Supplements for "Exponential tensor regression with covariates on multiple modes"

1 Proofs

Proof of Theorem 4.1. Define $\ell(\mathcal{B}) = \mathbb{E}(\mathcal{L}_{\mathcal{Y}}(\mathcal{B}))$, where the expectation is taken with respect to $\mathcal{Y} \sim \mathcal{B}_{\text{true}}$ under the model with true parameter $\mathcal{B}_{\text{true}}$. We first prove the following two conclusions:

C1. There exist two positive constants C_1, $C_2 > 0$, such that, with probability at least $1 - \exp(-C_1 \log K \sum_k p_k)$, the stochastic deviation, $\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})$, satisfies

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| = |\langle \mathcal{E},\ \mathcal{B} \times_1 \boldsymbol{X}_1 \times_2 \cdots \times_K \boldsymbol{X}_K \rangle| \le C_2 \|\mathcal{B}\|_F \log K \sqrt{\frac{\prod_k r_k}{\max_k r_k} \sum_k p_k}.$$

C2. The inequality $\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}}) \le -\frac{L}{2}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2$ holds, where $L > 0$ is the lower bound for $\min_{|\theta| \le \alpha} |b''(\theta)|$.

To prove C1, we note that the stochastic deviation can be written as:

$$\begin{aligned} \mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B}) &= \langle \mathcal{Y} - \mathbb{E}(\mathcal{Y}|\mathcal{X}),\ \Theta(\mathcal{B}) \rangle \\ &= \langle \mathcal{Y} - b'(\Theta^{\text{true}}),\ \Theta \rangle \\ &= \langle \mathcal{E} \times_1 \boldsymbol{X}_1^T \times_2 \cdots \times_K \boldsymbol{X}_K^T,\ \mathcal{B} \rangle, \end{aligned} \tag{1}$$

where $\mathcal{E} = [\![\varepsilon_{i_1,\ldots,i_K}]\!] \overset{\text{def}}{=} \mathcal{Y} - b'(\Theta^{\text{true}})$. Based on Lemma 1, $\varepsilon_{i_1,\ldots,i_K}$ is sub-Gaussian-(ϕU). Let $\check{\mathcal{E}} \overset{\text{def}}{=} \mathcal{E} \times_1 \boldsymbol{X}_1^T \times_2 \cdots \times_K \boldsymbol{X}_K^T$. By the property of sub-Gaussian r.v's, $\check{\mathcal{E}}$ is a $(p_1, \ldots, p_K)$-dimensional sub-Gaussian tensor with parameter bounded by $C_2 = \phi U c_2^K$. Here $c_2 > 0$ is the upper bound of $\sigma_{\max}(\boldsymbol{X}_k)$. Applying Cauchy-Schwarz inequality to (1) yields

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| \le \left\|\check{\mathcal{E}}\right\|_2 \|\mathcal{B}\|_*, \tag{2}$$

where $\|\cdot\|_2$ denotes the tensor spectral norm and $\|\cdot\|_*$ denotes the tensor nuclear norm. The nuclear norm $\|\mathcal{B}\|_*$ is bounded by $\|\mathcal{B}\|_* \le \sqrt{\frac{\prod_k r_k}{\max_k r_k}} \|\mathcal{B}\|_F$ (c.f. Wang and Li [2018], Wang et al. [2017]). The spectral norm $\left\|\check{\mathcal{E}}\right\|_2$ is bounded by $\left\|\check{\mathcal{E}}\right\|_2 \le C_1 U c^K \log K \sqrt{\sum_k p_k}$ with probability at least $1 - \exp(-C_2 \log K \sum_k p_k)$ (c.f. Wang and Li [2018], Tomioka and Suzuki [2014]). Combining these two bounds with (2), we have, with probability at least $1 - \exp(-C_2 \log K \sum_k p_k)$,

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| \le C_1 U c_2^K \|\mathcal{B}\|_F \log K \sqrt{\frac{\prod_k r_k}{\max_k r_k} \sum_k p_k}.$$

Next we prove C2. Applying Taylor expansion to $\ell(\mathcal{B})$ around $\mathcal{B}_{\text{true}}$,

$$\ell(\mathcal{B}) = \ell(\mathcal{B}_{\text{true}}) + \frac{1}{2} \text{vec}(\mathcal{B} - \mathcal{B}_{\text{true}})^T \mathcal{H}_{\mathcal{Y}}(\check{\mathcal{B}}) \text{vec}(\mathcal{B} - \mathcal{B}_{\text{true}}), \tag{3}$$

where $\mathcal{H}_{\mathcal{Y}}(\check{\mathcal{B}})$ is the (non-random) Hession of $\frac{\partial \ell^2(\mathcal{B})}{\partial^2 \mathcal{B}}$ evaluated at $\check{\mathcal{B}} = \alpha \text{vec}(\alpha\mathcal{B} + (1-\alpha)\mathcal{B}_{\text{true}})$ for some $\alpha \in [0, 1]$. Recall that $b''(\theta) = \text{Var}(y|\theta)$, because $y \in \mathbb{R}$ follows the exponential family

distribution with function $b(\cdot)$. By chain rule and the fact that $\Theta = \Theta(\mathcal{B}) = \mathcal{B} \times_1 \boldsymbol{X}_1 \cdots \times_K \boldsymbol{X}_K$, the equation (3) implies that

$$\ell(\mathcal{B}) - \ell(\mathcal{B}_{\text{true}}) = -\frac{1}{2} \sum_{i_1,\ldots,i_K} b''(\check{\theta}_{i_1,\ldots,i_K})(\theta_{i_1,\ldots,i_K} - \theta_{\text{true},i_1,\ldots,i_K})^2 \leq -\frac{L}{2}\|\Theta - \Theta^{\text{true}}\|_F^2, \tag{4}$$

holds for all $\mathcal{B} \in \mathcal{P}$, provided that $\min_{|\theta|\leq\alpha} |b''(\theta)| \geq L > 0$. In particular, the inequality (4) also applies to the constrained MLE $\hat{\mathcal{B}}$. So we have

$$\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}}) \leq -\frac{L}{2}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2. \tag{5}$$

Now we have proved both C1 and C2. Note that $\mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) \geq 0$ by the definition of $\hat{\mathcal{B}}$, This implies that

$$\begin{aligned}
0 &\leq \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) \\
&\leq \Big(\mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \ell(\hat{\mathcal{B}})\Big) - (\mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) - \ell(\mathcal{B}_{\text{true}})) + \Big(\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}})\Big) \\
&\leq \langle \mathcal{E},\ \Theta - \Theta^{\text{true}} \rangle - \frac{L}{2}\|\hat{\Theta} - \Theta^{\text{true}}\|_F^2,
\end{aligned}$$

where the second line follows from (5). Therefore,

$$\begin{aligned}
\|\hat{\Theta} - \Theta^{\text{true}}\|_F &\leq \frac{2}{L}\Big\langle \mathcal{E},\ \frac{\hat{\Theta} - \Theta^{\text{true}}}{\|\hat{\Theta} - \Theta^{\text{true}}\|_F} \Big\rangle \\
&\leq \frac{2}{L} \sup_{\Theta:\|\Theta\|_F=1, \Theta=\mathcal{B}\times_1\boldsymbol{X}_1\times_2\cdots\times_K\boldsymbol{X}_K} \langle \mathcal{E},\ \Theta \rangle \\
&\leq \frac{2}{L} \sup_{\mathcal{B}\in\mathcal{P}:\|\mathcal{B}\|_F\leq\prod_k \sigma_{\min}^{-1}(\boldsymbol{X}_k)} \langle \mathcal{E},\ \ \mathcal{B} \times_1 \boldsymbol{X}_1 \times_2 \cdots \times_K \boldsymbol{X}_K \rangle.
\end{aligned} \tag{6}$$

Combining (6) with C1 yields the desired conclusion.

□

Lemma 1 (sub-Gaussian residual)**.** *Define the residual tensor* $\mathcal{E} = [\![\varepsilon_{i_1,\ldots,i_K}]\!] = \mathcal{Y} - b'(\Theta) \in \mathbb{R}^{d_1\times\cdots\times d_K}$. *Under the Assumption A2,* $\varepsilon_{i_1,\ldots,i_K}$ *is a sub-Gaussian random variable with sub-Gaussian parameter bounded by* ϕU*, for all* $(i_1,\ldots,i_K) \in [d_1]\times\cdots\times[d_K]$.

Proof. The proof is similar to Lemma 3 in Fan et al. [2019]. For ease of presentation, we drop the subscript $(i_1,\ldots,i_K)$ and simply write ε $(= y - b'(\theta))$. For any given $t \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}(\exp(t\varepsilon|\theta) &= \int c(x) \exp\left(\frac{\theta x - b(\theta)}{\phi}\right) \exp\big(t(x - b'(\theta))\big)\, dx \\
&= \int c(x) \exp\left(\frac{(\theta+\phi t)x - b(\theta+\phi t) + b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) dx \\
&= \exp\left(\frac{b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) \\
&\leq \exp\left(\frac{\phi U t^2}{2}\right),
\end{aligned}$$

where $c(\cdot)$ and $b(\cdot)$ are known functions in the exponential family corresponding to y. Therefore, ε is sub-Gaussian-(ϕU). □

Proof of Theorem 4.2. The proof is similar to Baldin and Berthet [2018]. We sketch the main steps here for completeness. Recall that $\ell(\mathcal{B}) = \mathbb{E}(\mathcal{L}_{\mathcal{Y}}(\mathcal{B}))$. By the definition of KL divergence, we have that,

$$\begin{aligned}\ell(\hat{\mathcal{B}}) &= \ell(\mathcal{B}_{\text{true}}) - \sum_{(i_1,\ldots,i_K)} KL(\theta_{\text{true},i_1,\ldots,i_K}, \hat{\theta}_{i_1,\ldots,i_K}) \\ &= \ell(\mathcal{B}_{\text{true}}) - \text{KL}(\mathbb{P}_{\mathcal{Y}_{\text{true}}},\ \mathbb{P}_{\hat{\mathcal{Y}}}),\end{aligned}$$

where $\mathbb{P}_{\mathcal{Y}_{\text{true}}}$ denotes the distribution of $\mathcal{Y}|\mathcal{X}$ with true parameter $\mathcal{B}_{\text{true}}$, and $\mathbb{P}_{\hat{\mathcal{Y}}}$ denotes the distribution with estimated parameter $\hat{\mathcal{B}}$. Therefore

$$\begin{aligned}\text{KL}(\mathbb{P}_{\mathcal{Y}_{\text{true}}},\ \mathbb{P}_{\hat{\mathcal{Y}}}) &= \ell(\mathcal{B}_{\text{true}}) - \ell(\hat{\mathcal{B}}) \\ &= \frac{1}{2}\sum_{i_1\ldots,i_K} b''(\check{\theta}_{i_1,\ldots,i_K})(\theta_{i_1,\ldots,i_K} - \theta_{\text{true},i_1,\ldots,i_K})^2 \\ &\le \frac{U}{2}\|\Theta - \Theta^{\text{true}}\|_F^2 \\ &\le \frac{U}{2}c_2^{2K}\|\mathcal{B} - \mathcal{B}_{\text{true}}\|_F^2,\end{aligned}$$

where the second line comes from (3), and $c_2 > 0$ is the upper bound for the $\sigma_{\max}(\boldsymbol{X}_k)$. The result then follows from Theorem 4.1. □

Proof of Proposition 1. For notational convenience, we drop the subscript $\mathcal{Y}$ from the objective $\mathcal{L}_{\mathcal{Y}}(\cdot)$ and simply write as $\mathcal{L}(\cdot)$. Let $\mathcal{A} = (\mathcal{C}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_K) \in \mathbb{R}^{d_{\text{total}}}$ denote the collection of decision variables used in the alternating optimization, where $d_{\text{total}} = \prod_k r_k + \sum_k r_k d_k$. The objective function can be viewed either as a function of decision variables $\mathcal{A}$ or a function of coefficient tensor $\mathcal{B} := \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K$. With slight abuse of notation, we write both functions as $\mathcal{L}(\cdot)$ but the meaning should be clear given the context.

We use $S\colon \mathbb{R}^{d_{\text{total}}} \mapsto \mathbb{R}^{d_{\text{total}}}$ to denote the update mapping that sends the t-th iterate to the $(t+1)$-th iterate. Then, we have $S(\mathcal{A}^{(t)}) = \mathcal{A}^{(t+1)}$. According to the alternating algorithm, there are $(K+1)$ micro-steps for each block of decision variables in one iteration. That implies S is a composition of $(K+1)$ block-wise mappings. Each block-wise mapping is continuously differentiable, so the mapping S is also continuously differentiable.

Let $\mathcal{A}^* = (\mathcal{C}^*, \boldsymbol{M}_1^*, \ldots, \boldsymbol{M}_K^*) \in \mathbb{R}^{d_{\text{total}}}$ be a local maximum. By the definition of alternating optimization, $\mathcal{A}^*$ is also a fixed point for the mapping S; that is, $S(\mathcal{A}^*) = \mathcal{A}^*$. The Hessian of the objective function $\mathcal{L}(\cdot)$ at $\mathcal{A}^*$ is

$$H(\mathcal{A}^*) = \nabla^2\mathcal{L}\left(\mathcal{C}^*, \boldsymbol{M}_1^*, \ldots, \boldsymbol{M}_K^*\right) = \begin{pmatrix} \nabla^2_{\mathcal{C}\mathcal{C}}\mathcal{L} & \nabla^2_{\mathcal{C}\boldsymbol{M}_1}\mathcal{L} & \cdots & \nabla^2_{\mathcal{C}\boldsymbol{M}_K}\mathcal{L} \\ \nabla^2_{\boldsymbol{M}_1\mathcal{C}}\mathcal{L} & \nabla^2_{\boldsymbol{M}_1\boldsymbol{M}_1}\mathcal{L} & \cdots & \nabla^2_{\boldsymbol{M}_1\boldsymbol{M}_K}\mathcal{L} \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^2_{\boldsymbol{M}_K\mathcal{C}}\mathcal{L} & \nabla^2_{\boldsymbol{M}_K\boldsymbol{M}_1}\mathcal{L} & \cdots & \nabla^2_{\boldsymbol{M}_K\boldsymbol{M}_K}\mathcal{L} \end{pmatrix} =: L + D + L^\top,$$

where D collects the diagonal blocks and L collects the lower-diagonal blocks. By assumption, $H(\mathcal{A}^*)$ is strictly negative definite at every direction except the direction of orthogonal transformation. That implies that the diagonal block, D, is strictly negative definite and thus $(L+D)^{-1}$ is invertible. By [Bezdek and Hathaway, 2003, Lemma 2], we have $\nabla S(\mathcal{A}^*) = -(L+D)^{-1}L^T$.

Next, we construct the contraction relationship between iterates $\mathcal{B}^{(t+1)}$ and $\mathcal{B}^{(t)}$ using the property of ∇S in the neighborhood of $\mathcal{A}^*$.

We need to introduce some additional notations. Let $\|\mathcal{A}-\mathcal{A}'\|_F$ denote the Euclidean distance between two decision variables, where

$$\|\mathcal{A}-\mathcal{A}'\|_F^2 = \|\mathcal{C}-\mathcal{C}'\|_F^2 + \sum_{k=1}^{K}\|\boldsymbol{M}_k-\boldsymbol{M}_k'\|_F^2.$$

We introduce the equivalent relationship induced by orthogonal transformation. Let $\mathbb{O}_{d,r}$ be the collection of all d-by-r matrices with orthogonal columns, $\mathbb{O}_{d,r} := \{\boldsymbol{P}\in\mathbb{R}^{d\times r}\colon \boldsymbol{P}^T\boldsymbol{P}=\mathbf{1}_r\}$, where $\mathbf{1}_r$ is the r-by-r identity matrix.

Definition 1 (Equivalence relatonship)**.** Two decision variables $\mathcal{A}' = (\mathcal{C}', \boldsymbol{M}_1', \ldots, \boldsymbol{M}_k'), \mathcal{A} = (\mathcal{C}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_k)$ are called equivalent, denoted $\mathcal{A}\sim\mathcal{A}'$, if and only if there exist a set of orthogonal matrices $\boldsymbol{P}_k\in\mathbb{O}_{d_k,r_k}$ such that

$$\boldsymbol{M}_k'\boldsymbol{P}_k^T = \boldsymbol{M}_k,\ \forall k\in[K], \quad \text{and} \quad \mathcal{C}\times_1\boldsymbol{P}_1\times_2\cdots\times_K\boldsymbol{P}_K=\mathcal{C}.$$

Equivalently, two decision variables $\mathcal{A}$, $\mathcal{A}'$ are equivalent if the corresponding Tucker tensors are the same, $\mathcal{B}(\mathcal{A}) = \mathcal{B}'(\mathcal{A}')$. We use Ω_O to denote all decision variables that are equivalent to the local optimum $\mathcal{A}^*$, $\Omega_O := \{\mathcal{A}\in\mathbb{R}^{d_{\text{total}}}\colon \mathcal{A}\sim\mathcal{A}^*\}$. Here, we discuss two cases at a sufficiently-small neighborhood of $\mathcal{A}^*$.

Case 1: There exists an iteration number $t'\in\mathbb{N}_+$ such that $\mathcal{A}^{(t')}\in\Omega_O$. For such $\mathcal{A}^{(t')}$, we have $\mathcal{B}(\mathcal{A}^{(t')}) = \mathcal{B}(\mathcal{A}^*)$. Therefore,

$$0 = \|\mathcal{B}(\mathcal{A}^{(t')}) - \mathcal{B}(\mathcal{A}^*)\|_F \le \|\mathcal{A}^{(0)}-\mathcal{A}^*\|_F.$$

Case 2: The entire sequence of iterates $\mathcal{A}^{(t)}\in\mathbb{R}^{d_{\text{total}}}/\Omega_O$. By assumption, $H(\cdot)$ is strictly negative definite for all t large enough. For any such $\mathcal{A}^{(t)}$, we have

$$(\mathcal{A}^{(t)}-\mathcal{A}^*)^T H(\mathcal{A}^*)(\mathcal{A}^{(t)}-\mathcal{A}^*) < 0.$$

Recall that the differential map $\nabla S(\mathcal{A}^*) = -(L+D)^{-1}L^T$, where L, D are the lower- and diagonal-block of the Hession $H(\mathcal{A}^*)$, respectively. Define contraction coefficient

$$\rho = \max_{\boldsymbol{x}\in\mathbb{R}^{d_{\text{total}}}/\Omega_O,\ \|\boldsymbol{x}\|_2=1} \boldsymbol{x}^T\left[(L+D)^{-1}L\right] \in (0,1).$$

By the contraction principle, we have

$$\|\mathcal{A}^{(t)}-\mathcal{A}^*\|_F \le \rho^t\|\mathcal{A}^{(0)}-\mathcal{A}^*\|_F, \tag{7}$$

for $\mathcal{A}^{(0)}$ sufficiently closes to $\mathcal{A}^*$. By [Han et al., 2020, Lemma 3.1], there exists a constant $c>0$ such that

$$\|\mathcal{B}(\mathcal{A}^{(t)}) - \mathcal{B}(\mathcal{A}^*)\|_F \le c\|\mathcal{A}^{(t)}-\mathcal{A}^*\|_F, \quad \forall t\in\mathbb{N}_+. \tag{8}$$

Combining (7) and (8) gives

$$\|\mathcal{B}(\mathcal{A}^{(t)}) - \mathcal{B}(\mathcal{A}^*)\|_F \le c\rho^t\|\mathcal{A}^{(0)}-\mathcal{A}^*\|_F,$$

for initialization $\mathcal{A}^{(0)}$ sufficiently close to $\mathcal{A}^*$. Combining cases 1 and 2, we obtain that

$$\|\mathcal{B}(\mathcal{A}^{(t)}) - \mathcal{B}(\mathcal{A}^*)\|_F^2 \le c\rho^{2t}\left(\|\mathcal{C}^{(0)}-\mathcal{C}^*\|_F^2 + \sum_{k=1}^{K}\|\boldsymbol{M}_k^{(0)}-\boldsymbol{M}_k^*\|_F^2\right),$$

for some constant $c>0$ and any initialization $\mathcal{A}^{(0)} = (\mathcal{C}^{(0)}, \boldsymbol{M}_1^{(0)}, \ldots, \boldsymbol{M}_K^{(0)})$ sufficiently close to $\mathcal{A}^* = (\mathcal{C}^*, \boldsymbol{M}_1^*, \ldots, \boldsymbol{M}_K^*)$.

□

Proposition 2 (Global convergence)**.** *Assume the set* $\{\mathcal{A} \mid \mathcal{L}(\mathcal{A}) \geq \mathcal{L}(\mathcal{A}^{(0)})\}$ *is compact and the stationary points of* $\mathcal{L}(\mathcal{A})$ *are isolated module the equivalence relationship. Then any sequence* $\mathcal{A}^{(t)}$ *generated by alternating algorithm converges to a stationary point of* $\mathcal{L}(\mathcal{A})$ *up to equivalence.*

Proof. Pick an arbitrary iterate $\mathcal{A}^{(t)}$. Because of the compactness of set $\{\mathcal{A}\colon \mathcal{L}(\mathcal{A}) \geq \mathcal{L}(\mathcal{A}^{(0)})\}$ and the boundedness of the decision domain, there exist a sub-sequence of $\mathcal{A}^{(t)}$ that converges. Let $\mathcal{A}^*$ denote one of the limiting points of $\mathcal{A}^{(t)}$. Let $\mathcal{S} = \{\mathcal{A}^*\}$ denote the set of all the limiting points of $\mathcal{A}^{(t)}$. We have $\mathcal{S} \subset \{\mathcal{A}\colon \mathcal{L}(\mathcal{A}) \geq \mathcal{L}(\mathcal{A}^{(0)})\}$ and thus $\mathcal{S}$ is a compact set. By [Lange, 2012, Propositions 8.2.1 and 13.4.2], $\mathcal{S}$ is also connected. Note that all points in $\mathcal{S}$ are also stationary points of $\mathcal{L}(\cdot)$, because of the monotonic increase of $\mathcal{L}(\mathcal{A}^{(t)})$ as $t \to \infty$.
Consider the equivalence of Tucker tensor representation. We define the equivalent class of $\mathcal{A}$ as:

$$\mathcal{E}(\mathcal{A}) = \{\mathcal{A}'\colon \ \boldsymbol{M}_k' = \boldsymbol{M}_k \boldsymbol{P}_k^T, \mathcal{C}' = \mathcal{C} \times \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_K\}, \text{ where } \boldsymbol{P}_k^T \in \mathbb{O}_{d_k, r_k}, \forall k \in [K]\}.$$

We define an enlarged set $\mathcal{E}_S$ induced by the set $\mathcal{S}$,

$$\mathcal{E}_S = \bigcup \{\mathcal{E}(\mathcal{A}^*)\colon \mathcal{A}^* \in \mathcal{S}\}.$$

The enlarged set $\mathcal{E}_S$ satisfies the two properties below:

1. [Union of stationary points] The set $\mathcal{E}_S$ is an union of equivalent classes generated by the limiting points in $\mathcal{S}$.

2. [Connectedness module the equivalence] The set $\mathcal{E}_S$ is connected module the equivalence relationship. That property is obtained by the connectedness of S.

Now, note that the isolation of stationary points and Property 1 imply that $\mathcal{E}_S$ contains only finite number of equivalent classes. Otherwise, there is a subsequence of non-equivalent stationary points whose limit is not isolated, which contradicts the isolation assumption. Combining the finiteness with Property 2, we conclude that $\mathcal{E}_S$ contains only a single equivalent class; i.e. $\mathcal{E}_S = \mathcal{E}(\mathcal{A}^*)$, where $\mathcal{A}^*$ is a stationary point of $\mathcal{L}(\mathcal{A})$. Therefore, all the convergent sub-sequences of $\mathcal{A}^{(t)}$ converge to one stationary point $\mathcal{A}^*$ up to equivalence. We conclude that, any iterate $\mathcal{A}^{(t)}$ generated by Algorithm 1 converges to a stationary point of $\mathcal{L}(\mathcal{A})$ up to equivalence. □

2 Numerical implementation

2.1 Alternating Algorithm

The detailed alternating algorithm is organized in Algorithm 1.

2.2 Time complexity

The computational complexity of our tensor regression model is $O(d^3+d)$ for each loop of iterations, where $d = \prod_k d_k$ is the total size of the response tensor. More precisely, the update of core tensor costs $O(r^3 d^3)$, where $r = \sum_k r_k$ is the total size of the core tensor. The update of factor matrix $\boldsymbol{M}_k$ involves solving p_k separate GLMs. Solving those GLMs requires $O(r_k^3 p_k + p_k r_k^3 d d_k^{-1})$, and therefore the cost for updating K factors in total is $O(\sum_k r_k^3 p_k d_k + d \sum_k r_k^3 p_k d_k^{-1}) \approx O(\sum p_k d_k + d) \approx O(d)$.

Algorithm 1 Generalized tensor response regression with covariates on multiple modes

Input: Response tensor $\mathcal{Y} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$, covariate matrices $\boldsymbol{X}_k \in \mathbb{R}^{d_k \times p_k}$ for $k = 1, \ldots, K$, target Tucker rank $\boldsymbol{r} = (r_1, \ldots, r_K)$, link function f, infinity norm bound α

Output: Low-rank estimation for the coefficient tensor $\mathcal{B} \in \mathbb{R}^{p_1 \times \cdots \times p_K}$.

1: Calculate $\check{\mathcal{B}} = \mathcal{Y} \times_1 \left[(\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1} \boldsymbol{X}_1^T\right] \times_2 \cdots \times_K \left[(\boldsymbol{X}_K^T \boldsymbol{X}_K)^{-1} \boldsymbol{X}_K^T\right]$.

2: Initialize the iteration index $t = 0$. Initialize the core tensor $\mathcal{C}^{(0)}$ and factor matrices $\boldsymbol{M}_k^{(0)} \in \mathbb{R}^{p_k \times r_k}$ via rank-$\boldsymbol{r}$ Tucker approximation of $\check{\mathcal{B}}$, in the least-square sense.

3: **while** the relative increase in objective function $\mathcal{L}_{\mathcal{Y}}(\mathcal{B})$ is less than the tolerance **do**

4: Update iteration index $t \leftarrow t + 1$.

5: **for** $k = 1$ to K **do**

6: Obtain the factor matrix $\boldsymbol{M}_k^{(t+1)} \in \mathbb{R}^{p_k \times r_k}$ by solving p_k separate GLMs with link function f.

7: Update the columns of $\boldsymbol{M}_k^{(t+1)}$ by Gram-Schmidt orthogonalization.

8: **end for**

9: Obtain the core tensor $\mathcal{C}^{(t+1)} \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ by solving a GLM with $\mathrm{vec}(\mathcal{Y})$ as response, $\odot_{k=1}^{K} [\boldsymbol{X}_k \boldsymbol{M}_k^{(t)}]$ as covariates, and f as link function. Here $\odot$ denotes the Khatri-Rao product of matrices.

10: Rescale the core tensor subject to the infinity norm constraint.

11: Update $\mathcal{B}^{(t+1)} \leftarrow \mathcal{C}^{(t+1)} \times_1 \boldsymbol{M}_1^{(t+1)} \times_2 \cdots \times_K \boldsymbol{M}_K^{(t+1)}$.

12: **end while**

3 Simulation

3.1 Detailed simulation setting

In simulations, we use linear predictor which is simulated from $\mathcal{U} = [\![u_{ijk}]\!] = \mathcal{B} \times \{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}$. Here, we introduce the detailed setting to generate $\mathcal{U}$. The coefficient tensor $\mathcal{B}$ is generated using the Tucker factor representation $\mathcal{B} = \mathcal{C} \times \{\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3\}$, where both the core tensor $\mathcal{C}$ and factor matrix $\boldsymbol{M}_k$ are drawn i.i.d. from Uniform[-1,1]. The covariate matrix $\boldsymbol{X}_k$ is either an identity matrix (i.e. no covariate available) or Gaussian random matrix with i.i.d. entries from $N(0, \sigma_k)$. We set $\sigma_k = \sqrt{d_k}$ to ensure the singular values of $\boldsymbol{X}_k$ are bounded as d_k increases. The linear predictor $\mathcal{U}$ is also scaled such that $\|\mathcal{U}\|_\infty = 1$.

3.2 Simulation for rank selection

We provide the experiment results for assessing our BIC criterion (5.2). We consider the balanced situation where $d_k = d$, $p_k = 0.4 d_k$ for $k = 1, 2, 3$. We set $\alpha = 10$ and consider various combinations of dimension d and rank $\boldsymbol{r} = (r_1, r_2, r_3)$. For each combination, we simulate tensor data following Gaussian, Bernoulli, and Poisson models. We then minimize BIC using a grid search over three dimensions. The hyper-parameter α is set to infinity in the fitting, which essentially imposes no prior on the coefficient magnitude. Table S1 reports the selected rank averaged over $n_{\text{sim}} = 30$ replicates for Gaussian and Poisson models. We find that when $d = 20$, the selected rank is slightly smaller than the true rank, and the accuracy improves immediately when the dimension increases to $d = 40$. This agrees with our expectation, as in tensor regression, the sample size is related to the number of entries. A larger d implies a larger sample size, so the BIC selection becomes more accurate.

True Rank	Dimension (Gaussian tensors)		Dimension (Poisson tensors)	
$\boldsymbol{r}$	$d = 20$	$d = 40$	$d = 20$	$d = 40$
(3, 3, 3)	(2.1, 2.0, 2.0)	(**3**, **3**, **3**)	(2.0, 2.2, 2.1)	(**3**, **3**, **3**)
(4, 4, 6)	(3.2, 3.1, 5.0)	(**4**, **4**, **6**)	(**4.0**, **4.0**, 5.2)	(**4**, **4**, **6**)
(6, 8, 8)	(5.1, 7.0, 6.9)	(**6**, **8**, **8**)	(5.0, 6.1, 7.1)	(**6**, **8**, **8**)

Supplementary Table S1: Rank selection via BIC. Bold number indicates no significant difference between the estimate and the ground truth, based on a z-test with a level 0.05.

4 Additional results for real data analysis

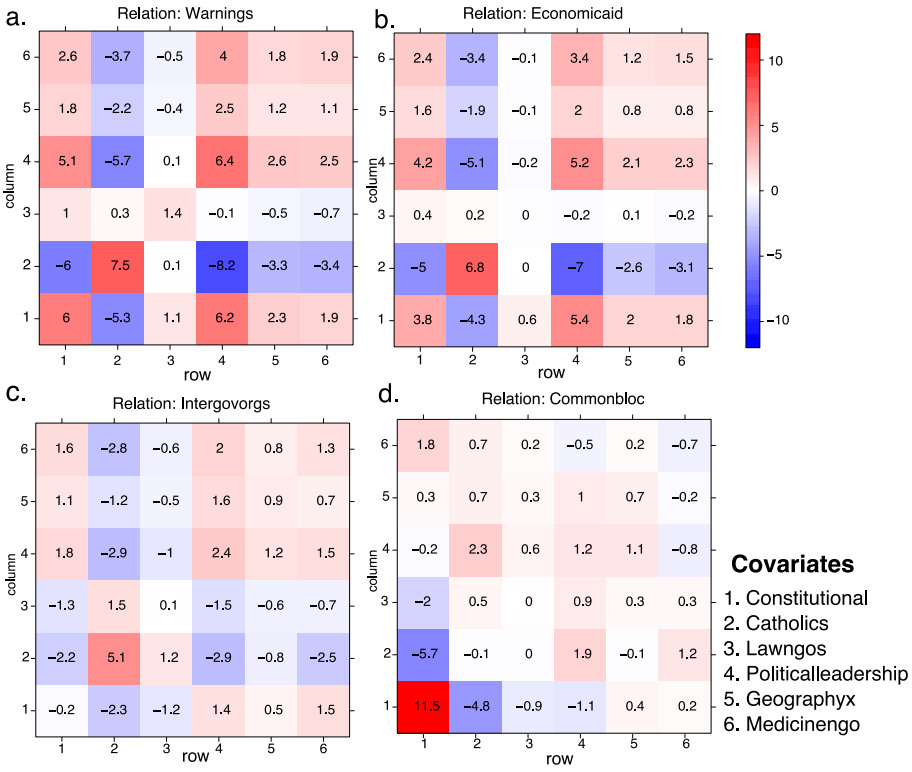
4.1 HCP data analysis

Supplement Figure S1 compares the estimated coefficients from our method (tensor regression) with those from classical GLM approach. A classical GLM is to regress the brain edges, one at a time, on the individual-level covariates, and this logistic model is repeatedly fitted for every edge $\in [68] \times [68]$. As we can see in the figure, our tensor regression shrinkages the coefficients towards center, thereby enforcing the sharing between coefficient entries.

4.2 Nations data analysis

We apply our tensor regression model to the *Nations* data. The multi-relationship networks are organized into a $14 \times 14 \times 56$ binary tensor, with each entry indicating the presence or absence of a connection, such as "sending tourist to", "export", "import", between countries. The 56 relations span the fields of politics, economics, military, religion, etc. The BIC criterion suggests a rank $\boldsymbol{r} = (4, 4, 4)$ for the coefficient tensor $\mathcal{B} \in \mathbb{R}^{6\times6\times56}$.

To investigate the effects of dyadic attributes towards connections, we depicte the estimated coefficients $\hat{\mathcal{B}} = [\![\hat{b}_{ijk}]\!]$ for several relation types (Supplement Figure S2). Note that entries $\hat{b}_{ijk}$ can be interpreted as the contribution, at the logit scale, of covariate pair (i, j) (ith covariate for the "sender" country and jth covariate for the "receiver" country) towards the connection of relation k. Several interesting findings emerge from the observation. We find that relations belonging to a same cluster tend to have similar covariate effects. For example, the relations *warnings* and *ecnomicaid* are classified into Cluster II, and both exhibit similar covariate pattern (Supplement Figure S2a-b). Moreover, the majority of the diagonal entries $\hat{\mathcal{B}}(i, i, k)$ positively contribute to the connection. This suggests that countries with coherent attributes tend to interact more often than others. We also find that the *constitutional* attribute is an important predictor for the *commonbloc* relation, whereas the effect is weaker for other relations (Supplement Figure S2d). This is not surprising, as the block partition during Cold War is associated with the *constitutional* attribute.

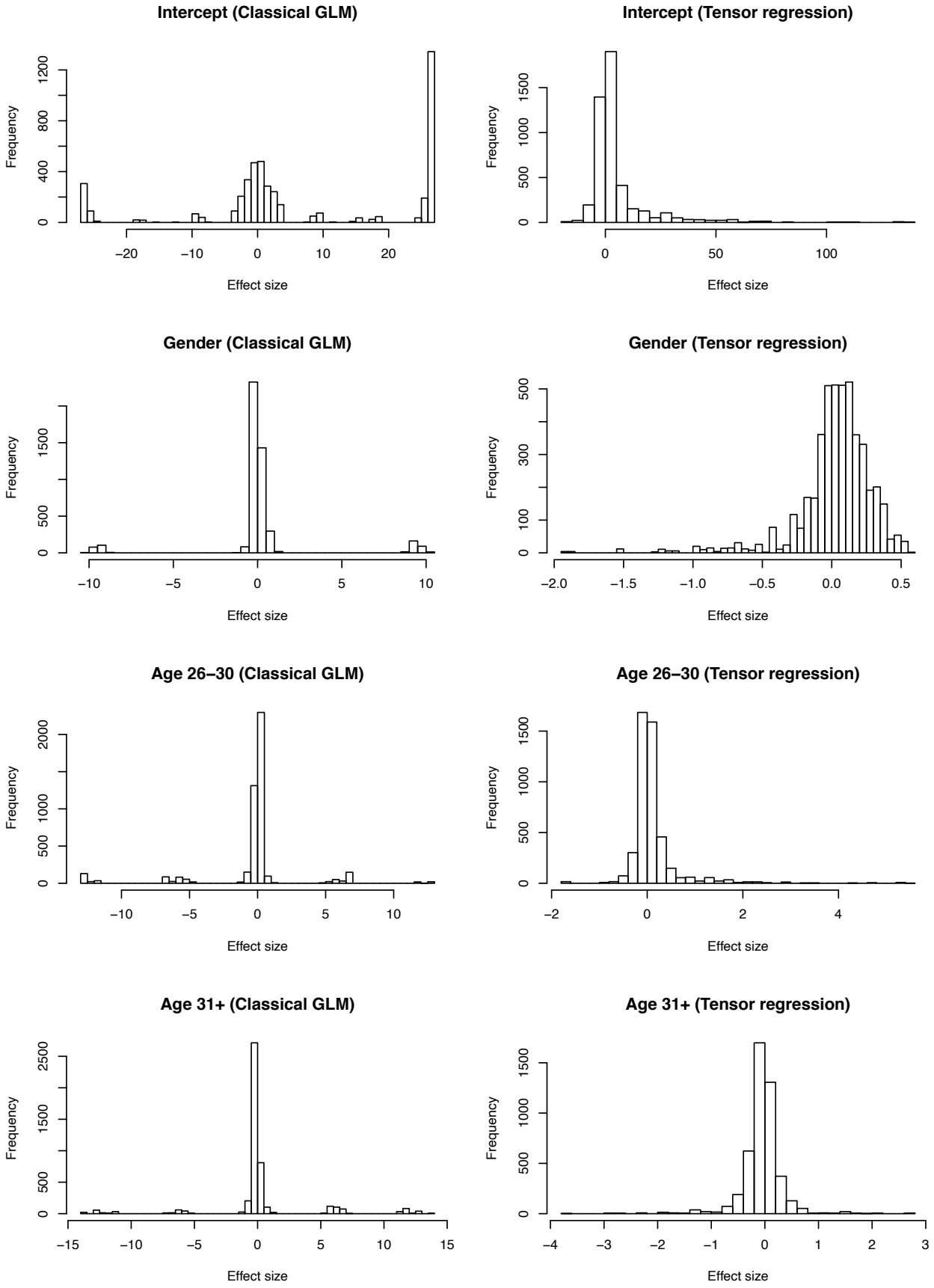


Supplementary Figure S2: Effect estimation in the *Nations* data. Panels (a)-(d) represent the estimated effects of country-level attributes towards the connection probability, for relations *warnning*, *economicaid*,*intergovorg*, and *commonblock*, respectively.

Supplement table S2 summarizes the K-means clustering of the 56 relations based on the 3$^{\text{rd}}$ mode factor $\boldsymbol{M}_3 \in \mathbb{R}^{56\times 4}$ in the tensor regression model.

Cluster I	officialvisits, intergovorgs, militaryactions, violentactions, duration, negativebehavior, boycottembargo, aidenemy, negativecomm, accusation, protestsunoffialacts, nonviolentbehavior, emigrants, relexports, timesincewar, commonbloc2, rintergovorgs3, relintergovorgs
Cluster II	economicaid, booktranslations, tourism, relbooktranslations, releconomicaid, conferences, severdiplomatic, expeldiplomats, attackembassy, unweightedunvote, reltourism, tourism3, relemigrants, emigrants3, students, relstudents, exports, exports3, lostterritory, dependent, militaryalliance, warning
Cluster III	treaties, reltreaties, exportbooks, relexportbooks, weightedunvote, ngo, relngo, ngoorgs3, embassy, reldiplomacy, timesinceally, independence, commonbloc1
Cluster IV	commonbloc0, blockpositionindex

Supplementary Table S2: K-means clustering of relations based on factor matrix in the coefficient tensor.



Supplementary Figure S1: Comparison of coefficient estimation in the HCP data.

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