

Smooth tensor estimation with unknown permutations

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Abstract

We consider the problem of structured tensor denoising in the presence of unknown permutations. Such data problems arise commonly in recommendation system, neuroimaging, community detection, and multiway comparison applications. Here, we develop a general family of smooth tensor models up to arbitrary index permutations; the model incorporates the popular tensor block models and Lipschitz hypergraphon models as special cases. We show that a constrained least-squares estimator in the block-wise polynomial family achieves the minimax error bound. A phase transition phenomenon is revealed with respect to the smoothness threshold needed for optimal recovery. In particular, we find that a polynomial of degree up to $(m - 2)(m + 1)/2$ is sufficient for accurate recovery of order- m tensors, whereas higher degree exhibits no further benefits. This phenomenon reveals the intrinsic distinction for smooth tensor estimation problems with and without unknown permutations. Furthermore, we provide an efficient polynomial-time Borda count algorithm that provably achieves optimal rate under monotonicity assumptions. The efficacy of our procedure is demonstrated through both simulations and Chicago crime data analysis.

Keywords: Tensor estimation, latent permutation, diverging dimensionality, phase transition, statistical-computational efficiency.

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1 Introduction

Higher-order tensor datasets are rising ubiquitously in modern data science applications, for instance, recommendation systems [4], social networks [5], and genomics [32]. Tensor provides effective representation of data structure that classical vector- and matrix-based methods fail to capture. For example, the music recommendation system [2] records ratings of songs from users on various contexts. This three-way tensor of user \times song \times context allows us to investigate interactions of users and songs in a context-specific manner.

Tensor estimation problem cannot be solved without imposing structures. An appropriate reordering of tensor entries often provides effective representation of the hidden structure. In the music recommendation example, suppose that we have certain criteria available (such as, similarities of music genres, ages of users, and importance of contexts) to reorder the songs, users, and contexts. Then, the sorted tensor will exhibit smooth structure, because entries from similar groups tend to have similar values.

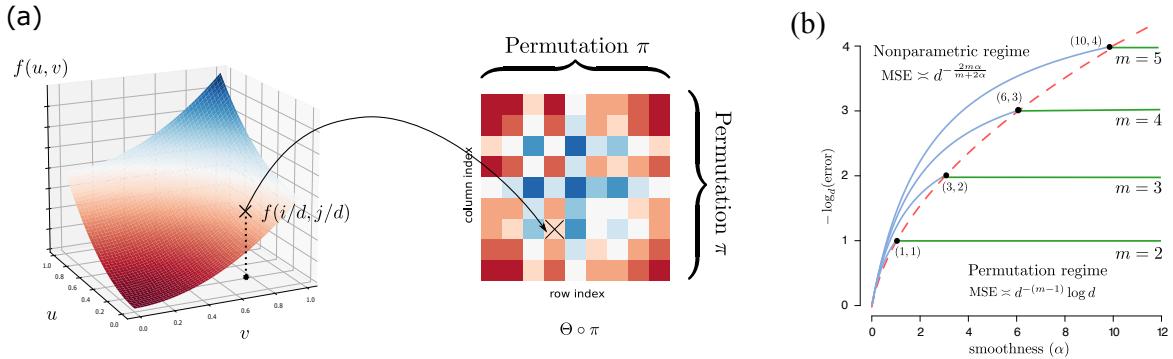


Figure 1: (a): Illustration of order- m d -dimensional permuted smooth tensor models with $m = 2$. (b): Phase transition of mean squared error (MSE) (on $-\log_d$ scale) as a function of smoothness α and tensor order m . Bold dots correspond to the critical smoothness level above which higher smoothness exhibits no further benefits to tensor estimation.

In this article, we develop a *permuted* smooth tensor model based on the aforementioned

motivation. We study a class of structured tensors, called *permuted smooth tensor model*, of the following form:

$$\mathcal{Y} = \Theta \circ \pi + \text{noise}, \quad \text{where } \Theta(i_1, \dots, i_m) = f\left(\frac{i_1}{d}, \dots, \frac{i_m}{d}\right), \quad (1)$$

where $\pi: [d] \rightarrow [d]$ is an *unknown* latent permutation, Θ is an *unknown* order- m d -dimensional signal tensor, and f is an *unknown* multivariate function with certain notion of smoothness, and $\Theta \circ \pi$ denotes the permuted tensor after reordering the indices along each of the m modes. Figure 1(a) shows an example of this generative model for the matrix case $m = 2$. Our primary goal is to estimate the permuted smooth signal tensor $\Theta \circ \pi$ from the noisy tensor observation \mathcal{Y} of arbitrary order m .

1.1 Our contributions

We develops a suite of statistical theory, efficient algorithms, and related applications for permuted smooth tensor models (1). Our contributions are summarized below.

	Pananjady and Samworth [26]	Balasubramanian [1]	Li et al. [22]	Ours (MLE)	Ours (Borda count)
Model structure	monotonic	Lipschitz	Lipschitz	α -smooth	α -smooth & monotonic
Fixed grid design	✓	✗	✗	✗	✓
Error rate for order- m tensor (e.g., when $(m, \alpha) = (3, 1)$)	d^{-1}	$d^{-\frac{2m}{m+2}}$	$d^{-\lfloor m/3 \rfloor}$	$d^{-\frac{2m\alpha}{m+2\alpha}} \vee d^{-(m-1)}$	$d^{-\frac{2m\alpha}{m+2\alpha}} \vee d^{-(m-1)}$
Minimax optimality	✓	✗	✗	✓	—*
Polynomial algorithm	✓	✗	✓	✗	✓

Table 1: Comparison of our results with previous work. For simplicity, we omit the log term in the rate. *The optimality is achieved under extra Lipschitz monotonicity conditions.

First, we develop a general permuted α -smooth tensor model of arbitrary smoothness level $\alpha > 0$. We establish the statistically optimal error rate and its dependence on model complexity. Specifically, we express the optimal rate as a function of tensor order m , tensor

dimension d , and the smoothness level α , given by

$$\text{Rate}(d) := d^{-\frac{2m\alpha}{m+2\alpha}} \vee d^{-(m-1)} \log d. \quad (2)$$

Table 1 summarizes the comparison of our work with previous results. Our framework substantially generalizes earlier works which focus on only matrices with $m = 2$ [16, 19] or Lipschitz function with $\alpha = 1$ [1, 22]. The generalization enables us to obtain results previously impossible: i) As tensor order m increases, we demonstrate the failure of previous clustering-based algorithms [1, 16], and we develop a new block-wise polynomial algorithm for tensors of order $m \geq 3$; ii) As smoothness α increases, we demonstrate that the error rate converges to a fast rate $\mathcal{O}(d^{-(m-1)})$, thereby disproving the conjectured lower bound $\mathcal{O}(d^{-2m/(m+2)})$ posed by earlier work [1]. The results showcase the accuracy gain of our new approach, as well as the intrinsic distinction between matrices and higher-order tensors.

Second, we discover a phase transition phenomenon with respect to the smoothness needed for optimal recovery in model (1). Figure 1(b) plots the dependence of estimation error in terms of smoothness level α for tensors of order m . We characterize two distinct error behaviors determined by a critical smoothness threshold; see Theorems 1-2 in Section 4. Specifically, the accuracy improves with α in the regime $\alpha \leq m(m-1)/2$, whereas the accuracy becomes a constant of α in the regime $\alpha > m(m-1)/2$. The results imply a polynomial of degree $(m-2)(m+1)/2 = [m(m-1)/2 - 1]$ is sufficient for accurate recovery of order- m tensors of arbitrary smoothness in model (1), whereas higher degree brings no further benefits. The phenomenon is distinctive from matrix problems [19, 16] and classical *non-permuted* smooth function estimation [31], thereby highlighting the fundamental challenges in our new setting. These statistical contributions, to our best knowledge, are new to the literature of general permuted smooth tensor problems.

Third, we propose two estimation algorithms with accuracy guarantees: the least-squares

estimation and Borda count estimation. The least-squares estimation, although being computationally hard, reveals the fundamental model complexity in the problem. The result serves as the benchmark and a useful guide to the algorithm design. Furthermore, we develop an efficient polynomial-time Borda count algorithm that provably achieves a minimax optimal rate under [an extra Lipschitz monotonic assumption](#). The algorithm handles a broad range of data types, including continuous and binary observations.

Lastly, we illustrate the efficacy of our method through both simulations and data applications. A range of practical settings are investigated in simulations, and we show the outperformance of our method compared to alternative approaches. Application to Chicago crime data is presented to showcase the usefulness of our method. We identify the key global pattern and pinpoint local smooth structure in the denoised tensor. Our method will help practitioners efficiently analyze tensor datasets in various areas. Toward this end, the package and all data used are released at CRAN.

1.2 Related work

Our work is closely related to but also clearly distinctive from several lines of existing research. We review related literature for comparison.

Structure learning with latent permutations. The estimation problem of (1) falls into the general category of structured learning with *latent permutation*. Models involving latent permutations have recently received a surge of interest, include graphons [7, 19], stochastic transitivity models [8, 28], statistical seriation [14], and graph matching [11]. These methods, however, are developed for matrices; the tensor counterparts are far less well understood. Table 1 summarizes the most related works to ours. Pananjady and Samworth [26] studied the permuted tensor estimation under isotonic constraints. We find that our smooth model

results in a much faster rate $\mathcal{O}(d^{-(m-1)})$ than the rate $\mathcal{O}(d^{-1})$ for isotonic models. The works [1, 22] studied similar smooth models as ours, but we gain substantial improvement in both statistics and computations. Balasubramanian [1] developed a (non-polynomial-time) clustering-based algorithm with a rate $\mathcal{O}(d^{-2m/(m+2)})$. Li et al. [22] developed a (polynomial-time) nearest neighbor estimation with a rate $\mathcal{O}(d^{-\lfloor m/3 \rfloor})$. Neither approach investigates the minimax optimality. By contrast, we develop a polynomial-time algorithm with a fast rate $\mathcal{O}(d^{-(m-1)})$ under mild conditions. The optimality of our estimator is safeguarded by matching a minimax lower bound.

Low-rank tensor models. There is a huge literature on structured tensor estimation under low-rank models, including CP models [20], Tucker models [37], and block models [34]. These models belong to parametric approaches, because they aim to explain the data with a finite number of parameters (i.e., decomposed factors). Our permuted smooth tensor model utilizes a different measure of model complexity than the usual low-rankness. We use *infinite* number of parameters (i.e., smooth functions) to allow growing model complexity. In this sense, our method belongs to nonparametric approaches. We emphasize that our permuted smooth tensor model does not necessarily include low-rank model. Compared to low-rank models, we utilize a different measure of *model complexity*. When the underlying signal is precisely low-rank, then low-rank methods are preferred. However, if the underlying signal is high rank but has certain shape structure, then our nonparametric approach better captures the intrinsic model complexity.

Nonparametric regression. Our model is also related to nonparametric regression [31]. One may view the problem (1) as a nonparametric regression, where the goal is to learn the function f based on scalar response $\mathcal{Y}(i_1, \dots, i_m)$ and design points $(\pi(i_1), \dots, \pi(i_m))$ in \mathbb{R}^m ;

see Figure 1(a). However, the *unknown* permutation π significantly influences the statistical and computational hardness of the problem. This latent π leads to a phase transition behavior in the estimation error; see Figure 1(b) and Section 4. We reveal two components of error for the problem, one for nonparametric error and the other for permutation error. The impact of unknown permutation hinges on tensor order and smoothness in an intriguing way, as shown in (2). This is clearly contrary to classical nonparametric regression.

Graphon and hypergraphon. Our work is also connected to graphons and hypergraphons. Graphon is a measurable function representing the limit of a sequence of exchangeable random graphs (matrices) [19, 16, 7]. Similarly, hypergraphon [38, 23] is introduced as a limiting function of m -uniform hypergraphs, i.e., a generalization of graphs in which edges can join m vertices with $m \geq 3$. While both our model (1) and hypergraphon focus on function representations, there are two remarkable differences. First, unlike the matrix case where graphon is represented as bivariate functions [23], hypergraphons for m -uniform hypergraphs should be represented as $(2^m - 2)$ -multivariate functions; see Zhao [38, Section 1.2]. Our framework (1) represents the function using m coordinates only, and in this sense, the model shares the common ground as *simple hypergraphons* [1]. We compare our method to earlier work in theory (Table 1 and Sections 4-5) and in numerical studies (Section 6). Second, [unlike typical simple hypergraphons where the design points are random, our generative model uses deterministic design points. The comparison of two approaches will be discussed in Sections 2 and 4.](#)

1.3 Notation and organization

Let \mathbb{N}, \mathbb{N}_+ denote the set of non-negative integers and positive integers, respectively. We use $[d] = \{1, \dots, d\}$ to denote the d -set for $d \in \mathbb{N}_+$. For a set S , $|S|$ denotes its cardinality and

$\mathbb{1}_S$ denotes the indicator function. For two positive sequences $\{a_d\}, \{b_d\}$, we denote $a_d \lesssim b_d$ if $\lim_{d \rightarrow \infty} a_d/b_d \leq c$ for some constant $c > 0$, and $a_d \asymp b_d$ if $c_1 \leq \lim_{d \rightarrow \infty} a_d/b_d \leq c_2$ for some constants $c_1, c_2 > 0$. Given a number $a \in \mathbb{R}$, the function $\lfloor a \rfloor$ is the largest integer strictly smaller than a and the ceiling function $\lceil a \rceil$ is the smallest integer no less than a . We use $\mathcal{O}(\cdot)$ to denote big-O notation hiding logarithmic factors, and \circ the function composition.

Let $\Theta \in \mathbb{R}^{d \times \dots \times d}$ be an order- m d -dimensional tensor, $\pi: [d] \rightarrow [d]$ be an index permutation, and $\Theta(i_1, \dots, i_m)$ the tensor entry indexed by (i_1, \dots, i_m) . We sometimes also use shorthand notation $\Theta(\omega)$ for tensor entries with indices $\omega = (i_1, \dots, i_m)$. We call a tensor *a binary-valued tensor* if its entries take value on $\{0, 1\}$ -labels, and *a continues-valued tensor* if its entries take values on a continuous scale. We define the Frobenius norm $\|\Theta\|_F^2 = \sum_{\omega \in [d]^m} |\Theta(\omega)|^2$ for a tensor Θ , and the ∞ -norm $\|\mathbf{x}\|_\infty = \max_{i \in [d]} |x_i|$ for a vector $\mathbf{x} = (x_1, \dots, x_d)^T$. We use $\Pi(d, d) = \{\pi: [d] \rightarrow [d]\}$ to denote all permutations on $[d]$, while $\Pi(d, k) = \{\pi: [d] \rightarrow [k]\}$ the collection of all onto mappings from $[d]$ to $[k]$. Given $\pi \in \Pi(d, k)$ and $\Theta \in \mathbb{R}^{k \times \dots \times k}$, we use $\Theta \circ \pi$ to denote the d -dimensional tensor such that $(\Theta \circ \pi)(i_1, \dots, i_m) = \Theta(\pi(i_1), \dots, \pi(i_m))$ for all $(i_1, \dots, i_m) \in [d]^m$. An event A is said to occur *with high probability* if $\mathbb{P}(A)$ tends to 1 as the tensor dimension $d \rightarrow \infty$.

The rest of the paper is organized as follows. Section 2 presents the permuted smooth tensor model and its connection to smooth function representation. In Section 3, we establish the approximation error based on block-wise polynomial approximation. Then, we develop two estimation algorithms with accuracy guarantees: the least-squares estimation and Borda count estimation. Section 4 presents a statistically optimal but computationally challenging least-squares estimator. Section 5 presents a polynomial-time Borda count algorithm with the same minimax optimal rate under an extra Lipschitz monotonic assumption. Simulations and data analyses are presented in Section 6. We conclude the paper with a discussion in

Section 7. All proofs and extensions are deferred to Appendix.

2 Smooth tensor model with unknown permutation

Suppose we observe an order- m d -dimensional data tensor from the following model,

$$\mathcal{Y} = \Theta \circ \pi + \mathcal{E}, \quad (3)$$

where $\pi: [d] \rightarrow [d]$ is an unknown latent permutation, $\Theta \in \mathbb{R}^{d \times \dots \times d}$ is an unknown signal tensor under certain smoothness (to be specified in next paragraph), and \mathcal{E} is a noise tensor consisting of zero-mean, independent sub-Gaussian entries with variance bounded by σ^2 . The general model allows continuous- and binary-valued tensors. For instance, in binary tensor problems, the entries in \mathcal{Y} are $\{0, 1\}$ -labels from Bernoulli distribution, and the entrywise variance of \mathcal{E} depends on the mean. For ease of presentation, we assume $\sigma = 1$ throughout the paper. We call (3) the Gaussian model if the \mathcal{E} consists of i.i.d. $\mathcal{N}(0, 1)$ entries, and call (3) the sub-Gaussian model if \mathcal{E} consists of independently (but not necessarily identically) distributed sub-Gaussian entries.

We now describe the smooth model on the signal Θ . Suppose that there exists a multivariate function $f: [0, 1]^m \rightarrow \mathbb{R}$ underlying the signal tensor, such that

$$\Theta(i_1, \dots, i_m) = f\left(\frac{i_1}{d}, \dots, \frac{i_m}{d}\right), \quad \text{for all } (i_1, \dots, i_m) \in [d]^m. \quad (4)$$

For a multi-index $\kappa = (\kappa_1, \dots, \kappa_m) \in \mathbb{N}^m$ and a vector $\mathbf{x} = (x_1, \dots, x_m)^T$, we denote $|\kappa| = \sum_{i \in [m]} \kappa_i$, $\kappa! = \prod_{i \in [m]} \kappa_i!$, $\mathbf{x}^\kappa = \prod_{i \in [m]} x_i^{\kappa_i}$, and the derivative operator $\nabla_\kappa = \frac{\partial^{|\kappa|}}{\partial x_1^{\kappa_1} \dots \partial x_m^{\kappa_m}}$. Assume the generative function f in (4) is in the α -Hölder smooth family [35, 31].

Definition 1 (α -Hölder smooth). Let $\alpha > 0$ and $L > 0$ be two positive constants. A function $f: [0, 1]^m \rightarrow \mathbb{R}$ is called α -Hölder smooth, denoted as $f \in \mathcal{F}(\alpha, L)$, if for every

$\mathbf{x}, \mathbf{x}_0 \in [0, 1]^m$, we have

$$\sum_{\kappa: |\kappa| = \lfloor \alpha \rfloor} \frac{1}{\kappa!} |\nabla_\kappa f(\mathbf{x}) - \nabla_\kappa f(\mathbf{x}_0)| \leq \begin{cases} L \|\mathbf{x} - \mathbf{x}_0\|_\infty, & \text{if } |\kappa| < \lfloor \alpha \rfloor, \\ L \|\mathbf{x} - \mathbf{x}_0\|_\infty^{\alpha - \lfloor \alpha \rfloor}, & \text{if } |\kappa| = \lfloor \alpha \rfloor. \end{cases}$$

The Hölder smooth function class is one of the most popular function classes considered in the nonparametric regression literature [19, 16]. In addition to the function class $\mathcal{F}(\alpha, L)$, we also define the smooth tensor class based on discretization (4),

$$\mathcal{P}(\alpha, L) = \left\{ \Theta \in \mathbb{R}^{d \times \dots \times d} : \Theta \text{ is generated from (4) and } f \in \mathcal{F}(\alpha, L) \right\}.$$

Combining (3) and (4) yields our proposed *permuted smooth tensor model*. The unknown parameters are the smooth tensor $\Theta \in \mathcal{P}(\alpha, L)$ and latent permutation $\pi \in \Pi(d, d)$. The generative model is visualized in Figure 1(a) for the case $m = 2$ (matrices). We give two examples to show the applicability of our permuted smooth tensor model.

Example 1 (Four-player game tensors). Consider the tournament of a four-player board game. Suppose there are in total d players, among which all combinations of four have played with each other. The tournament results are summarized as an order-4 (non-symmetric) tensor, with entries encoding the winner out of the four. Our model is then given by

$$\mathbb{E}\mathcal{Y}(i_1, \dots, i_4) = \mathbb{P}(\text{player } i_1 \text{ wins over } (i_2, i_3, i_4)) = f\left(\frac{\pi(i_1)}{d}, \dots, \frac{\pi(i_4)}{d}\right).$$

We interpret the permutation π as the unknown ranking among d players, and the function f as the unknown four-player interaction. Players with similar ranking may have similar performance reflected by the smoothness of f . For example, a variant of popular Plackett-Luce model [10] considers the parametric form $f(x_1, x_2, x_3, x_4) = \exp(\beta x_1) / \sum_{i=1}^4 \exp(\beta x_i)$. By contrast, our model leaves the form of f unspecified, and we learn the function from the data in a nonparametric approach.

Example 2 (Co-authorship networks). Consider a co-authorship network consisting of d nodes (authors) in total. We say there exists a hyperedge of size m between nodes (i_1, \dots, i_m) if the authors i_1, \dots, i_m have co-authored at least one paper. The resulting m -uniform hypergraph is represented as an order- m (symmetric) adjacency tensor. Our model is then expressed as

$$\mathbb{E}\mathcal{Y}(i_1, \dots, i_m) = \mathbb{P}(\text{authors } i_1, \dots, i_m \text{ co-authored}) = f\left(\frac{\pi(i_1)}{d}, \dots, \frac{\pi(i_m)}{d}\right).$$

We interpret the permutation π as the affinity measures of authors, and the function f represents the m -way interaction among authors. The parametric model [33] imposes logistic function $f(x_1, \dots, x_m) = (1 + \exp(-\beta x_1 x_2 \cdots x_m))^{-1}$. By contrast, our nonparametric model allows unknown f and learns the function directly from data.

Our model (4) assumes equally-spaced grid design $\{1/d, 2/d, \dots, d/d\}$ from the generative function f . One can also extend the model to non-equally-spaced designs. Specifically, suppose that the signal tensor is generated from f based on

$$\Theta(i_1, \dots, i_m) = f(x_{i_1}, \dots, x_{i_m}), \quad \text{for all } (i_1, \dots, i_m) \in [d]^m. \quad (5)$$

where the points $\{x_i\}_{i=1}^d$ can be modeled as either fixed latent variables or i.i.d. random variables from a probability distribution supported on $[0, 1]$. We refer to (5) as the relaxed smooth model; similar model has been developed in the literature of graphons and hypergraphons [7, 16, 19, 1]. Our paper will focus on the grid design (4), but we will also discuss the extension to (5) whenever possible as remarks of theorems.

We restrict ourselves to tensor models with equal dimension and same permutation along m modes. The results for non-symmetric tensors with m distinct permutations are similar but require extra notations; we assess this general case in Appendix A.2.

3 Block-wise tensor estimation

Our general strategy for estimating the permuted smooth tensor is based on the block-wise tensor approximation. In this section, we first introduce the tensor block model [34, 17]. Then, we extend this model to block-wise polynomial approximation.

3.1 Tensor block model

Tensor block models describe a checkerboard pattern in a signal tensor. The block model provides a meta structure to many popular models including the low-rankness [15] and isotonic tensors [26]. Here, we use tensor block models as a building block for estimating permuted smooth models.

Specifically, suppose that there are k clusters among d entities, and the cluster assignment is represented by a clustering function $z: [d] \rightarrow [k]$. Then, the tensor block model assumes that the entries of signal tensor $\Theta \in \mathbb{R}^{d \times \dots \times d}$ take values from a core tensor $\mathcal{S} \in \mathbb{R}^{k \times \dots \times k}$ according to the clustering function z ; that is,

$$\Theta(i_1, \dots, i_m) = \mathcal{S}(z(i_1), \dots, z(i_m)), \quad \text{for all } (i_1, \dots, i_m) \in [d]^m. \quad (6)$$

Here, the core tensor \mathcal{S} collects the entry values of m -way blocks; the core tensor \mathcal{S} and clustering function $z \in \Pi(d, k)$ are parameters of interest. A tensor Θ satisfying (6) is called a block- k tensor. Tensor block models allow various data types, as shown below.

Example 3 (Gaussian tensor block model). Let \mathcal{Y} be a continuous-valued tensor. The Gaussian tensor block model draws independent normal entries according to $\mathcal{Y}(i_1, \dots, i_m) \stackrel{\text{ind}}{\sim} N(\mathcal{S}(z(i_1), \dots, z(i_m)), \sigma^2)$. The mean model belongs to (6), and noises entries are i.i.d. $N(0, \sigma^2)$. The Gaussian tensor block model has served as the statistical foundation for many tensor clustering algorithms [34, 17].

Tensor block models have shown great success in discovering hidden group structures for many applications [34, 17]. Despite the popularity, the constant block assumption is insufficient to capture delicate structure when the signal tensor is complicated. This parametric model aims to explain data with a finite number of blocks; such an approach is useful when the sample outsizes the parameters. Our nonparametric model (4), by contrast, uses infinite number of parameters (i.e., smooth functions) to allow growing model complexity. We change the goal of tensor block model from clustering to approximating the generative function f in (4). In our setting, the number of blocks k should be interpreted as a resolution parameter (i.e., a bandwidth) of the approximation, similar to the notion of number of bins in histogram and polynomial regression [35].

3.2 Block-wise polynomial approximation

The block tensor (6) can be viewed as a discrete version of piece-wise *constant* function. This connection motivates us to use block-wise *polynomial* tensors to approximate α -Hölder functions. Now we extend (6) to block-wise polynomial models.

We introduce some additional notations. For a given block number k , we use $z \in \Pi(d, k)$ to denote the canonical clustering function that partitions $[d]$ into k equally-sized clusters such that $z(i) = \lceil ki/d \rceil$, for all $i \in [d]$. The collection of inverse images $\{z^{-1}(j) : j \in [k]\}$ is a partition of $[d]$ into k disjoint and equal-sized subsets. We use \mathcal{E}_k to denote the m -way partition, i.e., a collection of k^m disjoint and equal-sized subsets in $[d]^m$, such that

$$\mathcal{E}_k = \{z^{-1}(j_1) \times \cdots \times z^{-1}(j_m) : (j_1, \dots, j_m) \in [k]^m\}.$$

Let $\Delta \in \mathcal{E}_k$ denote the element in \mathcal{E}_k . We propose to approximate the signal tensor Θ in (4) by degree- ℓ polynomial tensors within each block $\Delta \in \mathcal{E}_k$. Specifically, let $\mathcal{B}(k, \ell)$ denote

the class of block- k degree- ℓ polynomial tensors,

$$\mathcal{B}(k, \ell) = \left\{ \mathcal{B} \in \mathbb{R}^{d \times \dots \times d} : \mathcal{B}(\omega) = \sum_{\Delta \in \mathcal{E}_k} \text{Poly}_{\ell, \Delta}(\omega) \mathbf{1}\{\omega \in \Delta\} \text{ for all } \omega \in [d]^m \right\}, \quad (7)$$

where $\text{Poly}_{\ell, \Delta}(\cdot)$ denotes a degree- ℓ polynomial function in \mathbb{R}^m , with coefficients depending on block Δ ; that is, a constant function $\text{Poly}_{0, \Delta}(\omega) = \beta_\Delta^0$ for $\ell = 0$, a linear function $\text{Poly}_{1, \Delta}(\omega) = \langle \boldsymbol{\beta}_\Delta, \omega \rangle + \beta_\Delta^0$ for $\ell = 1$, and so on so forth. Here β_Δ^0 and $\boldsymbol{\beta}_\Delta$ denote unknown coefficients in polynomial function. Note that the degree-0 polynomial block tensor reduces to the constant block model (6). We generalize the constant block model to degree- ℓ polynomial block tensor (7), analogous to the generalization from k -bin histogram to k -piece-wise polynomial regression in nonparametric statistics [35].

Smoothness of the function f in (4) plays an important role in the block-wise polynomial approximation. The following lemma explains the role of smoothness in the approximation.

Proposition 1 (Block-wise polynomial tensor approximation). *Suppose $\Theta \in \mathcal{P}(\alpha, L)$. Then, for every block number $k \leq d$ and degree $\ell \in \mathbb{N}$, we have the approximation error*

$$\inf_{\mathcal{B} \in \mathcal{B}(k, \ell)} \frac{1}{d^m} \|\Theta - \mathcal{B}\|_F^2 \lesssim \frac{L^2}{k^{2 \min(\alpha, \ell+1)}}.$$

Proposition 1 implies that we can always find a block-wise polynomial tensor close to the signal tensor generated from α -Hölder smooth function f . The approximation error decays with block number k and degree $\min(\alpha, \ell + 1)$.

4 Statistical limits via least-squares estimation

We develop two estimation methods based on the block-wise polynomial approximation. We first introduce a statistically optimal but computationally inefficient least-squares estimator. The least-squares estimation serves as a statistical benchmark because of its minimax

optimality. In Section 5, we will present a polynomial-time algorithm with a provably same optimal rate under monotonicity assumptions.

We propose the least-squares estimation for model (3) by minimizing the Frobenius loss over the block- k degree- ℓ polynomial tensor family $\mathcal{B}(k, \ell)$ up to permutations,

$$(\hat{\Theta}^{\text{LSE}}, \hat{\pi}^{\text{LSE}}) = \arg \min_{\Theta \in \mathcal{B}(k, \ell), \pi \in \Pi(d, d)} \|\mathcal{Y} - \Theta \circ \pi\|_F. \quad (8)$$

The least-squares estimator $(\hat{\Theta}^{\text{LSE}}, \hat{\pi}^{\text{LSE}})$ depends on two tuning parameters: the number of blocks k and the polynomial degree ℓ . The optimal choice (k^*, ℓ^*) will be provided below.

Theorem 1 establishes the error bound for the least-squares estimator (8). Note that Θ and π are in general not separably identifiable; for example, when the true signal is a constant tensor, then every permutation $\pi \in \Pi(d, d)$ gives equally good fit in statistics. We assess the estimation error on the composition $\Theta \circ \pi$ to avoid this identifiability issue. For two order- m d -dimensional tensors Θ_1, Θ_2 , define the mean squared error (MSE) by

$$\text{MSE}(\Theta_1, \Theta_2) = d^{-m} \|\Theta_1 - \Theta_2\|_F^2.$$

Theorem 1 (Least-squares estimation error). *Let $m \geq 2$. Consider the sub-Gaussian model (3) with $\Theta \in \mathcal{P}(\alpha, L)$. Let $(\hat{\Theta}^{\text{LSE}}, \hat{\pi}^{\text{LSE}})$ denote the least-squares estimator in (8) with a given (k, ℓ) . Then, for every $k \leq d$ and degree $\ell \in \mathbb{N}$, we have*

$$\text{MSE}(\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}}, \Theta \circ \pi) \lesssim \underbrace{\frac{L^2}{k^{2\min(\alpha, \ell+1)}}}_{\text{approximation error}} + \underbrace{\frac{k^m(\ell+m)^\ell}{d^m}}_{\text{nonparametric error}} + \underbrace{\frac{\log d}{d^{m-1}}}_{\text{permutation error}} \quad (9)$$

with high probability. In particular, setting $\ell^* = \min(\lfloor \alpha \rfloor, (m-2)(m+1)/2)$ and $k^* = c_1 d^{m/(m+2\min(\alpha, \ell^*+1))}$ yields the optimized error rate

$$(9) \lesssim \text{Rate}(d) = \begin{cases} c_2 d^{-\frac{2m\alpha}{m+2\alpha}}, & \text{when } \alpha < m(m-1)/2, \\ c_3 d^{-(m-1)} \log d, & \text{when } \alpha \geq m(m-1)/2. \end{cases} \quad (10)$$

Here, the function $\text{Rate}(d)$ is given in (2), and the constants $c_1, c_2, c_3 > 0$ depend on the model configuration (m, L, α) but not on the tensor dimension d .

We discuss the asymptotic error rate as $d \rightarrow \infty$ while treating other model configurations fixed. The final least-squares estimation rate (10) has two sources of error: the nonparametric error $d^{-\frac{2m\alpha}{m+2\alpha}}$ and the permutation error $d^{-(m-1)} \log d$. Intuitively, in the tensor data analysis problem, we can view each tensor entry as a data point, so sample size is the total number of entries, d^m . The unknown permutation results in $\log(d!) \approx d \log d$ complexity, whereas the unknown generative function results in $d^{-2m\alpha/(m+2\alpha)}$ nonparametric complexity. When the function f is smooth enough, estimating the function f becomes relatively easier compared to estimating the permutation π . This intuition coincides with the fact that the permutation error dominates the nonparametric error when $\alpha \geq m(m-1)/2$.

Remark 1 (Comparison to non-parametric regression). We now compare our results with existing work in the literature. In the vector case with $m = 1$, our model reduces to the one-dimensional regression problem such that $y_i = \theta_{\pi(i)} + \epsilon_i$, for all $i \in [d]$, where $\theta_i = f(i/d)$ and unknown $\pi \in \Pi(d, d)$. A similar analysis of Theorem 1 shows the error rate

$$\frac{1}{d} \sum_{i \in [d]} (\hat{\theta}_i^{\text{LSE}} - \theta_i)^2 \lesssim \left(d^{-\frac{2\alpha}{2\alpha+1}} + \log d \right), \quad (11)$$

under the choice of $\ell^* = 0$ and $k^* \asymp d^{\frac{1}{1+2\min(\alpha,1)}}$. Notice that $d^{-2\alpha/(2\alpha+1)}$ is the classical nonparametric minimax rate for α -Hölder smooth functions [31] with *known* permuted design points $\{\pi(i)\}_{i=1}^d$. By contrast, our model involves *unknown* π , which results in the non-vanishing permutation rate $\log d$ in (11).

Remark 2 (Breaking previous limits on matrices/tensors). In the matrix case with $m = 2$, Theorem 1 implies that the best rate is obtained under $\ell^* = 0$, i.e., the block-wise *constant* approximation. This result is consistent with existing literature on smooth graphons [5, 16, 19], where constant block model (see Section 3.1) is developed for estimation. In the tensor case with $m \geq 3$, earlier work [1] suggests that constant block approximation

$(\ell^* = 0)$ may remain minimax optima. Our Theorem 1 disproves this conjecture, and we reveal a much faster rate $d^{-(m-1)}$ compared to the conjectured lower bound $d^{-2m/(m+2)}$ [1] for sufficiently smooth tensors. We demonstrate that a polynomial up to degree $(m-2)(m+1)/2$ is sufficient (and necessary; see Theorem 2 below) for accurate estimation of order- m permuted smooth tensors. For example, permuted α -smooth tensors of order-3 require quadratic approximation ($\ell^* = 2$) with $k^* \asymp d^{1/3}$ blocks, for all $\alpha \geq 2$. The results show the clear difference between matrices and higher-order tensors.

We now show that the rate in (10) cannot be improved. The lower bound is information-theoretical and thus applies to all estimators including, but not limited to, the least-squares estimator (8) and Borda count estimator introduced in next section.

Theorem 2 (Minimax lower bound). *The estimation problem based on the Gaussian model (1) obeys the minimax lower bound*

$$\inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\Theta \in \mathcal{P}(\alpha, L), \pi \in \Pi(d, d)} \mathbb{P} \left(\text{MSE}(\hat{\Theta} \circ \hat{\pi}, \Theta \circ \pi) \gtrsim \text{Rate}(d) \right) \geq 0.8, \quad (12)$$

where the function $\text{Rate}(d)$ is given in (2).

The lower bound in (12) matches the upper bound in (10), demonstrating the statistical optimality of the convergence speed $\text{Rate}(d) = d^{-\frac{2m\alpha}{2\alpha+m}} \vee d^{-(m-1)} \log d$. The two-component error reveals the intrinsic model complexity. In particular, the permutation error $d^{-(m-1)}$ dominates nonparametric error $d^{-2m\alpha/(m+2\alpha)}$ for sufficiently smooth tensors. This phenomenon is contrary to classical nonparametric regression.

Remark 3 (Phase transition). Our Theorem 1 can be extended to relaxed smooth model (5). In the Appendix F, we show that the relaxed smooth model has the same convergence upper bound $\text{Rate}(d)$, with little modification of proofs. We conclude this section by summarizing an interesting phase transition phenomenon. Figure 1(b) plots the optimal convergence speed

$\text{Rate}(d)$. The impact of unknown permutation hinges on the tensor order and smoothness. The accuracy improves with respect to smoothness in the regime $\alpha \leq m(m-1)/2$; however, in the regime $\alpha > m(m-1)/2$, the accuracy becomes a constant with respect to smoothness. The result implies a polynomial of degree $\approx (m-2)(m+1)/2$ is sufficient for accurate recovery of order- m tensors, whereas higher degree brings no further benefits. This full picture of error dependence, to our best knowledge, is new to the literature.

5 Computational limits and polynomial-time algorithms

We should point out that the least-squares estimator $(\hat{\Theta}^{\text{LSE}}, \hat{\pi}^{\text{LSE}})$ requires exponential-time algorithms, even in the simple matrix case [16]. Specifically, in this section we show the non-existence of *polynomial-time algorithms* with rate (2) under the computational hardness conjecture for hypergraphic planted clique (HPC) [24]. The computational limit implies the necessity of extra assumptions to achieve the optimal rate (2) for any polynomial algorithm.

5.1 Computational limits under HPC detection conjecture

The hypergraphic planted clique detection conjecture plays an important role in constructing the computational limits of our problem. We briefly introduce the HPC hardness conjecture.

Consider an m -uniform hypergraph $G = (V, E)$, where V is a set of vertices, and E is a set of hyperedges. An Erdős-Rényi random hypograph, denoted by $\mathcal{G}_m(d, 1/2)$, is a random m -uniform hypergraph with d vertices and probability $1/2$ for each of the hyperedge connections. The hypergraphic planted clique (HPC) with clique size $\kappa > 0$, denoted by $\mathcal{G}_m(d, 1/2, \kappa)$, is generated from Erdős-Rényi random hypograph in the following way. First we generate an Erdős-Rényi random hypograph from $\mathcal{G}_m(d, 1/2)$. Then we independently pick κ vertices with uniform probability from d vertices. Finally, we obtain the $\mathcal{G}_m(d, 1/2, \kappa)$

by including only the hyperedges whose vertices all belong to the picked τ vertices. The HPC detection refers to the hypothesis testing problem,

$$H_0: G \sim \mathcal{G}_m(d, 1/2) \quad \text{v.s.} \quad H_1: G \sim \mathcal{G}_m(d, 1/2, \kappa). \quad (13)$$

The earlier work [24] presents the following hardness conjecture for testing (13).

Conjecture 1 (HPC detection conjecture [24]). *Consider the HPC problem in (13) and let $m \geq 2$ be fixed integer. Suppose $\limsup_{d \rightarrow \infty} \log \kappa / \log \sqrt{d} \leq 1 - \tau$, for any $\tau > 0$. Then, for any polynomial-time test sequence $\{\phi\}_d: G \mapsto \{0, 1\}$, we have*

$$\liminf_{d \rightarrow \infty} \{\mathbb{P}_{H_0}(\phi(G) = 1) + \mathbb{P}_{H_1}(\phi(G) = 0)\} \geq \frac{1}{2},$$

Now we construct the computational lower bound based on Conjecture 1.

Theorem 3 (Computational lower bound). *Assume Conjecture 1 holds. Define the relaxed smooth tensor class $\mathcal{P}_{\text{rel}} = \{\Theta : \Theta \text{ is generated from (5) with fixed latent variables } \{x_i\}_{i=1}^d \text{ and } f \in \mathcal{F}(\alpha, L)\}$. Consider the Gaussian model (3) with $\alpha > m/2$. There exists no polynomial algorithm that achieves the statistical optimal convergence Rate(d); i.e.,*

$$\frac{1}{\text{Rate}(d)} \inf_{\hat{\Theta} \text{ Polynomial-time}} \sup_{\Theta \in \mathcal{P}_{\text{rel}}} \text{MSE}(\hat{\Theta}, \Theta) \rightarrow \infty, \quad \text{as } d \rightarrow \infty. \quad (14)$$

Theorem 3 shows the *impossibility* of polynomial-time estimator to achieve the optimal statistical rate in the general model. The intuition in the proof is to show the best bound for *polynomial-time* tensor estimation as $d^{-m/2}$, in the absence of extra model structures. The condition $\alpha > m/2$ is a technical assumption to facilitate the proof. Theorem 3 is not a weakness of our proposed estimator; rather, (14) reveals the non-avoidable statistical-computational gap as a nature of the smooth tensor estimation problem. The fact has motivated us to introduce extra model structure to fill the gap.

5.2 Borda count algorithm

The earlier section has shown the impossibility of polynomial-time algorithms in the general model. In this section, we restrict ourself to a sub-model with extra monotonicity structures; this structure makes polynomial-time algorithm possible. We introduce a notion of marginal monotonicity for the generative functions.

Definition 2 (Marginal monotonicity). Let $\beta \geq 0$ be a non-negative constant. A function $f: [0, 1]^m \rightarrow \mathbb{R}$ is called β -monotonic, denoted as $f \in \mathcal{M}(\beta)$, if and only if,

$$\left(\frac{i-j}{d} \right)^{1/\beta} \lesssim g(i) - g(j), \quad \text{for all } j < i \in [d], \quad (15)$$

where we define the *score function* $g(i) = d^{-(m-1)} \sum_{(i_2, \dots, i_m) \in [d]^{m-1}} f\left(\frac{i}{d}, \frac{i_2}{d}, \dots, \frac{i_m}{d}\right)$ for $i \in [d]$.

We refer to $\mathcal{F}(\alpha, L) \cap \mathcal{M}(\beta)$ as the monotonic-plus-smooth function class. This class was initially proposed in previous literature of graphons. The work [7] proposes the Lipschitz monotonic function to facilitate the analysis of sorting-merging algorithm for matrix estimation; their setting is a special case of our Definition 2 with $(\alpha, \beta, m) = (1, 1, 2)$. Inspired by earlier work, we consider the similar monotonic-plus-smooth function class $\mathcal{F}(\alpha, L) \cap \mathcal{M}(\beta)$ under general configuration $\{(\alpha, \beta, m) : \alpha > 0, 0 < \beta \min(\alpha, 1) \leq 1, m \geq 2\}$. Note that the constraint $\beta \min(\alpha, 1) \leq 1$ is due to the natural relationship between joint smoothness and marginal smoothness. A large value of β in (15) implies the steepness of g .

Now we introduce the *Borda count* estimation that consists of two stages. The full procedure is illustrated in Figure 2. Define the empirical score function $\tau: [d] \rightarrow \mathbb{R}$ as

$$\tau(i) = \frac{1}{d^{m-1}} \sum_{(i_2, \dots, i_m) \in [d]^{m-1}} \mathcal{Y}(i, i_2, \dots, i_m).$$

1. **Sorting stage:** The sorting stage is to rearrange the observed tensor \mathcal{Y} so that the score function τ of sorted tensor is monotonically increasing. Define a permutation $\hat{\pi}^{\text{BC}}$ such that

$$\tau \circ (\hat{\pi}^{\text{BC}})^{-1}(1) \leq \tau \circ (\hat{\pi}^{\text{BC}})^{-1}(2) \leq \dots \leq \tau \circ (\hat{\pi}^{\text{BC}})^{-1}(d). \quad (16)$$

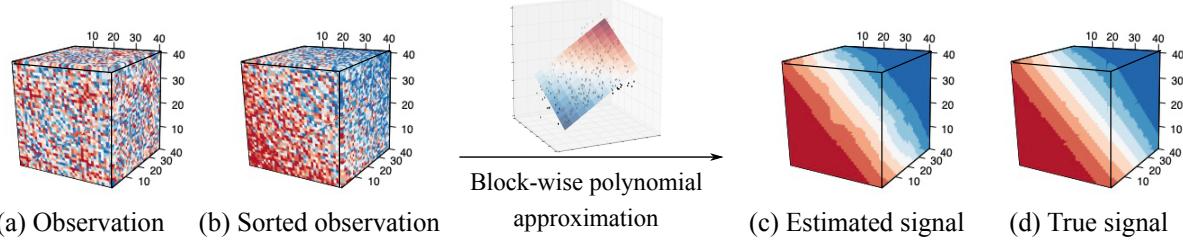


Figure 2: Illustration of Borda count estimation. We first sort tensor entries using the proposed procedure, and then estimate the signal by block-wise polynomial approximation.

Then, we obtain sorted observation $\tilde{\mathcal{Y}} = \mathcal{Y} \circ (\hat{\pi}^{\text{BC}})^{-1}$, illustrated in Figure 2(b).

2. Block-wise polynomial approximation stage: Given sorted observation $\tilde{\mathcal{Y}}$, we estimate the signal tensor by block-wise polynomial tensor based on the following optimization,

$$\hat{\Theta}^{\text{BC}} = \arg \min_{\Theta \in \mathcal{B}(k, \ell)} \|\tilde{\mathcal{Y}} - \Theta\|_F, \quad (17)$$

where $\mathcal{B}(k, \ell)$ denotes the block- k degree- ℓ tensor class in (7). An example of this procedure is shown in Figure 2(c). The estimator $\hat{\Theta}^{\text{BC}}$ depends on two tuning parameters: the number of blocks k and polynomial degree ℓ . The optimal choice of (k^*, ℓ^*) is provided in Theorem 4.

The following theorem ensures the statistical accuracy of the Borda count estimator.

Theorem 4 (Estimation error for Borda count algorithm under marginal monotonicity). *Consider the sub-Gaussian model (3) with $f \in \mathcal{F}(\alpha, L) \cap \mathcal{M}(\beta)$. Denote a constant $c(\alpha, \beta, m) := \frac{m(m-1)\beta \min(\alpha, 1)}{\max(0, 2(m-(m-1)\beta \min(\alpha, 1)))}$. Let $(\hat{\Theta}^{\text{BC}}, \hat{\pi}^{\text{BC}})$ be the Borda count estimator with $\ell^* = \min(\lfloor \alpha \rfloor, \lfloor c(\alpha, \beta, m) \rfloor)$ and $k^* \asymp d^{m/(m+2\min(\alpha, \ell^*+1))}$ in (17). Then, we have*

$$\text{MSE}(\hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}}, \Theta \circ \pi) \lesssim \begin{cases} d^{-\frac{2m\alpha}{m+2\alpha}} & \text{when } \alpha < c(\alpha, \beta, m), \\ \left(\frac{\log d}{d^{m-1}}\right)^{\beta \min(\alpha, 1)} & \text{when } \alpha \geq c(\alpha, \beta, m). \end{cases} \quad (18)$$

The estimation bound (18) comes from the approximation error (Proposition 1), non-parametric error (Theorem 1), and permutation error (Lemma 3).

Remark 4 (Sufficiently smooth tensors). When the generative function is infinitely smooth ($\alpha = \infty$) with Lipschitz monotonic score ($\beta = 1$), our estimation error (18) becomes

$$\text{MSE}(\hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}}, \Theta \circ \pi) \lesssim d^{-(m-1)} \log d, \quad (19)$$

under the choice of degree and block number

$$\ell^* = (m-2)(m+1)/2 \quad \text{and} \quad k^* \asymp d^{\frac{m}{m+2(\ell^*+1)}}. \quad (20)$$

Now, we compare the rate (19) with the classical low-rank estimation [33, 37, 20]. The low-rank tensor model with a constant rank is known to have MSE rate $\mathcal{O}(d^{-(m-1)})$ [33]. Our infinitely smooth tensor model achieves the same rate up to the negligible log term.

Hyperparameter tuning. Our algorithm has two tuning parameters (k, ℓ) . The theoretically optimal choices of (k, ℓ) are given in Theorems 1 and 4. In practice, since model configuration is unknown, we search (k, ℓ) via cross-validation. Based on our theorems, a polynomial of degree $\ell^* = (m-2)(m+1)/2$ is sufficient for accurate recovery of order- m tensors, whereas higher degree brings no further benefit. The practical impacts of hyperparameter tuning are investigated in Section 6.

5.3 Possible relaxation of monotonicity

We emphasize that our monotonicity assumption should be interpreted *up to permutation* in light of (3). The permutation relaxes the extent of stringency in the monotonic assumptions. In the univariate case ($m = 1$), *every smooth function is monotonic-plus-smooth up to permutations*. In particular, $f \in \mathcal{F}(1, L)$ implies the existence of permutation π such that $(f \circ \pi) \in \mathcal{F}(1, L) \cap \mathcal{B}(1)$. The monotonicity comes for free in this case, because the descending sorting, as a permutation, does not deteriorate the 1-d smoothness. For general

order m , the monotonicity imposes moderate constraints on the marginal structure. We provide two examples to illustrate flexibility allowed in our models.

Example 4 (Free monotonicity inherited from smoothness). Consider a quadratic function $f(x, y, z) = (x - 0.5)^2 + yz$. Although the function f is non-monotonic, we can show that f is nearly monotonic-plus-smooth up to permutations; i.e., f can be identified by $\bar{f} \circ \pi$ for some permutation π and $\bar{f} \in \mathcal{F}(\alpha, L) \cap \mathcal{B}(\beta)$ up to a negligible perturbation. See appendix B for the expression of (π, \bar{f}) . Therefore, our Borta count algorithm is applicable to f .

Example 5 (Decomposable monotonicity). Let $R \in \mathbb{N}_+$ be a constant, and $\{g_{r,i}(\cdot): [0, 1] \rightarrow \mathbb{R}\}$ be a set of 1-d smooth functions for $(r, i) \in [R] \times [m]$. Then, all decomposable smooth functions of the form $f(x_1, \dots, x_m) = \sum_{r \in [R]} g_{r,1}(x_1) \cdots g_{r,m}(x_m)$ are monotonic-plus-smooth up to permutations. In particular, low-rank tensors with smooth factors are also monotonic-plus-smooth up to permutations.

We show below that the additional monotonic assumptions do not change the minimax rate under Lipschitz (or equivalently, 1-smooth) condition.

Theorem 5 (Statistical minimax lower bound for Lipschitz and monotonic functions). *Consider Lipschitz monotonic functions such that $f \in \mathcal{F}(1, L) \cap \mathcal{M}(1)$. Define the Lipschitz monotonic tensor class $\mathcal{P}_{\text{mon}} = \{\Theta: \Theta \text{ is generated from (4) and } f \in \mathcal{F}(1, L) \cap \mathcal{M}(1)\}$. Then, the estimation problem based on Gaussian model (1) obeys the minimax lower bound*

$$\inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\Theta \in \mathcal{P}_{\text{mon}}, \pi \in \Pi(d, d)} \mathbb{P} \left(\text{MSE}(\hat{\Theta} \circ \hat{\pi}, \Theta \circ \pi) \gtrsim \text{Rate}(d) \right) \geq 0.8. \quad (21)$$

This result implies that the extra monotonicity assumption renders no changes to the minimax optimal rate of the problem, in the special Lipschitz situation when $\alpha = \beta = 1$. We conjecture that similar results hold true when the nonparametric error dominates the

rate (e.g., $\alpha < c(\alpha, \beta, m)$ in (18)). For general (α, β) , the optimality is unknown; we discuss the proof challenges in Appendix E.

Remark 5 (Other monotonicity assumptions). One may also consider other assumptions such as isotonic functions [26]. Define an isotonic function class \mathcal{M}

$$\mathcal{M} = \{f: [0, 1]^m \rightarrow \mathbb{R} \mid f(x_1, \dots, x_m) \leq f(x'_1, \dots, x'_m) \text{ when } x_i \leq x'_i \text{ for } i \in [m]\}. \quad (22)$$

The isotonic functions (22) concerns the *joint* monotonicity, where as our monotonicity (2) concerns the *marginal* monotonicity. The latter is a weaker assumption. The isotonic functions belong to $\mathcal{M}(0)$ based on our Definition 2. Therefore, all our upper bounds apply to isotonicity functions, although such extension is not necessarily sharp. The extension of sharp bounds in Theorems 4–5 to isotonic functions can be found in Appendix C.6.

We summarize the performance of our Borda count algorithm under various model assumptions in Figure 3. The minimax lower bounds are also illustrated for comparison. We find that the Borda count algorithm achieves computational and statistical optimality in the region $\mathcal{F}(1, L) \cap \mathcal{M}$, with \mathcal{M} being either isotonic or 1-monotonic. The optimality may not be attained in the absence of monotonicity, i.e., in the region $\mathcal{F}(\alpha, L)/\mathcal{M}$. Although the statistical-computational gap is non-avoidable in general (Theorem 3), the additional monotonicity assumptions fill in the gap in several cases.

6 Numerical analysis

6.1 Synthetic data

We simulate order-3 d -dimensional tensors based on the permuted smooth tensor model (4). Both symmetric and non-symmetric tensors are investigated. The symmetric tensors are

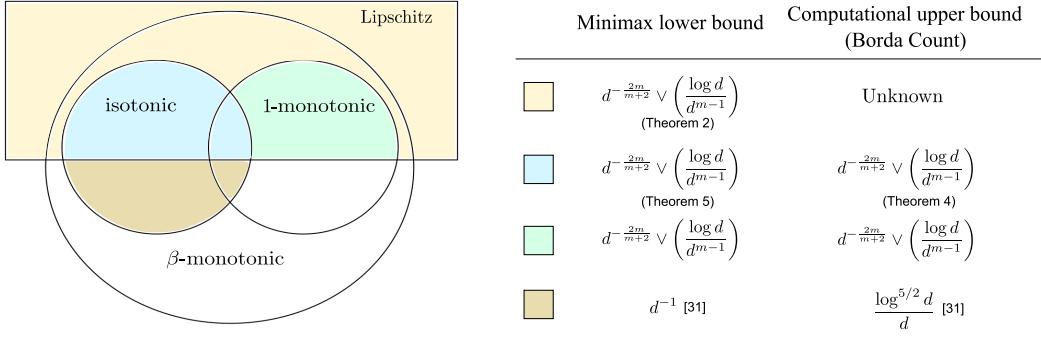


Figure 3: Comparison of statistical lower bound and computational upper bound of Borda count algorithm under special case $\alpha = 1$. The right table summarizes the statistical and computational rates corresponding to each colored region in the Venn diagram.

generated based on functions f in Table 2, and the non-symmetric set-up is described in Appendix A.2. The generative functions involve compositions of operations such as

Model ID	$f(x, y, z)$	CP rank	Tucker rank	$\geq (\alpha, \beta)$	Isotonic
1	xyz	1	(1, 1, 1)	$(\infty, 1)$	✓
2	$(x + y + z)/3$	3	(2, 2, 2)	$(\infty, 1)$	✓
3	$(1 + \exp(-(3x^2 + 3y^2 + 3z^2)))^{-1}$	9	(4, 4, 4)	$(\infty, 1/2)$	✓
4	$\log(1 + \max(x, y, z))$	≥ 100	$\geq (50, 50, 50)$	$(1, 1)$	✓
5	$\exp(-\max(x, y, z) - \sqrt{x} - \sqrt{y} - \sqrt{z})$	≥ 100	$\geq (50, 50, 50)$	$(1/2, 1)$	✓

Table 2: Smooth functions in simulation. We define the numerical CP/Tucker rank as the minimal rank r for which the relative approximation error is below 10^{-4} . The reported rank in the table is estimated from a $100 \times 100 \times 100$ signal tensor generated by (4).

polynomial, logarithm, exponential, square roots, etc. Notice that considered functions cover a reasonable range of model complexities from low rank to high rank. Two types of noise are considered: Gaussian noise and Bernoulli noise. For the Gaussian model, we simulate continuous-valued tensors with i.i.d. noises drawn from $N(0, 0.5^2)$. For the Bernoulli model, we generate binary tensors \mathcal{Y} using the success probability tensor $\Theta \circ \pi$. The permutation π

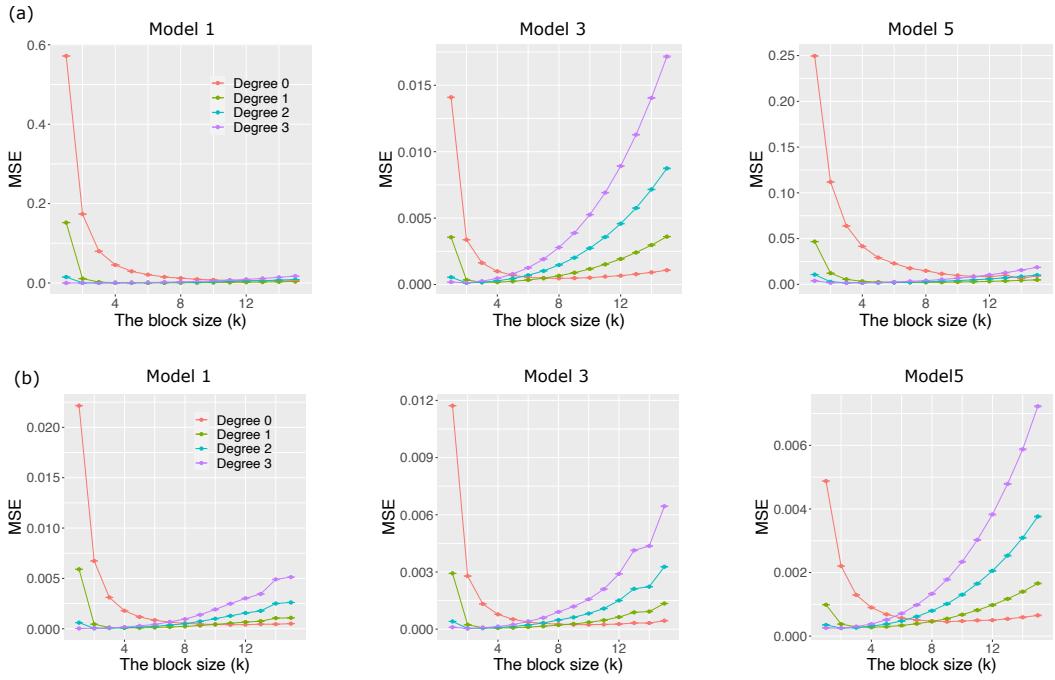


Figure 4: MSE versus the number of blocks based on different polynomial approximations.

Columns 1-3 consider the Models 1, 3, and 5 respectively. Panel (a) is for continuous tensors, whereas (b) is for the binary tensors.

is randomly chosen. For space consideration, only results for Models 1, 3, and 5 are presented in the main paper. The rest is presented in Appendix A.1. We first examine impacts of model complexity to estimation accuracy. We then compare Borda count estimation with alternative methods under a range of scenarios. Extensions to non-symmetric tensors and extra simulation results are provided in Appendix A.

Impacts of the number of blocks, tensor dimension, and polynomial degree. The first experiment examines the impact of the block number k and degree of polynomial ℓ for the approximation. We fix the tensor dimension $d = 100$, and vary the number of blocks $k \in \{1, \dots, 15\}$ and polynomial degree $\ell \in \{0, 1, 2, 3\}$. Figure 4 demonstrates the trade-off in accuracy determined by the number of groups for each polynomial degree. The results confirm our bias-variance analysis in Theorem 1. While a large block number k

provides less biased approximation, this large k renders the signal tensor estimation difficult within each block due to small sample size. In addition, we find that degree-2 polynomial approximation gives the smallest MSE among considered polynomial approximations for models 1-3. By Remark 4, plugging $(\alpha, m) = (\infty, 3)$ in (20) gives the theoretical choice $(k^*, \ell^*) = (d^{7/3}, 2)$. The results are consistent with our simulation.

The second experiment investigates the impact of the tensor dimension d for various polynomial degrees. We vary the tensor dimension $d \in \{10, \dots, 100\}$ and polynomial degree $\ell \in \{0, 1, 2, 3\}$ in each model configuration. We set optimal number of blocks as the one that gives the best accuracy. Figure S1 compares the estimation errors among different polynomial approximations. The result verifies that the degree-2 polynomial approximation performs the best under the sufficient tensor dimension, which is consistent with our theoretical results. We emphasize that this phenomenon is different from the matrix case where the degree-0 polynomial approximation gives the best results [16, 19].

Comparison with alternative methods. We compare our method (**Borda count**) with several popular alternative methods.

- SVD (**Spectral**) [36] performs universal singular-value thresholding [8] on unfolded tensors.
- Least-squares estimation (**LSE**) [17] uses spectral k -means to approximately solve the optimization problem (8) with constant block approximation ($\ell = 0$).
- Least-squares estimation (**BAL**) [1] uses count-based statistics to approximately solve the optimization problem (8) with constant block approximation ($\ell = 0$). This algorithm is only available for binary observations, so we only use it for comparison under Bernoulli model.

We choose degree-2 polynomial approximation as our theorems suggested, and vary tensor dimension $d \in \{10, \dots, 100\}$ under each model configuration. Hyperparameters are set to achieve the best performance in the outputs. Possible hyperparameters are the block

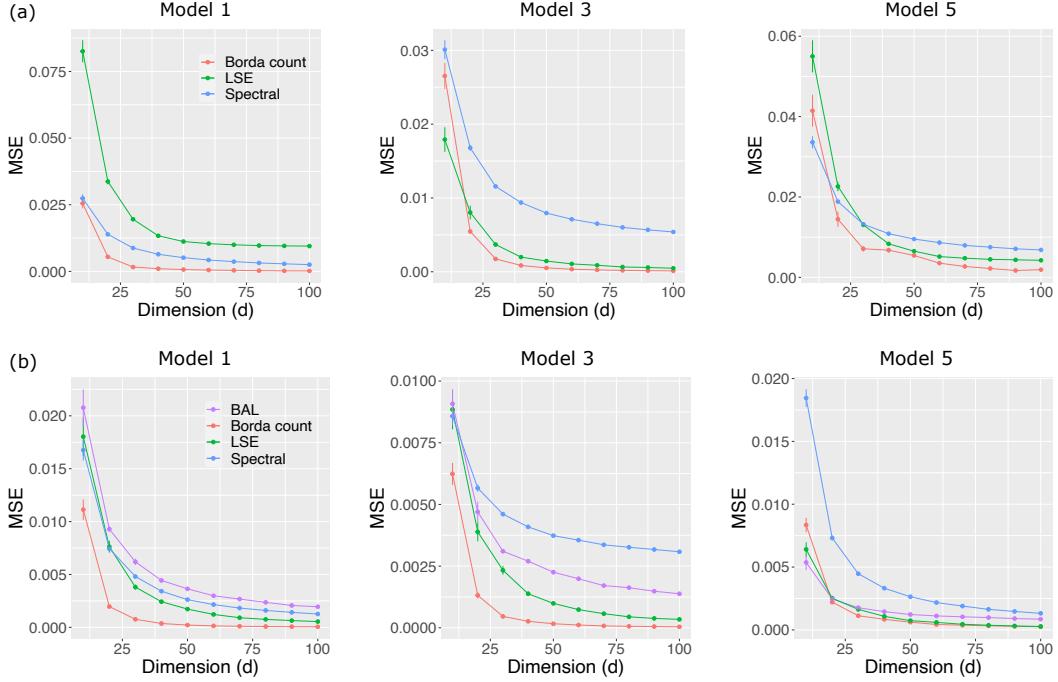


Figure 5: MSE versus the tensor dimension based on different estimation methods. Columns 1-3 consider the Models 1, 3, and 5 in Table 2 respectively. Panel (a) is for continuous tensors, whereas (b) is for the binary tensors.

number for **Borda count**, **LSE** and **BAL**, and the singular-value threshold for **Spectral**.

Figure 5 shows that our algorithm **Borda Count** achieves the best performance in all scenarios as the tensor dimension increases. The poor performance of **Spectral** can be explained by the loss of multilinear structure in the tensor unfolding procedure. The sub-optimality of **LSE** and **BAL** is possibly due to its limits in both statistics and computations. Statistically, constant block approximation results in sub-optimal rates compared to polynomial approximation. Computationally, the least-squares optimization (8) is computationally unstable. Figure 6 displays true signal tensors of three models and corresponding observed tensors of dimension $d = 80$ with Gaussian noise. We use oracle permutation π to obtain the estimated signal tensor from the estimated permuted signal tensor $\hat{\Theta} \circ \hat{\pi}$ for the better visualizations. We see that our **Borda count** achieves the best signal recovery, thereby

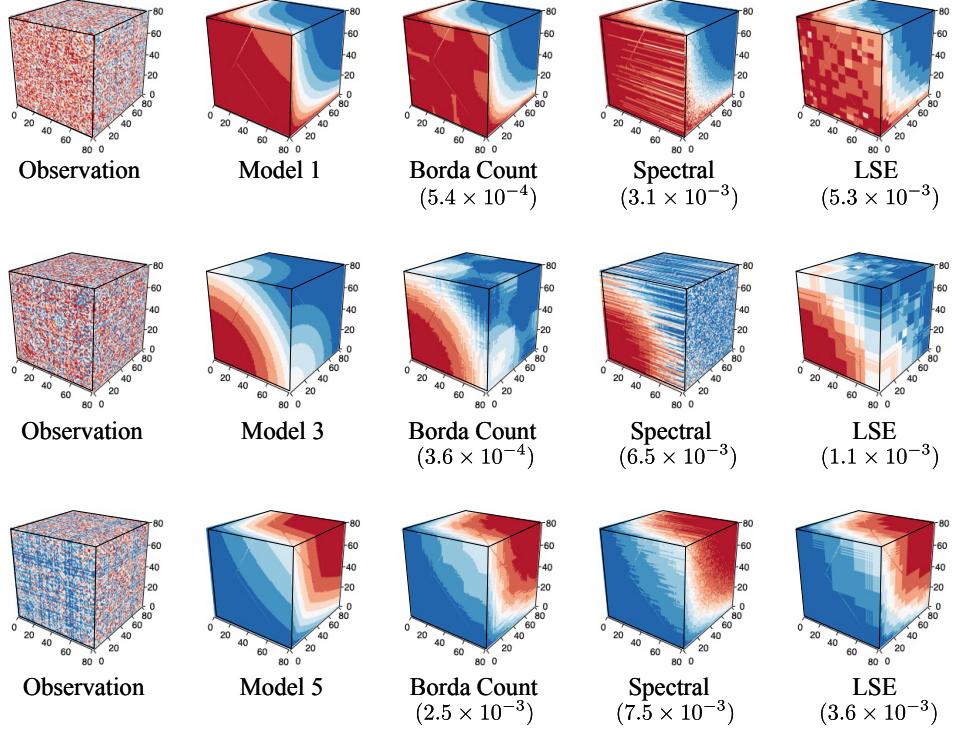


Figure 6: Performance comparison among different methods. The observed data tensors, true signal tensors, and estimated signal tensors are plotted for Models 1, 3 and 5 in Table 2 with fixed dimension $d = 80$. Numbers in parenthesis indicate the mean squared error.

supporting the numerical results in Figure 5.

6.2 Applications to Chicago crime data

The Chicago crime dataset consists of crime counts reported in the city of Chicago, ranging from January 1st, 2001 to December 11th, 2017. The observed tensor is an order-3 tensor with entries representing the log counts of crimes from 24 hours, 77 community areas, and 32 crime types. We apply our Borda count method to Chicago crime dataset. Because the data tensor is non-symmetric, we allow different number of blocks across the three modes. Cross validation result suggests the $(k_1, k_2, k_3) = (6, 4, 10)$, representing the block number for crime hours, community areas, and crime types, respectively.

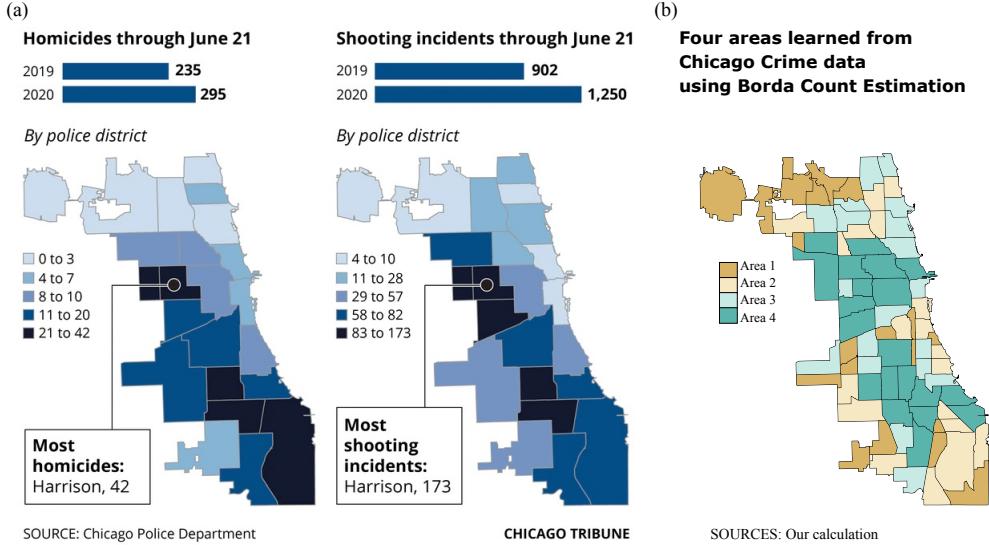


Figure 7: Chicago crime maps. Figure(a) is the benchmark map based on homicides and shooting incidents in community areas in Chicago [18]. Figure(b) shows the four clustered areas learned from 32 crime types using our method.

We first investigate the four community areas obtained from Borda count algorithm. Figure 7(b) shows the four areas overlaid on the Chicago map. Interestingly, we find that the clusters are consistent with actual locations, even though our algorithm did not take any geographic information such as longitude or latitude as inputs. In addition, we compare the cluster patterns with benchmark maps based on homicides and shooting incidents in Chicago shown in Figure 7(a). Our clusters share similar geographical patterns with Figure 7(a). The results demonstrate the power of our approach in detecting patterns from tensor data.

Then, we examine the denoised signal tensor obtained from our method and analyze the trends between crime types and crime hours by the four areas in Figure 7(b). Figure 8 shows the averaged log counts of crimes according to crime types and crime hours by four areas. We find that the major difference among four areas is the crime rates. Area 4 has the highest crime rates, and the crime rates monotonically decrease from Area 4 to Area 1. The variation in crime rates across hour and type, nevertheless, exhibits similarity among

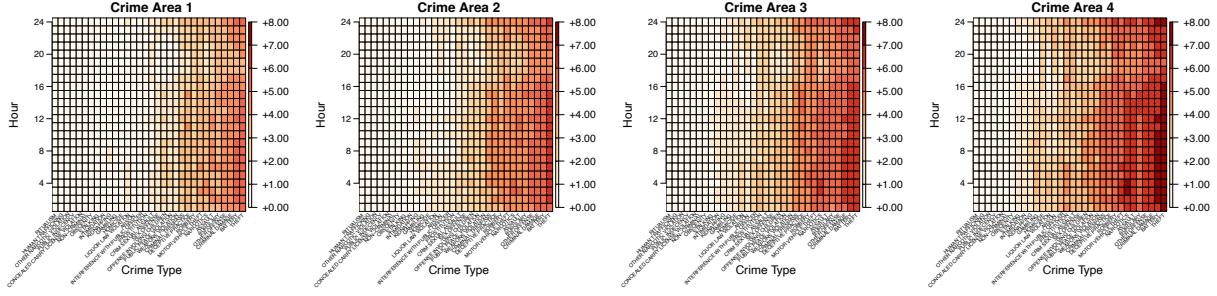


Figure 8: Averaged log counts of crimes according to crime types, hours, and the four areas estimated by our Borda count algorithm. We plot the estimated signal tensor entries averaged within four areas in the heatmap.

the four areas. Figure 8 shows that the number of crimes increases hourly from 8 p.m., peaks at night hours, and then drops to the lowest at 6 p.m. The identified patterns among the four community areas highlight the applicability of our method in real data.

Finally, we compare the prediction performance based on constant block model and our permuted smooth tensor model. Notice that constant block model uses $\ell = 0$ approximation, whereas our permuted smooth tensor model uses $\ell = 2$ approximation. We found that the mean squared prediction errors for our model vs. constant block model are 0.283 (0.006) vs. 0.399 (0.009), respectively. Here, the reported prediction errors are averaged over five runs of cross-validation, with standard errors in parentheses. The block number (k_1, k_2, k_3) with best prediction performance is $(6, 4, 10)$ for our models, and $(7, 11, 10)$ for constant block models. We see that the permuted smooth tensor model substantially outperforms the classical constant block models.

7 Conclusion and Discussions

We have presented a suite of statistical theory, estimation methods, and data applications for permuted smooth tensor models. Two estimation algorithms are proposed with accuracy

guarantees: the (statistically optimal) least-squares estimation and the (computationally tractable) Borda count estimation. In particular, we establish an interesting phase transition phenomenon with respect to the critical smoothness level. We demonstrate that a block-wise polynomial of order $(m - 2)(m + 1)/2$ is sufficient and necessary for accurate recovery of order- m tensors, in contrast to earlier beliefs on constant block approximation. Experiments demonstrate the effectiveness of both theoretical findings and algorithms.

We discuss several possible extensions from our work. One limitation of our model is that we consider the 1-dimensional latent space embedding. The extension from 1-dimensional latent space model to general dimensional latent model is analogous to the extension from the block model to the mixed membership model. Our parallel work [21] considers the general dimensional latent variable model, by assuming a set of s -dimensional vectors $\mathbf{a}_{i_k}^{(k)} \in \mathbb{R}^s$ with $s \geq 1$ and a latent function $f: [0, 1]^s \times \dots \times [0, 1]^s \rightarrow \mathbb{R}$ such that

$$\Theta(i_1, \dots, i_m) = f(\mathbf{a}_{i_1}^{(1)}, \dots, \mathbf{a}_{i_m}^{(m)}), \text{ for all } (i_1, \dots, i_m) \in [d_1] \times \dots \times [d_m].$$

This generalization extends the latent permutation $\pi \in \Pi(d, d)$ to the set of latent vectors in $[0, 1]^s$. However, we find that this extension is not free. We need a stronger analytic function class with ∞ -smoothness for the theoretical analysis. Compared to this current paper, the analysis of analytic functions uses different techniques and yields new results of its own. We refer readers to [21] for independent interest.

Another limitation of our algorithm is the requirement of hyperparameter tuning. There is a vast literature on nonparametric estimation that focuses on adaptivity. For example, spatially adaptive methods have been developed in the contexts of wavelets [12], splines [25], and trend filtering [30]; tuning-free algorithms have been proposed for several shape-constrained functions [9, 13, 3]; see [6] for a review. Our work is orthogonal to these advances, and in principle we can combine these tools in our tensor estimation. In this

paper, we choose standard polynomial algorithm because of its simplicity. The parsimony leads to an easier analysis on the critical smoothness level $(m - 2)(m + 1)/2$. Exploiting various nonparametric techniques for tensor models warrants future research.

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SUPPLEMENTARY MATERIALS

Appendix: The appendix includes the proofs and extra simulation results.

Software and data: R-package contains code to perform the methods described in the article. The package also contains all datasets used as examples in the article.

A Extra numerical results

A.1 Results for Models 2 and 4 in Table 1

We first present simulation results for Models 2 and 4 omitted in Section 6. Figure S2 compares the estimation performance among the **Borda count**, **LSE**, and **Spectral methods**. We find that our Borda count algorithm outperforms others in both models. The first two columns in Figure S3 show the impact of the number of blocks k and degree of polynomial ℓ for the approximation with fixed dimension $d = 100$. Similar to results for Models 1, 3 and 5 in the main paper, we find the optimal k balances the trade-off between approximation error and signal tensor estimation error within each block. The last two columns compare our **Borda count** with other alternative methods. We find that our method still outperforms **LSE** and **Spectral** in all scenarios under Models 2 and 4.

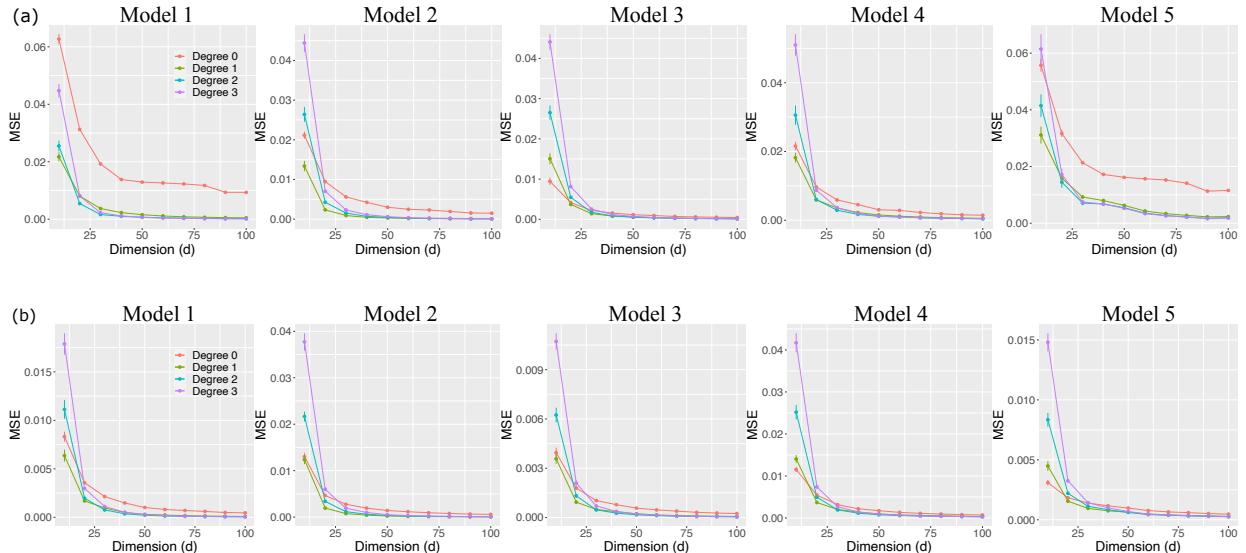


Figure S1: MSE versus the tensor dimension based on different polynomial approximations. Columns 1-5 consider the Models 1-5 in Table 2 respectively. Panel (a) is for continuous tensors, whereas (b) is for the binary tensors.

Suggested by one reviewer, we consider another competing algorithm (denoted as cluster + poly ℓ algorithm) for comparison. The algorithm performs clustering first and then

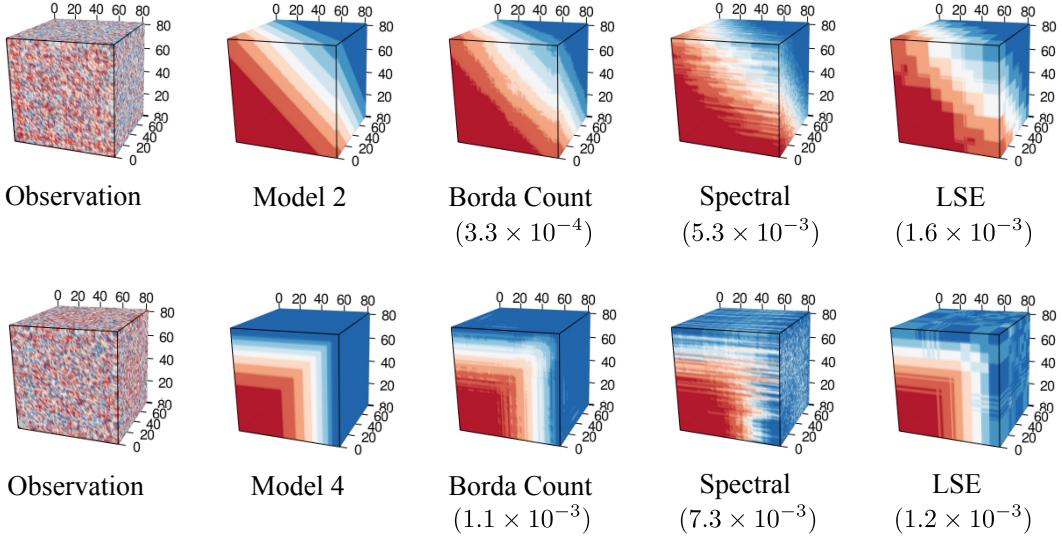


Figure S2: Performance comparison between different methods. The observed data tensors, true signal tensors, and estimated signal tensors are plotted for Models 2 and 4 in Table 2 with fixed dimension $d = 80$. Numbers in parenthesis indicate the mean squared error.

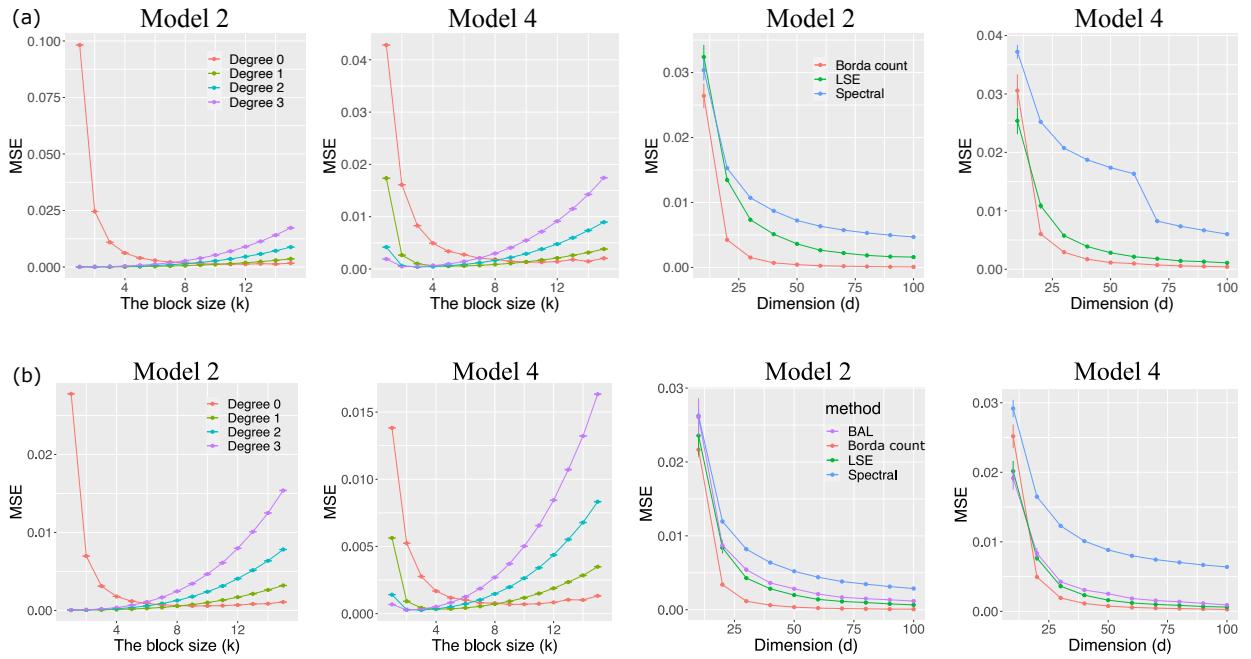


Figure S3: Simulation results for Models 2 and 4 in Table 2. Columns 1-2 plots MSE versus the number of blocks for different polynomial approximation, while Columns 3-4 shows the MSE versus the tensor dimension according to estimation methods. Panel (a) is for continuous tensors, whereas (b) is for the binary tensors.

polynomial approximation within clusters. Since we do not know the permutation, we randomly assign the order of nodes within a block. We perform the simulation using the same setting in Section 6. Figure S4 shows the comparison among all methods, including the new cluster + poly ℓ algorithm for $\ell \in \{1, 2, 3\}$. Notice that cluster + poly0 algorithm is equivalent to LSE algorithm. The result shows that the cluster + poly ℓ algorithms are unstable except Model 3.

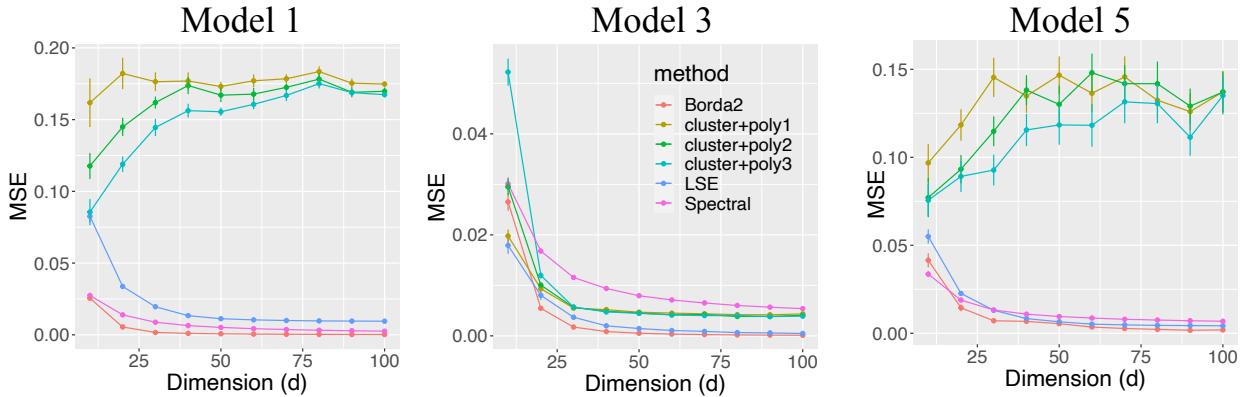


Figure S4: MSE versus the tensor dimension based on different estimation methods. Columns 1-3 consider the Models 1, 3, and 5 with continuous case in Table 2 respectively. cluster+poly ℓ means polynomial ℓ -approximation

A.2 Investigation of non-symmetric tensors

Our models and techniques easily extend to non-symmetric tensors. We describe the simulation set-up for non-symmetric tensors and results. We simulate order-3 tensors based on the non-symmetric functions in Table S1.

Model ID	$f(x, y, z)$
1	$xy + z$
2	$x^2 + y + yz^2$
3	$x(1 + \exp(-3(x^2 + y^2 + z^2)))^{-1}$
4	$\log(1 + \max(x, y, z) + x^2 + yz)$
5	$\exp(-x - \sqrt{y} - z^3)$

Table S1: List of non-symmetric smooth functions in simulation.

We fix the tensor dimension $30 \times 40 \times 50$ and assume that the noise tensors are from Gaussian distribution. Similar to other simulations, we evaluate the accuracy of the estimation by MSE and report the summary statistics across $n_{\text{sim}} = 20$ replicates. The

Method	Model 1	Model 2	Model 3	Model 4	Model 5
Borda count	(2,1,2)	(1,2,2)	(1,3,3)	(2,1,2)	(1,4,4)
LSE	(6,2,3)	(8,5,8)	(6,9,6)	(9,5,6)	(7,9,3)
Spectral	(1,24)	(3,48)	(1,48)	(1,28)	(1,22)

Table S2: Hyperparameters for the methods under Models 1-5 in Table S1. For **Borda count** and **LSE** methods, the values in the table indicate the number of blocks. For **Spectral** method, the first value indicates the tensor unfolding mode, while the second one represents the singular value threshold.

hyperparameters are chosen via cross-validation that give the best accuracy for each method. Table S2 summarizes the choice of hyperparameters.

Table S3 compares the MSEs from repeated simulations based on different methods under Models 1-5 (see Table S1). We find that Borda count estimation outperforms all alternative methods for non-symmetric tensors. The results demonstrate the applicability of our method to general tensors

Method	Model 1	Model 2	Model 3	Model 4	Model 5
Borda count	0.57 (0.01)	0.51 (0.02)	0.87 (0.02)	1.02 (0.02)	2.56 (0.21)
LSE	23.58 (0.03)	7.70 (0.04)	9.45 (0.05)	3.29 (0.05)	9.93 (0.03)
Spectral	10.76 (0.06)	10.64 (0.05)	6.27 (0.05)	10.90 (0.06)	5.24 (0.04)

Table S3: MSEs from 20 repeated simulations based on different methods. All numbers are displayed on the scales 10^{-3} . Standard errors are reported in parenthesis.

A.3 Extra results on Chicago crime data analysis

We investigate the ten groups of crime types clustered by our method. Table S4 shows that the clustering captures the similar type of crimes. For example, group 2 consists of misdemeanors such as public indecency, non-criminal, and concealed carry license violation, while group 6 represents sex-related offenses such as prostitution, sex offense, and crime sexual assault.

B Details on Example 4

We show that Borda count algorithm works perfectly in the Example 4. For simplicity, assume that d is an even number. The signal tensor is represented by

$$\Theta(i, j, k) = \left(\frac{i}{d} - 0.5 \right)^2 + \left(\frac{j}{d} \right) \left(\frac{k}{d} \right), \quad \text{for all } (i, j, k) \in [d]^3.$$

GROUP	I	II	III
CRIME TYPE	RITUALISM, HUMAN TRAFFICKING, OTHER NARCOTIC VIOLATION	PUBLIC INDECENCY, NON-CRIMINAL, CONCEALED CARRY LICENSE VIOLATION	OBScenity, STALKING, INTIMIDATION
GROUP	IV	V	VI
CRIME TYPE	KIDNAPPING, GAMBLING, HOMICIDE	LIQUOR LAW VIOLATION, ARSON, INTERFERENCE WITH PUBLIC OFFICER	PROSTITUTION, SEX OFFENSE, CRIM SEXUAL ASSAULT
GROUP	VII	VIII	VIII
CRIME TYPE	OTHER OFFENSE, CRIMINAL DAMAGE, BATTERY, THEFT, BURGLARY	CRIMINAL TRESPASS, ROBBERY, DECEPTIVE PRACTICE	NARCOTICS, ASSAULT, MOTOR VEHICLE THEFT
GROUP	X		
CRIME TYPE	PUBLIC PEACE VIOLATION, WEAPONS VIOLATION, OFFENSE INVOLVING CHILDREN		

Table S4: Groups of crime types learned based on the Borda count estimation.

We construct permutations π_1, π_2, π_3 and a multivariate function $\bar{f}: [0, 1]^3 \rightarrow \mathbb{R}$ such that

$$\Theta(i, j, k) = \bar{f} \left(\frac{\pi_1(i)}{d}, \frac{\pi_2(j)}{d}, \frac{\pi_3(k)}{d} \right) \pm \frac{1}{d^2}, \quad \text{for all } (i, j, k) \in [d]^3. \quad (23)$$

Define the permutations by

$$\pi_1(i) = \begin{cases} 2i - d, & \text{if } i > \frac{d}{2}, \\ d + 1 - 2i, & \text{if } i \leq \frac{d}{2}, \end{cases} \quad \text{and} \quad \pi_2(i) = \pi_3(i) = i. \quad (24)$$

Define the function $\bar{f}: [0, 1]^3 \rightarrow \mathbb{R}$ by

$$\begin{aligned} \bar{f}: [0, 1]^3 &\rightarrow [0, 1] \\ (x, y, z) &\mapsto \frac{1}{4}x^2 + yz. \end{aligned} \quad (25)$$

One can verify that \bar{f} is monotonic-and-smooth such that $\bar{f} \in \mathcal{F}(2, 1/4) \cap \mathcal{B}(1/2)$. Furthermore, the construction (24)-(25) satisfies (23), where the perturbation term $1/d^2$ can be absorbed into the approximation error by Proposition 1:

$$\frac{1}{d^m} \inf_{\mathcal{B} \in \mathcal{B}(k, \ell)} \|\Theta - \mathcal{B}\|_F \leq \frac{L}{k^{\min(2, \ell+1)}} + \frac{1}{d^2} \leq \frac{L+1}{k^{\min(2, \ell+1)}}.$$

In conclusion, the signal tensor constructed in the Example 4 can be regarded as a tensor generated from $\mathcal{F}(2, 5/4) \cap \mathcal{B}(1/2)$. Therefore, our Borda count algorithm is applicable to this case with the claimed accuracy in our paper. This example highlights that our marginal monotonicity assumption is weaker than the usual sense thanks to the unknown permutations.

C Proofs of main theorems

C.1 Proof of Proposition 1

Proof of Proposition 1. We denote \mathcal{E}_k as the m -way partition by

$$\mathcal{E}_k = \left\{ \bigtimes_{a=1}^m z^{-1}(j_a) : (j_1, \dots, j_m) \in [k]^m \right\},$$

where $z: [d] \rightarrow [k]$ is the canonical clustering function such that $z(i) = \lceil ki/d \rceil$, for all $i \in [d]$, and we use the shorthand $\times_{a=1}^m$ to denote the Cartesian product of m sets. For a given $\mathbf{j} := (j_1, \dots, j_m)$ and the related partition $\times_{a=1}^m z^{-1}(j_a) \in \mathcal{E}_k$, fix any index $\omega_{\mathbf{j}} \in \times_{a=1}^m z^{-1}(j_a)$. Then, we have

$$\|\omega - \omega_{\mathbf{j}}\|_\infty \leq \frac{d}{k}, \quad \text{for all } \omega \in \bigtimes_{a=1}^m z^{-1}(j_a).$$

We define the block-wise degree- ℓ polynomial tensor \mathcal{B} based on the partition \mathcal{E}_k as

$$\mathcal{B}(\omega) = \sum_{\mathbf{j} \in [k]^m} \text{Poly}_{\min(\lfloor \alpha \rfloor, \ell), \omega_{\mathbf{j}}} \left(\frac{\omega}{d} \right) \mathbb{1}\{\omega \in \bigtimes_{a=1}^m z^{-1}(j_a)\},$$

where $\text{Poly}_{\min(\lfloor \alpha \rfloor, \ell), \omega_{\mathbf{j}}}$ denotes a degree- ℓ polynomial function defined by

$$\text{Poly}_{\min(\lfloor \alpha \rfloor, \ell), \omega_{\mathbf{j}}}(\mathbf{x}) = \sum_{\kappa: |\kappa| \leq \min(\lfloor \alpha \rfloor, \ell+1)} \frac{1}{\kappa!} \left(\mathbf{x} - \frac{\omega_{\mathbf{j}}}{d} \right)^\kappa \nabla_\kappa f \left(\frac{\omega_{\mathbf{j}}}{d} \right).$$

It follows from Taylor's theorem that

$$\left| f \left(\frac{\omega}{d} \right) - \text{Poly}_{\min(\lfloor \alpha \rfloor, \ell), \omega_{\mathbf{j}}} \left(\frac{\omega}{d} \right) \right| \leq L \left\| \frac{\omega - \omega_{\mathbf{j}}}{d} \right\|_\infty^{\min(\alpha, \ell+1)} \leq L \left(\frac{1}{k} \right)^{\min(\alpha, \ell+1)}, \quad (26)$$

for all $\omega \in \times_{a=1}^m z^{-1}(j_a)$.

Based on the construction of block-wise degree- ℓ polynomial tensor \mathcal{B} , we have

$$\begin{aligned} & \frac{1}{d^m} \|\Theta - \mathcal{B}\|_F^2 \\ &= \frac{1}{d^m} \sum_{\omega \in [d]^m} |\Theta(\omega) - \mathcal{B}(\omega)|^2 \\ &= \frac{1}{d^m} \sum_{\mathbf{j} \in [k]^m} \sum_{\omega \in \times_{a=1}^m z^{-1}(j_a)} \left| f \left(\frac{\omega}{d} \right) - \text{Poly}_{\min(\lfloor \alpha \rfloor, \ell), \omega_{\mathbf{j}}} \left(\frac{\omega}{d} \right) \right|^2 \\ &\leq \frac{L^2}{k^{2\min(\alpha, \ell+1)}}, \end{aligned}$$

where the last inequality uses (26). \square

C.2 Proof of Theorem 1

Proof of Theorem 1. By Proposition 1, there exists a block-wise polynomial tensor $\mathcal{B} \in \mathcal{B}(k, \ell)$ such that

$$\|\mathcal{B} - \Theta\|_F^2 \lesssim \frac{L^2 d^m}{k^{2\min(\alpha, \ell+1)}}. \quad (27)$$

By the triangle inequality,

$$\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \Theta \circ \pi\|_F^2 \leq 2\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \mathcal{B} \circ \pi\|_F^2 + 2\underbrace{\|\mathcal{B} \circ \pi - \Theta \circ \pi\|_F^2}_{\text{Proposition 1}}. \quad (28)$$

Therefore, it suffices to bound $\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \mathcal{B} \circ \pi\|_F^2$. By the global optimality of least-square estimator, we have

$$\begin{aligned}\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \mathcal{B} \circ \pi\|_F &\leq \left\langle \frac{\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \mathcal{B} \circ \pi}{\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \mathcal{B} \circ \pi\|_F}, \mathcal{E} + (\Theta \circ \pi - \mathcal{B} \circ \pi) \right\rangle \\ &\leq \sup_{\pi, \pi': [d] \rightarrow [d]} \sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle + \underbrace{\|\mathcal{B} \circ \pi - \Theta \circ \pi\|_F}_{\text{Proposition 1}}.\end{aligned}$$

Now we bound inner product term. For fixed π, π' , let \mathbf{P} and \mathbf{P}' be permutation matrices corresponding to permutations π and π' respectively. We express vectorized block-wise degree- ℓ polynomial tensors, $\text{vec}(\mathcal{B})$ and $\text{vec}(\mathcal{B}')$, by discrete polynomial functions. Specifically, denote $\text{vec}(\mathcal{B}) = \mathbf{X}\boldsymbol{\beta}$ and $\text{vec}(\mathcal{B}') = \mathbf{X}\boldsymbol{\beta}'$, where $\mathbf{X} \in \mathbb{R}^{d^m \times k^m(\ell+m)^\ell}$ is a design matrix consisting of m -multivariate degree- ℓ polynomial basis over grid design $\{1/d, \dots, d/d\}$ (or $\{x_i\}_{i=1}^d$ for relaxed smooth model (5)), $\boldsymbol{\beta}$ and $\boldsymbol{\beta}' \in \mathbb{R}^{k^m(\ell+m)^\ell}$ are corresponding coefficient vectors. Notice that the number of coefficients for m -multivariate polynomial of degree- ℓ is $\binom{\ell+m}{\ell} \approx (\ell+m)^\ell$; we choose to use $(\ell+m)^\ell$ for each block for notational simplicity. Therefore, we rewrite the inner product by

$$\begin{aligned}\left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle &= \left\langle \frac{(\mathbf{P}')^{\otimes m} \text{vec}(\mathcal{B}') - (\mathbf{P})^{\otimes m} \text{vec}(\mathcal{B})}{\|(\mathbf{P}')^{\otimes m} \text{vec}(\mathcal{B}') - (\mathbf{P})^{\otimes m} \text{vec}(\mathcal{B})\|_F}, \mathcal{E} \right\rangle \\ &= \left\langle \frac{(\mathbf{P}')^{\otimes m} \mathbf{X}\boldsymbol{\beta}' - (\mathbf{P})^{\otimes m} \mathbf{X}\boldsymbol{\beta}}{\|(\mathbf{P}')^{\otimes m} \mathbf{X}\boldsymbol{\beta}' - (\mathbf{P})^{\otimes m} \mathbf{X}\boldsymbol{\beta}\|_F}, \mathcal{E} \right\rangle \\ &= \left\langle \frac{\mathbf{A}\mathbf{c}}{\|\mathbf{A}\mathbf{c}\|_2}, \mathcal{E} \right\rangle\end{aligned}$$

where we define $\mathbf{A} := (\mathbf{P}' \ - \mathbf{P}) \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{X} \end{pmatrix} \in \mathbb{R}^{d^m \times 2k^m(\ell+m)^\ell}$ and $\mathbf{c} := \begin{pmatrix} \boldsymbol{\beta}' \\ \boldsymbol{\beta} \end{pmatrix} \in \mathbb{R}^{2k^m(\ell+m)^\ell}$.

By Lemma 8, we have

$$\sup_{\mathbf{c} \in \mathbb{R}^{2k^m(\ell+m)^\ell}} \left\langle \frac{\mathbf{A}\mathbf{c}}{\|\mathbf{A}\mathbf{c}\|_2}, \mathcal{E} \right\rangle \leq \sup_{\mathbf{c} \in \mathbb{R}^{2k^m(\ell+m)^\ell}} \left\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, \mathbf{e} \right\rangle, \quad (29)$$

where $\mathbf{e} \in \mathbb{R}^{2k^m(\ell+m)^\ell}$ is a sub-Gaussian random vector with variance proxy bounded by σ^2 . By the union bound of sub-Gaussian maxima over countable set $\{\pi, \pi': [d] \rightarrow [d]\}$, we obtain

$$\begin{aligned}&\mathbb{P} \left(\sup_{\pi, \pi': [d] \rightarrow [d]} \sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle \geq t \right) \\ &\leq \sum_{\pi, \pi' \in [d]^d} \mathbb{P} \left(\sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle \geq t \right)\end{aligned}$$

$$\begin{aligned}
&\leq d^d \mathbb{P} \left(\sup_{\mathbf{c} \in \mathbb{R}^{2k^m(\ell+m)^\ell}} \left\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, \mathbf{e} \right\rangle \geq t \right) \\
&\leq \exp \left(-\frac{t^2}{8\sigma^2} + k^m(\ell+m)^\ell \log 6 + d \log d \right), \tag{30}
\end{aligned}$$

where the second inequality is from (29) and the last inequality is from Lemma 9. Setting $t = C\sigma\sqrt{k^m(\ell+m)^\ell + d \log d}$ in (30) for a sufficiently large $C > 0$ gives

$$\sup_{\pi, \pi': [d] \rightarrow [d]} \sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi}{\|\mathcal{B}' \circ \pi' - \mathcal{B} \circ \pi\|_F}, \mathcal{E} \right\rangle \lesssim \sigma \sqrt{k^m(\ell+m)^\ell + d \log d},$$

with high probability.

Combining the inequalities (27), (28) and (30) yields the desired conclusion

$$\|\hat{\Theta}^{\text{LSE}} \circ \hat{\pi}^{\text{LSE}} - \Theta \circ \pi\|_F^2 \lesssim \sigma^2 (k^m(\ell+m)^\ell + d \log d) + \frac{L^2 d^m}{k^{2 \min(\alpha, \ell+1)}}. \tag{31}$$

Finally, optimizing (31) with respect to (k, l) gives that

$$(31) \lesssim \begin{cases} L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}}, & \text{when } \alpha < m(m-1)/2, \\ \sigma^2 d^{-(m-1)} \log d, & \text{when } \alpha \geq m(m-1)/2, \end{cases}$$

under the choice

$$\ell^* = \min(\lfloor \alpha \rfloor, (m-2)(m+1)/2), \quad k^* = \left\lceil \left(d^m L^2 / \sigma^2 \right)^{\frac{1}{m+2 \min(\alpha, \ell^*+1)}} \right\rceil.$$

□

C.3 Proof of Theorem 2

Proof of Theorem 2. By the definition of the tensor space, we seek the minimax rate ε^2 in the following expression

$$\inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\Theta \in \mathcal{P}(\alpha, L)} \sup_{\pi \in \Pi(d, d)} \mathbb{P} \left(\frac{1}{d^m} \|\Theta \circ \pi - \hat{\Theta} \circ \hat{\pi}\|_F^2 \geq \varepsilon^2 \right).$$

On one hand, if we fix a permutation $\pi \in \Pi(d, d)$, the problem can be viewed as a classical m -dimensional α -smooth nonparametric regression with d^m sample points. The minimax lower bound is known to be $\varepsilon^2 = L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}}$. On the other hand, if we fix $\Theta \in \mathcal{P}(\alpha, L)$, the problem become a new type of convergence rate due to the unknown permutation. We refer to the resulting error as the permutation rate, and we will prove that $\varepsilon^2 = \sigma^2 d^{-(m-1)} \log d$. Since our target is the sum of the two rates, it suffice to prove the two different rates separately. In the following arguments, we will proceed by this strategy.

Nonparametric rate. The nonparametric rate for α -smooth function is readily available in the literature. We cite the results here for self-completeness.

Lemma 1 (Minimax rate for α -smooth function estimation; see Section 2 in Stone [29]). *Consider a sample of N data points, $(\mathbf{x}_1, Y_1), \dots, (\mathbf{x}_N, Y_N)$, where $\mathbf{x}_n = (\frac{i_1}{d}, \dots, \frac{i_m}{d}) \in [0, 1]^m$ is the m -dimensional predictor and $Y_n \in \mathbb{R}$ is the scalar response for $n \in [N]$. Consider the observation model*

$$Y_n = f(\mathbf{x}_n) + \varepsilon_n, \quad \text{with } \varepsilon_n \sim \text{i.i.d. } \mathcal{N}(0, 1), \quad \text{for all } n \in [N].$$

Assume f is in the α -Hölder smooth function class, denoted by $\mathcal{F}(\alpha, L)$. Then,

$$\inf_{\hat{f}} \sup_{f \in \mathcal{F}(\alpha, L)} \mathbb{P} \left(\|f - \hat{f}\|_2 \geq \sigma^{\frac{4\alpha}{m+2\alpha}} L^{\frac{2m}{m+2\alpha}} N^{-\frac{2\alpha}{m+2\alpha}} \right) \geq 0.9.$$

Our desired nonparametric rate readily follows from Lemma 1 by taking sample size $N = d^m$ and function norm $\|f - \hat{f}\|_2 = \frac{1}{d^m} \|\Theta - \hat{\Theta}\|_F^2$. In summary, for a given permutation $\pi \in \Pi(d, d)$, we have

$$\inf_{\hat{\Theta}} \sup_{\Theta \in \mathcal{P}(\alpha, L)} \mathbb{P} \left(\frac{1}{d^m} \|\hat{\Theta} \circ \pi - \Theta \circ \pi\|_F^2 \geq L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}} \right) \geq 0.9. \quad (32)$$

Permutation rate. Since nonparametric rate dominates permutation rate when $\alpha \leq 1$, it is sufficient to prove the permutation rate lower bound for $\alpha \geq 1$. We first show the minimax permutation rate for k -block degree-0 tensor family $\mathcal{B}(k, 0)$, and then construct a smooth $f \in \mathcal{F}(\alpha, L)$ to mimic the constant block tensors.

Let $\Pi(d, k)$ denote the collection of all possible onto mappings from $[d]$ to $[k]$. Lemma 2 shows the permutation rate over k -block degree-0 tensor family $\mathcal{B}(k, 0)$ is $\sigma^2 d^{-(m-1)} \log k$.

Lemma 2 (Permutation error for tensor block model). *Consider the problem of estimating d -dimensional, block- k signal tensors from the Gaussian tensor block model. For every given integer $k \in [d]$, there exists a core tensor $\mathcal{S} \in \mathbb{R}^{k \times \dots \times k}$ satisfying*

$$\inf_{\hat{\Theta}} \sup_{z \in \Pi(d, k)} \mathbb{P} \left(\frac{1}{d^m} \|\hat{\Theta} - \mathcal{S} \circ z\|_F^2 \gtrsim \frac{\sigma^2 \log k}{d^{m-1}} \right) \geq 0.9. \quad (33)$$

The proof of Lemma 2 is constructive and deferred to Section G. The core tensor \mathcal{S} in Lemma 2 has a special pattern of zero's that will be used in the proof of Theorem 2; see the Section G for details.

To prove the permutation rate, we fix a core tensor $\mathcal{S} \in \mathbb{R}^{k \times \dots \times k}$ satisfying (33), and use it to construct the smooth tensor. we construct a function $f \in \mathcal{F}(\alpha, L)$ that mimics the core tensor \mathcal{S} in block tensor family $\mathcal{B}(k, 0)$. For notational simplicity, we do not distinguish the fractional number and its rounding integer. For example, we simply write $d/2$ (instead of

$\lceil d/2 \rceil$ or $\lfloor d/2 \rfloor$) to represent its rounding integer. Define $k = d^\delta$ for some $\delta \in (0, 1)$, which will be specified later. Consider a smooth function $K(x)$ that is infinitely differentiable,

$$K(x) = C_k \exp\left(-\frac{1}{1 - 64x^2}\right) \mathbb{1}\left\{|x| < \frac{1}{8}\right\},$$

where $C_k > 0$ satisfies $\int K(x)dx = 1$. Then, we define a smooth convolution function as

$$\psi(x) = \int_{-3/8}^{3/8} K(x-y)dy.$$

The smooth convolution function has support $[-1/2, 1/2]$ and takes value 1 on the interval $[-1/4, 1/4]$. For a given core tensor \mathcal{S} in Lemma 2, we define an α -smooth function

$$f(x_1, \dots, x_m) = \sum_{(a_1, \dots, a_m) \in [k]^m} \left(\mathcal{S}(a_1, \dots, a_m) - \frac{1}{2} \right) \prod_{i=1}^m \psi\left(kx_i - a_i + \frac{1}{2}\right) + \frac{1}{2}. \quad (34)$$

One can verify that $f \in \mathcal{F}(\alpha, L)$ as long as we choose sufficiently small δ depending on α and L . Notice that for any $(a_1, \dots, a_m) \in [k]^m$,

$$f(x_1, \dots, x_m) = \mathcal{S}(a_1, \dots, a_m), \quad \text{if } (x_1, \dots, x_m) \in \bigtimes_{i=1}^m \left[\frac{a_i - 3/4}{k}, \frac{a_i - 1/4}{k} \right]. \quad (35)$$

From this observation, we define a sub-domain $I \subset [d]$ such that

$$I = \left(\bigcup_{a=1}^k \left[\frac{d(a - 3/4)}{k}, \frac{d(a - 1/4)}{k} \right] \right) \cap [d]. \quad (36)$$

We have that $|I| = d/2$ by definition. Let $\Theta(\mathcal{S}) \in \mathbb{R}^{d \times \dots \times d}$ denote the tensor induced by f in (34). We use subscript I to denote objects when restricted in the indices set I . For example, $\Theta_I(\mathcal{S}) \in \mathbb{R}^{d/2 \times \dots \times d/2}$ denotes the sub-tensor with indices in I , and $\|\cdot\|_{F,I}$ denotes the sum of squares over indices in I . Based on (35) and (36), $\Theta_I(\mathcal{S})$ has block structure with the core tensor \mathcal{S} . We use $\Pi(d/2, d/2)$ to denote the set of all permutations on I while fixing indices on $[d] \setminus I$; that is, $\Pi(d/2, d/2) = \{\pi: I \rightarrow I\} \cong \{\pi \in \Pi(d, d): \pi(i) = i \text{ for } i \in [d] \setminus I\}$. Then, we have

$$\begin{aligned} & \inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\pi \in \Pi(d, d)} \mathbb{P} \left(\frac{1}{d^m} \|\hat{\Theta} \circ \hat{\pi} - \Theta(\mathcal{S}) \circ \pi\|_F^2 \geq \varepsilon^2 \right) \\ & \stackrel{(*)}{=} \inf_{\hat{\Theta}} \sup_{\pi \in \Pi(d, d)} \mathbb{P} \left(\frac{1}{d^m} \|\hat{\Theta} - \Theta(\mathcal{S}) \circ \pi\|_F^2 \geq \varepsilon^2 \right) \\ & \geq \inf_{\hat{\Theta}} \sup_{\pi \in \Pi(d, d)} \mathbb{P} \left(\frac{1}{d^m} \|\hat{\Theta} - \Theta(\mathcal{S}) \circ \pi\|_{F,I}^2 \geq \varepsilon^2 \right) \\ & \stackrel{(**)}{\geq} \inf_{\hat{\Theta}} \sup_{\pi \in \Pi(d/2, d/2)} \mathbb{P} \left(\frac{1}{(d/2)^m} \|\hat{\Theta}_I - \Theta_I(\mathcal{S}) \circ \pi\|_F^2 \geq 2^m \varepsilon^2 \right) \end{aligned}$$

$$= \inf_{\hat{\Theta}} \sup_{z \in \Pi(d/2, k)} \mathbb{P} \left(\frac{1}{(d/2)^m} \|\hat{\Theta}_I - \mathcal{S} \circ z\|_F^2 \geq 2^m \varepsilon^2 \right), \quad (37)$$

where (*) absorbs the estimate $\hat{\pi}$ into the estimate $\hat{\Theta}$, and (**) uses the constructed function (34) and the permutation collections $\Pi(d/2, d/2)$. Based on the construction of \mathcal{S} in Lemma 2, the cross terms in $(\mathcal{S} \circ z)(I, I^c, \dots)$ are zero. Therefore, we reduce the problem of estimating $\pi: [d] \rightarrow [d]$ to estimating $z: I \rightarrow [k]$ in the sub-tensor. Applying Lemma 2 to (37) by using $d/2$ in the place of d and $k = d^\delta$ for a constant $\delta > 0$ yields the desired conclusion.

Combining two rates. Now, we combine (32) and (37) to get the desired lower bound. For any Θ generated as in (4) with $f \in \mathcal{F}(\alpha, L)$, by union bound, we have

$$\begin{aligned} & \mathbb{P} \left\{ \frac{1}{d^m} \|\hat{\Theta} - \Theta\|_F^2 \gtrsim L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}} + \frac{\sigma^2 \log d}{d^{m-1}} \right\} \\ & \geq \mathbb{P} \left\{ \frac{1}{d^m} \|\hat{\Theta} - \Theta\|_F^2 \gtrsim L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}} \right\} + \mathbb{P} \left\{ \frac{1}{d^m} \|\hat{\Theta} - \Theta\|_F^2 \gtrsim \frac{\sigma^2 \log d}{d^{m-1}} \right\} - 1. \end{aligned}$$

Taking sup on both sides with the property

$$\sup_{\substack{\Theta \in \mathcal{P}(\alpha, L) \\ \pi \in \Pi(d, d)}} (f(\pi) + g(\Theta)) = \sup_{\pi \in \Pi(d, d)} f(\pi) + \sup_{\Theta \in \mathcal{P}(\alpha, L)} g(\Theta)$$

yields the desired rate (12). \square

C.4 Proof of Theorem 3

The proof of Theorem 3 leverages results of hypergraphic planted clique and constant higher-order clustering problems. We first briefly explain the constant higher-order clustering problems. We then prove the main result.

C.4.1 Constant higher-order clustering and computational lower bound

Let $\mathbf{k} = (k_1, \dots, k_m)$ and $\mathbf{d} = (d_1, \dots, d_m)$. We introduce the constant high-order clustering (CHC) problem [24]. Consider a data tensor $\mathcal{Y} \in \mathbb{R}^{d_1 \times \dots \times d_m}$ generated from the signal plus noise model

$$\mathcal{Y} = \Theta + \mathcal{E}, \quad (38)$$

where the entries in \mathcal{E} are i.i.d. drawn from Gaussian distribution, and the signal tensor Θ contains a constant planted structure:

$$\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda) := \{ \lambda' \mathbb{1}_{I_1} \otimes \dots \otimes \mathbb{1}_{I_m} : |I_i| = k_i, \text{ for all } i \in [m], \lambda' \geq \lambda \}.$$

Here $I_i \subset [d_i]$ denotes a subset of indices, $|\cdot|$ denotes the cardinality of the set, $\mathbb{1}_{I_i}$ is the d_i -dimensional indicator vector such that $(\mathbb{1}_{I_i})_j = 1$ if $j \in I_i$ and 0 otherwise. The CHC detection problem is to test the following hypothesis based on the observed tensor \mathcal{Y} ,

$$H_0: \Theta = 0 \quad \text{v.s.} \quad H_1: \Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda). \quad (39)$$

The following proposition provides the asymptotic regime for impossible polynomial-time detection of CHC under Conjecture 1. This proposition plays important role to prove our Theorem 3.

Proposition 2 (Theorem 15 in [24]). *Consider CHC detection problem in (39) in the Gaussian noise model (38) under the asymptotic regime $d \rightarrow \infty$ satisfying*

$$d = d_1 = \cdots = d_m, \quad k = k_1 = \cdots = k_m = d^\delta, \quad \lambda = d^{-\gamma},$$

with $0 \leq \delta \leq 1$ and $\gamma > (m\delta - m/2) \vee 0$. Then, under Conjecture 1, for any polynomial-time test sequence $\{\phi\}_d: \mathcal{Y} \mapsto \{0, 1\}$, we have

$$\liminf_{d \rightarrow \infty} \left\{ \mathbb{P}_{\mathcal{H}_0}(\phi(\mathcal{Y}) = 1) + \sup_{\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)} \mathbb{P}_{\Theta}(\phi(\mathcal{Y}) = 0) \right\} \geq \frac{1}{2}.$$

C.4.2 Proof of Theorem 3

Proof of Theorem 3. Assume that the true signal $\Theta \in \mathbb{R}^{d \times \cdots \times d}$ has constant planted structure with a given $\mathbf{k} = (k, \dots, k)$ such that

$$\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda) = \{\lambda \mathbb{1}_I \otimes \cdots \otimes \mathbb{1}_I : |I| = k\}.$$

We have $\Theta_{\text{CHC}} \subset \mathcal{P}_{\text{rex}}$, because we can set the infinitely smooth function $f: [0, 1]^m \rightarrow [0, 1]$ by

$$f(x_1, \dots, x_m) = \lambda \prod_{i \in [m]} x_i. \quad (40)$$

Then, there is one-to-one correspondence between tensors in $\Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)$ and tensors generated by the above f under the choice $x_i = \mathbb{1}_{i \in I}$ for all $i \in [m]$.

We consider the regime where polynomial-time solvable test is impossible based on Proposition 2. We set $\delta = 1/2$, $k = c_1 d^\delta$, and $\lambda = c_2 d^{-\gamma}$ for any fixed $\gamma \in (0, \frac{2\alpha-m}{m})$, so that any polynomial-time test sequence ϕ satisfies

$$\liminf_{d \rightarrow \infty} \left\{ \mathbb{P}_{\mathcal{H}_0}(\phi(\mathcal{Y}) = 1) + \sup_{\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)} \mathbb{P}_{\Theta}(\phi(\mathcal{Y}) = 0) \right\} \geq \frac{1}{2}.$$

The choice of λ is possible given that $\alpha > m/2$. Notice that the choice $\lambda \lesssim O(1)$ ensures the function (40) satisfies the definition (1) for all $\alpha > 0$.

We prove by contradiction. Assume that there exists a hypothetical estimator $\hat{\Theta}$ from a polynomial-time algorithm that attains the rate $\text{Rate}(d)$. Specifically, there exists a constant $b > 0$ such that

$$\limsup_{d \rightarrow \infty} \frac{1}{\text{Rate}(d)} \sup_{\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)} \frac{1}{d^m} \mathbb{E} \|\hat{\Theta} - \Theta\|_F^2 \leq b. \quad (41)$$

By Markov's inequality, the inequality (41) implies that, when d is sufficiently large, for all $\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)$ and all $u > 0$, we have

$$\|\hat{\Theta} - \Theta\|_F \leq u \sqrt{\text{Rate}(d)d^m}, \quad (42)$$

with probability at least $1 - b/u$. Consider the hypothesis test in (39). We employ the following test

$$\phi(\mathcal{Y}) = \mathbf{1}(\|\hat{\Theta}\|_F \geq u \sqrt{\text{Rate}(d)d^m}).$$

The Type I error of the test ϕ is controlled by

$$\mathbb{P}_0(\|\hat{\Theta}\|_F \geq u \sqrt{\text{Rate}(d)d^m}) = \mathbb{P}_0(\|\hat{\Theta} - \Theta\|_F \geq u \sqrt{\text{Rate}(d)d^m}) \leq b/u.$$

For Type II error, we obtain,

$$\begin{aligned} \sup_{\Theta \in \Theta_{\text{CHC}}(k, d, \lambda)} \mathbb{P}_{\Theta}(\phi(\mathcal{Y}) = 0) &= \sup_{\Theta \in \Theta_{\text{CHC}}(k, d, \lambda)} \mathbb{P}_{\Theta}(\|\hat{\Theta}\|_F < u \sqrt{\text{Rate}(d)d^m}) \\ &\leq \sup_{\Theta \in \Theta_{\text{CHC}}(k, d, \lambda)} \mathbb{P}_{\Theta}(\|\hat{\Theta} - \Theta\|_F^2 > \|\Theta\|_F^2 - u^2 \text{Rate}(d)d^m) \\ &\stackrel{(*)}{\leq} \sup_{\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)} \mathbb{P}_{\Theta}(\|\hat{\Theta} - \Theta\|_F^2 > u^2 \text{Rate}(d)d^m) \\ &\stackrel{(**)}{\leq} b/u. \end{aligned}$$

The inequality $(*)$ holds because

$$\|\Theta\|_F^2 \geq \lambda^2 k^m = c_1^m c_2^2 d^{\frac{m}{2} - \gamma} \geq 2u^2 \text{Rate}(d)d^m \asymp d^{\frac{m}{2} - \frac{2\alpha-m}{m}}$$

where the last inequality is true under the regime $c_1^m c_2 > 2u^2$. We can always choose constants c_1 and c_2 given the value u . The inequality $(**)$ holds because of the statement (42). Putting Type I and II errors together, we obtain

$$\mathbb{P}_{\mathcal{H}_0}(\phi(\mathcal{Y}) = 1) + \sup_{\Theta \in \Theta_{\text{CHC}}(\mathbf{k}, \mathbf{d}, \lambda)} \mathbb{P}_{\Theta}(\phi(\mathcal{Y}) = 0) \leq 2b/u < 1/2,$$

for $u > 4b$. This fact contradicts the Proposition 2. Therefore, there is no polynomial-time $\hat{\Theta}$ satisfying (41). \square

C.5 Proof of Theorem 4

We first present a lemma to show the estimation error of $\hat{\pi}$. The exponent β measures the difficulty for permutation estimation. We find that a larger β guarantees a faster consistency rate of $\hat{\pi}^{\text{BC}}$, which is represented below 2.

Lemma 3 (Permutation error). *Consider the sub-Gaussian tensor model (3) with $f \in \mathcal{M}(\beta)$. With high probability, we have*

$$\text{Loss}(\pi, \hat{\pi}^{\text{BC}}) := \frac{1}{d} \max_{i \in [d]} |\pi(i) - \hat{\pi}^{\text{BC}}(i)| \lesssim \left(d^{-(m-1)/2} \sqrt{\log d} \right)^{\beta}.$$

The proof of Lemma 3 is provided in Section G.

Proof of Theorem 4. By Proposition 1, there exists a block-wise polynomial tensor $\mathcal{B} \in \mathcal{B}(k, \ell)$ satisfying (27). By the triangle inequality, we decompose estimation error into three terms,

$$\begin{aligned} & \| \hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi \|_F \\ & \leq \| \hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}} - \mathcal{B} \circ \hat{\pi}^{\text{BC}} \|_F + \| \mathcal{B} \circ \hat{\pi}^{\text{BC}} - \Theta \circ \hat{\pi}^{\text{BC}} \|_F + \| \Theta \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi \|_F \\ & = \underbrace{\| \hat{\Theta}^{\text{BC}} - \mathcal{B} \|_F}_{\text{Nonparametric error}} + \underbrace{\| \Theta \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi \|_F}_{\text{Permutation error}} + \underbrace{\| \mathcal{B} - \Theta \|_F}_{\text{Proposition 1}}. \end{aligned} \quad (43)$$

Therefore, it suffices to bound two terms $\| \Theta \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi \|_F$ and $\| \hat{\Theta}^{\text{BC}} - \mathcal{B} \|_F$ separately.

Permutation error. For any $(i_1, \dots, i_m) \in [d]^m$, we have

$$\begin{aligned} & | \Theta(\hat{\pi}^{\text{BC}}(i_1), \dots, \hat{\pi}^{\text{BC}}(i_m)) - \Theta(\pi(i_1), \dots, \pi(i_m)) | \\ & \leq \left\| \left(\frac{\hat{\pi}^{\text{BC}}(i_1)}{d}, \dots, \frac{\hat{\pi}^{\text{BC}}(i_m)}{d} \right) - \left(\frac{\pi(i_1)}{d}, \dots, \frac{\pi(i_m)}{d} \right) \right\|_{\infty}^{\min(\alpha, 1)} \\ & \leq \left[\frac{1}{d} \max_{i \in [d]} |\hat{\pi}^{\text{BC}}(i) - \pi(i)| \right]^{\min(\alpha, 1)} \\ & \lesssim \left(\sigma d^{-(m-1)/2} \sqrt{\log d} \right)^{\beta \min(\alpha, 1)}, \end{aligned}$$

where the first inequality is from the α -Hölder smoothness of Θ , and the last inequality is from Lemma 3. Therefore, we obtain the upper bound of the permutation error

$$\frac{1}{d^m} \| \Theta \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi \|_F^2 \lesssim \left(\sigma^2 \frac{\log d}{d^{m-1}} \right)^{\beta \min(\alpha, 1)}. \quad (44)$$

Nonparametric error. Recall that Borda count estimation is defined by $\hat{\Theta}^{\text{BC}} := \arg \min_{\Theta \in \mathcal{B}(k, \ell)} \|\tilde{\mathcal{Y}} - \Theta\|_F^2$, where $\tilde{\mathcal{Y}} = \mathcal{Y} \circ (\hat{\pi}^{\text{BC}})^{-1}$. By the optimality of least-square estimator, we have

$$\| \hat{\Theta}^{\text{BC}} - \mathcal{B} \|_F \leq \left\langle \frac{\hat{\Theta}^{\text{BC}} - \mathcal{B}}{\| \hat{\Theta}^{\text{BC}} - \mathcal{B} \|_F}, \mathcal{Y} \circ \pi \circ (\hat{\pi}^{\text{BC}})^{-1} - \mathcal{B} \right\rangle$$

$$\begin{aligned}
&\equiv \left\langle \frac{\hat{\Theta}^{\text{BC}} - \mathcal{B}}{\|\hat{\Theta}^{\text{BC}} - \mathcal{B}\|_F}, \mathcal{E} + (\Theta \circ \pi \circ (\hat{\pi}^{\text{BC}})^{-1} - \mathcal{B}) \right\rangle \\
&\leq \sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' - \mathcal{B}}{\|\mathcal{B}' - \mathcal{B}\|_F}, \mathcal{E} \right\rangle + \|\Theta \circ \pi - \mathcal{B} \circ \hat{\pi}^{\text{BC}}\|_F \\
&\leq \sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \underbrace{\left\langle \frac{\mathcal{B}' - \mathcal{B}}{\|\mathcal{B}' - \mathcal{B}\|_F}, \mathcal{E} \right\rangle}_{\text{Permutation error (44)}} + \underbrace{\|\Theta \circ \pi - \Theta \circ \hat{\pi}^{\text{BC}}\|_F}_{\text{Proposition 1}} + \underbrace{\|\Theta - \mathcal{B}\|_F}_{\text{Proposition 1}}
\end{aligned}$$

Now we bound the inner product term. By the same argument in the proof of Theorem 1, the space embedding $\mathcal{B}(k, \ell) \subset \mathbb{R}^{(\ell+m)^{\ell} k^m}$ implies the space embedding $\{(\mathcal{B}' - \mathcal{B}): \mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)\} \subset \mathbb{R}^{2(\ell+m)^{\ell} k^m}$. Therefore, we have

$$\sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' - \mathcal{B}}{\|\mathcal{B}' - \mathcal{B}\|_F}, \mathcal{E} \right\rangle \leq \sup_{\mathbf{c} \in \mathbb{R}^{2k^m(\ell+m)^\ell}} \left\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, e \right\rangle, \quad (45)$$

where $e \in \mathbb{R}^{2k^m(\ell+m)^\ell}$ is a sub-Gaussian random vector with variance proxy σ^2 . Combining (45) and Lemma 9 yields

$$\begin{aligned}
\mathbb{P} \left(\sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' - \mathcal{B}}{\|\mathcal{B}' - \mathcal{B}\|_F}, \mathcal{E} \right\rangle \geq t \right) &\leq \mathbb{P} \left(\sup_{\mathbf{c} \in \mathbb{R}^{2k^m(\ell+m)^\ell}} \left\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, e \right\rangle \geq t \right) \\
&\leq \exp \left(-\frac{t^2}{8\sigma^2} + k^m(\ell+m)^\ell \log 6 \right),
\end{aligned}$$

Setting $t = C\sigma\sqrt{k^m(\ell+m)^\ell}$ for sufficiently large $C > 0$ gives

$$\sup_{\mathcal{B}, \mathcal{B}' \in \mathcal{B}(k, \ell)} \left\langle \frac{\mathcal{B}' - \mathcal{B}}{\|\mathcal{B}' - \mathcal{B}\|_F}, \mathcal{E} \right\rangle \lesssim \sigma\sqrt{k^m(\ell+m)^\ell}, \quad (46)$$

with high probability.

Finally, combining all sources of error from Proposition 1 and inequalities (44), (46), (43) yields

$$\frac{1}{d^m} \|\hat{\Theta}^{\text{BC}} \circ \hat{\pi}^{\text{BC}} - \Theta \circ \pi\|_F \lesssim \left(\sigma^2 \frac{\log d}{d^{m-1}} \right)^{\beta \min(\alpha, 1)} + \sigma^2 \frac{k^m(\ell+m)^\ell}{d^m} + \frac{L^2}{k^{2 \min(\alpha, \ell+1)}}. \quad (47)$$

Finally, optimizing (47) with respect to (k, l) gives that

$$(47) \lesssim \begin{cases} L^2 \left(\frac{\sigma}{L} \right)^{\frac{4\alpha}{m+2\alpha}} d^{-\frac{2m\alpha}{m+2\alpha}}, & \text{when } \alpha < c(\alpha, \beta, m), \\ \left(\frac{\sigma^2 \log d}{d^{m-1}} \right)^{\beta \min(\alpha, 1)}, & \text{when } \alpha \geq c(\alpha, \beta, m), \end{cases}$$

under the choice

$$\ell^* = \min(\lfloor \alpha \rfloor, \lfloor c(\alpha, \beta, m) \rfloor), \quad k^* = c_1 d^{m/(m+2 \min(\alpha, \ell^*+1))},$$

where $c(\alpha, \beta, m) := \frac{m(m-1)\beta \min(\alpha, 1)}{\max(0, 2(m-(m-1)\beta \min(\alpha, 1)))}$.

□

C.6 Proof of Theorem 5

We prove a stronger result than Theorem 5 in the main paper. Specifically, we show that the conclusion (21) holds for either 1-monotonic in Definition 2 or isotonic functions defined in (22). Let \mathcal{M} denote either 1-monotonic or isotonic function class. Define the monotonic Lipschitz tensor class

$$\mathcal{P}_{M,L} = \left\{ \Theta \in \mathbb{R}^{d \times \dots \times d} : \Theta(\omega) = f\left(\frac{\omega}{d}\right) \text{ for all } \omega = (i_1, \dots, i_m) \in [d]^m \text{ and } f \in \mathcal{H}(1, L) \cap \mathcal{M} \right\}.$$

We will show the estimation problem under the Gaussian model (1) obeys the minimax lower bound

$$\inf_{(\hat{\Theta}, \hat{\pi})} \sup_{\Theta \in \mathcal{P}_{M,L}, \pi \in \Pi(d,d)} \mathbb{P} \left(\text{MSE}(\hat{\Theta} \circ \hat{\pi}, \Theta \circ \pi) \gtrsim d^{-\frac{2m}{m+2}} + d^{-(m-1)} \log d \right) \geq 0.8.$$

Proof of Theorem 5. We prove the nonparametric rate and permutation rate separately.

Nonparametric rate. We first show the nonparametric rate for a given permutation $\pi \in \Pi(d, d)$

$$\inf_{\hat{\Theta}} \sup_{\Theta \in \mathcal{P}_{M,L}} \mathbb{P} \left(\text{MSE}(\hat{\Theta} \circ \hat{\pi}, \Theta \circ \pi) \geq cd^{-\frac{2m}{m+2}} \right) \geq 0.9. \quad (48)$$

We introduce the following function

$$K(x_1, \dots, x_m) = \prod_{i=1}^m (1 - 2|x_i|) \mathbf{1}\{|x_i| \leq 1/2\}.$$

Let us take $k = \lceil c_1 d^{\frac{m}{2+m}} \rceil$ for some constant $c_1 > 0$ to be determined later. For any $\mathbf{a} = (a_1, \dots, a_m) \in [k]^m$, define the function

$$\phi_{\mathbf{a}}(x_1, \dots, x_m) = Mk^{-1}K \left(kx_1 - a_1 + \frac{1}{2}, \dots, kx_m - a_m + \frac{1}{2} \right), \quad (49)$$

where $M > 0$ is a constant only depending on L . By the construction of the function, we have the following lemma

Lemma 4. *For some $M > 0$, depending on L , the function (49) satisfies*

1. $\phi_{\mathbf{a}}(x_1, \dots, x_m) \in \mathcal{F}(1, L)$
2. $\sum_{(i_1, \dots, i_m) \in [d]^m} \phi_{\mathbf{a}}(i_1/d, \dots, i_m/d) \geq \frac{M^2}{3^m} d^m k^{-2-m}$

The proof of Lemma 4 is deferred to Section G. Let $\Gamma = \{0, 1\}^N$ be the set of all binary sequences of length $N = \binom{k+m-1}{m}$. For any $\gamma = \{\gamma_{\mathbf{a}}\}_{1 \leq a_1 \leq \dots \leq a_m \leq k} \in \Gamma$, we define the symmetric function h^{γ} by

$$h^{\gamma}(x_1, \dots, x_m) = \frac{1}{2} + \sum_{1 \leq a_1 \leq \dots \leq a_m \leq k} \gamma_{\mathbf{a}} \phi_{\mathbf{a}}(x_1, \dots, x_m), \quad \text{for } x_1 \geq \dots \geq x_m.$$

We further define monotonically increasing function given ω by

$$f^\gamma = g + \delta h^\gamma,$$

where $g(x_1, \dots, x_m) = \frac{1}{2} + \frac{1}{m} \sum_{i=1}^m x_i$. We choose small constant $\delta > 0$ so that f^γ is a monotonic function for all $\gamma \in \Gamma$ with respect to both isotonic and 1-monotonic definition.

Notice that f^γ is continuous and has repeating structure across each block $\times_{i=1}^m [\frac{a_i-1}{k}, \frac{a_i}{k}]$. It suffices to check the monotonicity for any $(x_1, x_2, \dots, x_m) \in \times_{i=1}^m [\frac{a_i-1}{k}, \frac{a_i}{k}]$ where $a_i \in [k]$ for all $i \in [m]$. For fixed (x_2, \dots, x_m) , we have

$$f^\gamma(x_1, x_2, \dots, x_m) = \frac{1+\delta}{2} + \frac{1}{m} \sum_{i=1}^m x_i + \frac{\delta \gamma_a M}{k} \left(1 - 2 \left| kx_1 - a_1 + \frac{1}{2} \right| \right) \prod_{i=2}^m \left(1 - 2 \left| kx_i - a_i + \frac{1}{2} \right| \right)$$

Since $\gamma_a = 0$ gives trivial result, we assume that $\gamma_a = 1$. When $x_1 \in \left[\frac{(a_1-1)}{k}, \frac{(a_1-1/2)}{k} \right]$, we have linear function whose coefficient greater than $1/m$ with respect to x_1 . When $x_1 \in \left[\frac{(a_1-1/2)}{k}, \frac{a_1}{k} \right]$, we have

$$f^\gamma(x_1, \dots, x_m) = \frac{1+\delta}{2} + \frac{1}{m} x_1 - \delta M C_1 x_1 + C_2,$$

where C_1 and C_2 are constant only depending on m, a_1 and (x_1, \dots, x_m) . Therefore, choosing sufficiently small constant $\delta > 0$ always makes f^γ monotonic with respect to both isotonic and 1-monotonic definition. Since both g and h^γ are Lipschitz, we show that $f^\gamma \in \mathcal{F}(1, L) \cap \mathcal{M}$ for all $\gamma \in \Gamma$.

Now we consider the subspace $\mathcal{F} = \{f^\gamma : \gamma \in \Gamma\} \subset \mathcal{F}(1, L) \cap \mathcal{M}$. To apply Lemma 11, we first upper bound $\sup_{f, f' \in \mathcal{F}} D(\mathbb{P}_f, \mathbb{P}_{f'})$. For any $f \in \mathcal{F}$, denote $f(\omega) = f(i_1/d, \dots, i_m/d)$ where $\omega = (i_1/d, \dots, i_m/d)$. Then for any $f, f' \in \mathcal{F}$, we have

$$\begin{aligned} D(\mathbb{P}_f | \mathbb{P}_{f'}) &\leq \frac{1}{2\sigma^2} \sum_{\omega \in [d]^m} (f(\omega) - f'(\omega))^2 \\ &\leq \frac{1}{2\sigma^2} \sum_{\omega \in [d]^m} \delta^2 (h(\omega) - h'(\omega))^2 \\ &\leq \frac{1}{2\sigma^2} \delta^2 M^2 d^m k^{-2}. \end{aligned}$$

where the first inequality holds by Lemma 12.

Next we lower bound the packing number of \mathcal{F} . For any $f^\gamma, f^{\gamma'} \in \mathcal{F}$, we have

$$\begin{aligned} \rho^2(f^\gamma, f^{\gamma'}) &\geq \frac{1}{d^m} \sum_{\omega \in [d]^m} \sum_{1 \leq a_1 \leq \dots \leq a_m \leq k} \delta^2 (\gamma_a - \gamma'_a)^2 \phi_a^2(\omega) \\ &\geq \frac{\delta^2 M^2}{3^m} k^{-2-m} \rho_H(\gamma, \gamma'), \end{aligned}$$

where the last inequality uses Lemma 4 and ρ_H is defined in Lemma 10. By Lemma 10, we select a subset $S \subset \Gamma$ such that $|S| \geq \exp(k^m/16)$ and $\rho_H(\gamma, \gamma') \geq k^m/8$ for any $\gamma \neq \gamma' \in S$. Therefore setting $\epsilon^2 = ck^{-2\alpha}$ for some constant $c > 0$, we have

$$\log \mathcal{M}(\epsilon, \mathcal{F}, \rho) \geq k^m/16.$$

By the choice of $k = \lceil c_1 d^{\frac{m}{2+m}} \rceil$ and Lemma 11, we have

$$\inf_{\hat{\Theta}} \sup_{f \in \mathcal{F}} \mathbb{P} \left(\frac{1}{d^m} \sum_{\omega \in [d]^m} \left(\hat{\Theta}(\omega) - f(\omega/d) \right)^2 \geq cd^{-\frac{2m}{m+2}} \right) \geq 0.9,$$

which completes the proof of (48).

Permutation rate. Similar to the proof of Theorem 2, we first show the minimax permutation rate for k -block isotonic tensor family, and then construct a monotonic (with respect to both isotonic and 1-monotonic definition) and a function $f \in \mathcal{F}(1, L) \cap \mathcal{M}(1)$ to mimic the constant isotonic block tensors.

The following lemma serves the same role as Lemma 2, except that now we has an isotonic tensor $\mathcal{S} \in \mathbb{R}^{k \times \dots \times k}$ satisfying (33). The proof of Lemma 5 is constructive and deferred to the end of this subsection.

Lemma 5 (Permutation error for isotonic tensor block model). *Consider the problem of estimating d -dimensional, block- k signal tensors from Gaussian tensor block models. For every given integer $k \in [d]$, there exists an isotonic core tensor $\mathcal{S} \in \mathbb{R}^{k \times \dots \times k}$ satisfying*

$$\inf_{\hat{\Theta}} \sup_{z \in \Pi(d, k)} \mathbb{P} \left\{ \frac{1}{d^m} \sum_{(i_1, \dots, i_m) \in [d]^m} \left[\hat{\Theta}(i_1, \dots, i_m) - \mathcal{S}(z(i_1), \dots, z(i_m)) \right]^2 \gtrsim \frac{\sigma^2 \log k}{d^{m-1}} \right\} \geq 0.9.$$

The remaining proof is the same as Theorem 2, and we omit the proof here for brevity. \square

D Sharp extension of Theorem 4 to isotonic functions

We provide a sharp upper bound under isotonic assumption similar to Theorem 4. Define isotonic function class \mathcal{M}

$$\mathcal{M} = \{f: [0, 1]^m \rightarrow \mathbb{R} \mid f(x_1, \dots, x_m) \leq f(x'_1, \dots, x'_m) \text{ when } x_i \leq x'_i \text{ for } i \in [m]\}.$$

Unlike β -monotonic functions, the permutation π per se may not be accurately estimated for isotonic functions. However, we find that the composition $\Theta \circ \pi$ can still be accurately estimated due to the joint monotonicity.

Lemma 6 (Permutation error of Borda count algorithm under isotonic assumption). Consider the sub-Gaussian tensor model (3) with $f \in \mathcal{M}$. Let $\hat{\pi}$ be the permutation such that the permuted empirical score function $\tau \circ \hat{\pi}^{-1}$ is monotonically increasing. Then with high probability

$$\frac{1}{d^m} \|\Theta \circ \hat{\pi} - \Theta \circ \pi\|_F^2 \lesssim \frac{\log d}{d^{m-1}}.$$

The proof of Lemma 6 is provided in Section G. Now, we obtain the same statistical accuracy of the Borda count estimator for the isotonic functions as in Theorem 4.

Theorem 6 (Estimation error for Borda count algorithm under isotonic assumption). Consider the sub-Gaussian tensor model with $f \in \mathcal{H}(\alpha, L) \cap \mathcal{M}$. Let $(\hat{\Theta}^{\text{BC}}, \hat{\pi}^{\text{BC}})$ be the Borda count estimator in (16)-(17) with $\ell^* = \min(\lfloor \alpha \rfloor, (m-2)(m+1)/2)$ and $k^* \asymp d^{m/(m+2\min(\alpha, \ell^*+1))}$. Then, we have in high probability,

$$(9) \quad \lesssim \begin{cases} d^{-\frac{2m\alpha}{m+2\alpha}}, & \text{when } \alpha < m(m-1)/2, \\ d^{-(m-1)} \log d, & \text{when } \alpha \geq m(m-1)/2. \end{cases}$$

Proof of Theorem 6. The proof of Theorem 6 is the exactly same as in Theorem 4, except that we now use Lemma 6 in place of Lemma 3. We omit the proof for brevity. \square

E Challenges for extending Theorem 5 to arbitrary α

Our Theorem 5 is based on Lipschitz monotone ($\alpha = \beta = 1$). Here we provide a high level explanation for the technical challenges of constructing minimax lower bound for the arbitrary $\alpha > 0$ under monotonicity assumption.

Recall the proof of minimax lower bound for arbitrary α without monotonicity in Theorem 2. The main idea is to fix the permutation first and construct a subset of $\mathcal{F}(\alpha, L)$ to get nonparametric rate. Then we fix the smooth function and let the permutation vary to get the permutation rate. Finally we combine two rates to obtain the minimax lower bound. To obtain the nonparametric and permutation rates, we use Fano's method in Lemma 11 via a finite subset of the function class $\mathcal{F}(\alpha, L)$ for each case. Technical challenges come from constructing a subset of the α -smooth function class with extra monotonicity structure. We cannot directly use function classes constructed for Theorem 2.

Challenge in nonparametric rates. In the general proof of the nonparametric rate, we partition the domain into k^m uniform blocks and consider a set of α -smooth functions $\{\phi_{\mathbf{a}}\}_{\mathbf{a} \in [k]^m}$ whose support is one of the partition in $[0, 1]^m$. The function class consists of functions defined as

$$h^\gamma(x_1, \dots, x_m) = \frac{1}{2} + \sum_{1 \leq a_1 \leq \dots \leq a_m \leq k} \gamma_{\mathbf{a}} \phi_{\mathbf{a}}(x_1, \dots, x_m), \quad \text{for } x_1 \geq \dots \geq x_m,$$

where $\gamma_a \in \{0, 1\}^{\binom{k+m-1}{m}}$.

However, the extra monotonicity constraints make this function class non-applicable because h^γ is not monotonic in general. One possible remedy is to construct the function class of the form

$$f^\gamma = g + \delta h^\gamma,$$

where $g(x_1, \dots, x_m) = 1/2 + \frac{1}{m} \sum_{i=1}^m x_i$ and δ is small constant not depending on tensor dimension d . This new construction allows to have moderate sized function class while keeping both α -smoothness and monotonicity. When $\alpha \geq 1$, we are able to set small constant δ to make the function f^γ α -smooth and monotonic. When $0 < \alpha < 1$, however, we find that δ cannot be a constant but depends on k (or equivalently, on d) if f^γ is α -smooth. This dependency of δ severely worsens the minimax lower bound resulting in the loose bound. How to construct such function classes is unknown.

Challenge in permutation rate. In permutation rate, we construct the smooth function that mimics the constant block tensor over $d/2$ points. This construction reduces our problem to that of estimating a block constant tensor. Then we obtain the permutation rate applying minimax lower bound of estimating constant block tensor (Lemma 2). Since we now consider the monotonic α -smooth function, we construct the smooth function that mimics the isotonic constant block tensor and provide minimax lower bound Lemma 5 corresponding to Lemma 2. The issue we have is how to construct the monotonic α -smooth function that mimics the isotonic constant tensor block over $d/2$ points. Without monotonic constraint, we use the infinitely differentiable function called smooth convolution function whose support is a partition of $[0, 1]^m$. This allows the constructed function to have constant block structure over at least $d/2$ points, smoothly connecting the other points.

However, the monotonicity constraint makes the application of the smooth convolution function impossible because the smooth convolution functions are not monotonic. We overcome this challenge using piece-wise linear functions when $\alpha = 1$. We construct Lipschitz monotonic function where $d/2$ points mimic isotonic constant block tensor and the other half points are connected by piecewise linear function. How to construct the monotonic smooth function for general $\alpha \neq 1$ is unknown.

F Extension to relaxed smooth models

We now extend the grid design (4) to random design (5). Specifically, suppose that the signal tensor is generated from the following random design model

$$\Theta(i_1, \dots, i_m) = f(x_{i_1}, \dots, x_{i_m}), \quad (50)$$

where the sequence $\{x_i\}_{i=1}^d$ are i.i.d. random variables sampled from a probability distribution supported on $[0, 1]$.

We briefly discuss the extension of Theorem 1 to random design. The proof of Theorem 1 is based on Proposition 1 and the union bound over (\mathcal{B}, π) . For random designs, we change the union bound over the set $\{\pi: [d] \rightarrow [d]\}$ to union bound over the set $\{(x_1, \dots, x_d) : x_i \in [0, 1] \text{ for all } i \in [d]\}$. These two sets have the same order of complexity up to log terms. Therefore, it suffices to present the extension of Proposition 1 to random designs.

Lemma 7 (Block-wise polynomial tensor approximation under random design). *Consider the random design model (50). For every block number $k \leq d$ and degree $\ell \in \mathbb{N}$, we have the approximation error*

$$\inf_{\mathcal{B} \in \mathcal{B}(k, \ell)} \frac{1}{d^m} \mathbb{E} \|\Theta - \mathcal{B}\|_F^2 \lesssim \frac{L^2}{k^{2 \min(\alpha, \ell+1)}}.$$

Proof of Lemma 7. The proof is similar to that of Proposition 1 except that we now construct the block-wise degree- ℓ polynomial tensor \mathcal{B} conditional on the random variables $\{x_i\}_{i \in [d]}$. In Proposition 1, we use the canonical clustering function such that $z(i) = \lceil ki/d \rceil$ for all $i \in [d]$. Here, we use the clustering function $z^*: [d] \rightarrow [k]$ conditional on $\{x_i\}_{i \in [d]}$ such that

$$z^*(i) = \lceil k * x_i \rceil.$$

The remaining proof is exactly the same as in Proposition 1 and thus omitted. \square

G Technical lemmas

Proof of Lemma 2. We provide the proof for $m = 3$ only. The extension to higher orders ($m \geq 4$) uses exactly the same techniques and thus is omitted. For notational simplicity, we do not distinguish the fractional number and its rounding integer. For example, we simply write $k/3$, instead of $\lceil k/3 \rceil$, to represent its rounding integer.

Let us pick $\omega_1, \dots, \omega_{k/3} \in \{0, 1\}^{k^2/9}$ such that $\rho_H(\omega_p, \omega_q) \geq k^2/36$ for all $p \neq q \in [k/3]$. This selection is possible by lemma 10. Fixing such $\omega_1, \dots, \omega_{k/3}$, we define a symmetric core tensor $\mathcal{S} \in \mathbb{R}^{k \times k \times k}$ for $p < q < r$,

$$\mathcal{S}(p, q, r) = \begin{cases} s_{p,q,r} & \text{if } p \in \{1, \dots, k/3\}, q \in \{k/3 + 1, \dots, 2k/3\}, \\ & r \in \{2k/3 + 1, \dots, k\}, \\ 0 & \text{Otherwise,} \end{cases}$$

where $\{s_{p,q,r} : p \in \{1, \dots, k/3\}, q \in \{k/3 + 1, \dots, 2k/3\}, r \in \{2k/3 + 1, \dots, k\}\}$ satisfies

$$\mathbf{s}(r) := \text{vec} \left(\mathcal{S} \left(1 : \frac{k}{3}, \left(\frac{k}{3} + 1 \right) : \frac{2k}{3}, r \right) \right)$$

$$= \sqrt{\frac{c\sigma^2 \log k}{d^2}} \omega_{r-2k/3} \quad \text{for any } r \in \{2k/3 + 1, \dots, k\}. \quad (51)$$

The choice of constant $c > 0$ is deferred to a later part of the proof. Notice that for any $r_1, r_2 \in \{2k/3 + 1, \dots, k\}$, we have

$$\|\mathbf{s}(r_1) - \mathbf{s}(r_2)\|_F^2 \geq \frac{c\sigma^2 k^2 \log k}{36d^2}. \quad (52)$$

Define a subset of permutation set $\Pi(d, k)$ by

$$\mathcal{Z} = \left\{ z \in \Pi(d, k) : |z^{-1}(a)| = \frac{d}{k} \text{ for all } a \in [k], z^{-1}(a) = \left\{ \frac{(a-1)d}{k} + 1, \dots, \frac{ad}{k} \right\} \text{ for } a \in [2k/3] \right\}.$$

Each $z \in \mathcal{Z}$ induces a block structure in $\mathcal{B}(k, 0)$, with known positions for the $2k/3$ indices and unknown positions for the remaining $k/3$ indices. We consider the collection of block tensors induced by \mathcal{Z} and \mathcal{S} ; i.e.,

$$\mathcal{B}(\mathcal{Z}) = \{ \Theta^z \in \mathbb{R}^{d \times d \times d} : \Theta^z(i, j, k) = \mathcal{S}(z(i), z(j), z(k)) \text{ for } z \in \mathcal{Z} \}.$$

To apply Lemma 11, we seek for the upper bound for $\sup_{\Theta, \Theta' \in \mathcal{B}(\mathcal{Z})} D(\mathbb{P}_\Theta || \mathbb{P}_{\Theta'})$ and the lower bound for $\log \mathcal{M}(\epsilon, \mathcal{B}(\mathcal{Z}), \rho)$, where ρ is defined by $\rho(\Theta, \Theta') = \frac{1}{d^3} \|\Theta - \Theta'\|_F^2$. For the Gaussian model, we have

$$D(\mathbb{P}_\Theta || \mathbb{P}_{\Theta'}) \leq \frac{1}{2\sigma^2} \|\Theta - \Theta'\|_F \leq \frac{1}{2\sigma^2} d^3 \frac{c\sigma^2 \log k}{d^2} = \frac{cd \log k}{2}, \quad (53)$$

where the first inequality holds for any $\Theta, \Theta' \in \mathcal{B}(\mathcal{Z})$ by Lemma 12.

Next we provide a lower bound of the packing number $\log \mathcal{M}(\epsilon, \mathcal{B}(\mathcal{Z}), \rho)$ with $\epsilon^2 \asymp \frac{\sigma^2 \log k}{d^2}$. From the construction of \mathcal{S} in (51), we have one-to-one correspondence between \mathcal{Z} and $\mathcal{B}(\mathcal{Z})$. Thus $\mathcal{M}(\epsilon, \mathcal{B}(\mathcal{Z}), \rho) = \mathcal{M}(\epsilon, \mathcal{Z}, \rho')$ for the metric ρ' on \mathcal{Z} defined by $\rho'(z_1, z_2) = \rho(\Theta^{z_1}, \Theta^{z_2})$. Let P be the packing set in \mathcal{Z} with the same cardinality of $\mathcal{M}(\epsilon, \mathcal{Z}, \rho')$. Given any $z \in \mathcal{Z}$, define its ϵ -neighbor by $\mathcal{N}(z, \epsilon) = \{z' \in \mathcal{Z} : \rho'(z, z') \leq \epsilon\}$. Then, we have $\cup_{z \in P} \mathcal{N}(z, \epsilon) = \mathcal{Z}$, because the cardinality of P is same as packing number $\mathcal{M}(\epsilon, \mathcal{Z}, \rho')$. Therefore, we have

$$|\mathcal{Z}| \leq \sum_{z \in P} |\mathcal{N}(z, \epsilon)| \leq |P| \max_{z \in P} |\mathcal{N}(z, \epsilon)|. \quad (54)$$

It remains to find the upper bound of $\max_{z \in P} |\mathcal{N}(z, \epsilon)|$. For any $z_1, z_2 \in \mathcal{Z}$, $z_1(i) = z_2(i)$ for $i \in [2d/3]$ and $|z_1^{-1}(p)| = d/k$ for all $p \in [k]$. Therefore,

$$\begin{aligned} \rho'^2(z_1, z_2) &\geq \frac{1}{d^3} \sum_{1 \leq i_1 \leq d/3 < i_2 \leq 2d/3 \leq i_3 \leq d} (\mathcal{S}(z_1(i_1), z_1(i_2), z_1(i_3)) - \mathcal{S}(z_2(i_1), z_2(i_2), z_2(i_3)))^2 \\ &= \frac{1}{d^3} \sum_{2n/3 < i_3 \leq n} \sum_{1 \leq p \leq k/3 < q \leq 2k/3} \sum_{i_1 \in z_1^{-1}(p), i_2 \in z_1^{-1}(q)} (\mathcal{S}(p, q, z_1(i_3)) - \mathcal{S}(p, q, z_2(i_3)))^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d^3} \sum_{2n/3 < i_3 \leq n} \sum_{1 \leq p \leq k/3 < q \leq 2k/3} \left(\frac{d}{k} \right)^2 (\mathcal{S}(p, q, z_1(i_3)) - \mathcal{S}(p, q, z_2(i_3)))^2 \\
&= \frac{1}{d^3} \sum_{2n/3 < i_3 \leq n} \left(\frac{d}{k} \right)^2 \| \mathbf{s}(z_1(i_3) - \mathbf{s}(z_2(i_3))) \|_F^2 \\
&\geq \frac{c\sigma^2 \log k}{36d^3} |\{j : z_1(j) \neq z_2(j)\}|,
\end{aligned}$$

where the last inequality is from (52). Hence with the choice of $\epsilon^2 = \frac{c\sigma^2 \log k}{288d^2}$, we have $|\{j : z(j) \neq z'(j)\}| \leq d/8$ for any $z' \in \mathcal{N}(z, \epsilon)$. This implies

$$|\mathcal{N}(z, \epsilon)| \leq \binom{d}{d/8} k^{d/8} \leq (8e)^{d/8} k^{d/8} \leq \exp\left(\frac{1}{5}d \log k\right), \quad (55)$$

for sufficiently large k . Next we find the lower bound of $|\mathcal{Z}|$ based on Stirling's formula,

$$|\mathcal{Z}| = \frac{(d/3)!}{[(d/k)!]^{k/3}} = \exp\left(\frac{1}{3}d \log k + o(d \log k)\right) \geq \exp\left(\frac{1}{4}d \log k\right). \quad (56)$$

Plugging (55) and (56) into (54) yields

$$\mathcal{M}(\epsilon, \mathcal{B}(\mathcal{Z}), \rho) = |P| \geq \frac{\max_{z \in P} \mathcal{N}(z, \epsilon)}{|\mathcal{Z}|} \geq \exp\left(\frac{1}{20}d \log k\right). \quad (57)$$

Finally, applying Lemma 11 based on (53) and (57) gives

$$\inf_{\hat{\Theta}} \sup_{\Theta \in \mathcal{B}(\mathcal{Z})} \mathbb{P}\left(\frac{1}{d^3} \|\hat{\Theta} - \Theta\|_F^2 \geq \frac{C\sigma^2 \log k}{d^2}\right) = \inf_{\hat{\Theta}} \sup_{z \in \mathcal{Z}} \mathbb{P}\left(\frac{1}{d^3} \|\hat{\Theta} - \mathcal{S} \circ z\|_F^2 \geq \frac{C\sigma^2 \log k}{d^2}\right) \geq 0.9,$$

with some constant $C > 0$ for sufficiently small $c > 0$ in (51). \square

Proof of Lemma 3. Without loss of generality, assume that π is the identity permutation. Notice that $g(i) - \tau(i)$ is the sample average of d^{m-1} independent mean-zero Gaussian random variables with the variance proxy σ^2 . Based on the maxima of sub-Gaussian random variables, we have

$$\max_{i \in [d]} |g(i) - \tau(i)| < 2\sigma d^{-(m-1)/2} \sqrt{\log d}, \quad (58)$$

with probability $1 - \frac{2}{d^2}$.

By the β -monotonicity of the function g , we have

$$g(1) \leq g(2) \leq \dots \leq g(d-1), \quad (59)$$

The estimated permutation $\hat{\pi}$ is defined for which

$$\tau \circ \hat{\pi}^{-1}(1) \leq \tau \circ \hat{\pi}^{-1}(2) \leq \dots \leq \tau \circ \hat{\pi}^{-1}(d-1) \leq \tau \circ \hat{\pi}^{-1}(d). \quad (60)$$

For any given index i , we examine the error $|i - \hat{\pi}(i)|$. By (59) and (60), we have

$$i = \underbrace{|\{j: g(j) \leq g(i)\}|}_{=:I}, \quad \text{and} \quad \hat{\pi}(i) = \underbrace{|\{j: \tau(j) \leq \tau(i)\}|}_{=:II},$$

where $|\cdot|$ denotes the cardinality of the set. We claim that the sets I and II differ only in at most $d^{(m-1)\beta/2}$ elements. To prove this, we partition the indices in $[d]$ in two cases.

1. Long-distance indices in $\{j: |j - i|/d \geq C(\sigma d^{-(m-1)/2} \sqrt{\log d})^\beta\}$ for some sufficient large constant $C > 0$. In this case, the ordering of (i, j) remains the same in (59) and (60), i.e.,

$$g(i) < g(j) \iff \tau(i) < \tau(j). \quad (61)$$

We only prove the right side direction in (61) here. The other direction can be similarly proved. Suppose that $g(i) < g(j)$. Then we have

$$\begin{aligned} \tau(j) - \tau(i) &\geq -|g(j) - \tau(j)| - |g(i) - \tau(i)| + g(j) - g(i) \\ &> -4\sigma d^{-(m-1)/2} \sqrt{\log d} + g(j) - g(i) \\ &\geq 0, \end{aligned}$$

where the second inequality is from (58) with probability at least $(1 - 2/d^2)^d$ and the last inequality uses β -monotonicity of $g(\cdot)$, and the assumption $|j - i|/d \geq C(\sigma d^{-(m-1)/2} \sqrt{\log d})^\beta$. Therefore we show that $g(i) < g(j)$ implies $\tau(i) < \tau(j)$. In this case, we conclude that none of long-distance indices belongs to $I\Delta II$.

2. Short-distance indices in $\{j: |j - i|/d < (\sigma d^{-(m-1)/2} \sqrt{\log d})^\beta\}$. In this case, (59) and (60) may yield different ordering of (i, j) .

Combining the above two cases gives that

$$\left\{ j: \frac{1}{d}|j - i| \leq (4\sigma d^{-(m-1)/2} \sqrt{\log d})^\beta \right\} \supset I\Delta II.$$

Finally, we have

$$\text{Loss}(\pi, \hat{\pi}) := \frac{1}{d} \max_{i \in [d]} |\pi(i) - \hat{\pi}(i)| \leq \frac{1}{d} |I\Delta II| \leq 2 (4\sigma d^{-(m-1)/2} \sqrt{\log d})^\beta,$$

with high probability. \square

Proof of Lemma 4. Here we provide the proof of Lemma 4 for the general $\alpha \in (0, 1]$.

1. $\phi_a \in \mathcal{F}(\alpha, L)$: We first claim that

$$|K(x_1, \dots, x_m) - K(y_1, \dots, y_m)| \leq 2 \sum_{i=1}^m |x_i - y_i|. \quad (62)$$

We prove this by induction. When $m = 2$, we have

$$|K(x_1, x_2) - K(y_1, y_2)| \leq 2(|x_1 - y_1| + |x_2 - y_2|).$$

Suppose $m = \ell$, we have

$$|K(x_1, \dots, x_\ell) - K(y_1, \dots, y_\ell)| \leq 2 \sum_{i=1}^{\ell} |x_i - y_i|. \quad (63)$$

Then we have

$$\begin{aligned} & |K(x_1, \dots, x_\ell, x_{\ell+1}) - K(y_1, \dots, y_\ell, y_{\ell+1})| \\ &= |K(x_1, \dots, x_\ell)(1 - 2|x_{\ell+1}|)\mathbb{1}\{|x_{\ell+1}| \leq 1/2\} - K(y_1, \dots, y_\ell)(1 - 2|y_{\ell+1}|)\mathbb{1}\{|y_{\ell+1}| \leq 1/2\}| \\ &\leq 2K(x_1, \dots, x_\ell)|x_{\ell+1} - y_{\ell+1}| + (1 - 2|y_{\ell+1}|)|K(x_1, \dots, x_\ell) - K(y_1, \dots, y_\ell)| \\ &\leq 2|x_{\ell+1} - y_{\ell+1}| + 2 \sum_{i=1}^{\ell} |x_i - y_i| \\ &\leq 2 \sum_{i=1}^{\ell+1} |x_i - y_i|, \end{aligned}$$

where the second inequality uses (63). By mathematical induction, (62) holds true for any integer $m > 0$.

For any $\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_m)$ in the support of $\phi_{\mathbf{a}}$, we have

$$\begin{aligned} |\phi_{\mathbf{a}}(x_1, \dots, x_m) - \phi_{\mathbf{a}}(y_1, \dots, y_m)| &\leq 2Mk^{1-\alpha} \sum_{i=1}^m |x_i - y_i| \\ &\leq 2Mmk^{1-\alpha} \|\mathbf{x} - \mathbf{y}\|_{\infty} \\ &\leq 2Mm \|\mathbf{x} - \mathbf{y}\|_{\infty}^{\alpha}, \end{aligned}$$

where the last inequality uses the fact that $\|\mathbf{x} - \mathbf{y}\|_{\infty} \leq 1/k$ and $\alpha \in (0, 1]$. Setting $M = L/2m$, we prove $\phi_{\mathbf{a}} \in \mathcal{F}(\alpha, L)$.

2. $\sum_{(i_1, \dots, i_m) \in [d]^m} \phi_{\mathbf{a}}(i_1/d, \dots, i_m/d) \geq \frac{M^2}{3^m} d^m k^{-2\alpha-m}$: Notice that

$$\begin{aligned} \sum_{(i_1, \dots, i_m) \in [d]^m} \phi_{\mathbf{a}}^2 \left(\frac{i_1}{d}, \dots, \frac{i_m}{d} \right) &= M^2 k^{-2\alpha} \sum_{(i_1, \dots, i_m) \in [d]^m} K^2 \left(\frac{ki_1}{d} - a_1 + \frac{1}{2}, \dots, \frac{ki_m}{d} - a_m + \frac{1}{2} \right) \\ &= \left(\sum_{\frac{d(a-1)}{k} < i \leq \frac{da}{k}} \left(1 - 2 \left| \frac{ki}{d} - a + \frac{1}{2} \right| \right)^2 \right)^m, \end{aligned}$$

and

$$\begin{aligned}
\sum_{\frac{d(a-1)}{k} < i \leq \frac{na}{k}} \left(1 - 2 \left| \frac{ki}{d} - a + \frac{1}{2} \right| \right)^2 &= \sum_{\frac{d(a-1)}{k} < i \leq \frac{na}{k}} \left(1 - \frac{2k}{n} \left| i - \frac{(a-1/2)n}{k} \right| \right)^2 \\
&= 4 \sum_{0 \leq t \leq \frac{d}{2k}} \left(1 - \frac{2k}{d} t \right)^2 \\
&\geq 2 \int_0^{d/2k} \left(1 - \frac{2k}{d} t \right)^2 dt \\
&= \frac{d}{3k}.
\end{aligned}$$

Therefore, we complete the proof. \square

Proof of Lemma 5. We follow the proof of Lemma 2. Major difference from the proof of Lemma 2 is that we construct an isotonic block tensor instead of the block tensor in Lemma 2. Therefore, we only provide how to construct isotonic block tensor \mathcal{S} which shares the same property of the block tensor used in Lemma 2.

Let us pick $\omega_1, \dots, \omega_{k/3} \in \{0, 1\}^{k/3 \times k/3}$ such that $\rho_H(\omega_p, \omega_q) \geq k^2/36$ for all $p \neq q \in [k/3]$. This selection is possible by lemma 10. For the simplicity, assume that k is a multiple of 3. Fixing such $\omega_1, \dots, \omega_{k/3}$, we define a symmetric core tensor $\mathcal{S} \in \mathbb{R}^{k \times k \times k}$ for $p < q < r$,

$$\mathcal{S}(p, q, r) = \begin{cases} s_{p,q,r} & \text{if } p \in \{1, \dots, k/3\}, q \in \{k/3 + 1, \dots, 2k/3\}, \\ & r \in \{2k/3 + 1, \dots, k\}, \\ \lfloor \frac{p-1}{3} \rfloor + \lfloor \frac{q-1}{3} \rfloor + \lfloor \frac{r-1}{3} \rfloor & \text{Otherwise.} \end{cases}$$

Here we define a cubic $\{s_{p,q,r}\}$ as

$$s_{p,q,r} = \lfloor \frac{p-1}{3} \rfloor + \lfloor \frac{q-1}{3} \rfloor + \lfloor \frac{r-1}{3} \rfloor + \delta \sqrt{\frac{c\sigma^2 \log k}{d^2}} \left(\frac{p+q+r}{k} - 1 \right) + \sqrt{\frac{c\sigma^2 \log k}{d^2}} [\omega_{r-2k/3}]_{p,q-k/3},$$

where a constant $\delta > 0$ is set to make \mathcal{S} isotonic and the choice of constant $c > 0$ is deferred to a later part of the proof. For a given $r \in \{2k/3 + 1, \dots, k\}$, define

$$\mathbf{s}(r) := \text{vec} \left(\mathcal{S} \left(1 : \frac{k}{3}, \left(\frac{k}{3} + 1 \right) : \frac{2k}{3}, r \right) \right). \quad (64)$$

By the construction, we have, for any $r_1, r_2 \in \{2k/3 + 1, \dots, k\}$

$$\|\mathbf{s}(r_1) - \mathbf{s}(r_2)\|_F^2 \geq \frac{C_{c,\delta} \sigma^2 k^2 \log k}{36d^2}, \quad (65)$$

for some constant $C_{c,\delta} > 0$.

Define a subset of permutation set $\Pi(d, k)$ by

$$\mathcal{Z} = \left\{ z \in \Pi(d, k) : |z^{-1}(a)| = \frac{d}{k} \text{ for } a \in [k], z^{-1}(a) = \left\{ \frac{(a-1)d}{k} + 1, \dots, \frac{ad}{k} \right\} \text{ for } a \in [2k/3] \right\}.$$

Each $z \in \mathcal{Z}$ induces a block tensor in $\mathcal{B}(k, 0)$. We consider the collection of block tensors induced by \mathcal{Z} ; i.e.,

$$\mathcal{B}(\mathcal{Z}) = \{ \Theta^z \in \mathbb{R}^{d \times d \times d} : \Theta^z(i, j, k) = \mathcal{S}(z(i), z(j), z(k)) \text{ for } z \in \mathcal{Z} \}.$$

For the Gaussian model, we have

$$D(\mathbb{P}_\Theta || \mathbb{P}_{\Theta'}) \leq \frac{1}{2\sigma^2} \|\Theta - \Theta'\|_F \leq \frac{1}{2\sigma^2} d^3 \frac{c' \sigma^2 \log k}{d^2} = \frac{c' d \log k}{2}, \quad (66)$$

where the first inequality holds for any $\Theta, \Theta' \in \mathcal{B}(\mathcal{Z})$ by Lemma 12. Based on (65) and (66), we verify that all the bounds for Lemma 11 are the same as those in the proof of Lemma 2. Therefore, using the same argument in the proof of Lemma 2 we obtain

$$\inf_{\hat{\Theta}} \sup_{z \in \Pi(d, k)} \mathbb{P} \left(\frac{1}{d^3} \|\hat{\Theta} - \mathcal{S} \circ z\|_F^2 \geq \frac{C \sigma^2 \log k}{d^2} \right) \geq 0.9,$$

with some constant $C > 0$ for sufficiently small $c > 0$ in (64). \square

Proof of Lemma 6. Without loss of generality, assume that π is the identity permutation. Notice that $g(i) - \tau(i)$ is the sample average of d^{m-1} independent mean-zero sub-Gaussian random variables with the variance proxy σ^2 . Based on the maxima of sub-Gaussian random variables, we have

$$\max_{i \in [d]} |g(i) - \tau(i)| \lesssim 2\sigma \left(\frac{\log d}{d^{m-1}} \right)^{1/2},$$

with probability $1 - \frac{2}{d^2}$. Denote $\delta = 2\sigma \sqrt{\log d / d^{m-1}}$. The estimated permutation $\hat{\pi}$ is defined for which

$$\tau \circ \hat{\pi}^{-1}(1) \leq \tau \circ \hat{\pi}^{-1}(2) \leq \dots \leq \tau \circ \hat{\pi}^{-1}(d-1) \leq \tau \circ \hat{\pi}^{-1}(d).$$

By definition of the $\hat{\pi}$, we have for any $i > j$ but $\hat{\pi}(i) = j$,

$$\frac{1}{d^{m-1}} \sum_{(i_2, \dots, i_m) \in [d]^{m-1}} |\Theta(i, i_2, \dots, i_m) - \Theta(\hat{\pi}(i), i_2, \dots, i_m)| \lesssim \delta. \quad (67)$$

Similarly for any $i < j$ but $\hat{\pi}(i) = j$, we have

$$\frac{1}{d^{m-1}} \sum_{(i_2, \dots, i_m) \in [d]^{m-1}} |\Theta(\hat{\pi}(i), i_2, \dots, i_m) - \Theta(i, i_2, \dots, i_m)| \lesssim \delta. \quad (68)$$

Therefore we obtain

$$\begin{aligned}
& \frac{1}{d^m} \sum_{(i_1, \dots, i_m) \in [d]^m} |\Theta(\hat{\pi}(i_1), \hat{\pi}(i_2), \dots, \hat{\pi}(i_m)) - \Theta(i_1, i_2, \dots, i_m)| \\
& \leq \frac{1}{d^m} \sum_{(i_1, \dots, i_m) \in [d]^m} \left(|\Theta(\hat{\pi}(i_1), \hat{\pi}(i_2), \dots, \hat{\pi}(i_m)) - \Theta(i_1, \hat{\pi}(i_2), \dots, \hat{\pi}(i_m))| \right. \\
& \quad + |\Theta(i_1, \hat{\pi}(i_2), \dots, \hat{\pi}(i_m)) - \Theta(i_1, i_2, \dots, \hat{\pi}(i_m))| \\
& \quad \left. + \dots + |\Theta(i_1, i_2, \dots, \hat{\pi}(i_m)) - \Theta(i_1, i_2, \dots, i_m)| \right) \\
& \leq \frac{m}{d^m} \sum_{i_1 \in [d]} \sum_{(i_2, \dots, i_m)} |\Theta(\hat{\pi}(i_1), i_2, \dots, i_m) - \Theta(i_1, i_2, \dots, i_m)| \\
& \lesssim \delta,
\end{aligned}$$

where the second inequality uses the symmetricity of the tensor and the last inequality uses (67) and (68) in the case of wrong permutations. Since the Frobenius norm is bounded by ℓ_1 norm, we complete the proof. \square

Definition 3 (sub-Gaussian random vectors). A random vector $\mathbf{e} \in \mathbb{R}^d$ is said to be sub-Gaussian with variance proxy σ^2 if $\mathbb{E}\mathbf{e} = 0$ and $\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, \mathbf{e} \rangle$ is sub-Gaussian with variance proxy σ^2 for any vector $\mathbf{c} \in \mathbb{R}^d$. Note that the entries in \mathbf{e} need not be independently nor identically distributed.

Lemma 8 (Sub-Gaussian maxima under embedding). Let $\mathbf{A} \in \mathbb{R}^{d_1 \times d_2}$ be a deterministic matrix with rank $r \leq \min(d_1, d_2)$. Let $\mathbf{y} \in \mathbb{R}^{d_1}$ be a vector consisting of independent sub-Gaussian entries with variance proxy σ^2 . Then, there exists a sub-Gaussian random vector $\mathbf{x} \in \mathbb{R}^r$ with variance proxy σ^2 such that

$$\max_{\mathbf{p} \in \mathbb{R}^{d_2}} \left\langle \frac{\mathbf{Ap}}{\|\mathbf{Ap}\|_2}, \mathbf{y} \right\rangle = \max_{\mathbf{q} \in \mathbb{R}^r} \left\langle \frac{\mathbf{q}}{\|\mathbf{q}\|_2}, \mathbf{x} \right\rangle.$$

Proof of Lemma 8. Let $\mathbf{u}_i \in \mathbb{R}^{d_1}$, $\mathbf{v}_j \in \mathbb{R}^{d_2}$ singular vectors and $\lambda_i \in \mathbb{R}$ be singular values of \mathbf{A} such that $\mathbf{A} = \sum_{i=1}^r \lambda_i \mathbf{u}_i \mathbf{v}_i^T$. Then for any $\mathbf{p} \in \mathbb{R}^{d_2}$, we have

$$\mathbf{Ap} = \sum_{i=1}^r \lambda_i \mathbf{u}_i \mathbf{v}_i^T \mathbf{p} = \sum_{i=1}^r \lambda_i (\mathbf{v}_i^T \mathbf{p}) \mathbf{u}_i = \sum_{i=1}^r \alpha_i \mathbf{u}_i,$$

where $\boldsymbol{\alpha}(\mathbf{p}) = (\alpha_1, \dots, \alpha_r)^T := (\lambda_1(\mathbf{v}_1^T \mathbf{p}), \dots, \lambda_r(\mathbf{v}_r^T \mathbf{p}))^T \in \mathbb{R}^r$. Notice that $\boldsymbol{\alpha}(\mathbf{p})$ covers \mathbb{R}^r in the sense that $\{\boldsymbol{\alpha}(\mathbf{p}) : \mathbf{p} \in \mathbb{R}^{d_2}\} = \mathbb{R}^r$. Therefore, we have

$$\begin{aligned}
\max_{\mathbf{p} \in \mathbb{R}^{d_2}} \left\langle \frac{\mathbf{Ap}}{\|\mathbf{Ap}\|_2}, \mathbf{y} \right\rangle &= \max_{\mathbf{p} \in \mathbb{R}^{d_2}} \sum_{i=1}^r \frac{\alpha_i}{\|\boldsymbol{\alpha}(\mathbf{p})\|_2} \mathbf{u}_i^T \mathbf{y} \\
&= \max_{\mathbf{p} \in \mathbb{R}^{d_2}} \left\langle \frac{\boldsymbol{\alpha}(\mathbf{p})}{\|\boldsymbol{\alpha}(\mathbf{p})\|_2}, \mathbf{x} \right\rangle
\end{aligned}$$

$$= \max_{\mathbf{q} \in \mathbb{R}^r} \left\langle \frac{\mathbf{q}}{\|\mathbf{q}\|_2}, \mathbf{x} \right\rangle,$$

where we define $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r] \in \mathbb{R}^{d_2 \times r}$ and $\mathbf{x} = \mathbf{U}^T \mathbf{y} \in \mathbb{R}^r$. Since \mathbf{U} has orthonormal columns, the vector \mathbf{x} is a sub-Gaussian random vector with variance proxy σ^2 . \square

Lemma 9 (Theorem 1.19 in Rigollet and Hütter [27]). *Let $\mathbf{e} \in \mathbb{R}^d$ be a sub-Gaussian vector with variance proxy σ^2 . Then,*

$$\mathbb{P} \left(\max_{\mathbf{c} \in \mathbb{R}^d} \left\langle \frac{\mathbf{c}}{\|\mathbf{c}\|_2}, \mathbf{e} \right\rangle \geq t \right) \leq \exp \left(-\frac{t^2}{8\sigma^2} + d \log 6 \right).$$

Lemma 10 (Varshamov-Gilbert bound). *There exists a sequence of subset $\omega_1, \dots, \omega_N \in \{0, 1\}^d$ such that*

$$\rho_H(\omega_i, \omega_j) := \|\omega_i - \omega_j\|_F^2 \geq \frac{d}{4} \text{ for any } i \neq j \in [N],$$

for some $N \geq \exp(d/8)$.

For any two probability measures \mathbb{P} and \mathbb{Q} , define the Kullback-Leibler divergence by

$$D(\mathbb{P}|\mathbb{Q}) = \int \left(\log \frac{d\mathbb{P}}{d\mathbb{Q}} \right) d\mathbb{P}.$$

Lemma 11 (Proposition 4.1 in Gao et al. [16]). *Let (Ξ, ρ) be a metric space and $\{\mathbb{P}_\xi : \xi \in \Xi\}$ be a collection of probability measure. For any totally bounded $T \subset \Xi$, define the Kullback-Leibler diameter of T by $d_{KL}(T) = \sup_{\xi, \xi' \in T} D(\mathbb{P}_\xi | \mathbb{P}_{\xi'})$. Then, for any $\epsilon > 0$,*

$$\inf_{\hat{\xi}} \sup_{\xi \in \Xi} \mathbb{P}_\xi \left\{ \rho^2(\hat{\xi}, \xi) \geq \frac{\epsilon^2}{4} \right\} \geq 1 - \frac{d_{KL}(T) + \log 2}{\log \mathcal{M}(\epsilon, T, \rho)},$$

where $\mathcal{M}(\epsilon, T, \rho)$ is the packing number of T by ϵ -radius balls with respect to the metric ρ .

Lemma 12 (Proposition 17 in Gao et al. [15]). *Let \mathbb{P}_Θ (respectively, $\mathbb{P}_{\Theta'}$) denote the distribution of a Gaussian tensor with mean Θ (respectively, Θ') and entrywise i.i.d. noise from $N(0, \sigma^2)$. Then we have*

$$D(\mathbb{P}_\Theta | \mathbb{P}_{\Theta'}) \leq \frac{1}{2\sigma^2} \sum_{\omega \in [d]^m} (\Theta(\omega) - \Theta'(\omega))^2.$$