Nonparametric Tensor Completion via Sign Series

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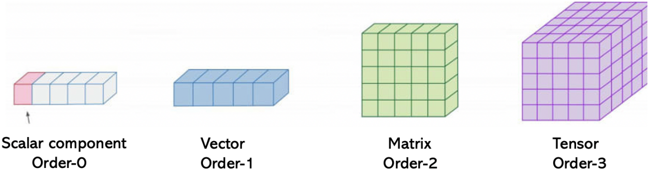
What is a tensor?

- Tensors are generalizations of vectors and matrices:

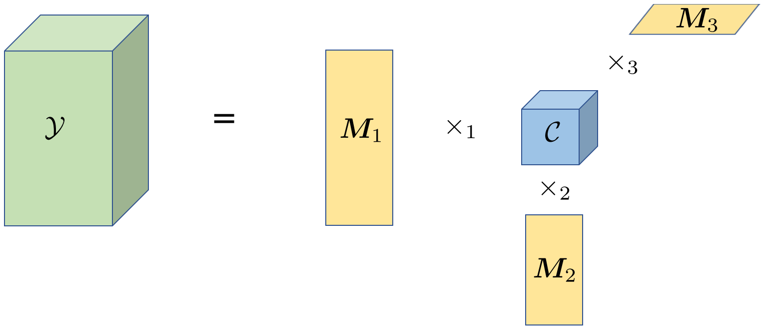


- We focus on tensors of order 3 or greater, also called higher-order tensors.
- Denote an order-$K(d_1, \cdots, d_K)$ dimensional tensor as $\mathcal{Y} = [\![y_\omega]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ where $\omega \in [d_1] \times \cdots \times [d_K]$.

Tensor decomposition (Tucker decomposition)

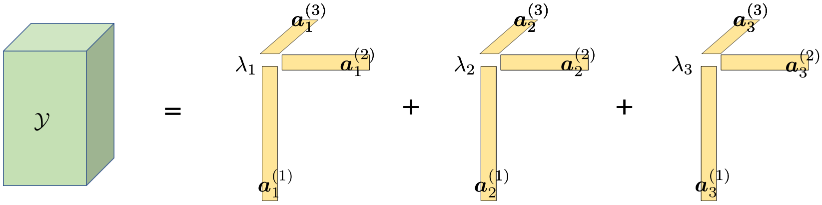
- Tucker decomposition (De Lathauwer et al., 2000).
  - $\mathcal{Y} = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \boldsymbol{M}_2 \times_3 \boldsymbol{M}_3$.
  - Generalization of matrix SVD to higher orders: $\boldsymbol{Y} = \boldsymbol{U}\Sigma\boldsymbol{V}^T (= \Sigma \times_1 \boldsymbol{U} \times_2 \boldsymbol{V})$
  - Tucker rank of an order-3 tensor is defined as

$$r(\mathcal{Y}) = (r_1, r_2, r_3).$$

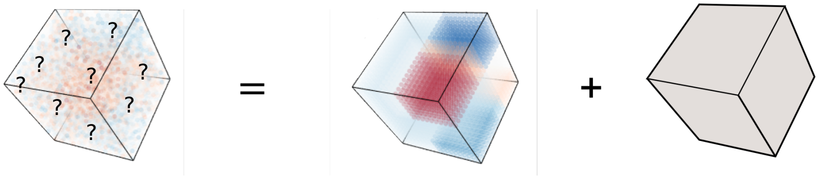


Tensor decomposition (CP decomposition)

- Canonical Polyadic (CP) decomposition (Hitchcock, 1927).
  - $\mathcal{Y} = \sum_{s=1}^{r} \lambda_s \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$.
  - Generalization of matrix SVD to higher orders: $\boldsymbol{Y} = \sum_{s=1}^{r} \lambda_s \boldsymbol{u}_s \otimes \boldsymbol{v}_s$
  - CP rank is defined as the minimal r for which the above equation holds.
  - Today, we use the tensor rank as CP rank.

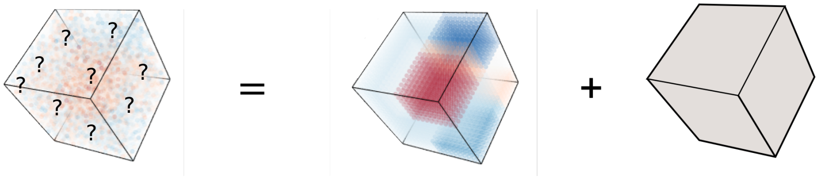


Main problems: the signal plus noise model



$$\overbrace{\mathcal{Y}}^{}\!\!\!\!\underbrace{}_{\text{Data tensor}} = \underbrace{\Theta}_{\text{Signal tensor}} + \underbrace{\mathcal{E}}_{\text{Noise tensor}}$$

Main problems: the signal plus noise model



$$\underbrace{\mathcal{Y}}_{\text{Data tensor}} \qquad = \qquad \underbrace{\Theta}_{\text{Signal tensor}} \qquad + \qquad \underbrace{\mathcal{E}}_{\text{Noise tensor}}$$

We focus on the two problems

1. Nonparametric tensor estimation: How to estimate the signal tensor Θ?
2. Complexity of tensor completion: How many observed tensor entries do we need?

Tensor based learning is challenging

- Matrix based model (Cai and Zhou, 2013; Davenport et al., 2014; Ganti et al., 2015; Fan et al., 2019)

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  - Applying matrix methods to higher-order tensor destroys structural information.
  - Tensor is different from matrix and more challenging.
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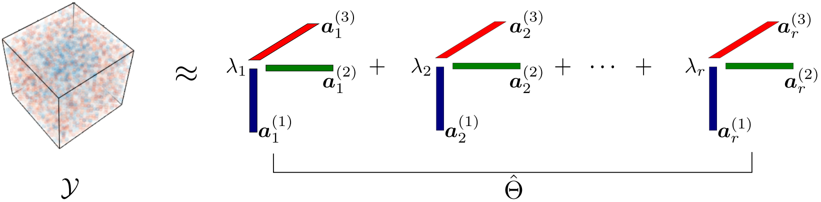
  - Low rank models are inadequate in many cases.

Inadequacies of low rank models

- Low rank models.



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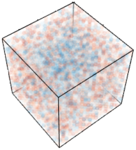
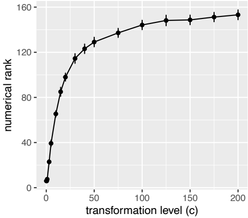


$$\mathcal{Y} \approx \underbrace{\lambda_1 \, \boldsymbol{a}_1^{(1)} \circ \boldsymbol{a}_1^{(2)} \circ \boldsymbol{a}_1^{(3)} + \lambda_2 \, \boldsymbol{a}_2^{(1)} \circ \boldsymbol{a}_2^{(2)} \circ \boldsymbol{a}_2^{(3)} + \cdots + \lambda_r \, \boldsymbol{a}_r^{(1)} \circ \boldsymbol{a}_r^{(2)} \circ \boldsymbol{a}_r^{(3)}}_{\hat{\Theta}}$$

- There are two limitations of the model
  1. The sensitivity to order-preserving transformation.
  2. The inadequacy for special structures.

Inadequacies of low rank models

- The sensitivity to order-preserving transformation.

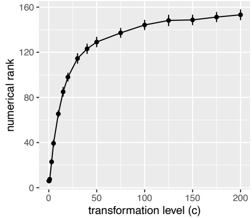


$$\Theta = \frac{1}{1+\exp(-c\,(\mathcal{Z}))}, \qquad \text{where}$$

$$\mathcal{Z} = \boldsymbol{a}^{\otimes 3} + \boldsymbol{b}^{\otimes 3} + \boldsymbol{c}^{\otimes 3}.$$

Inadequacies of low rank models

- The sensitivity to order-preserving transformation.



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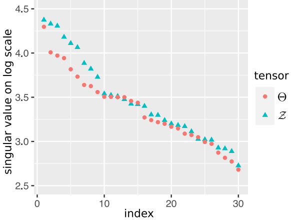
- The inadequacy for special structures.



$$\Theta = \log(1+\mathcal{Z}), \quad \text{where}$$
$$\mathcal{Z}(i,j,k) = \frac{1}{d}\max(i,j,k).$$

Motivating toy example in noiseless case

$$\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\Theta} \xrightarrow[\pi \in \{-\underbrace{1, -0.}_{\frac{1}{H}}5, 0, 0.5, 1\}]{\text{Dichotomization}} \underbrace{\left(\begin{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{\pi = -1} \\ \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{\pi = -0.5, 0} \\ \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{\pi = 0.5, 1} \end{matrix} \right)}_{\operatorname{sgn}(\Theta - \pi)} \xrightarrow{\text{Aggregation}} \begin{pmatrix} -3/5 & 1/5 & 1 \\ 1/5 & 1/5 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\tilde{\Theta} = \frac{1}{2H+1} \sum_{\pi \in \Pi} \operatorname{sgn}(\Theta - \pi)$$

where $\text{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1 & \text{otherwise.} \end{cases}$

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- Signed tensors are much simpler in the sense of rank.
- With a series of sign tensors, we can successfully preserve all information in the original signals.
- In noise case, we estimate $\operatorname{sgn}(\Theta-\pi)$ from the tensor data $\operatorname{sgn}(\mathcal{Y}-\pi)$.

Sign rank

- Two tensors are sign equivalent denoted as $\Theta \simeq \Theta'$ if $\text{sgn}(\Theta) = \text{sgn}(\Theta')$.
- Sign rank is defined as

$$\text{srank}(\Theta) = \min\{\text{rank}(\Theta')\colon \Theta' \simeq \Theta, \Theta' \in \mathbb{R}^{d_1\times\cdots\times d_K}\}.$$

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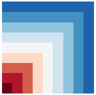
ex) $\Theta =$  , $\mathrm{sgn}(\Theta) =$  $\Longrightarrow \mathrm{rank}(\Theta) = d, \mathrm{srank}(\Theta) = 2.$

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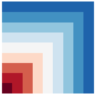
ex) $\Theta =$  , $\mathrm{sgn}(\Theta) =$  $\Longrightarrow \mathrm{rank}(\Theta) = d, \mathrm{srank}(\Theta) = 2.$

- More generally, for any strictly monotonic function $g \colon \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$,

$$\mathsf{srank}(\Theta) \leq \mathsf{rank}(g(\Theta)).$$

Sign representable tensors

Sign representable tensors

A tensor Θ is called r-sign representable if the tensor $(\Theta - \pi)$ has sign rank bounded by r for all $\pi \in [-1, 1]$. The collection $\{\text{sgn}(\Theta - \pi)\colon \pi \in [-1, 1]\}$ is called the sign tensor series.

ex1) $\Theta = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is 2-sign representable.

ex2) $\Theta(i_1, \ldots, i_K) = \log(1 + \max(i_1, \ldots, i_K))$ is 2-sign representable.

ex3) Θ such that $\text{rank}(\Theta) \leq r$, is $(r + 1)$-sign representable.

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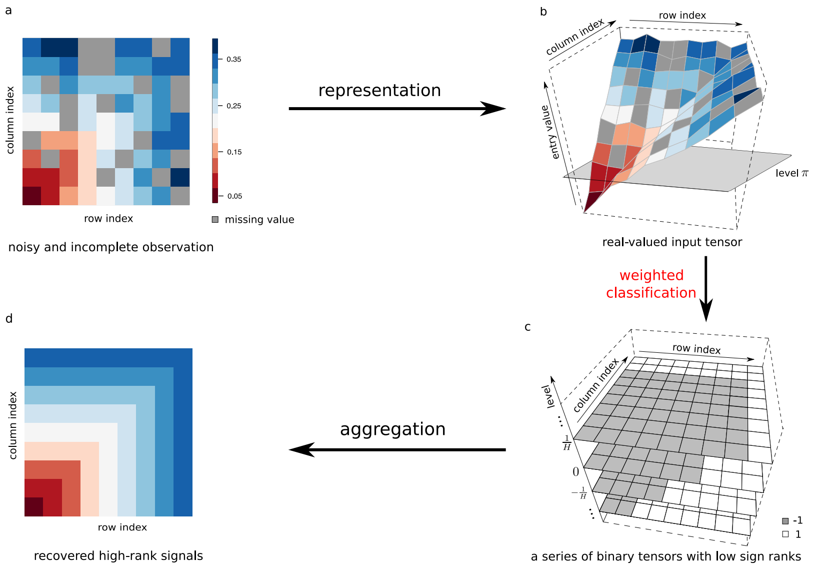
- Instead of the classical low rank assumption, we assume

$$\Theta \in \mathscr{P}_{\text{sgn}}(r) := \{\Theta\colon \text{srank}(\Theta - \pi) \le r \text{ for all } \pi \in [-1,1]\}.$$

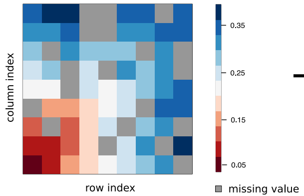
Our new approach



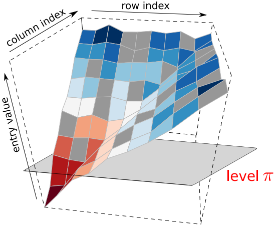
Step 1: representation

- We are given the observed tensor $\mathcal{Y}_\Omega \in [-1,1]^{d_1\times\cdots\times d_K}$ with observed index set $\Omega \subset [d_1]\times\cdots\times[d_K]$.
- We obtain sign tensor series

$$\{\text{sgn}(\mathcal{Y}_\Omega - \pi)\}_{\pi\in\mathcal{H}}, \quad \text{where } \mathcal{H} = \left\{-1,\ldots,-\frac{1}{H},0,\frac{1}{H},\ldots,1\right\}.$$



noisy and incomplete observation



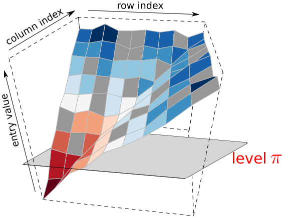
real-valued input tensor

Step 2: weighted classification

- We estimate $\text{sgn}(\Theta - \pi)$ through $\text{sgn}(\mathcal{Y}_\Omega - \pi)$ via weighted classification.
- Objective function of weighted classification is

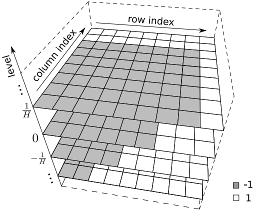
$$L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi) = \frac{1}{|\Omega|} \sum_{\pi \in \Omega} \underbrace{|\mathcal{Y}(\omega) - \pi|}_{\text{weight}} \times \underbrace{|\text{sgn}(\mathcal{Z}(\omega)) - \text{sgn}(\mathcal{Y}(\omega) - \pi)|}_{\text{classification loss}}$$

- Magnitude $|\Theta(\omega) - \pi|$ plays important role in estimation.



real-valued input tensor

weighted classification



a series of binary tensors with low sign ranks

Step 2: weighted classification

α smoothness

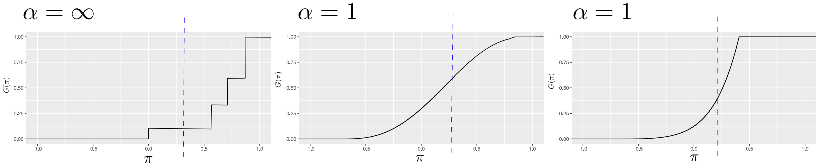
For fixed π, we call Θ is α smooth if there exist $\alpha = \alpha(\pi) > 0, c = c(\pi) > 0$, such that

$$\sup_{0 \le t < \rho(\pi, \mathcal{N})} \frac{\mathbb{P}_{\omega \sim \Pi}[|\Theta(\omega) - \pi| \le t]}{t^{\alpha}} \le c,$$

where $\rho(\pi, \mathcal{N}) = \min_{\pi' \in \mathcal{N}} |\pi - \pi'|$ and $\mathcal{N} = \{\pi \colon \mathbb{P}(\Theta(\omega) = \pi) \neq 0\}$. If α and c are global constants for all* π's, we call Θ is α-globally smooth.

* except for a finite number of π's.

Rate depends on the behavior of CDF function $G(\pi) = \mathbb{P}_{\omega \in \Pi}[\Theta(\omega) \le \pi]$.



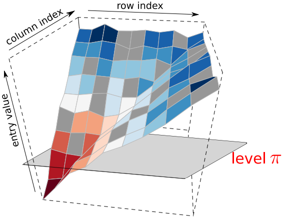
Step 2: weighed classification

- If Θ is α smooth ($\alpha > 0$), we have a unique optimizer such that

$$\text{sgn}(\Theta - \pi) = \mathop{\arg\min}_{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \leq r} \mathbb{E}_{\mathcal{Y}_\Omega} L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$

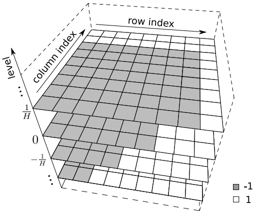
- So we obtain a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi \in \mathcal{H}}$ as

$$\hat{\mathcal{Z}}_\pi = \mathop{\arg\min}_{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \leq r} L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$



real-valued input tensor

weighted classification

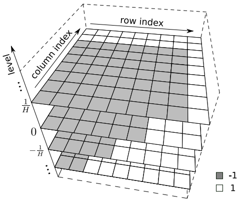


a series of binary tensors with low sign ranks

Step 3: aggregation

- From a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi\in\mathcal{H}}$ in the weighted classification, we propose the tensr estimate

$$\hat{\Theta} = \frac{1}{2H+1}\sum_{\pi\in\mathcal{H}} \text{sgn}\hat{\mathcal{Z}}_\pi.$$



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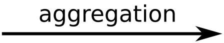

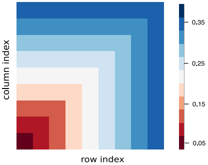

recovered high-rank signals

Sign tensor estimation error from weighted classification

- For two tensor Θ_1, Θ_2, define $\text{MAE}(\Theta_1, \Theta_2) = \mathbb{E}_{\omega\in\Pi}|\Theta_1(\omega) - \Theta_2(\omega)|$.

Sign tensor estimation for fixed π (L. and Wang, 2021)

Suppose $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ and $\Theta(\omega)$ is α smooth for fixed π. Denote $d_{\max} = \max_{k\in[K]} d_k$. Then, with very high probability over $\mathcal{Y}_\Omega$,

$$\text{MAE}(\text{sgn}\hat{\mathcal{Z}}_\pi, \text{sgn}(\Theta - \pi)) \lesssim \left(\frac{d_{\max} r}{|\Omega|}\right)^{\frac{\alpha}{\alpha+2}}.$$

- Sign estimation error shows a polynomial decay with the number of observed entries.

Tensor estimation error

Tensor estimation error (L. and Wang 2021)

Suppose $\Theta \in \mathscr{P}_{\textsf{sgn}}(r)$ and $\Theta(\omega)$ is α-globally smooth. Then, with very high probability over $\mathcal{Y}_\Omega$,

$$\textsf{MAE}(\hat{\Theta}, \Theta) \lesssim \left(\frac{d_{\max} r}{|\Omega|}\right)^{\frac{\alpha}{\alpha+2}} + \frac{1}{H} + \frac{H d_{\max} r}{|\Omega|}.$$

In particular, setting $H \asymp \left(\frac{|\Omega|}{d_{\max} r}\right)^{1/2}$ yields the error bound

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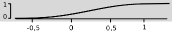
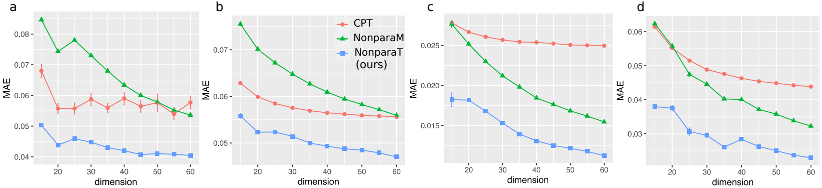
- Tensor estimation is generally no better than sign tensor estimation.
- Sample complexity:

$$\text{MAE}(\hat{\Theta}, \Theta) \to 0, \text{ as } \frac{|\Omega|}{d_{\max} r} \to \infty.$$

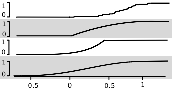
Simulations for estimation error vs tensor dimension

Simulation	Signal Tensor Θ	Rank	Sign Rank	Global α	CDF of Tensor Entries	Noise
1	$\mathcal{C} \times \boldsymbol{M}_1 \times \boldsymbol{M}_2 \times \boldsymbol{M}_3$	3^3	$\leq 3^3$	∞		Uniform $[-0.3, 0.3]$
2	$\lvert \boldsymbol{a} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{a} \otimes \mathbf{1} \rvert$	d	≤ 3	1		Normal $\mathcal{N}(0, 0.15)$
3	$\log(0.5 + Z_{\max})$	$\geq d$	2	1		Uniform $[-0.1, 0.1]$
4	$2.5 - \exp(\mathcal{Z}_{\min}^{1/3})$	$\geq d$	2	1		Normal $\mathcal{N}(0, 0.15)$



- **NonPraT**: Our nonparametric tensor method, **CPT**: low rank tensor CP decomposition, **NonPraraM**: the matrix version of our method.
- Our method (NonparaT) achives the best performance.

Simulations for estimation error vs the observation fraction

Simulation	Signal Tensor Θ	Rank	Sign Rank	Global α	CDF of Tensor Entries	Noise
1	$\mathcal{C} \times \boldsymbol{M}_1 \times \boldsymbol{M}_2 \times \boldsymbol{M}_3$	3^3	$\leq 3^3$	∞		Uniform $[-0.3, 0.3]$
2	$\lvert \boldsymbol{a} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{a} \otimes \mathbf{1} \rvert$	d	≤ 3	1		Normal $\mathcal{N}(0, 0.15)$
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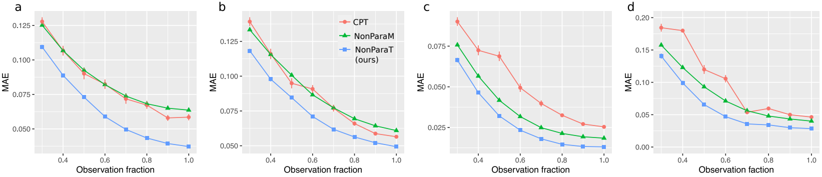

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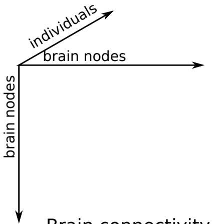
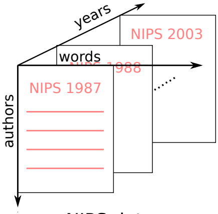
Data application



Brain connectivity



NIPS data

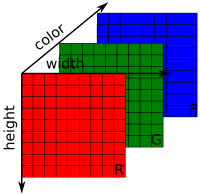
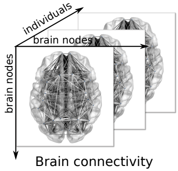


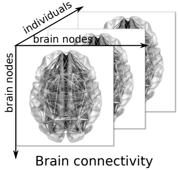
Image data

Data application: Brain connectivity



- The MRN-114 human brain connectivity data consists of 68 brain regions for 114 individuals along with their IQ scores (Wang et al., 2017).
- $\mathcal{Y} \in \{0, 1\}^{68\times 68\times 114}$.

Data application: Brain connectivity



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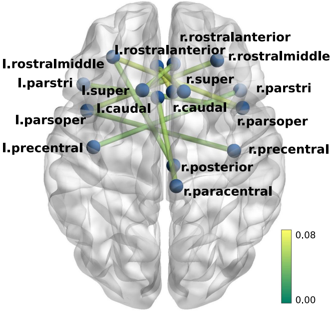
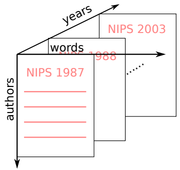
- We examine the estimated signal tensor $\hat{\Theta}$.
- Top 10 brain edges based on regression analysis show inter-hemisphere connections.



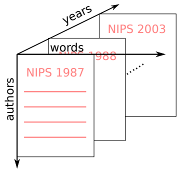
Data application: NIPS



NIPS data

- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003 (Globerson et al., 2007).
- Log transformation yields the dataset $\mathcal{Y} \in \mathbb{R}^{100\times200\times17}$.

Data application: NIPS



NIPS data

- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003 (Globerson et al., 2007).
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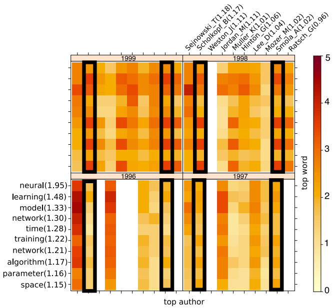
- We examine the estimated signal tensor $\hat{\Theta}$.
- Most frequent words is consistent with the active topics
- There are strong heterogeneity among word occurrences across authors and years.
- Similar word patterns (B. Schölkopf and A. Smola).

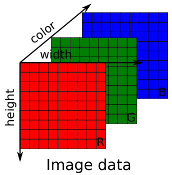


Data application: Brain connectivity + NIPS

MRN-114 brain connectivity dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.001)	**0.14**(0.001)	**0.12**(0.001)	**0.12**(0.001)	**0.11**(0.001)
Low-rank CPT	0.26(0.006)	0.23(0.006)	0.22(0.004)	0.21(0.006)	0.20(0.008)
NIPS word occurrence dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.002)	**0.16**(0.002)	**0.15**(0.001)	**0.14**(0.001)	**0.13**(0.001)
Low-rank CPT	0.22(0.004)	0.20(0.007)	0.19(0.007)	0.17(0.007)	0.17(0.007)
Naive imputation (Baseline)	0.32(.001)				

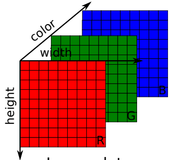
Table: MAE comparison in the brain data and NIPS data on cross-validation (5 repetitions 5 folds). Standard errors are reported in parenthesis.

Data application: Image



- The original data is from licensed google image file.
- $\mathcal{Y} \in [0,1]^{217\times 217\times 3}$.
- We sample 50% entries in the original image tensor and check completion performance.

Data application: Image



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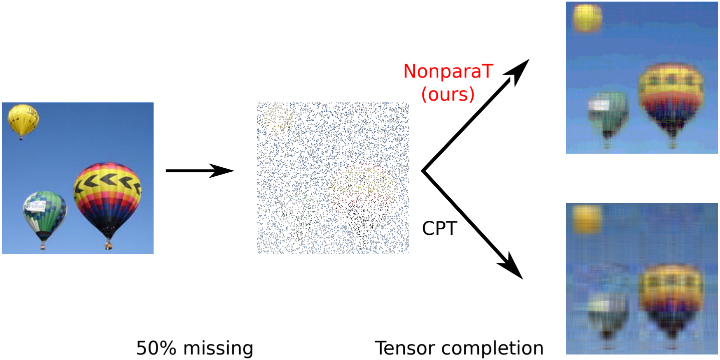

Summary

- We have developed a completion method that address both low- and high-ranknesss based on sign series representation.
- Estimation error rates and sample complexities are established.
- Our approach has good interpretation and prediction performance in both simulations and data applications.

Summary

- We have developed a completion method that address both low- and high-ranknesss based on sign series representation.
- Estimation error rates and sample complexities are established.
- Our approach has good interpretation and prediction performance in both simulations and data applications.
- Thank you!

Appendix

Algorithm 1 Nonparametric tensor completion

Input: Noisy and incomplete data tensor $\mathcal{Y}_\Omega$, rank r, resolution parameter H.

for $\pi \in \mathcal{H} = \{-1, \ldots, -\frac{1}{H}, 0, \frac{1}{H}, \ldots, 1\}$ **do**

  Random initialization of tensor factors $\boldsymbol{A}_k = [\boldsymbol{a}_1^{(k)}, \ldots, \boldsymbol{a}_r^{(k)}] \in \mathbb{R}^{d_k \times r}$ for all $k \in [K]$.

  while not convergence **do**

    for $k = 1, \ldots, K$ **do**

      Update $\boldsymbol{A}_k$ while holding others fixed:

      $\boldsymbol{A}_k \leftarrow \arg\min_{\boldsymbol{A}_k \in \mathbb{R}^{d_k \times r}} \sum_{\omega \in \Omega} |\mathcal{Y}(\omega) - \pi| F(\mathcal{Z}(\omega) \text{sgn}(\mathcal{Y}(\omega) - \pi))$,

      where $F(\cdot)$ is the large-margin loss, and $\mathcal{Z} = \sum_{s \in [r]} \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$ is a rank-r tensor.

    end for

  end while

  Return $\mathcal{Z}_\pi \leftarrow \sum_{s \in [r]} \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$.

end for

Output: Estimated signal tensor $\hat{\Theta} = \frac{1}{2H+1} \sum_{\pi \in \mathcal{H}} \text{sgn}(\mathcal{Z}_\pi)$.

References I

Cai, C., Li, G., Poor, H. V., and Chen, Y. (2019). Nonconvex low-rank tensor completion from noisy data. In *Advances in Neural Information Processing Systems*, pages 1863–1874.

Cai, T. and Zhou, W.-X. (2013). A max-norm constrained minimization approach to 1-bit matrix completion. *The Journal of Machine Learning Research*, 14(1):3619–3647.

Davenport, M. A., Plan, Y., Van Den Berg, E., and Wootters, M. (2014). 1-bit matrix completion. *Information and Inference: A Journal of the IMA*, 3(3):189–223.

De Lathauwer, L., De Moor, B., and Vandewalle, J. (2000). A multilinear singular value decomposition. *SIAM Journal on Matrix Analysis and Applications*, 21(4):1253–1278.

References II

Fan, J., Gong, W., and Zhu, Z. (2019). Generalized high-dimensional trace regression via nuclear norm regularization. *Journal of Econometrics*, 212(1):177–202.

Ganti, R. S., Balzano, L., and Willett, R. (2015). Matrix completion under monotonic single index models. In *Advances in Neural Information Processing Systems*, pages 1873–1881.

Globerson, A., Chechik, G., Pereira, F., and Tishby, N. (2007). Euclidean embedding of co-occurrence data. *Journal of Machine Learning Research*, 8:2265–2295.

Hitchcock, F. L. (1927). The expression of a tensor or a polyadic as a sum of products. *Journal of Mathematics and Physics*, 6(1-4):164–189.

References III

Jain, P. and Oh, S. (2014). Provable tensor factorization with missing data. In *Advances in Neural Information Processing Systems*, volume 27, pages 1431–1439.

Montanari, A. and Sun, N. (2018). Spectral algorithms for tensor completion. *Communications on Pure and Applied Mathematics*, 71(11):2381–2425.

Wang, L., Durante, D., Jung, R. E., and Dunson, D. B. (2017). Bayesian network–response regression. *Bioinformatics*, 33(12):1859–1866.