Necessary condition for matrix-valued kernels

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Theorem 0.1 (Necessary condition)**.** *Suppose* $\boldsymbol{K}\colon \mathbb{R}^{d'\times d}\times\mathbb{R}^{d'\times d}\mapsto\mathbb{R}^{d\times d}$ *is a function that takes as input a pair of matrices and produces a matrix. Let* $\{\boldsymbol{X}_i\in\mathbb{R}^{d'\times d}\colon i\in[n]\}$ *denote a set of input matrices, and let* $\mathcal{K}$ *denote an order-4* (d,d,n,n)*-dimensional array,*

$$\mathcal{K}=[\![\mathcal{K}(i,i',p,p')]\!],\quad \textit{where } \mathcal{K}(i,i',p,p') \textit{ is the } (p,p')\textit{-th entry of the matrix } \boldsymbol{K}(\boldsymbol{X}_i,\boldsymbol{X}_{i'}).$$

Then, the factorization $\boldsymbol{K}(\boldsymbol{X},\boldsymbol{X}')=\boldsymbol{h}(\boldsymbol{X})^T\boldsymbol{h}(\boldsymbol{X}')$ *exists for some mapping* $\boldsymbol{h}$*, only if both of the following conditions hold:*

(1) For every index $i\in[n]$*, the slide* $\mathcal{K}(i,i,\cdot,\cdot)\in\mathbb{R}^{d\times d}$ *is a symmetric, positive definite matrix.*

(2) For every index $p\in[d]$*, the slide* $\mathcal{K}(\cdot,\cdot,p,p)\in\mathbb{R}^{n\times n}$ *is a symmetric, positive definite matrix.*