Compare two semi-supervised tensor regression model with covariate matrix

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1 Show the equivalence between two models

In the Gaussian-response case, consider the response tensor $Y \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ and a covariate matrix $A \in \mathbb{R}^{d_1 \times p}, d_1 \ll p$. The model 3 in my previous is:

$$\mathbb{E}Y = U \times_1 M_1 \times_2 M_2 \times_3 M_3, A = M_1 W \tag{1}$$

where $U \in \mathbb{R}^{r_1 \times r_2 \times r_3}, M_1 \in \mathbb{R}^{d_1 \times r_1}, M_2 \in \mathbb{R}^{d_2 \times r_2}, M_3 \in \mathbb{R}^{d_3 \times r_3}, W \in \mathbb{R}^{r_1 \times p}$.

WTS model 3 is equal to the model below:

$$\mathbb{E}Y = B \times_1 A = Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A \tag{2}$$

where $B \in \mathbb{R}^{p \times d_2 \times d_3}, Q \in \mathbb{R}^{r_1 \times r_2 \times r_3}, N_1 \in \mathbb{R}^{p \times r_1}, N_2 \in \mathbb{R}^{d_2 \times r_2}, N_3 \in \mathbb{R}^{d_3 \times r_3}$.

First: Notice that the estimate of the expectation response tensor would be the same, so to show the equivalence, we should find Q, N_1, N_2, N_3 subject to:

$$Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A = U \times_1 M_1 \times_2 M_2 \times_3 M_3, \ where \ A = M_1 W$$

Second: Notice that the model should be the same on mode 2 and mode 3, so we can easily conclude that $M_2 = N_2, M_3 = N_3$.

Third: Notice that in (1), the expectation response tensor is decomposed by tucker with r_1 on mode 1, which means the expectation response tensor should be at least has a rank r_1 on mode 1. Therefore, it is reasonable to say that $rank(A)$ is at least equal or larger than r_1. And because $A = M_1 W, rank(M_1) = r_1$, so $rank(W) \geq r_1$.

Therefore, we can find a row operation matrix $P \in \mathbb{R}^{p \times p}$ and decompose PW^T as :

$$PW^T = \begin{bmatrix} W_b \\ W_n \end{bmatrix}$$

where $W_b \in \mathbb{R}^{r_1 \times r_1}, W_n \in \mathbb{R}^{(p-r_1) \times r_1}$, and $rank(W_b) = r_1$. Then we can construct N_1^T as:

$$N_1^T = [W_b^{-1}, 0]P$$

Therefore,

$$N_1^T W^T = [W_b^{-1}, 0]PP^{-1} \begin{bmatrix} W_b \\ W_n \end{bmatrix} = I_{r_1 \times r_1}$$

And let $Q = U$, we can get:

$$\begin{aligned}
\mathbb{E}Y &= Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A \\
&= U \times_1 N_1 \times_2 M_2 \times_3 M_3 \times_1 (M_1 W) \\
&= U \times_1 (M_1 W) N_1 \times_2 M_2 \times_3 M_3 \\
&= U \times_1 M_1 \times_2 M_2 \times_3 M_3, \; where \; A = M_1 W
\end{aligned}$$