Smooth tensor estimation and completion

Miaoyan Wang, April 26, 2020

1 Model

Let $\mathcal{Y} = [\![y_{i_1,\ldots,i_K}]\!] \in \{0,1\}^{d\times\cdots\times d}$ be an order-K, $(d,\ldots,d)$-dimensional binary tensor. Assume that the entries of $\mathcal{Y}$ are independent Bernoulli random variables with success probabilities

$$\mathbb{P}(y_{i_1,\ldots,i_K}=1) = f(\xi^{(1)}_{i_1},\ldots,\xi^{(K)}_{i_K}), \quad \text{for all } (i_1,\ldots,i_K)\in[d]\times\cdots\times[d],$$

where $f\colon [0,1]^K \mapsto [0,1]$ is an unknown, multivariate function belonging to a function class $f \in \mathcal{F}_\alpha(M)$, and $\boldsymbol{\xi}^{(k)} = (\xi^{(k)}_1,\ldots,\xi^{(k)}_d) \in [0,1]^d$ are unknown, independent random vectors following distributions $\mathbb{P}_{\boldsymbol{\xi}^{(k)}}$ for all $k\in[K]$. Furthermore, we define the function class

$$\mathcal{F}_\alpha(M) = \{f\colon \mathrm{Im}(f)\in[0,1] \text{ and } \|f\|_{\mathcal{H}_\alpha}\leq M\},$$

where $\alpha>0$ is the smoothness parameter and $M>0$ is a constant upper bound for the Hölder norm $\|\cdot\|_{\mathcal{H}_\alpha}$ in the class. Specifically,

$$\|f\|_{\mathcal{H}_\alpha} \stackrel{\text{def}}{=} \max_{|\boldsymbol{i}|\leq\lfloor\alpha\rfloor}\sup_{\boldsymbol{x}\in\mathcal{D}}|\nabla_{\boldsymbol{i}}f(\boldsymbol{x})| + \max_{|\boldsymbol{i}|=\lfloor\alpha\rfloor}\sup_{\boldsymbol{x}\neq\boldsymbol{x}'\in\mathcal{D}}\frac{\nabla_{\boldsymbol{i}}|f(\boldsymbol{x})-\nabla_{\boldsymbol{i}}f(\boldsymbol{x}')|}{\|\boldsymbol{x}-\boldsymbol{x}'\|_1^{\alpha-\lfloor\alpha\rfloor}},$$

where we have used the short-hand notion

$$\nabla_{\boldsymbol{i}}f(\boldsymbol{x}) = \frac{\partial^{i_1+\cdots+i_K}}{\partial x_1^{i_1}\cdots\partial x_K^{i_K}}f(x_1,\ldots,x_k),$$

for multi-indices $\boldsymbol{i}=(i_1,\ldots,i_K)$ with $|\boldsymbol{i}| = i_1+\cdots+i_K$, and $\boldsymbol{x}=(x_1,\ldots,x_K)$ in the domain.

Our goal is to estimate the function values $f(\cdot)$ evaluated at realizations $\{(\xi^{(1)}_{i_1},\ldots,\xi^{(K)}_{i_K})\colon (i_1,\ldots,i_K)\in[d]\times\cdots\times[d]\}$.