Identifiability for sparse Tucker-1 models

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Theorem 0.1 (Identifiability). *Let $\Omega_k \in \mathbb{R}^{d\times d}$ denote the precision matrix for tissue $k \in [K]$. Assume that $\{\Omega_k\}$ admits the rank-r Tucker-1 decomposition model with $r \leq \min(K, d^2)$,*

$$\Omega_k = \Theta_0 + \sum_{l=1}^{r} u_{lk}\Theta_l, \quad \text{for all} \quad k = 1, \ldots, K. \tag{1}$$

Define the vector $\mathbf{u}_l = (u_{l1}, \ldots, u_{lK})^T \in \mathbb{R}^K$ and Supp$(\mathbf{u}_l) = \{i \in [K] \colon \mu_{lk} \neq 0\}$. Suppose the vectors $\{\mathbf{u}_l\}$ and Θ_0 satisfy the following conditions,

1. *The vectors $\{\mathbf{u}_l\}$ have mutually non-overlapping supports; i.e, Supp$(\mathbf{u}_l) \cap$ Supp$(\mathbf{u}_l') = 0$, for all $l \neq l' \in [r]$.*
2. *The vectors $\{\mathbf{u}_l\}$ are normalized such that $\mathbf{u}_l^T\mathbf{u}_l = 1$ for all $l \in [r]$.*
3. *$\frac{1}{n}\sum_k \Omega_k = \Theta_0$.*

Then the decomposition (2) is unique. Specifically, if $\{\Omega_k\}$ admits another rank-r decomposition that satisfying the above conditions 1-3,

$$\Omega_k = \Theta_0' + \sum_{l=1}^{r} u_{lk}'\Theta_l', \quad \text{for all} \quad k = 1, \ldots, K, \tag{2}$$

then we must have $\Theta_l = \Theta_l'$ for $l = 0, 1, \ldots, r$, and $\mathbf{u}_l = \pm\mathbf{u}'_{\pi(l)}$ up to sign and permutation $\pi : [r] \to [r]$.