Beyond low-rankness: nonparametric models for tensor completion and regression

Nonparametric signal tensor estimation via sign series

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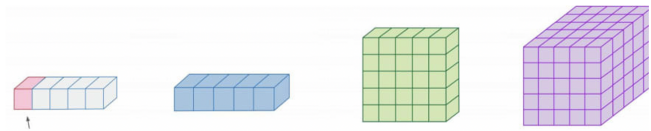
Preliminary Exam

What is tensor?

- Tensors are generalizations of vectors and matrices:



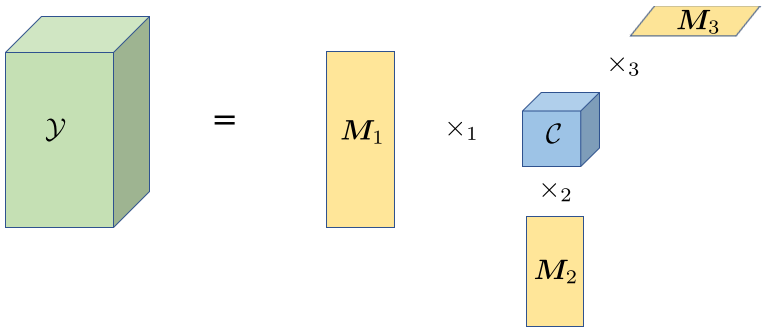
- We focus on tensors of order 3 or greater, also called higher-order tensors.
- Denote an order-K $(d_1, \cdots, d_K)$ dimensional tensor as $\mathcal{Y} = [\![y_\omega]\!] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ where $\omega \in [d_1] \times \cdots \times [d_K]$.

Tensor decomposition (Tucker decomposition)

Delete

- Tucker decomposition (De Lathauwer et al., 2000).
  - $\mathcal{Y} = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \boldsymbol{M}_2 \times_3 \boldsymbol{M}_3$.
  - Generalization of matrix SVD to higher orders: $\boldsymbol{Y} = \boldsymbol{U}\Sigma\boldsymbol{V}^T (= \Sigma \times_1 \boldsymbol{U} \times_2 \boldsymbol{V})$
  - Tucker rank of an order-3 tensor is defined as

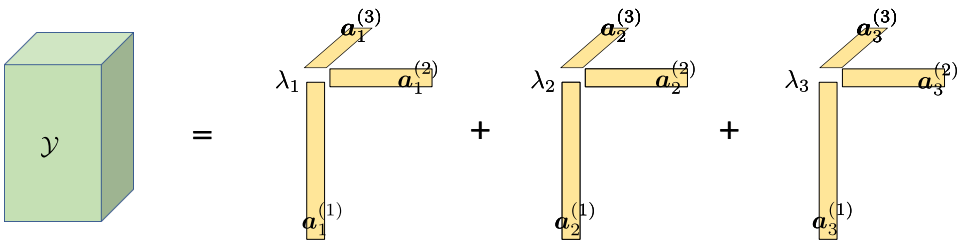
$$r(\mathcal{Y}) = (r_1, r_2, r_3).$$



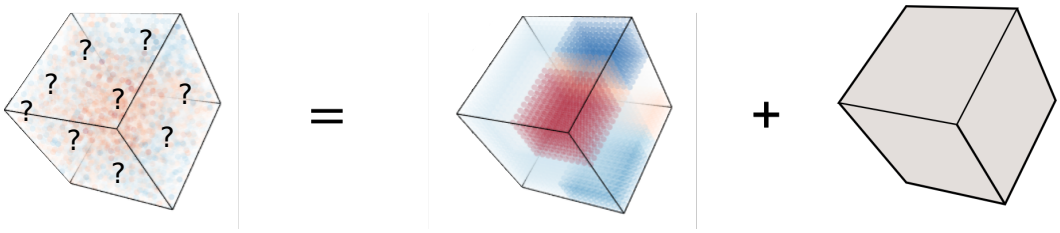
Tensor decomposition (CP decomposition)

Delete

- Canonical Polyadic (CP) decomposition (Hitchcock, 1927).
  - $\mathcal{Y} = \sum_{s=1}^{r} \lambda_s \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$.
  - Generalization of matrix SVD to higher orders: $\boldsymbol{Y} = \sum_{s=1}^{r} \lambda_s \boldsymbol{u}_s \otimes \boldsymbol{v}_s$
  - CP rank is defined as the minimal r for which the above equation holds.
  - Today, we use the tensor rank as CP rank.



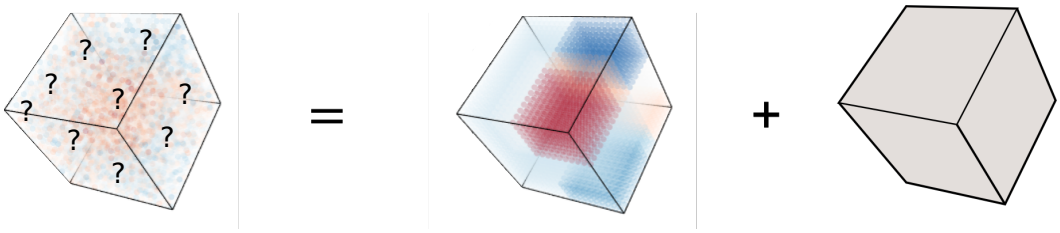
Main problems: the signal plus noise model



Rewrite pages 1-5:
- My research (my p2)
- What is tensor (your p2 + p5)
- Challenges (my p6. Structure + Data type)
- Talk outline (three bullets)

1. Previous work. Parametric tensor models for ordinal observations. (p5+1~2 slides).
2. Current work. Nonparametric models for tensor completion and regression.
3. Future work. Other approaches for nonparametric tensor modeling.

Main problems: the signal plus noise model



We focus on the two problems

1. Nonparametric tensor estimation: How to estimate the signal tensor Θ?
2. Complexity of tensor completion: How many observed tensor entries do we need?

Tensor based learning is challenging

High-rank matrix model

- **Matrix based model** (Cai and Zhou, 2013; Davenport et al., 2014; Ganti et al., 2015; Fan et al., 2019)

update citation as well

Tensor based learning is challenging

- **Matrix based model** (Cai and Zhou, 2013; Davenport et al., 2014; Ganti et al., 2015; Fan et al., 2019)
  - Applying matrix methods to higher-order tensor destroys structural information.
  - Tensor is different from matrix and more challenging.

Tensor based learning is challenging

- **Matrix based model** (Cai and Zhou, 2013; Davenport et al., 2014; Ganti et al., 2015; Fan et al., 2019)
  - Applying matrix methods to higher-order tensor destroys structural information.
  - ~~Tensor is different from matrix and more challenging.~~ (vage)

  tensors are more challenging because tensor rank may exceed dimension
- **Low rank tensor model** (Jain and Oh, 2014; Montanari and Sun, 2018; Cai et al., 2019)

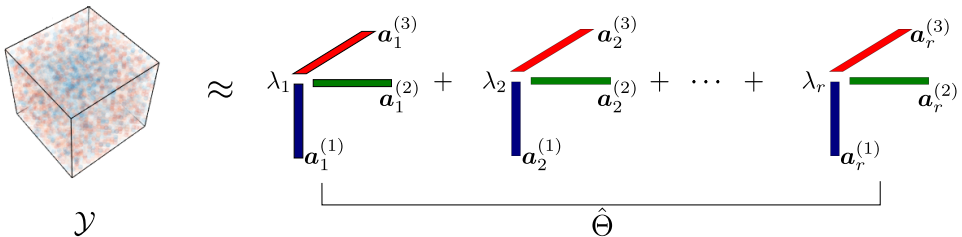
Tensor based learning is challenging

D

- **Matrix based model** (Cai and Zhou, 2013; Davenport et al., 2014; Ganti et al., 2015; Fan et al., 2019)
  - Applying matrix methods to higher-order tensor destroys structural information.
  - Tensor is different from matrix and more challenging.
- **Low rank tensor model** (Jain and Oh, 2014; Montanari and Sun, 2018; Cai et al., 2019)
  - Low rank models are inadequate in many cases.

Inadequacies of low rank models

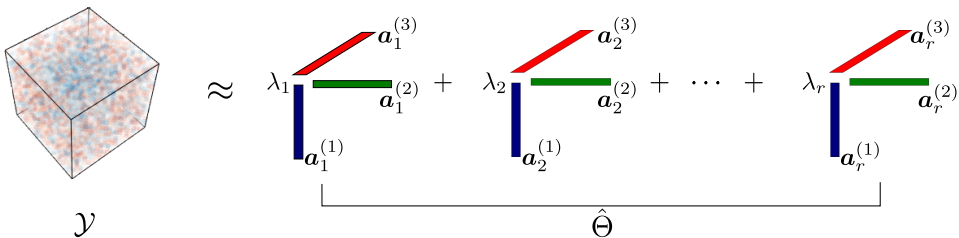
- Low rank models. (citation ....)



Inadequacies of low rank models

Delete.

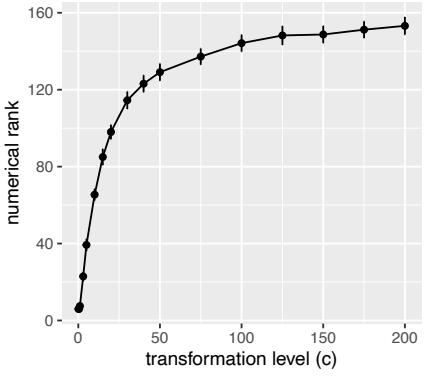
- Low rank models.



- There are two limitations of the model
  1. The sensitivity to order-preserving transformation.
  2. The inadequacy for special structures.

Inadequacies of low rank models

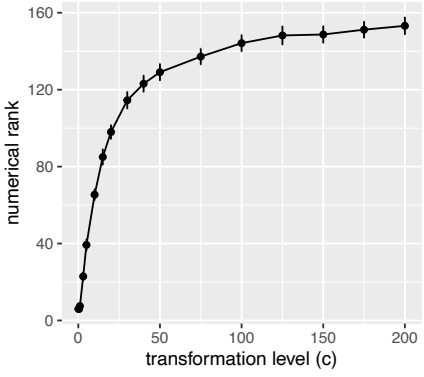
- The sensitivity to order-preserving transformation



$$\Theta = \frac{1}{1+\exp(-c\,(\mathcal{Z}))}, \quad \text{where}$$
$$\mathcal{Z} = \boldsymbol{a}^{\otimes 3} + \boldsymbol{b}^{\otimes 3} + \boldsymbol{c}^{\otimes 3}.$$

Inadequacies of low rank models

- The sensitivity to order-preserving transformation



$$\Theta = \frac{1}{1+\exp(-c(\mathcal{Z}))}, \quad \text{where}$$
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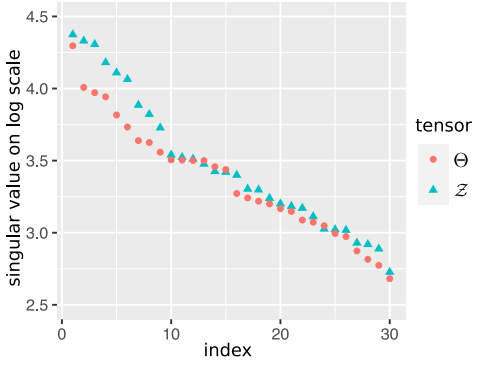
- The inadequacy for special structures.



$$\Theta = \log(1+\mathcal{Z}), \quad \text{where}$$
$$\mathcal{Z}(i,j,k) = \frac{1}{d}\max(i,j,k).$$

Motivating toy example in noiseless case

Delete.

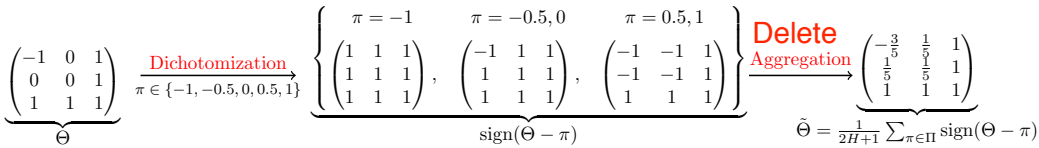
$$\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\Theta} \xrightarrow[\pi \in \{-1, -0.5, 0, 0.5, 1\}]{\text{Dichotomization}} \underbrace{\left\{ \begin{array}{ccc} \pi = -1 & \pi = -0.5, 0 & \pi = 0.5, 1 \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \right\}}_{\text{sign}(\Theta - \pi)} \xrightarrow{\text{Aggregation}} \underbrace{\begin{pmatrix} -\frac{3}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{1}{5} & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\tilde{\Theta} = \frac{1}{2H+1} \sum_{\pi \in \Pi} \text{sign}(\Theta - \pi)}$$

~~Motivating toy example in noiseless case~~

Why sign matters?

* For a bounded tensor Theta in [-1,1]^{d1…dK}

Theta ~ {1\over H}\sum_{\pi \in \tH} sign(Theta-pi), where H = {-1,…-1/H, 0, 1/H, … 1}



Sign (*not signed*) tensors are invariant to order-preserving transformation.

- Signed tensors are much simpler in the sense of rank.
- With a series of sign tensors, we successfully preserve all information in the original signals.

More flexible signal tensors allowed by using sign tensor series representation.

- In noise case, we estimate $\mathrm{sgn}(\Theta - \pi)$ from the tensor data $\mathrm{sgn}(\mathcal{Y} - \pi)$.

noisy

Sign rank

* Key idea: we use a local notion of low-rankness to allow a richer family of signal tensors
  - Two tensors are sign equivalent denoted as $\Theta \simeq \Theta'$ if $\mathrm{sgn}(\Theta) = \mathrm{sgn}(\Theta')$.
  - Sign rank is defined as

$$\mathrm{srank}(\Theta) = \min\{\mathrm{rank}(\Theta') \colon \Theta' \simeq \Theta, \Theta' \in \mathbb{R}^{d_1 \times \cdots \times d_K}\}.$$

Sign rank

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ex) $\Theta =$  , $\mathrm{sgn}(\Theta) =$  $\Longrightarrow \mathrm{rank}(\Theta) = d, \mathrm{srank}(\Theta) = 2.$

Sign rank

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- Sign rank is defined as

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remove (ex)

highlight d and 2

ex) $\Theta =$  , $\mathrm{sgn}(\Theta) =$  $\implies \mathrm{rank}(\Theta) = d, \mathrm{srank}(\Theta) = 2.$

- ~~More generally,~~ for any strictly monotonic function $g \colon \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$,

$$\mathrm{srank}(\Theta) \leq \mathrm{rank}(g(\Theta)).$$

Sign representable tensors

Sign representable tensors

A tensor Θ is called *r*-sign representable if the tensor $(\Theta - \pi)$ has sign rank bounded by r for all $\pi \in [-1, 1]$.

- Most existing structure tensors belong to sign representable family:
  - Low-rank CP tensors, Tucker tensors, stochastic block models.
  - High-rank tensors from GLM, single index models,
  - Tensors with repeating patterns, e.g. $\Theta(i_1, \ldots, i_K) = \log(1 + \max(i_1, \ldots, i_K))$ is 2-sign representable.

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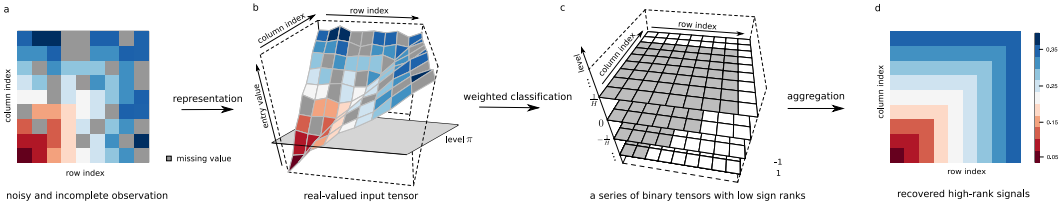
  - Tensors with repeating patterns, e.g. $\Theta(i_1, \ldots, i_K) = \log(1 + \max(i_1, \ldots, i_K))$ is 2-sign representable.
- Instead of the classical low rank assumption, we propose the sign representable tensor family

$$\Theta \in \mathscr{P}_{\text{sgn}}(r) := \{\Theta \colon \text{srank}(\Theta - \pi) \leq r \text{ for all } \pi \in [-1, 1]\}.$$

~~Our new approach~~ Our solution: sign signal helps!

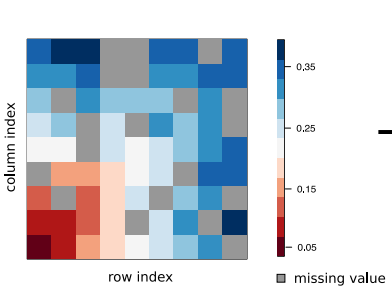
general principle: slide title should be informative



your earlier clock layout in 992 is better.
(current figure too small to see).

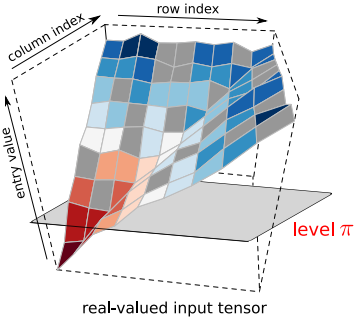
two slides:
first figure shows a and d (with a question mark on top of question mark)
second shows a-d (four pictures in a clock).

Step 1: representation



noisy and incomplete observation

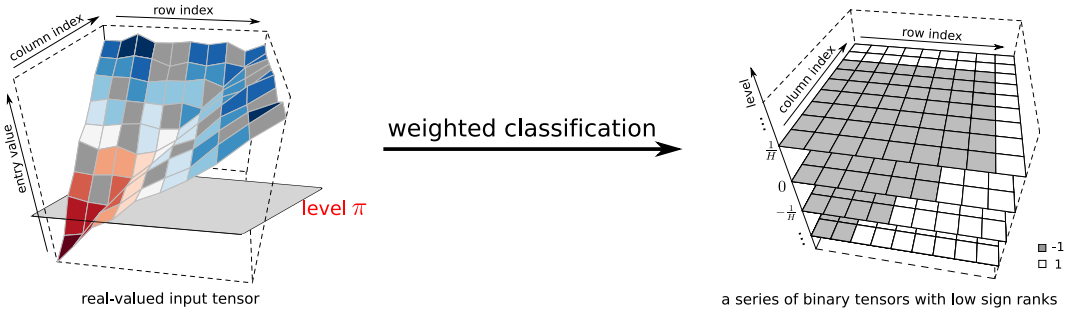
representation



- We observe a noisy incomplete tensor $\mathcal{Y}_\Omega \in [-1,1]^{d_1 \times \cdots \times d_K}$ with observed index set $\Omega \subset [d_1] \times \cdots \times [d_K]$.
- We dichotomize the data into a series of sign tensors:

$$\{\mathrm{sgn}(\mathcal{Y}_\Omega - \pi)\}_{\pi \in \mathcal{H}}, \qquad \text{where } \mathcal{H} = \left\{-1, \ldots, -\frac{1}{H}, 0, \frac{1}{H}, \ldots, 1\right\}.$$

Step 2: weighted classification



- We estimate $\text{sgn}(\Theta - \pi)$ through $\text{sgn}(\mathcal{Y}_\Omega - \pi)$ via weighted classification.
- Objective function of weighted classification is

$$L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi) = \frac{1}{|\Omega|} \sum_{\pi \in \Omega} \underbrace{|\mathcal{Y}(\omega) - \pi|}_{\text{weight}} \times \underbrace{|\text{sgn}(\mathcal{Z}(\omega)) - \text{sgn}(\mathcal{Y}(\omega) - \pi)|}_{\text{classification loss}}$$

Do you mean weight IY-\pil?

- Magnitude $|\Theta(\omega) - \pi|$ plays important role in estimation.

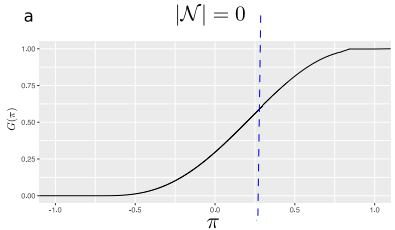
Main assumption

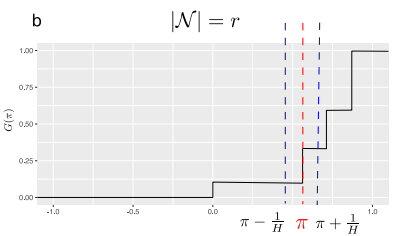
S2

Combine S2+S2'. No need to explain much on N.

- We quantify difficulty of the problem using CDF $G(\pi) = \mathbb{P}_{\omega \in \Pi}[\Theta(\omega) \leq \pi]$.
- We define classification hard region $\mathcal{N}$ at which point mass concentrates,

$$\mathcal{N} = \left\{ \pi : \frac{G(\pi + \Delta s) - G(\pi - \Delta s)}{\Delta s} \geq C \right\}, \text{ where } \Delta s = 1 / \prod_{k}^{K} d_k.$$





α smoothness

For fixed $\pi \in \mathcal{N}^c$, we call Θ is α smooth if there exist $\alpha = \alpha(\pi) > 0, c = c(\pi) > 0$, such that

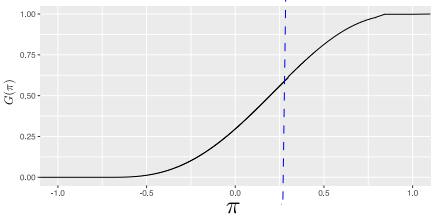
$$\sup_{\Delta \le t \le \rho(\pi, \mathcal{N})} \frac{G(\pi + t) - G(\pi - t)}{t^\alpha} \le c,$$

where $\rho(\pi, \mathcal{N}) = \min_{\pi' \in \mathcal{N}} |\pi - \pi'| + \Delta$. If α and c are global constants for all π's, we call Θ is α-globally smooth.

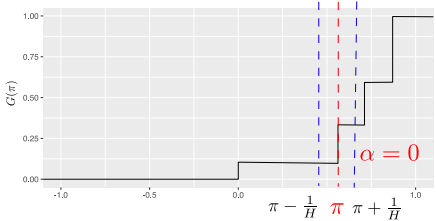
* Partition [-1,1] = N+N^c, where N^c consists of levels whose pseudo density is uniformly bounded, and N otherwise.
* G is globally alpha-smooth in that for all pi \in N^c,
....
for two constants alpha, c > 0, where rho = ...

Keep only global.
no need to say local.

a $|\mathcal{N}| = 0$ $\alpha = 1$

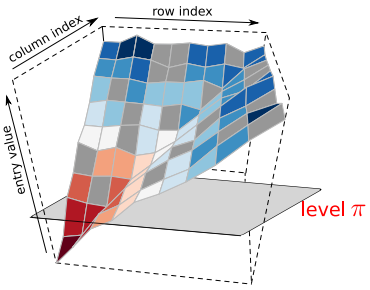


b $|\mathcal{N}| = r$ $\alpha = \infty$

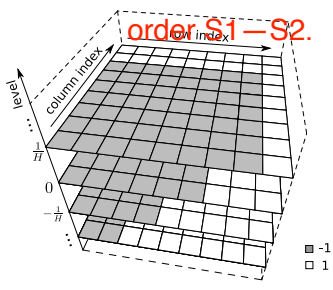


Step 2: weighed classification

S1



weighted classification



highlight

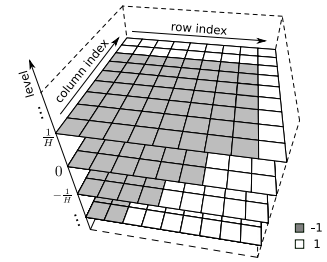
- If $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ is α smooth ($\alpha > 0$), we have a unique optimizer such that

$$\text{sgn}(\Theta - \pi) = \underset{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \le r}{\arg\min} \ \mathbb{E}_{\mathcal{Y}_\Omega} L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$

- So we obtain a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi \in \mathcal{H}}$ as

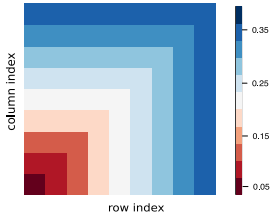
$$\hat{\mathcal{Z}}_\pi = \underset{\mathcal{Z}:\ \text{rank}(\mathcal{Z}) \le r}{\arg\min} \ L(\mathcal{Z}, \mathcal{Y}_\Omega - \pi).$$

Step 3: aggregation



a series of binary tensors with low sign ranks

aggregation



recovered high-rank signals

- From a series of optimizers $\{\hat{\mathcal{Z}}_\pi\}_{\pi\in\mathcal{H}}$ in the weighted classification, we obtain the tensor estimate

$$\hat{\Theta} = \frac{1}{2H+1}\sum_{\pi\in\mathcal{H}} \mathrm{sgn}\hat{\mathcal{Z}}_\pi.$$

Sign tensor estimation error

For two tensor Θ_1, Θ_2, define $\mathrm{MAE}(\Theta_1, \Theta_2) = \mathbb{E}_{\omega \in \Pi} |\Theta_1(\omega) - \Theta_2(\omega)|$.

Sign tensor estimation for fixed π (L. and Wang, 2021)

Suppose $\Theta \in \mathscr{P}_{\mathrm{sgn}}(r)$ is α smooth for fixed π. Denote $d_{\max} = \max_{k \in [K]} d_k$. Then, with very high probability over $\mathcal{Y}_\Omega$,

, and d1=…=dk=d

add footnote on \lesssim (footnote: log term suppressed)

$$\mathrm{MAE}(\mathrm{sgn}\hat{\mathscr{Z}}_\pi, \mathrm{sgn}(\Theta - \pi)) \lesssim \left(\frac{d_{\max} r \log |\Omega|}{|\Omega|} \right)^{\frac{\alpha}{\alpha+2}}.$$

- Sign estimation error shows a polynomial decay with the number of observed entries.

principle: always present the simplest case in talk.

Tensor estimation error

Tensor estimation error (L. and Wang 2021)

Suppose $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ is α-globally smooth. Then, with very high probability over $\mathcal{Y}_\Omega$,

with bounded INI

simplify statements

$$\text{MAE}(\hat{\Theta}, \Theta) \lesssim \left(\frac{d_{\max} r \log |\Omega| \log H}{|\Omega|} \right)^{\frac{\alpha}{\alpha+2}} + \frac{1 + |\mathcal{N}|}{H} + \frac{d_{\max} r \log |\Omega| \log H}{|\Omega|}.$$

In particular, setting $H \asymp \left(\frac{(1+|\mathcal{N}|)|\Omega|}{d_{\max} r \log |\Omega|} \right)^{1/2}$ yields the tightest error bound.

add Bracket to each of the term:
errors inherited from sign estimation
Bias
Variance

* see paper for general case that allows unbounded IN^cI and sub-Gaussian noise.

Tensor estimation error

Delete. Add a table for implications in common models.

Tensor estimation error (L. and Wang, 2021)

Model | Quiz | Rate (power of d) | Previous results

Suppose $\Theta \in \mathscr{P}_{\text{sgn}}(r)$ is α-globally smooth. Then, with very high probability over $\mathcal{Y}_\Omega$,

$$\text{MAE}(\hat{\Theta}, \Theta) \lesssim \left(\frac{d_{\max} r \log |\Omega| \log H}{|\Omega|} \right)^{\frac{\alpha}{\alpha+2}} + \frac{1 + |\mathcal{N}|}{H} + \frac{d_{\max} r \log |\Omega| \log H}{|\Omega|}.$$

In particular, setting $H \asymp \left(\frac{(1+|\mathcal{N}|)|\Omega|}{d_{\max} r \log |\Omega|} \right)^{1/2}$ yields the tightest error bound.

- For full observation case with equal dimension $d_1 = \cdots = d_K = d$ and bounded $|\mathcal{N}| \leq C$,

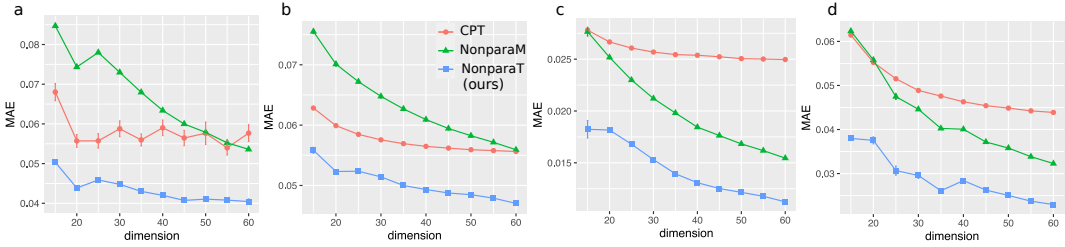
$$\text{MAE}(\hat{\Theta}, \Theta) \lesssim \tilde{\mathcal{O}} \left(d^{-(K-1)\min\left(\frac{\alpha}{\alpha+2}, \frac{1}{2}\right)} \right).$$

- Tensor estimation is generally no better than sign tensor estimation.
- Sample complexity: $\text{MAE}(\hat{\Theta}, \Theta) \to 0$, as $\frac{|\Omega|}{d_{\max} r \log^2 |\Omega|} \to \infty$.

Simulations for estimation error vs tensor dimension

Delete

Simulation	Signal Tensor Θ	Rank	Sign Rank	α	$\lvert\mathcal{N}\rvert$	CDF of Tensor Entries	Noise
1	$\mathcal{C} \times \boldsymbol{M}_1 \times \boldsymbol{M}_2 \times \boldsymbol{M}_3$	3^3	$\leq 3^3$	∞	$\leq 3^3$		Uniform $[-0.3, 0.3]$
2	$\lvert \boldsymbol{a} \otimes \boldsymbol{1} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{a} \otimes \boldsymbol{1} \rvert$	d	≤ 3	1	0		Normal $\mathcal{N}(0, 0.15)$
3	$\log(0.5 + Z_{\max})$	$\geq d$	2	∞	d		Uniform $[-0.1, 0.1]$
4	$2.5 - \exp(Z_{\min}^{1/3})$	$\geq d$	2	∞	d		Normal $\mathcal{N}(0, 0.15)$

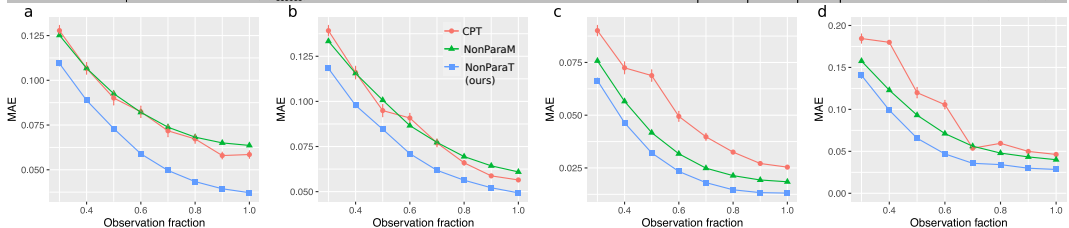


- **NonPraT**: Our nonparametric tensor method, **CPT**: low rank tensor CP decomposition, **NonPraraM**: the matrix version of our method.
- Our method (NonparaT) achives the best performance.

Simulations for estimation error vs the observation fraction

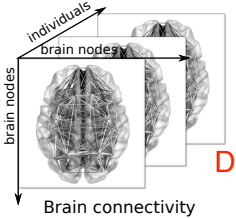
Delete

Simulation	Signal Tensor Θ	Rank	Sign Rank	α	$\lvert\mathcal{N}\rvert$	CDF of Tensor Entries	Noise
1	$\mathcal{C} \times \boldsymbol{M}_1 \times \boldsymbol{M}_2 \times \boldsymbol{M}_3$	3^3	$\leq 3^3$	∞	$\leq 3^3$		Uniform $[-0.3, 0.3]$
2	$\lvert \boldsymbol{a} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{a} \otimes \mathbf{1} \rvert$	d	≤ 3	1	0		Normal $\mathcal{N}(0, 0.15)$
3	$\log(0.5 + Z_{\max})$	$\geq d$	2	∞	d		Uniform $[-0.1, 0.1]$
4	$2.5 - \exp(Z_{\min}^{1/3})$	$\geq d$	2	∞	d		Normal $\mathcal{N}(0, 0.15)$



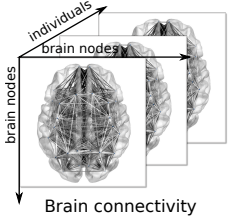
- Our method (NonparaT) achives the best performance in completion.

Data application: Brain connectivity



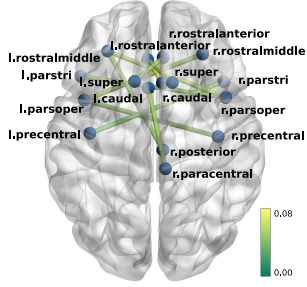
- The MRN-114 human brain connectivity data consists of 68 brain regions for 114 individuals along with their IQ scores (Wang et al., 2017).
- Data tensor $\mathcal{Y} \in \{0,1\}^{68\times 68\times 114}$.

Data application: Brain connectivity

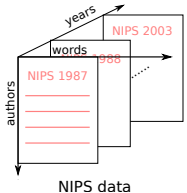


- The MRN-114 human brain connectivity data consists of 68 brain regions for 114 individuals along with their IQ scores (Wang et al., 2017).
- $\mathcal{Y} \in \{0,1\}^{68\times68\times114}$.

- We examine the estimated signal tensor $\hat{\Theta}$.
- Top 10 brain edges based on regression analysis show inter-hemisphere connections.

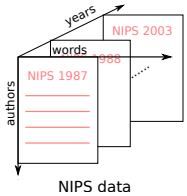


Data application: NIPS



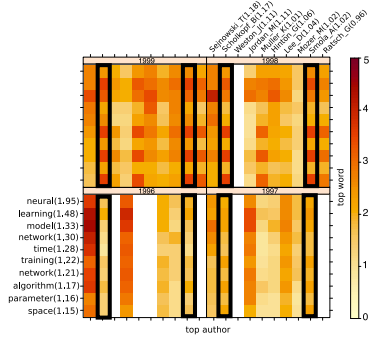
- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003 (Globerson et al., 2007).
- Log transformation yields the dataset $\mathcal{Y} \in \mathbb{R}^{100 \times 200 \times 17}$.

Data application: NIPS



- The NIPS dataset consists of word occurrence counts in papers published from 1987 to 2003 (Globerson et al., 2007).
- Log transformation yields the dataset $\mathcal{Y} \in \mathbb{R}^{100\times200\times17}$.

- We examine the estimated signal tensor $\hat{\Theta}$.
- Most frequent words are consistent with the active topics
- There are strong heterogeneity among word occurrences across authors and years.
- Similar word patterns (B. Schölkopf and A. Smola).



Data application: Brain connectivity + NIPS

MRN-114 brain connectivity dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.001)	**0.14**(0.001)	**0.12**(0.001)	**0.12**(0.001)	**0.11**(0.001)
Low-rank CPT	0.26(0.006)	0.23(0.006)	0.22(0.004)	0.21(0.006)	0.20(0.008)
NIPS word occurrence dataset					
Method	$r = 3$	$r = 6$	$r = 9$	$r = 12$	$r = 15$
NonparaT (Ours)	**0.18**(0.002)	**0.16**(0.002)	**0.15**(0.001)	**0.14**(0.001)	**0.13**(0.001)
Low-rank CPT	0.22(0.004)	0.20(0.007)	0.19(0.007)	0.17(0.007)	0.17(0.007)
Naive imputation (Baseline)	0.32(.001)				

5-folded cross-validation.

Table: MAE comparison in the brain data and NIPS data on ~~cross-validation (5 repetitions 5 folds).~~ ~~Standard errors are reported in parenthesis.~~

- Our method substantially outperforms the low-rank CP method for every configuration under consideration.

Nonparametric trace regression via sign series

Add transition slide:
Outline (three bullet points)
Current bullet (visible).
Other bullet in gray fonts (less visible)

L26-28. Write in terms of tensor.

- $\boldsymbol{X} \in \mathcal{X} \subset \mathbb{R}^{d_1 \times d_2}$ denotes the matrix predictor, $Y \in \mathbb{R}$ the scalar response. We consider the regression model,

$$Y = f(\boldsymbol{X}) + \epsilon,$$

where $f : \mathcal{X} \to \mathbb{R}$ is an unknown regression function of interest, and ϵ is mean-zero noise.

- Trace regression (Fan et al., 2019; Hamidi and Bayati, 2019) assumes that

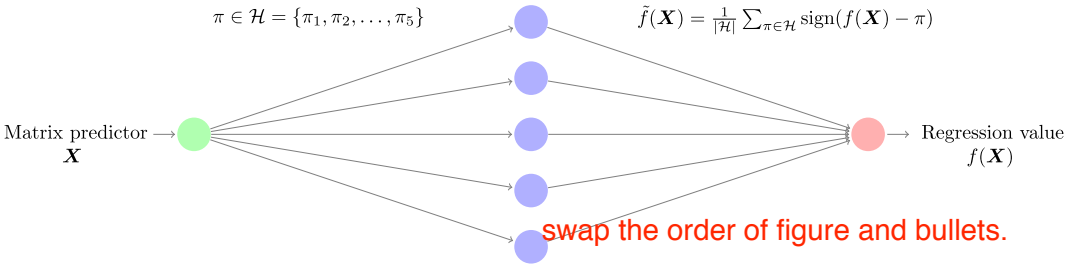
$$f(\boldsymbol{X}) = \langle \boldsymbol{B}, \boldsymbol{X} \rangle + b, \text{ for all } \boldsymbol{X} \in \mathcal{X},$$

where $\boldsymbol{B}$ is usually a low rank matrix.

Functional form of f(X)

- Low rank assumption on $\boldsymbol{B}$ is not adequate in many cases.

Nonparametric trace regression via sign series



swap the order of figure and bullets.

(model first, and then explain by figure.)

- We apply our framework for the regression problem.
- We assume that for given π, sign of regression function is represented by

$$\mathrm{sign}(f(\boldsymbol{X}) - \pi) = \mathrm{sign}\left(\langle \boldsymbol{B}_\pi, \boldsymbol{X} \rangle + b_\pi\right),$$

where $\boldsymbol{B}_\pi$ is a low rank matrix.

Add a slide after this slide:
two examples:
1. single index model
2. tensor completion.

Nonparametric trace regression via sign series

- Based on the sign rank assumption, we estimate the series of sign functions $\{\text{sign}(f-\pi)\}_{\pi\in\mathcal{H}}$ from the weighted classification,

change to L(phi; data) =

$$L\left(\phi; \{(\boldsymbol{X}_i, \text{sign}(Y_i-\pi))\}_{i=1}^n\right) = \frac{1}{n}\sum_{i=1}^n |Y_i-\pi| \times |\text{sign}(Y_i-\pi) - \text{sign}(\phi(\boldsymbol{X}_i))|.$$

where phi is

(add one sentence about learning reduction + meta/base algorithm + no need to reinvent the wheel)

- Our framework enjoys theoretical guarantees.
- We detailed the trace regression problem in Lee et al. (2021).

Our extend the broad nonparametric paradigm to many important matrix/tensor learning problems, including regression, completion, multi-task learning, and compressed sensing

Nonparametric probability tensor estimation from hypergraph

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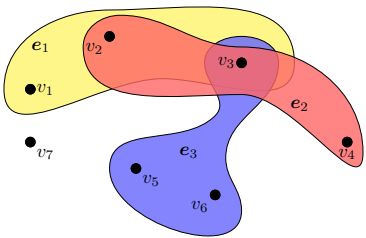

- Hypergraph considers higher-way interaction among nodes.
- An observed adjacency tensor $\mathcal{A} \in \{0,1\}^{d \times \cdots \times d}$ corresponding to a hypergraph is generated by

$$\mathcal{A}_\omega \sim \text{Bernoulli}(\Theta_\omega), \text{ for all } \omega \in \underbrace{[d] \times \cdots \times [d]}_{K}.$$

- The block model (Ahn et al., 2018; Wang and Zeng, 2019; Han et al., 2020) has

$$\Theta_\omega = \mathcal{Q}_{z(\omega_1), z(\omega_2), \ldots z(\omega_K)}, \text{ for all } \omega \in [d] \times \cdots \times [d],$$

where $\mathcal{Q} \in [0,1]^{m \times \cdots \times m}$ and $z \colon [d] \to [m]$ is a hidden partition.

- We will take nonparametric approach considering smooth function f such that

$$\Theta_\omega = f(\xi_{\omega_1}, \xi_{\omega_2}, \cdots, \xi_{\omega_K}), \text{ for all } \omega \in [d] \times \cdots \times [d],$$

where $\{\xi_i\}_{i=1}^d$ are i.i.d. random variables sampled from Unif$[0,1]$.

Thank you!

Appendix: Algorithm

Algorithm 1 Nonparametric tensor completion

Input: Noisy and incomplete data tensor $\mathcal{Y}_\Omega$, rank r, resolution parameter H.

for $\pi \in \mathcal{H} = \{-1, \ldots, -\frac{1}{H}, 0, \frac{1}{H}, \ldots, 1\}$ **do**

  Random initialization of tensor factors $\boldsymbol{A}_k = [\boldsymbol{a}_1^{(k)}, \ldots, \boldsymbol{a}_r^{(k)}] \in \mathbb{R}^{d_k \times r}$ for all $k \in [K]$.

  while not convergence **do**

    for $k = 1, \ldots, K$ **do**

      Update $\boldsymbol{A}_k$ while holding others fixed: $\boldsymbol{A}_k \leftarrow \arg\min_{\boldsymbol{A}_k \in \mathbb{R}^{d_k \times r}} \sum_{\omega \in \Omega} |\mathcal{Y}(\omega) - \pi| F(\mathcal{Z}(\omega) \mathrm{sgn}(\mathcal{Y}(\omega) - \pi))$, where $F(\cdot)$ is the large-margin loss, and $\mathcal{Z} = \sum_{s \in [r]} \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$ is a rank-r tensor.

    end for

  end while

  Return $\mathcal{Z}_\pi \leftarrow \sum_{s \in [r]} \boldsymbol{a}_s^{(1)} \otimes \cdots \otimes \boldsymbol{a}_s^{(K)}$.

end for

Output: Estimated signal tensor $\hat{\Theta} = \frac{1}{2H+1} \sum_{\pi \in \mathcal{H}} \mathrm{sgn}(\mathcal{Z}_\pi)$.

Theoretical guarantees for a large-margin loss classification

- we consider the estimation

$$\hat{\mathcal{Z}}_\pi = \underset{\text{rank}(\mathcal{Z})\le r}{\arg\min} \sum_{\omega\in\Omega} |\mathcal{Y}(\omega) - \pi| \times F(\mathcal{Z}(\omega)\text{sign}(\mathcal{Y}(\omega) - \pi) + \lambda\|\mathcal{Z}\|_F^2,$$

where $\lambda > 0$ is the penalty parameter and F is a large-margin loss satisfying the following assumption,

Assumption 1

(a) (Approximation error) For any given $\pi \in [-1, 1]$, there exists a sequence of tensors $\mathcal{Z}_\pi^{(n)} \in \mathscr{P}_{\text{sgn}}(r)$, such that $\text{Risk}_F(\mathcal{Z}_\pi^{(n)}) - \text{Risk}_F(\Theta - \pi) \le a_n$, for some sequence $a_n \to 0$ as $n \to \infty$. Furthermore, assume $\|\mathcal{Z}_\pi^{(n)}\|_F \le J$ for some constant $J > 0$.

(b) $F(z) = (1 - z)_+$ is hinge loss.

Theoretical guarantees for a large-margin loss classification

Estimation error for a large margin loss (L. and Wang 2021)

Denote $t_n = \frac{d_{\max} r K \log n}{n}$. Suppose the surrogate loss F satisfies Assumption 1 with $a_n \lesssim t_n^{(\alpha+1)/(\alpha+2)}$. Set $\lambda \asymp t_n^{(\alpha+1)/(\alpha+2)} + t_n/\rho(\pi, \mathcal{N})$. Then, with high probability, we have:

1. (Sign tensor estimation). For all $\pi \in [-1, 1]$ except for a finite number of levels,

$$\text{MAE}(\text{sign}(\hat{\mathcal{Z}}_\pi, \text{sign}(\Theta - \pi)) \lesssim t_n^{\frac{\alpha}{2+\alpha}} + \frac{1}{\rho^2(\pi, \mathcal{N})} t_n.$$

2. (Tensor estimation).

$$\text{MAE}(\hat{\Theta}, \Theta) \lesssim (t_n \log H)^{\frac{\alpha}{2+\alpha}} + \frac{1 + |\mathcal{N}|}{H} + t_n H \log H.$$

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