

# **Compare two semi-supervised tensor regression model with covariate matrix**

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# 1 Show the equivalence between two models

In the Gaussian-response case, consider the response tensor  $Y \in \mathbb{R}^{d_1 \times d_2 \times d_3}$  and a covariate matrix  $A \in \mathbb{R}^{d_1 \times p}$ ,  $d_1 \ll p$ . The model 3 in my previous is:

$$\mathbb{E}Y = U \times_1 M_1 \times_2 M_2 \times_3 M_3, A = M_1 W \quad (1)$$

where  $U \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ ,  $M_1 \in \mathbb{R}^{d_1 \times r_1}$ ,  $M_2 \in \mathbb{R}^{d_2 \times r_2}$ ,  $M_3 \in \mathbb{R}^{d_3 \times r_3}$ ,  $W \in \mathbb{R}^{r_1 \times p}$ .

**WTS model 3 is equal to the model below:**

$$\mathbb{E}Y = B \times_1 A = Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A \quad (2)$$

where  $B \in \mathbb{R}^{p \times d_2 \times d_3}$ ,  $Q \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ ,  $N_1 \in \mathbb{R}^{p \times r_1}$ ,  $N_2 \in \mathbb{R}^{d_2 \times r_2}$ ,  $N_3 \in \mathbb{R}^{d_3 \times r_3}$ .

**First:** Notice that the estimate of the expectation response tensor would be the same, so to show the equivalence, we should find  $Q, N_1, N_2, N_3$  subject to:

$$Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A = U \times_1 M_1 \times_2 M_2 \times_3 M_3, \text{ where } A = M_1 W$$

**Second:** Notice that the model should be the same on mode 2 and mode 3, so we can easily conclude that  $M_2 = N_2, M_3 = N_3$ .

**Third:** Notice that in (1), the expectation response tensor is decomposed by tucker with  $r_1$  on mode 1. Therefore, it is reasonable to say that  $\text{rank}(A)$  is equal to  $r_1$  because the decomposition of expectation response tensor should have the same rank. And because  $A = M_1 W, \text{rank}(M_1) = r_1$ , so  $\text{rank}(W) = r_1$ .

Therefore, we can find a row operation matrix  $P \in \mathbb{R}^{p \times p}$  and decompose  $PW^T$  as :

$$PW^T = \begin{bmatrix} W_b \\ W_n \end{bmatrix}$$

where  $W_b \in \mathbb{R}^{r_1 \times r_1}$ ,  $W_n \in \mathbb{R}^{(p-r_1) \times r_1}$ , and  $\text{rank}(W_b) = r_1$ . Then we can construct  $N_1^T$  as:

$$N_1^T = [W_b^{-1}, 0]P$$

Therefore,

$$N_1^T W^T = [W_b^{-1}, 0]P P^{-1} \begin{bmatrix} W_b \\ W_n \end{bmatrix} = I_{r_1 \times r_1}$$

And let  $Q = U$ , we can get:

$$\begin{aligned}
\mathbb{E}Y &= Q \times_1 N_1 \times_2 N_2 \times_3 N_3 \times_1 A \\
&= U \times_1 N_1 \times_2 M_2 \times_3 M_3 \times_1 (M_1 W) \\
&= U \times_1 (M_1 W) N_1 \times_2 M_2 \times_3 M_3 \\
&= U \times_1 M_1 \times_2 M_2 \times_3 M_3, \text{ where } A = M_1 W
\end{aligned}$$

So we can conclude that these two model are equivalent.