter}}{n_{sample}}} = \sqrt{\frac{270}{30^3}} \approx 0.1 \tag{23}$$

5 Algorithm codes

5.1 Algorithms for tensor svd.

```r
library(rTensor)

```

```r
StepAm = function(A,r,p){
  m = nrow(A); n = ncol(A)
  l = r+p
  omega = matrix(rnorm(n*l),nrow = n, ncol = l)
  Y = A%*%omega
  Q = qr.Q(qr(Y))
  return(Q)
}

StepA = function(A,r,p,d1,d2){
  l = r + p
  P1 = matrix(rnorm(d1*l),nrow = d1, ncol = l)
  P2 = matrix(rnorm(d2*l),nrow = d2, ncol = l)
  omega = khatri_rao(P1,P2)
  Y = A%*%omega
  Q = qr.Q(qr(Y))
  return(Q)
}

# Algorithm 1.
tensor_svd = function(tnsr,k1,k2,k3,p){
  App = list(Z=NULL,U=NULL)
  mat1 <- k_unfold(tnsr,m=1)
  mat2 <- k_unfold(tnsr,m=2)
  mat3 <- k_unfold(tnsr,m=3)
  Q1 <- StepAm(mat1@data,k1,p)
  Q2 <- StepAm(mat2@data,k2,p)
  Q3 <- StepAm(mat3@data,k3,p)
  Coreten <- ttm(ttm(ttm(tnsr,t(Q1),1),t(Q2),2),t(Q3),3)
  App$Z = Coreten
  App$U = list(Q1,Q2,Q3)
  return(App)
}



# Algorithm 2.
tensor_svd2 = function(tnsr,k1,k2,k3,p){
  App = list(Z=NULL,U=NULL)
  rk = c(k1,k2,k3)
  a = c(1,2,3,1,2,3)
  Omega = list()
  Q = list()
  for (i in 1:3) {

    Omega[[i]] <- matrix(rnorm(tnsr@modes[i]*(rk[i]+p)),ncol = rk[i]+p)
  }
  for (i in 1:3) {
    ing <- matrix(rnorm(prod(rk+p)),ncol = rk[i]+p)
    tmp <- k_unfold(ttm(ttm(tnsr,t(Omega[[a[i+1]]]),a[i+1]),t(Omega[[a[i
   +2]]]),a[i+2]),m =i)@data%*%ing
    Q[[i]] <- qr.Q(qr(tmp))
  }
  Coreten <- ttm(ttm(ttm(tnsr,t(Q[[1]]),1),t(Q[[2]]),2),t(Q[[3]]),3)
  App$Z <- Coreten
  App$U <- Q
  return(App)
}



# Algorithm 3.
tensor_svd3 = function(tnsr,k1,k2,k3,p){
  App = list(Z=NULL,U=NULL)
  mode <- tnsr@modes
  mat1 <- k_unfold(tnsr,m=1)
  mat2 <- k_unfold(tnsr,m=2)
  mat3 <- k_unfold(tnsr,m=3)
  Q1 <- StepA(mat1@data,k1,p,mode[2],mode[3])
  Q2 <- StepA(mat2@data,k2,p,mode[3],mode[1])
  Q3 <- StepA(mat3@data,k3,p,mode[1],mode[2])
  Coreten <- ttm(ttm(ttm(tnsr,t(Q1),1),t(Q2),2),t(Q3),3)
  App$Z = Coreten
  App$U = list(Q1,Q2,Q3)
  return(App)
}

# Algorithm 4.
```

```r
tensor_svd4 = function(tnsr,k1,k2,k3,p,nrep=5){
  App = list(Z=NULL ,U=NULL)
  mat1 <- k_unfold(tnsr,m=1)
  mat2 <- k_unfold(tnsr,m=2)
  mat3 <- k_unfold(tnsr,m=3)
  Q1_rep=Q2_rep=Q3_rep=NULL
  mode <- tnsr@modes

  for(s in 1:nrep){
    Q1 <- StepA(mat1@data,k1,p,mode[2],mode[3])
    Q2 <- StepA(mat2@data,k2,p,mode[3],mode[1])
    Q3 <- StepA(mat3@data,k3,p,mode[1],mode[2])

    Q1_rep=cbind(Q1_rep,Q1)
    Q2_rep=cbind(Q2_rep,Q2)
    Q3_rep=cbind(Q3_rep,Q3)
  }

  Q1=as.matrix(svd(Q1_rep)$u[,1:k1])
  Q2=as.matrix(svd(Q2_rep)$u[,1:k2])
  Q3=as.matrix(svd(Q3_rep)$u[,1:k3])

  Coreten <- ttm(ttm(ttm(tnsr,t(Q1),1),t(Q2),2),t(Q3),3)

  App$Z = Coreten
  App$U = list(Q1,Q2,Q3)
  return(App)
}
```

5.2 Simulation for comparison of tensor algorithms.

```r


## Recovery from a noise simulation.

sd = 0.01*1:50
result = data.frame(matrix(nrow = 200, ncol =3))
names(result) <- c("sd","tensor_resid","method")
```

```r
for (i in 1:50) {
  s=sd[i]
  result[i,1] = s
  result[i+50,1] = s
  result[i+100,1] = s
  result[i+150,1] = s
  e = as.tensor(array(rnorm(1000000,mean =0,sd = s),dim = c(100,100,100)))
  D = B+e
  est1 = tensor_svd(D,20,20,20,5)
  est2 = tensor_svd2(D,20,20,20,5)
  est3 = tensor_svd3(D,20,20,20,5)
  est4 = tensor_svd4(D,20,20,20,5)
  result[i,2] = tensor_resid(B,est1)
  result[i+50,2] = tensor_resid(B,est2)
  result[i+100,2]= tensor_resid(B,est3)
  result[i+150,2]= tensor_resid(B,est4)
  result[i,3] = "1st"
  result[i+50,3] = "2nd"
  result[i+100,3]= '3rd'
  result[i+150,3]= "4th"

}


ggplot(data = result, aes(x = sd, y = tensor_resid,col = method))+geom_
    point()+
  geom_line()


## Angle simulation.
a <- as.matrix(rnorm(100))
b <- as.matrix(rnorm(100))
c <- as.matrix(rnorm(100))
tnsr <- as.tensor(array(1,dim = c(1,1,1)))
X <- ttm(ttm(ttm(tnsr,a,1),b,2),c,3)
sd = 0.01*1:50
result = data.frame(matrix(0,nrow = 200, ncol =6))
names(result) <- c("sd","angle1","angle2","angle3","method","MSE")

```

```r

for (i in 1:50) {
  s=sd[i]
  result[i,1] = s
  result[i+50,1] = s
  result[i+100,1] = s
  result[i+150,1]= s
  for (j in 1:100) {
    set.seed(j)
    e = as.tensor(array(rnorm(1000000,mean =0,sd = s),dim = c(100,100,100)
   ))
    D = X+e
    est1 = tensor_svd(D,1,1,1,0)
    est2 = tensor_svd2(D,1,1,1,0)
    est3 = tensor_svd3(D,1,1,1,0)
    est4 = tensor_svd4(D,1,1,1,0)

    result[i,2] <- result[i,2]+angle(est1$U[[1]],a)
    result[i,3] <- result[i,3]+angle(est1$U[[2]],b)
    result[i,4] <- result[i,4]+angle(est1$U[[3]],c)
    result[i,6] <- result[i,6]+tensor_resid(X,est1)
    result[i+50,2] <- result[i+50,2]+angle(est2$U[[1]],a)
    result[i+50,3] <- result[i+50,3]+angle(est2$U[[2]],b)
    result[i+50,4] <- result[i+50,4]+angle(est2$U[[3]],c)
    result[i+50,6] <- result[i+50,6]+tensor_resid(X,est2)
    result[i+100,2] <- result[i+100,2]+angle(est3$U[[1]],a)
    result[i+100,3] <- result[i+100,3]+angle(est3$U[[2]],b)
    result[i+100,4] <- result[i+100,4]+angle(est3$U[[3]],c)
    result[i+100,6] <- result[i+100,6]+tensor_resid(X,est3)
    result[i+150,2] <- result[i+150,2]+angle(est4$U[[1]],a)
    result[i+150,3] <- result[i+150,3]+angle(est4$U[[2]],b)
    result[i+150,4] <- result[i+150,4]+angle(est4$U[[3]],c)
    result[i+150,6] <- result[i+150,6]+tensor_resid(X,est4)

  }
  result[i,5] = "1st"
  result[i+50,5] = "2nd"
  result[i+100,5]= '3rd'
  result[i+150,5] = "4th"

}
result[,2:4] <- result[,2:4]/100
result[,6] <- result[,6]/100

write.table(result, file = "4comparison.csv")


g1 <- ggplot(data = result, aes(x=sd,y = MSE ,color = method))+
  geom_point(aes(x=sd, y = MSE))+geom_line(aes(x=sd, y = MSE))
g2 <- ggplot(data = result, aes(x=sd,y = abs(angle1) ,color = method))+
  geom_point(aes(x=sd, y = abs(angle1)))+geom_line(aes(x=sd, y = abs(
    angle1)))+ylab("1st axis")
g3 <- ggplot(data = result, aes(x=sd,y = abs(angle2) ,color = method))+
  geom_point(aes(x=sd, y = abs(angle2)))+geom_line(aes(x=sd, y = abs(
    angle2)))+ylab("2nd axis")
g4 <- ggplot(data = result, aes(x=sd,y = abs(angle3) ,color = method))+
  geom_point(aes(x=sd, y =  abs(angle3)))+geom_line(aes(x=sd, y =  abs(
    angle3)))+ylab("3rd axis")
library(gridExtra)
grid.arrange(g1,g2,g3,g4)
```

6 Algorithms for ordinal tensor analysis(complete version)

```r
library(MASS)
library(rTensor)
library(pracma)
library(ggplot2)
library(ggthemes)
library(gridExtra)

# Some functions needed for Algorithm 4 and 5.
realization = function(tnsr,alpha){
  thet <- k_unfold(tnsr,1)@data
  theta1 <- thet + alpha[1]
  theta2 <- thet + alpha[2]
  result <- k_unfold(tnsr,1)@data
  p1 <- logistic(theta1)
  p2 <- logistic(theta2)-logistic(theta1)
  p3 <- matrix(1,nrow = nrow(thet),ncol = ncol(thet))-logistic(theta2)
  for (i in 1:nrow(thet)) {
    for(j in 1:ncol(thet)){
      result[i,j] <- sample(c(1,2,3),1,prob= c(p1[i,j],p2[i,j],p3[i,j]))
    }
  }
  return(k_fold(result,1,modes = tnsr@modes))
}


h1 = function(A_1,W1,ttnsr,omega){
  thet =W1%*%c(A_1)
  p1 = logistic(thet + omega[1])
  p2 = logistic(thet + omega[2])
  p = cbind(p1,p2-p1,1-p2)
  return(-sum(log(c(p[which(c(ttnsr)==1),1],p[which(c(ttnsr)==2),2],p[
   which(c(ttnsr)==3),3]))))
}
g1 = function(A_1,W1,ttnsr,omega){
  thet =W1%*%c(A_1)
  p1 = logistic(thet + omega[1])
  p2 = logistic(thet + omega[2])
  q1 <- p1-1
  q2 <- (p2*(1-p2)-p1*(1-p1))/(p1-p2)
  q3 <- p2
  gd = apply(diag(q1[which(c(ttnsr)==1)])%*%W1[which(c(ttnsr)==1),],2,sum)
   +
    apply(diag(q2[which(c(ttnsr)==2)])%*%W1[which(c(ttnsr)==2),],2,sum)+
    apply(diag(q3[which(c(ttnsr)==3)])%*%W1[which(c(ttnsr)==3),],2,sum)
  return(gd)
}


comb = function(A,W,ttnsr,k,omega,alph=TRUE){
  nA = A
  tnsr1 <- k_unfold(as.tensor(ttnsr),k)@data
  if (alph==TRUE) {
    l <- lapply(1:nrow(A),function(i){optim(A[i,],
                             function(x) h1(x,W,tnsr1[i,],omega),
                             function(x) g1(x,W,tnsr1[i,],omega),
                             method = "BFGS")$par})
    nA <- matrix(unlist(l),nrow = nrow(A),byrow = T)
  }else{
    l <- lapply(1:nrow(A),function(i){constrOptim(A[i,],
                             function(x) h1(x,W,tnsr1[i,],omega),function(x)
   g1(x,W,tnsr1[i,],omega),
                             ui = rbind(W,-W),ci = rep(-alph,2*nrow(W)),
   method = "BFGS")$par})
    nA <- matrix(unlist(l),nrow = nrow(A),byrow = T)
  }
  return(nA)
}


corecomb = function(C,W,ttnsr,omega,alph=TRUE){
  Cvec <- c(C@data)
  h <- function(x) h1(x,W,ttnsr,omega)
  g <- function(x) g1(x,W,ttnsr,omega)
  if (alph==TRUE) {
    d <- optim(Cvec,h,g,method = "BFGS")
    C <- new("Tensor",C@num_modes,C@modes,data =d$par)
  }else{
    d <- constrOptim(Cvec,h,g,ui = rbind(W,-W),ci = rep(-alph,2*nrow(W)),
   method = "BFGS")
    C <- new("Tensor",C@num_modes,C@modes,data =d$par)
  }
  return(C)
}


## Algorithm 4.
fit_ordinal = function(ttnsr,C,A_1,A_2,A_3,omega,alph = TRUE){

  alphbound <- alph+10^-4
  result = list()
```

```
error<- 3
iter = 0
d1 <- nrow(A_1); d2 <- nrow(A_2); d3 <- nrow(A_3)
r1 <- ncol(A_1); r2 <- ncol(A_2); r3 <- ncol(A_3)
if (alph == TRUE) {
  while ((error > 10^-4)&(iter<50) ) {
    iter = iter +1


    #update A_1
    prevtheta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
    prev <- likelihood(ttnsr,prevtheta,omega)
    W1 =kronecker(A_3,A_2)%*%t(k_unfold(C,1)@data)
    A_1 <- comb(A_1,W1,ttnsr,1,omega)


    # update A_2
    W2 <- kronecker(A_3,A_1)%*%t(k_unfold(C,2)@data)
    A_2 <- comb(A_2,W2,ttnsr,2,omega)

    # update A_3
    W3 <- kronecker(A_2,A_1)%*%t(k_unfold(C,3)@data)
    A_3 <- comb(A_3,W3,ttnsr,3,omega)

    # update C
    W4 <- kronecker(kronecker(A_3,A_2),A_1)
    C <- corecomb(C,W4,ttnsr,omega)
    theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
    new <- likelihood(ttnsr,theta,omega)
    error <- abs((new-prev)/prev)
  }
}else{
  while ((error > 10^-4)&(iter<50) ) {
    iter = iter +1


    #update A_1
    prevtheta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
    prev <- likelihood(ttnsr,prevtheta,omega)
```

```
      W1 =kronecker(A_3,A_2)%*%t(k_unfold(C,1)@data)
      A_1 <- comb(A_1,W1,ttnsr,1,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break


      # update A_2
      W2 <- kronecker(A_3,A_1)%*%t(k_unfold(C,2)@data)
      A_2 <- comb(A_2,W2,ttnsr,2,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break

      # update A_3
      W3 <- kronecker(A_2,A_1)%*%t(k_unfold(C,3)@data)
      A_3 <- comb(A_3,W3,ttnsr,3,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break

      # update C
      W4 <- kronecker(kronecker(A_3,A_2),A_1)
      C <- corecomb(C,W4,ttnsr,omega,alph)
      theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      new <- likelihood(ttnsr,theta,omega)
      error <- abs((new-prev)/prev)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break
    }
  }

  result$C <- C; result$A_1 <- A_1; result$A_2 <- A_2; result$A_3 <- A_3
  result$iteration <- iter
  return(result)
}

## Algorithm 5.
fit_ordinal2 = function(ttnsr,C,A_1,A_2,A_3,omega=TRUE,alph = TRUE){
  omega <- sort(rnorm(2))
  alphbound <- alph+10^-4
  result = list()
  error<- 3
  iter = 0
  d1 <- nrow(A_1); d2 <- nrow(A_2); d3 <- nrow(A_3)
  r1 <- ncol(A_1); r2 <- ncol(A_2); r3 <- ncol(A_3)
  if (alph == TRUE) {
    while ((error > 10^-4)&(iter<50) ) {
      iter = iter +1


      #update A_1
      prevtheta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      prev <- likelihood(ttnsr,prevtheta,omega)
      W1 =kronecker(A_3,A_2)%*%t(k_unfold(C,1)@data)
      A_1 <- comb(A_1,W1,ttnsr,1,omega)


      # update A_2
      W2 <- kronecker(A_3,A_1)%*%t(k_unfold(C,2)@data)
      A_2 <- comb(A_2,W2,ttnsr,2,omega)

      # update A_3
      W3 <- kronecker(A_2,A_1)%*%t(k_unfold(C,3)@data)
      A_3 <- comb(A_3,W3,ttnsr,3,omega)

      # update C
      W4 <- kronecker(kronecker(A_3,A_2),A_1)
      C <- corecomb(C,W4,ttnsr,omega)

      #update omega
      theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      m <- polr(as.factor(c(ttnsr))~offset(-c(theta@data)))
      omega <- m$zeta



      theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      new <- likelihood(ttnsr,theta,omega)
      error <- abs((new-prev)/prev)
    }
  }else{
    while ((error > 10^-4)&(iter<50) ) {
      iter = iter +1


      #update A_1
      prevtheta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      prev <- likelihood(ttnsr,prevtheta,omega)
      W1 =kronecker(A_3,A_2)%*%t(k_unfold(C,1)@data)
      A_1 <- comb(A_1,W1,ttnsr,1,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break


      # update A_2
      W2 <- kronecker(A_3,A_1)%*%t(k_unfold(C,2)@data)
      A_2 <- comb(A_2,W2,ttnsr,2,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break

      # update A_3
      W3 <- kronecker(A_2,A_1)%*%t(k_unfold(C,3)@data)
      A_3 <- comb(A_3,W3,ttnsr,3,omega,alphbound)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break

      # update C
      W4 <- kronecker(kronecker(A_3,A_2),A_1)
      C <- corecomb(C,W4,ttnsr,omega,alph)
      if(max(abs(ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)@data))>=alph) break

      #update omega
      theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      m <- polr(as.factor(c(ttnsr))~offset(-c(theta@data)))
      omega <- m$zeta


      theta <- ttm(ttm(ttm(C,A_1,1),A_2,2),A_3,3)
      new <- likelihood(ttnsr,theta,omega)
      error <- abs((new-prev)/prev)
    }
  }

  result$C <- C; result$A_1 <- A_1; result$A_2 <- A_2; result$A_3 <- A_3
  result$iteration <- iter; result$omega <- omega
  return(result)
```

```
}
```

References

[1] L De Lathauwer, B De Moor, J Vandewalle *A multilinear singular value decomposition* SIAM J. Matrix Anal. Appl., 21(4), 1253–1278.