Beyond the Signs: Nonparametric Tensor Completion via Sign Series

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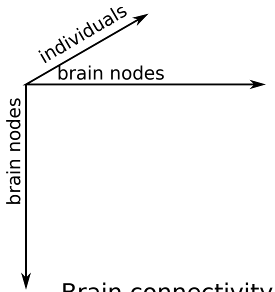
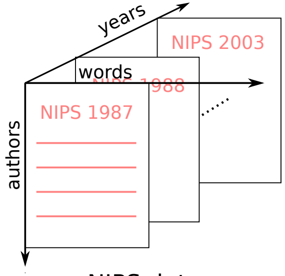
Tensors in application



Brain connectivity



NIPS data

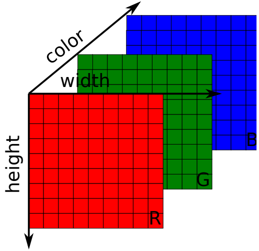
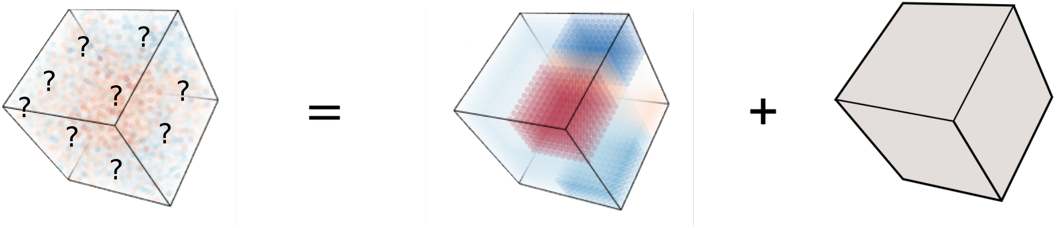


Image data

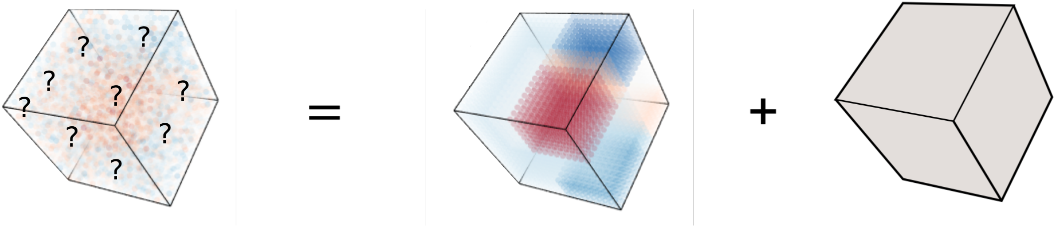
1. The human brain connectivity dataset consists of 68 brain regions for 114 individuals (Wang et al., 2017).
2. The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003 along with author information (Globerson et al., 2007).
3. An RGB image consists of pixel values across three channels.

Main problems: the signal plus noise model



$$\underbrace{\mathcal{Y}}_{\text{Data tensor}} = \underbrace{\Theta}_{\text{Signal tensor}} + \underbrace{\mathcal{E}}_{\text{Noise tensor}}$$

Main problems: the signal plus noise model



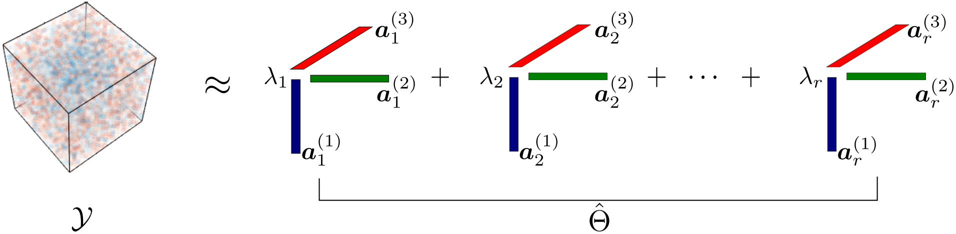
$$\underbrace{\mathcal{Y}}_{\text{Data tensor}} \qquad \underbrace{\Theta}_{\text{Signal tensor}} \qquad \underbrace{\mathcal{E}}_{\text{Noise tensor}}$$

We focus on the two problems

1. Signal tensor estimation: How to estimate the signal tensor Θ?
2. Complexity of tensor completion: How many observed tensor entries do we need?

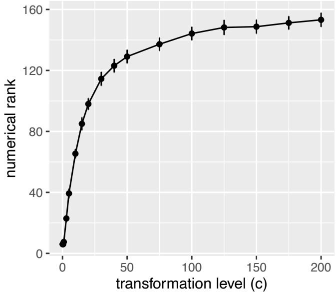
Inadequacies of low-rank models

- Low-rank models (Anandkumar et al., 2014; Montanari and Sun, 2018; Cai et al., 2019).



Inadequacies of low-rank models

- Sensitivity to order-preserving transformation



$$\Theta = \frac{1}{1+\exp(-c\,(\mathcal{Z}))}, \quad \text{where}$$
$$\mathcal{Z} = \boldsymbol{a}^{\otimes 3} + \boldsymbol{b}^{\otimes 3} + \boldsymbol{c}^{\otimes 3}.$$

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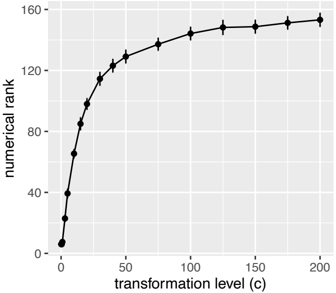

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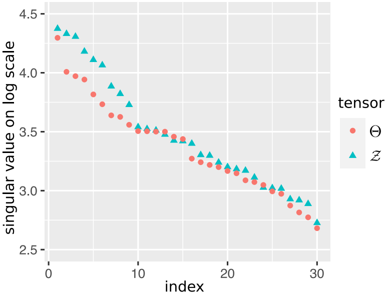
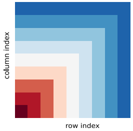
- Inadequacy for special structures.



$$\Theta = \log(1+\mathcal{Z}), \quad \text{where}$$
$$\mathcal{Z}(i,j,k) = \frac{1}{d}\max(i,j,k).$$



Why sign matters?

For a bounded tensor $\Theta \in [-1,1]^{d_1\times\cdots\times d_K}$,

$$\Theta \approx \frac{1}{|\mathcal{H}|}\sum_{\pi\in\mathcal{H}} \text{sgn}(\Theta-\pi), \quad \text{where } \mathcal{H} = \left\{-1.\ldots,-\frac{1}{H},0,\frac{1}{H},\ldots,1\right\}.$$

- Sign tensors are invariant to order-preserving transformation.
- More flexible signal tensors are allowed by using sign tensor series representation.
- In noisy case, we estimate $\text{sgn}(\Theta-\pi)$ from the tensor data $\text{sgn}(\mathcal{Y}-\pi)$.

Sign rank

- Key idea: we use a local notion of low-rankness to allow a richer family of signal tensors.
- Two tensors are sign equivalent denoted $\Theta \simeq \Theta'$ if $\text{sgn}(\Theta) = \text{sgn}(\Theta')$, where

$$[\text{sgn}(\Theta)]_\omega := \begin{cases} 1 & \text{if } \Theta_\omega \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

.

- Sign rank is defined as

$$\boxed{\text{srank}(\Theta) = \min\{\text{rank}(\Theta') \colon \Theta' \simeq \Theta, \Theta' \in \mathbb{R}^{d_1 \times \cdots \times d_K}\}}.$$

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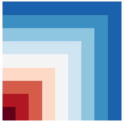
$\Theta =$  , $\quad \mathrm{sgn}(\Theta) =$  $\Longrightarrow \begin{array}{c} \mathrm{rank}(\Theta) = d \\ \mathrm{srank}(\Theta) = 2 \end{array}$

Sign representable tensors

Sign representable tensors

A tensor Θ is called r-sign representable if the tensor $(\Theta - \pi)$ has sign rank bounded by r for all $\pi \in [-1, 1]$.

- Most existing structure tensors belong to sign representable family:
  - Low-rank CP tensors, Tucker tensors, stochastic block models.
  - High-rank tensors from GLM, single index models,
  - Tensors with repeating patterns, e.g. $\Theta(i_1, \ldots, i_K) = \log(1 + \max(i_1, \ldots, i_K))$ is 2-sign representable.

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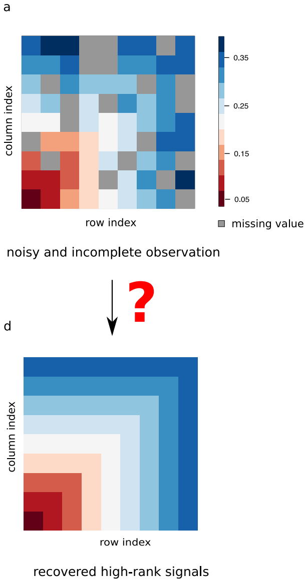
  - Tensors with repeating patterns, e.g. $\Theta(i_1, \ldots, i_K) = \log(1 + \max(i_1, \ldots, i_K))$ is 2-sign representable.
- Instead of the classical low-rank assumption, we propose the sign representable tensor family

$$\Theta \in \mathscr{P}_{\text{sgn}}(r) := \{\Theta \colon \text{srank}(\Theta - \pi) \leq r \text{ for all } \pi \in [-1, 1]\}.$$

Our solution: sign signal helps!



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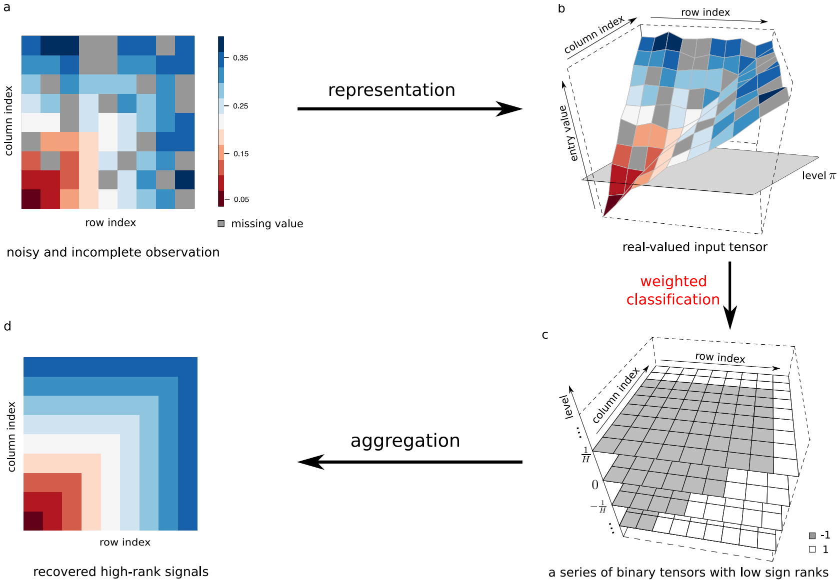
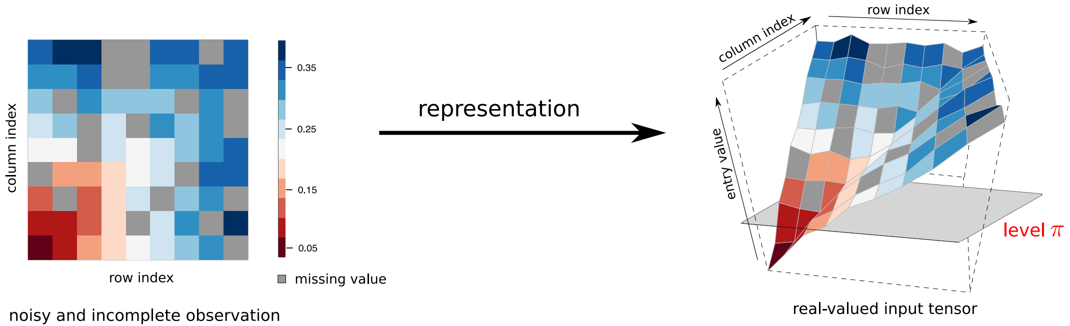


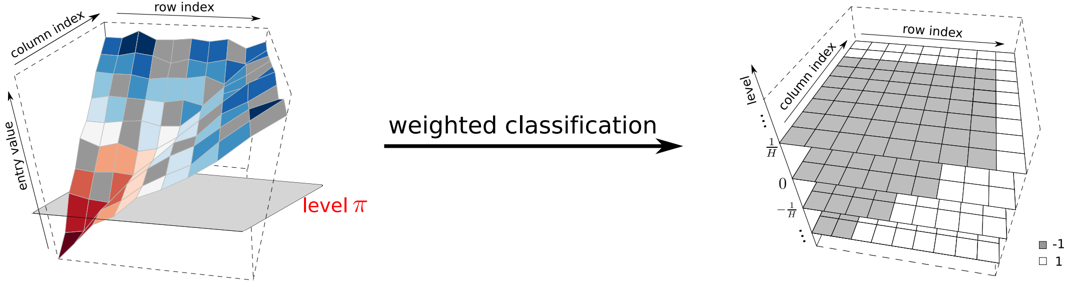
Step 1: representation



- We observe a noisy incomplete tensor $\mathcal{Y}_\Omega \in [-1,1]^{d_1\times\cdots\times d_K}$ with observed index set $\Omega \subset [d_1]\times\cdots\times[d_K]$.
- We dichotomize the data into a series of sign tensors:

$$\{\mathrm{sgn}(\mathcal{Y}_\Omega - \pi)\}_{\pi\in\mathcal{H}}, \quad \text{where } \mathcal{H} = \left\{-1, \ldots, -\frac{1}{H}, 0, \frac{1}{H}, \ldots, 1\right\}.$$

Step 2: weighted classification



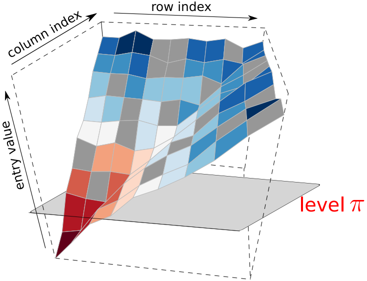
real-valued input tensor

a series of binary tensors with low sign ranks

- We estimate $\text{sgn}(\Theta - \pi)$ through $\text{sgn}(\mathcal{Y}_\Omega - \pi)$ via weighted classification.
- Objective function of weighted classification is

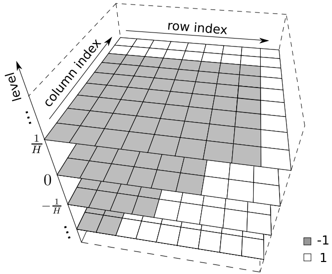
$$L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \underbrace{|\mathcal{Y}(\omega) - \pi|}_{\text{weight}} \times \underbrace{|\text{sgn}(\mathcal{Z}(\omega)) - \text{sgn}(\mathcal{Y}(\omega) - \pi)|}_{\text{classification loss}}$$

Step 2: weighed classification



real-valued input tensor

weighted classification



a series of binary tensors with low sign ranks

- If $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ is α-smooth ($\alpha > 0$), we have a unique optimizer such that

$$\text{sgn}(\Theta - \pi) = \underset{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \leq r}{\arg\min}\ \mathbb{E}_{\mathcal{Y}_\Omega} L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$

- We obtain a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi \in \mathcal{H}}$ as

$$\hat{\mathcal{Z}}_\pi = \underset{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \leq r}{\arg\min}\ L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$

Identification for sign tensor estimation

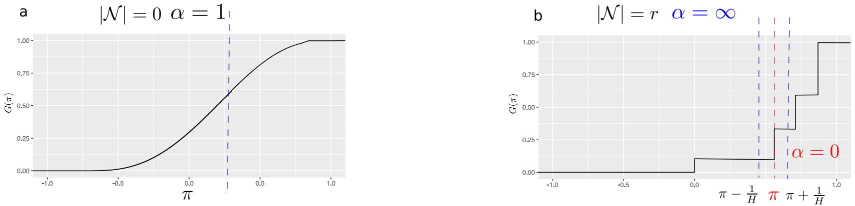
- We quantify difficulty of the problem using CDF $G(\pi) = \mathbb{P}_{\omega \in \Pi}[\Theta(\omega) \le \pi]$.

α-smoothness

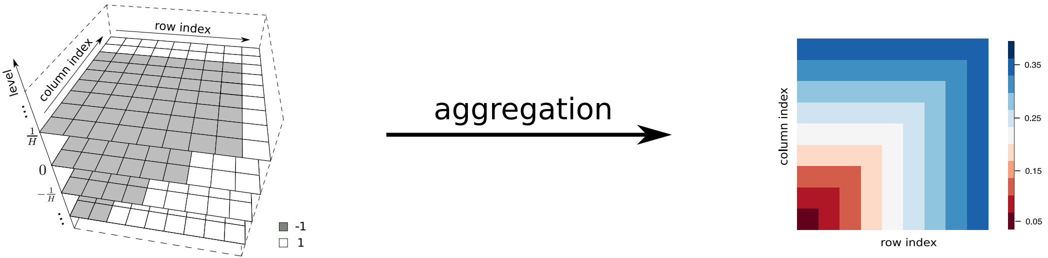
- Partition $[-1, 1] = \mathcal{N} \cup \mathcal{N}^c$, where $\mathcal{N}^c$ consists of levels whose pseudo density (histogram with bin size $\Delta s = d^{-K}$) is uniformly bounded, and $\mathcal{N}$ otherwise.
- $G(\pi)$ is globally α-smooth in that for all $\pi \in \mathcal{N}^c$,

$$\sup_{\Delta s \le t < \rho(\pi, \mathcal{N})} \frac{G(\pi + t) - G(\pi - t)}{t^{\alpha}} \le c,$$

for two constants $\alpha, c > 0$, where $\rho(\pi, \mathcal{N}) = \min_{\pi' \in \mathcal{N}} |\pi - \pi'| + \Delta s$.



Step 3: aggregation



a series of binary tensors with low sign ranks

recovered high-rank signals

- From a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi\in\mathcal{H}}$ in the weighted classification, we obtain the tensor estimate

$$\hat{\Theta} = \frac{1}{2H+1}\sum_{\pi\in\mathcal{H}} \text{sgn}\hat{\mathcal{Z}}_\pi.$$

Estimation error

For two tensor Θ_1, Θ_2, define $\text{MAE}(\Theta_1, \Theta_2) = \mathbb{E}_{\omega \in \Pi} |\Theta_1(\omega) - \Theta_2(\omega)|$.

Estimation error (L. and Wang 2021)

Suppose $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ is α-smooth with bounded $|\mathcal{N}|$, and $d_1 = \cdots = d_K = d$.

1. (Sign tensor estimation) For all $\pi \in \mathcal{N}^c$, with high probability,

$$\text{MAE}(\text{sgn}\hat{\mathcal{Z}}_\pi, \text{sgn}(\Theta - \pi)) \lesssim^* \left(\frac{dr}{|\Omega|}\right)^{\frac{\alpha}{\alpha+2}}.$$

2. (Tensor estimation)

$$\text{MAE}(\hat{\Theta}, \Theta) \lesssim^* \underbrace{\left(\frac{dr}{|\Omega|}\right)^{\frac{\alpha}{\alpha+2}}}_{\text{Error inherited from sign estimation}} + \underbrace{\frac{1}{H}}_{\text{Bias}} + \underbrace{\frac{Hdr}{|\Omega|}}_{\text{Variance}} \asymp^{**} \left(\frac{dr}{|\Omega|}\right)^{\min\left(\frac{\alpha}{\alpha+2}, \frac{1}{2}\right)}.$$

*log term suppressed, **$H \asymp (|\Omega|/dr)^{1/2}$

- Tensor estimation is generally no better than sign tensor estimation.
- See paper for general case that allows unbounded $|\mathcal{N}|$ and sub-Gaussian noise.

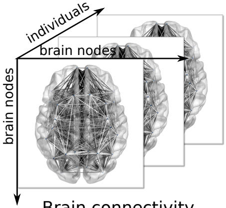
Comparison to existing results

Special case with full observation:

Model	Our rate* (power of d)	Previous results
Tensor block model	$-(K-1)/2$	$\alpha=\infty$; minimax rate in Wang & Zeng '19
Single index model	$-(K-1)/3$	$\alpha=1$; conjecture on the optimality; matrix rate $d^{-1/3}$ improves $\mathcal{O}(d^{-1/4})$ by Ganti et al. '18
Generalized linear model	$-(K-1)/3$	$\alpha=1$; close to parametric rate in L.& Wang '20
α-smooth $\mathscr{P}_{\text{sgn}}(r)$	$-(K-1)\min(\frac{\alpha}{\alpha+2}\wedge\frac{1}{2})$	faster rate as α increases

*Reference: C. Lee and M. Wang. Beyond the Signs: Nonparametric tensor completion via sign series. arXiv:2102.00384, 2021.

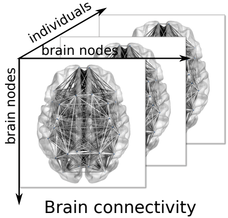
Data application: Brain connectivity



Brain connectivity

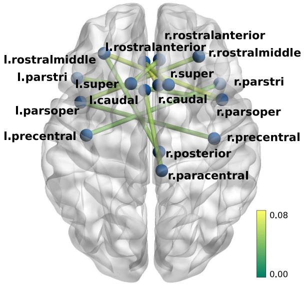
- The human brain connectivity dataset consists of 68 brain regions for 114 individuals with their IQ scores.
- Data tensor $\mathcal{Y} \in \{0, 1\}^{68\times 68\times 114}$.

Data application: Brain connectivity

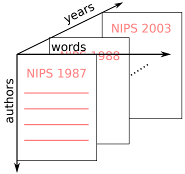


- The human brain connectivity dataset consists of 68 brain regions for 114 individuals with their IQ scores.
- Data tensor $\mathcal{Y} \in \{0,1\}^{68\times 68\times 114}$.

- We examine the estimated signal tensor $\hat{\Theta}$.
- Top 10 brain edges based on regression analysis show inter-hemisphere connections.



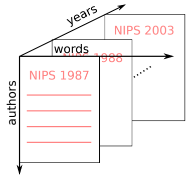
Data application: NIPS



NIPS data

- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003.
- Data tensor $\mathcal{Y} \in \mathbb{R}^{100\times 200\times 17}$.

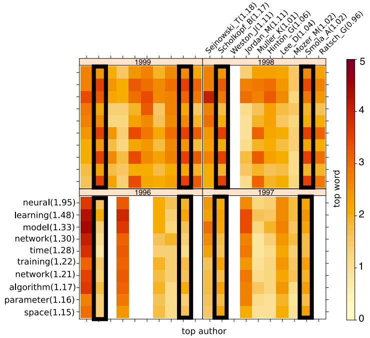
Data application: NIPS



NIPS data

- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003.
- Data tensor $\mathcal{Y} \in \mathbb{R}^{100 \times 200 \times 17}$.

- We examine the estimated signal tensor $\hat{\Theta}$.
- Most frequent words are consistent with the active topics
- Strong heterogeneity among word occurrences across authors and years.
- Similar word patterns (B. Schölkopf and A. Smola).



Data application: Brain connectivity + NIPS

MRN-114 brain connectivity dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.001)	**0.14**(0.001)	**0.12**(0.001)	**0.12**(0.001)	**0.11**(0.001)
Low-rank CPT	0.26(0.006)	0.23(0.006)	0.22(0.004)	0.21(0.006)	0.20(0.008)
NIPS word occurrence dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.002)	**0.16**(0.002)	**0.15**(0.001)	**0.14**(0.001)	**0.13**(0.001)
Low-rank CPT	0.22(0.004)	0.20(0.007)	0.19(0.007)	0.17(0.007)	0.17(0.007)
Naive imputation (Baseline)	0.32(.001)				

Table: MAE comparison in the brain data and NIPS data on 5-folded cross-validation

- Our method outperforms the low-rank CP method in applications.

Summary

- We have developed a completion method that address both low- and high-rankness based on sign series representation.
- Estimation error rates and sample complexities are established.
- Our approach has good interpretation and prediction performance in both simulations and data applications.
- Software: https://cran.r-project.org/web/packages/TensorComplete/index.html

Thank you!

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