Compatibility of smooth tensor and low-rank tensor

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Let $\boldsymbol{S}^r = \{\boldsymbol{x} \in \mathbb{R}^r \colon \|\boldsymbol{x}\|_2 \leq 1\}$ be an r-dimensional unit ball. We also use $\boldsymbol{x} = (\boldsymbol{x}(1), \ldots, \boldsymbol{x}(r))^T$ to denote the vector in $\mathbb{R}^r$, where $\boldsymbol{x}(i) \in \mathbb{R}$ denotes i-th element in the vector. Consider the following generative process:

Step 1. Fix an m-variate function

$$\begin{aligned} f \colon \boldsymbol{S}^r \times \cdots \times \boldsymbol{S}^r &\to [0,1], \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) &\mapsto f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) = \sum_{i=1}^r \boldsymbol{x}_1(i) \cdots \boldsymbol{x}_m(i). \end{aligned}$$

By the multilinearlity of f and boundedness of $\boldsymbol{S}^r$, f is a 1-Lipschitz function.

Step 2. Draw either random or fixed samples $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ from $\boldsymbol{S}^r$.

Step 3. Define the signal tensor

$$\Theta(i_1, \ldots, i_m) = f(\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}), \quad \text{for all } (i_1, \ldots, i_m) \in [d]^m.$$

By construction, Θ is a rank-r tensor.

Theorem 0.1 (Block approximation to low-rank tensors)**.** *Let $\Theta \in [0,1]^m$ be an order-m rank-r tensor generated from steps 1-3. Then, for every $K \in \mathbb{N}_+$, there exists a block-K tensor $\bar{\Theta}$ such that*

$$\|\Theta - \bar{\Theta}\|_F^2 \lesssim \frac{d^m}{K^{2/r}}.$$

Proof. Because $\boldsymbol{S}^r$ is a compact set, $\boldsymbol{S}^r$ can be covered by K Euclidean balls with radius $K^{-1/r}$. Let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K$ be the center of these balls. Without loss of generality, assume $\boldsymbol{y}_k \in \boldsymbol{S}^r$ for all $k \in [K]$. Then, by definition of covering sets, for every $i \in [d]$, there exists $k = k(i) \in \{1, \ldots, K\}$ such that

$$\|\boldsymbol{x}_i - \boldsymbol{y}_{k(i)}\|_2 \lesssim \frac{1}{K^{1/r}}.$$

By the smoothness of f, we have

$$|f(\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}) - f(\boldsymbol{y}_{k(i_1)}, \ldots, \boldsymbol{y}_{k(i_m)})|^2 \lesssim \frac{1}{K^{2/r}}. \tag{1}$$

Notice that there are in total K distinct points $\boldsymbol{y}_{k(i)}$. Therefore, the tensor $\bar{\Theta}$ defined by

$$\bar{\Theta}(i_1, \ldots, i_m) := f(\boldsymbol{y}_{k(i_1)}, \ldots, \boldsymbol{y}_{k(i_m)})$$

has at most K blocks on each model. Based on (), we have constructed a block-K tensor $\bar{\Theta}$ such that

$$\|\Theta - \bar{\Theta}\|_F^2 \leq \frac{d^m}{K^{2/r}}.$$

$\square$