# Supplements for "Generalized tensor regression with covariates on multiple modes"

## 1 Proofs

*Proof of Theorem 4.1.* Define $\ell(\mathcal{B}) = \mathbb{E}(\mathcal{L}_{\mathcal{Y}}(\mathcal{B}))$, where the expectation is taken with respect to $\mathcal{Y} \sim \mathcal{B}_{\text{true}}$ under the model with true parameter $\mathcal{B}_{\text{true}}$. We first prove the following two conclusions:

C1. There exists two positive constants $C_1$, $C_2 > 0$, such that, with probability at least $1 - \exp(-C_1 \log K \sum_k p_k)$, the stochastic deviation, $\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})$, satisfies

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| = |\langle \mathcal{E},\ \mathcal{B} \times_1 \boldsymbol{X}_1 \times_2 \cdots \times_K \boldsymbol{X}_K \rangle| \le C_2 \|\mathcal{B}\|_F \log K \sqrt{\frac{\prod_k r_k}{\max_k r_k} \sum_k p_k}.$$

C2. The inequality $\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}}) \le -\frac{L}{2}\|\hat{\Theta} - \Theta_{\text{true}}\|_F^2$ holds, where $L > 0$ is the lower bound for $\min_{|\theta| \le \alpha} |b''(\theta)|$.

To prove C1, we note that the stochastic deviation can be written as:

$$\begin{aligned} \mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B}) &= \langle \mathcal{Y} - \mathbb{E}(\mathcal{Y}|\mathcal{X}),\ \Theta(\mathcal{B}) \rangle \\ &= \langle \mathcal{Y} - b'(\Theta_{\text{true}}),\ \Theta \rangle \\ &= \langle \mathcal{E} \times_1 \boldsymbol{X}_1^T \times_2 \cdots \times_K \boldsymbol{X}_K^T,\ \mathcal{B} \rangle, \end{aligned} \tag{1}$$

where $\mathcal{E} = [\![\varepsilon_{i_1,\ldots,i_K}]\!] \overset{\text{def}}{=} \mathcal{Y} - b'(\Theta_{\text{true}})$. Based on Proposition 1, $\varepsilon_{i_1,\ldots,i_K}$ is sub-Gaussian-$(\phi U)$. Let $\check{\mathcal{E}} \overset{\text{def}}{=} \mathcal{E} \times_1 \boldsymbol{X}_1^T \times_2 \cdots \times_K \boldsymbol{X}_K^T$. By the property of sub-Gaussian r.v's, $\check{\mathcal{E}}$ is a $(p_1, \ldots, p_K)$-dimensional sub-Gaussian tensor with parameter bounded by $C_2 = \phi U c_2^K$. Here $c_2 > 0$ is the upper bound of $\sigma_{\max}(\boldsymbol{X}_k)$. Applying Cauchy-Schwarz inequality to (1) yields

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| \le \left\|\check{\mathcal{E}}\right\|_2 \|\mathcal{B}\|_*, \tag{2}$$

where $\|\cdot\|_2$ denotes the tensor spectral norm and $\|\cdot\|_*$ denotes the tensor nuclear norm. The nuclear norm $\|\mathcal{B}\|_*$ is bounded by $\|\mathcal{B}\|_* \le \sqrt{\frac{\prod_k r_k}{\max_k r_k}} \|\mathcal{B}\|_F$ (c.f. [1, 2]). The spectral norm $\left\|\check{\mathcal{E}}\right\|_2$ is bounded by $\left\|\check{\mathcal{E}}\right\|_2 \le C_1 U c^K \log K \sqrt{\sum_k p_k}$ with probability at least $1 - \exp(-C_2 \log K \sum_k p_k)$ (c.f. [1, 3]). Combining these two bounds with (2), we have, with probability at least $1 - \exp(-C_2 \log K \sum_k p_k)$,

$$|\mathcal{L}_{\mathcal{Y}}(\mathcal{B}) - \ell(\mathcal{B})| \le C_1 U c_2^K \|\mathcal{B}\|_F \log K \sqrt{\frac{\prod_k r_k}{\max_k r_k} \sum_k p_k}.$$

Next we prove C2. Applying Taylor expansion to $\ell(\mathcal{B})$ around $\mathcal{B}_{\text{true}}$,

$$\ell(\mathcal{B}) = \ell(\mathcal{B}_{\text{true}}) - \frac{1}{2} \text{vec}(\mathcal{B} - \mathcal{B}_{\text{true}})^T \mathcal{H}_{\mathcal{Y}}(\check{\mathcal{B}}) \text{vec}(\mathcal{B} - \mathcal{B}_{\text{true}}), \tag{3}$$

where $\mathcal{H}_{\mathcal{Y}}(\check{\mathcal{B}})$ is the (non-random) Hession of $\frac{\partial \ell^2(\mathcal{B})}{\partial^2 \mathcal{B}}$ evaluated at $\check{\mathcal{B}} = \alpha \text{vec}(\alpha \mathcal{B} + (1-\alpha)\mathcal{B}_{\text{true}})$ for some $\alpha \in [0, 1]$. Recall that $b''(\theta) = \text{Var}(y|\theta)$, because $y \in \mathbb{R}$ follows the exponential family

distribution with function $b(\cdot)$. By chain rule and the fact that $\Theta = \Theta(\mathcal{B}) = \mathcal{B} \times_1 \boldsymbol{X}_1 \cdots \times_K \boldsymbol{X}_K$, the equation (4) implies that

$$\ell(\mathcal{B}) - \ell(\mathcal{B}_{\text{true}}) = -\frac{1}{2} \sum_{i_1,\ldots,i_K} b''(\check{\theta}_{i_1,\ldots,i_K})(\theta_{i_1,\ldots,i_K} - \theta_{\text{true},i_1,\ldots,i_K})^2 \leq -\frac{L}{2}\|\Theta - \Theta_{\text{true}}\|_F^2, \tag{4}$$

holds for all $\mathcal{B} \in \mathcal{P}$, provided that $\min_{|\theta|\leq\alpha} |b''(\theta)| \geq L > 0$. In particular, the inequality (4) also applies to the constrained MLE $\hat{\mathcal{B}}$. So we have

$$\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}}) \leq -\frac{L}{2}\|\hat{\Theta} - \Theta_{\text{true}}\|_F^2. \tag{5}$$

Now we have proved both C1 and C2. Note that $\mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) \geq 0$ by the definition of $\hat{\mathcal{B}}$, This implies that

$$\begin{aligned}
0 &\leq \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) \\
&\leq \Big(\mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{B}}) - \ell(\hat{\mathcal{B}})\Big) - (\mathcal{L}_{\mathcal{Y}}(\mathcal{B}_{\text{true}}) - \ell(\mathcal{B}_{\text{true}})) + \Big(\ell(\hat{\mathcal{B}}) - \ell(\mathcal{B}_{\text{true}})\Big) \\
&\leq \langle \mathcal{E},\ \Theta - \Theta_{\text{true}} \rangle - \frac{L}{2}\|\hat{\Theta} - \Theta_{\text{true}}\|_F^2,
\end{aligned}$$

where the second line follows from (5). Therefore,

$$\begin{aligned}
\|\hat{\Theta} - \Theta_{\text{true}}\|_F &\leq \frac{2}{L}\Big\langle \mathcal{E},\ \frac{\hat{\Theta} - \Theta_{\text{true}}}{\|\hat{\Theta} - \Theta_{\text{true}}\|_F} \Big\rangle \\
&\leq \frac{2}{L} \sup_{\Theta:\|\Theta\|_F=1, \Theta=\mathcal{B}\times_1\boldsymbol{X}_1\times_2\cdots\times_K\boldsymbol{X}_K} \langle \mathcal{E},\ \Theta \rangle \\
&\leq \frac{2}{L} \sup_{\mathcal{B}\in\mathcal{P}:\|\mathcal{B}\|_F\leq\prod_k \sigma_{\min}^{-1}(\boldsymbol{X}_k)} \langle \mathcal{E},\ \ \mathcal{B} \times_1 \boldsymbol{X}_1 \times_2 \cdots \times_K \boldsymbol{X}_K \rangle.
\end{aligned} \tag{6}$$

Combining (6) with C1 yields the desired conclusion.

□

**Proposition 1** (sub-Gaussian residual)**.** *Define the residual tensor* $\mathcal{E} = [\![\varepsilon_{i_1,\ldots,i_K}]\!] = \mathcal{Y} - b'(\Theta) \in \mathbb{R}^{d_1\times\cdots\times d_K}$. *Under the Assumption A2,* $\varepsilon_{i_1,\ldots,i_K}$ *is a sub-Gaussian random variable with sub-Gaussian parameter bounded by* $\phi U$*, for all* $(i_1,\ldots,i_K) \in [d_1]\times\cdots\times[d_K]$.

*Proof.* The proof is similar to Lemma 3 in [4]. For ease of presentation, we drop the subscript $(i_1,\ldots,i_K)$ and simply write $\varepsilon$ $(= y - b'(\theta))$. For any given $t \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}(\exp(t\varepsilon|\theta) &= \int c(x) \exp\left(\frac{\theta x - b(\theta)}{\phi}\right) \exp\big(t(x - b'(\theta))\big)\, dx \\
&= \int c(x) \exp\left(\frac{(\theta+\phi t)x - b(\theta+\phi t) + b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) dx \\
&= \exp\left(\frac{b(\theta+\phi t) - b(\theta) - \phi t b'(\theta)}{\phi}\right) \\
&\leq \exp\left(\frac{\phi U t^2}{2}\right),
\end{aligned}$$

where $c(\cdot)$ and $b(\cdot)$ are known functions in the exponential family corresponding to $y$. Therefore, $\varepsilon$ is sub-Gaussian-$(\phi U)$. □

*Proof of Theorem 4.2.* The proof is similar to [5]. We sketch the main steps here for completeness. Recall that $\ell(\mathcal{B}) = \mathbb{E}(\mathcal{L}_{\mathcal{Y}}(\mathcal{B}))$. By the definition of KL divergence, we have that,

$$\begin{aligned}\ell(\hat{\mathcal{B}}) &= \ell(\mathcal{B}_{\text{true}}) - \sum_{(i_1,\ldots,i_K)} KL(\theta_{\text{true},i_1,\ldots,i_K}, \hat{\theta}_{i_1,\ldots,i_K}) \\ &= \ell(\mathcal{B}_{\text{true}}) - \text{KL}(\mathbb{P}_{\mathcal{Y}_{\text{true}}},\ \mathbb{P}_{\hat{\mathcal{Y}}}),\end{aligned}$$

where $\mathbb{P}_{\mathcal{Y}_{\text{true}}}$ denotes the distribution of $\mathcal{Y}|\mathcal{X}$ with true parameter $\mathcal{B}_{\text{true}}$, and $\mathbb{P}_{\hat{\mathcal{Y}}}$ denotes the distribution with estimated parameter $\hat{\mathcal{B}}$. Therefore

$$\begin{aligned}\text{KL}(\mathbb{P}_{\mathcal{Y}_{\text{true}}},\ \mathbb{P}_{\hat{\mathcal{Y}}}) &= \ell(\mathcal{B}_{\text{true}}) - \ell(\hat{\mathcal{B}}) \\ &= \frac{1}{2}\sum_{i_1\ldots,i_K} b''(\check{\theta}_{i_1,\ldots,i_K})(\theta_{i_1,\ldots,i_K} - \theta_{\text{true},i_1,\ldots,i_K})^2 \\ &\leq \frac{U}{2}\|\Theta - \Theta_{\text{true}}\|_F^2 \\ &\leq \frac{U}{2}c_2^{2K}\|\mathcal{B} - \mathcal{B}_{\text{true}}\|_F^2,\end{aligned}$$

where the second line comes from (4), and $c_2 > 0$ is the upper bound for the $\sigma_{\max}(\boldsymbol{X}_k)$. The result then follows from Theorem 4.1. □

## 2 Time complexity

The computational complexity of our tensor regression model is $O(d^3+d)$ for each loop of iterations, where $d = \prod_k d_k$ is the total size of the response tensor. More precisely, the update of core tensor costs $O(r^3d^3)$, where $r = \sum_k r_k$ is the total size of the core tensor. The update of factor matrix $\boldsymbol{M}_k$ involves solving $p_k$ separate GLMs. Solving those GLMs requires $O(r_k^3p_k + p_kr_k^3dd_k^{-1})$, and therefore the cost for updating $K$ factors in total is $O(\sum_k r_k^3p_kd_k + d\sum_k r_k^3p_kd_k^{-1}) \approx O(\sum p_kd_k + d) \approx O(d)$.

## 3 Additional results for real data analysis

Here we provide additional results for the real data analysis.

### 3.1 HCP data analysis

The Supplement Figur S1 compares the estimated coefficients from our method (tensor regression) with those from classical GLM approach. A classical GLM is to regress the brain edges, one at a time, on the individual-level covariates, and this logistic model is repeatedly fitted for every edge $\in [68] \times [68]$. As we can see in the figure, our tensor regression shrinkages the coefficients towards center, thereby enforcing the sharing between coefficient entries.

### 3.2 Nations data analysis

Supplement table S1 summarizes the $K$-means clustering of the 56 relations based on the 3$^{\text{rd}}$ mode factor $\boldsymbol{M}_3 \in \mathbb{R}^{56\times 4}$ in the tensor regression model.

| | |
|---|---|
| Cluster I | officialvisits, intergovorgs, militaryactions, violentactions, duration, negativebehavior, boycottembargo, aidenemy, negativecomm, accusation, protestsunoffialacts, nonviolentbehavior, emigrants, relexports, timesincewar, commonbloc2, rintergovorgs3, relintergovorgs |
| Cluster II | economicaid, booktranslations, tourism, relbooktranslations, releconomicaid, conferences, severdiplomatic, expeldiplomats, attackembassy, unweightedunvote, reltourism, tourism3, relemigrants, emigrants3, students, relstudents, exports, exports3, lostterritory, dependent, militaryalliance, warning |
| Cluster III | treaties, reltreaties, exportbooks, relexportbooks, weightedunvote, ngo, relngo, ngoorgs3, embassy, reldiplomacy, timesinceally, independence, commonbloc1 |
| Cluster IV | commonbloc0, blockpositionindex |

Supplementary Table S1: $K$-means clustering of relations based on factor matrix in the coefficient tensor.

# References


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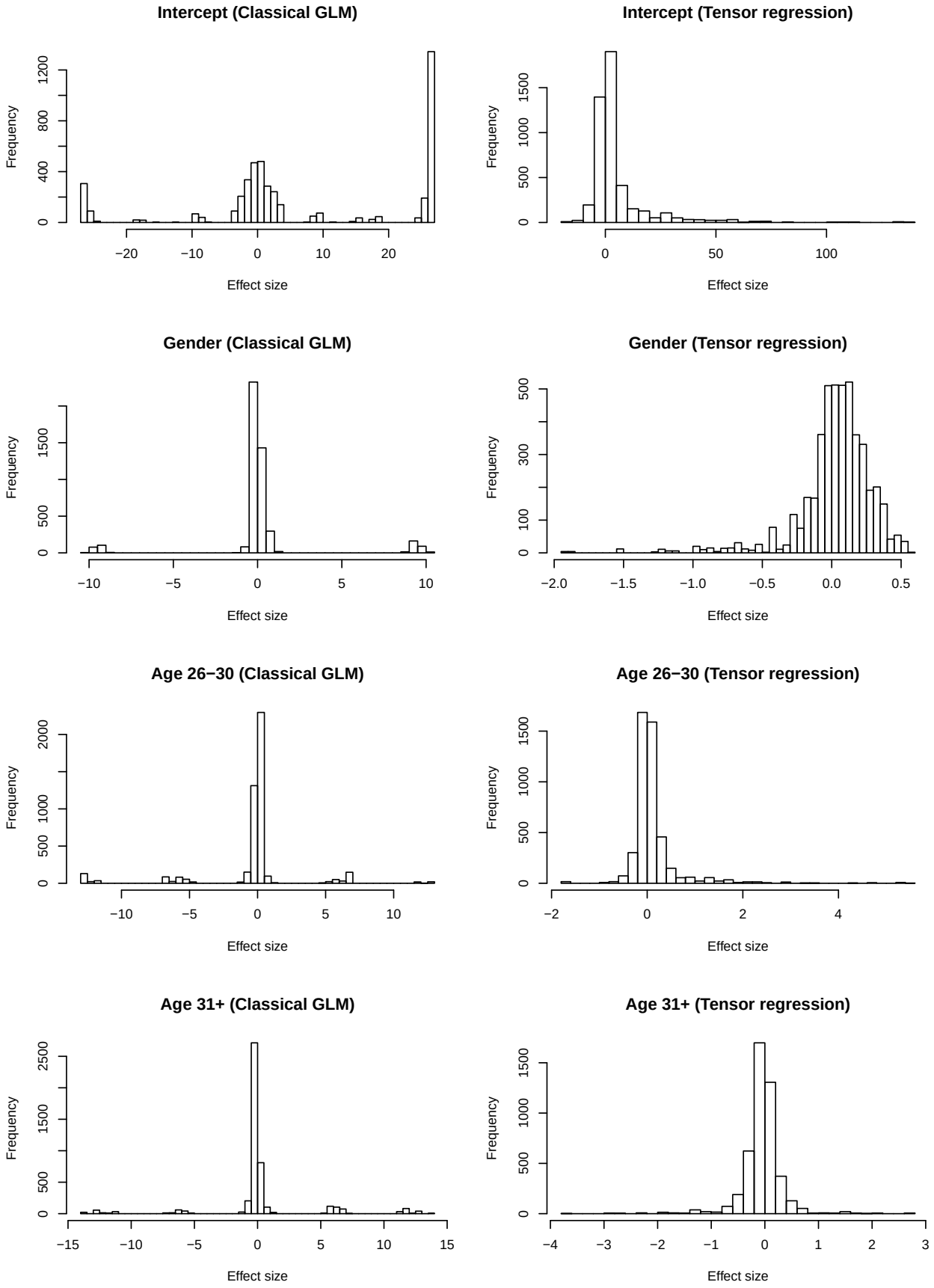
[4] Jianqing Fan, Wenyan Gong, and Ziwei Zhu. Generalized high-dimensional trace regression via nuclear norm regularization. *Journal of Econometrics*, 2019.

[5] Nicolai Baldin and Quentin Berthet. Optimal link prediction with matrix logistic regression. *arXiv preprint arXiv:1803.07054*, 2018.






Supplementary Figure S1: Comparison of coefficient estimation in the HCP data.