Sub-Gaussianity of GLM tensor with bounded variance

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Assumption 1 (GLM tensor)**.** We call the tensor $\mathcal{Y} = [\![y_\omega]\!]$ is exponential family tensor with bounded variance, if the following two assumptions are met.

1. [GLM density] Conditional on canonical parameter tensor $\Theta = [\![\theta_\omega]\!]$, the tensor entries y_ω's are independent of each other, and $y_\omega|\theta_\omega$ follows a generalized linear model (GLM) with density

$$p(y_\omega|\theta_\omega) = c(y_\omega, \phi) \exp\left(\frac{y_\omega\theta_\omega - b(\theta_\omega)}{\phi}\right),$$

where $b(\cdot)$ is a known function depending on the distribution family of y_ω, $\phi > 0$ is the dispersion parameter, and $c(\cdot)$ is a known normalizing function.

2. [Boundedness] The parameter tensor Θ is bounded, i.e, $\|\Theta\|_\infty \leq \alpha$ for some constant $\alpha > 0$.

Proposition 1 (sub-Gaussian residuals under bounded variance)**.** Let $\mathcal{Y}$ be a GLM data tensor, and $\mathcal{E} = \mathcal{Y} - b'(\Theta)$ be the residual tensor, where $b'(\cdot)$ denotes the first-order derivative. Under Assumption 1, the entries of $\mathcal{E}$ are independent sub-Gaussian entries with parameter (ϕU), where $U = \max_{|\theta|\leq\alpha} b''(\theta) < \infty$ and $\phi > 0$ is the dispersion parameter in the GLM density.

Proof. It is easy to see that the entries of $\mathcal{E} = [\![\varepsilon_\omega]\!]$ are independent conditional on $\Theta = [\![\theta_\omega]\!]$. Furthermore, we show that ε_ω is a sub-Gaussian random variable under the boundedness condition on θ_ω. For notational convinene, we drop the subscript ω, and simply write ε and θ. By the definition of sub-Gaussian random variable, it suffices to show

$$\mathbb{E}\left[\exp(t\varepsilon|\theta)\right] \leq \exp\left(\frac{\phi U t^2}{2}\right), \quad \text{for all } t \in \mathbb{R}.$$

By the definition of GLM density, we have

$$\begin{aligned}
\mathbb{E}[\exp(t\varepsilon|\theta)] &= \int c(y,\phi)\exp\left(\frac{y\theta - b(\theta)}{\phi}\right)\exp\left[t(y - b'(\theta))\right] dy \\
&= \int c(y,\phi)\exp\left(\frac{y(\theta+\phi t) - b(\theta+\phi t) + b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) dy \\
&= \exp\left(\frac{b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) \\
&\leq \exp\left(\frac{\phi U t^2}{2}\right),
\end{aligned}$$

where the last inequality follows from Taylor expansion and the definition of U. Therefore, ε is sub-Gaussian-(ϕU). $\square$