Graphic Lasso: Clustering accuracy for precision matrix model

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1 Accuracy

Consider the optimization problem.

$$\begin{aligned}
\min_{\boldsymbol{U},\{\Theta^l\}} \quad & Q(\boldsymbol{U},\{\Theta^l\}) = \sum_{k=1}^{K} \operatorname{tr}(S^k\Omega^k) - \log\det\left(\Omega^k\right), \qquad (1)\\
s.t. \quad & \Omega^k = \sum_{l=1}^{r} u_{kl}\Theta^l,\\
& |\Theta^l|_1 < b, \quad \text{for} \quad l \in [r],\\
& \boldsymbol{U} \in \{0,1\}^{k\times r} \text{ is a membership matrix},\\
& \{\Theta^l\} \text{ are irreducible and invertible.}
\end{aligned}$$

Notations

Note that the definitiona of confusion matrix and MCR differ with the definitions in tensor block model (TBM) by a factor K(d in TBM). Therefore, the final result includes a K in the dominator in this model.

1. Σ^l for $l = 1,...,r$: the true covariance matrices, where $(\Theta^l)^{-1} = \Sigma^l$
2. $D = [\![D_{ij}]\!] \in [0,1]^{r\times r}$: the confusion matrix between the true membership matrix $\boldsymbol{U}$ and the estimation $\hat{\boldsymbol{U}}$, where $D_{ij} = \sum_{k=1}^{K} \boldsymbol{I}(u_{ki} = \hat{u}_{kj} = 1)$.
3. $I_l = \{k : u_{kl} = 1\}$ for $l = 1,...,r$: the index set of the categories belong to the group l. The sets I_l rely on the membership $\boldsymbol{U}$. Note that $\sum_{l=1}^{r} |I_l| = K$. Let $\hat{I}_l$ denote the index sets according to the estimation $\hat{\boldsymbol{U}}$.
4. $MCR(\hat{\boldsymbol{U}},\boldsymbol{U}) = \max_{l,a\neq a'\in[r]} \min\{D_{al}, D_{a'l}\}$: the misclassification rate.
5. $\delta = \min_{k\neq l\in[r]} \min_{(i,j)}(\Sigma_{ij}^k - \Sigma_{ij}^l)^2$: the minimal gap between the true covariance matrices.
6. $\varphi\min(\cdot)$: the minimal singular value of the matrix.
7. $\varphi_{\max}(\cdot)$: the maximal singular value of the matrix,
8. $n_k > 0$ for $k \in [K]$: the n_k is the sample size for the k-th category.

Theorem 1.1. *Suppose the singular value for the true precision matrices are bounded, i.e.,* $0 < s < \min_{l\in[r]} \varphi_{\min}(\Theta^l) \leq \max_{l\in[r]} \varphi_{\max}(\Theta^l) < \tau < \infty$, *where* $s < \tau$ *are positive constants. The minimal gap between the precision matrices* $\delta = \min_{k\neq l\in[r]} \min_{(i,j)}(\Sigma^k_{ij} - \Sigma^l_{ij})^2$ *is larger than 0. Let* $\hat{\boldsymbol{U}}$ *denote the minimizer of the objective function* (1)*. Then, for any* $\epsilon \in [0, 1]$, *we have*

$$\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) \leq C_1 \exp\left(-\frac{C_2 \min_{k\in[K]} n_k \epsilon^2\delta^2}{s^2K^2r^2b^2 \max_{l\in[r]} \|\Theta^l\|^2_{\max}}\right),$$

where C_1, C_2 *are two constants only depend on* τ.

Proof. Recall the objective function

$$Q(\boldsymbol{U}, \{\Theta^l\}) = \sum_{k=1}^{K} \mathrm{tr}(S^k\Omega^k) - \log\det(\Omega^k) = \sum_{k=1}^{K}\langle S^k, \Omega^k\rangle - \log\det(\Omega^k).$$

The deviation between the true parameters $\{\boldsymbol{U}, \{\Theta^l\}\}$ and estimations $\{\hat{\boldsymbol{U}}, \{\hat{\Theta}^l\}\}$ comes from two aspects: the estimation of $\{\Theta^l\}$ and the misclassification (estimation of $\boldsymbol{U}$). We tease apart these two parts.

1. First, we suppose the membership $\boldsymbol{U}$ is given. We now assess the stochastic error due to the estimation of $\{\Theta^l\}$, conditional on $\boldsymbol{U}$. Note that the objective function is convex. The optimal $\{\Theta^l\}$ satisfies the first order condition.

$$\frac{\partial Q}{\partial \Theta^l_{ij}} = \sum_{k\in I_l} S^k_{ij} - |I_l|\frac{C(\Theta^l)_{ij}}{\det(\Theta^l)} = 0,$$

where $C(A)_{ij}$ is the cofactor of matrix A corresponding to the element A_{ij}. Note that Θ^l should be a symmetric matrix. The cofactor matrix $C(\Theta^l) = C^T(\Theta^l)$. Then, the derivation of the matrix Θ^l is equal to

$$\frac{\partial Q}{\partial \Theta^l} = \sum_{k\in I_l} S^k - |I_l|\frac{C^T(\Theta^l)}{\det(\Theta^l)} = 0,$$

which implies that

$$\hat{\Theta}^l = \left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}, \quad \text{for} \quad l \in [r].$$

Therefore, the estimation of $\{\Theta^l\}$ is a function of $\boldsymbol{U}$. Consider the function $F(\boldsymbol{U}) = Q(\boldsymbol{U}, \{\hat{\Theta}^l\})$. By a straightforward calculation, we have

$$\begin{aligned} F(\boldsymbol{U}) &= \sum_{l=1}^{r}\sum_{k\in I_l}\langle S^k, \left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}\rangle - |I_l|\log\det\left(\left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}\right) \\ &= \sum_{l=1}^{r}\langle\sum_{k\in I_l} S^k, \left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}\rangle - |I_l|\log\det\left(\left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}\right) \\ &= \sum_{l=1}^{r} |I_l|p - |I_l|\log\det\left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}. \end{aligned}$$

Note that $\sum_{l=1}^r |I_l| p = Kp$ is independent with the membership. We only need to consider the second term. For simplicity, we define

$$F(\boldsymbol{U}) = -\sum_{l=1}^{r} |I_l| \log\det\left(\left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right)^{-1}\right).$$

Note that $\mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|}\right] = \frac{\sum_{a=1}^r D_{al}\Sigma^l}{|I_l|}$. Correspondingly, we define the population version of $F(\boldsymbol{U})$ as following.

$$G(\boldsymbol{U}) = -\sum_{l=1}^{r} |I_l| \log\det\left(\left(\frac{\sum_{a=1}^r D_{al}\Sigma^a}{|I_l|}\right)^{-1}\right),$$

where $\Sigma^k = \mathbb{E}[S^k]$ is the true covariance matrices. Therefore, the deviation $F(\boldsymbol{U}) - G(\boldsymbol{U})$ quantifies the stochastic error due to the estimation of $\{\Theta^l\}$.

2. Next, we free $\boldsymbol{U}$ and quantify the total deviation. Considering the maximizer,

$$\hat{\boldsymbol{U}} = \arg\min_{\boldsymbol{U}} F(\boldsymbol{U}).$$

The corresponding $G(\hat{\boldsymbol{U}})$ is

$$G(\hat{\boldsymbol{U}}) = -\sum_{l=1}^{r} |\hat{I}_l| \log\det\left(\left(\frac{\sum_{a=1}^r D_{al}\Sigma^a}{|\hat{I}_l|}\right)^{-1}\right),$$

and the function $G(\boldsymbol{U})$ with true membership is

$$G(\boldsymbol{U}) = -\sum_{l=1}^{r} |I_l| \log\det\left(\left(\frac{\sum_{k\in I_l} \Sigma^l}{|I_l|}\right)^{-1}\right) = \sum_{l=1}^{r} |I_l| \log\det(\Sigma^l).$$

Then, the deviation $G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U})$ measures the stochastic error of the misclassification.

Now back to the probability of misclassification rate. By Lemma 1, we have

$$\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) \leq \mathbb{P}(G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) \leq -\frac{1}{4s}\epsilon\delta).$$

Notice that the total deviation between $\hat{\boldsymbol{U}}$ and $\boldsymbol{U}$ is able to be decomposed into three parts.

$$\begin{aligned} F(\hat{\boldsymbol{U}}) - F(\boldsymbol{U}) &= \left[F(\hat{\boldsymbol{U}}) - G(\hat{\boldsymbol{U}})\right] + \left[G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U})\right] + [G(\boldsymbol{U}) - F(\boldsymbol{U})] \\ &\leq 2m - \frac{1}{4s}\epsilon\delta, \end{aligned}$$

where $m = sup_{\boldsymbol{U}}|F(\boldsymbol{U}) - G(\boldsymbol{U})|$. Since $\hat{\boldsymbol{U}}$ is the minimizer of the objective function, we know that $F(\hat{\boldsymbol{U}}) - F(\boldsymbol{U}) \leq 0$. Therefore, we obtain the accuracy of misclassification rate

$$
\begin{aligned}
\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) &\leq \mathbb{P}(F(\hat{\boldsymbol{U}}) - F(\boldsymbol{U}) \leq 2m - \frac{1}{4s}\epsilon\delta) \\
&\leq \mathbb{P}(m \geq \frac{1}{8s}\epsilon\delta) \\
&\leq \mathbb{P}\left(\max_{k,(i,j)} |S^k_{(i,j)} - \mathbb{E}[S^k_{(i,j)}]| \geq \frac{\epsilon\delta}{8sKrb\max_{l\in[r]} \|\Theta^l\|_{\max}}\right), \\
&\leq C_1 \exp\left(-\frac{C_2 \min_{k\in[K]} n_k \epsilon^2\delta^2}{s^2K^2r^2b^2 \max_{l\in[r]} \|\Theta^l\|^2_{\max}}\right)
\end{aligned}
$$

where the third inequality follow by Lemma 2, the last inequality follows by the Lemma 3, and C_1, C_2 are two constants.

$\square$

Lemma 1. *Assume the minimal singular-value of the true precision matrices is lower bounded $\min_{l\in[r]} \varphi_{\min}(\Theta^l) > s$, where s is a positive constant, and the minimal gap between covariance matrices $\delta > 0$. For any fixed $\epsilon > 0$, suppose the misclassification rate $MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon$, we have*

$$
G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) \leq -\frac{1}{4s}\epsilon\delta.
$$

Proof. Note that for an invertible matrix A, $\det(A^{-1}) = \frac{1}{\det(A)}$. Recall the formula of $G(\boldsymbol{U})$ and $G(\hat{\boldsymbol{U}})$. We have

$$
G(\boldsymbol{U}) = \sum_{l=1}^{r} |I_l| \log\det(\Sigma^l), \quad \text{and} \quad G(\hat{\boldsymbol{U}}) = \sum_{l=1}^{r} |\hat{I}_l| \log\det\left(\frac{\sum_{a=1}^r D_{al}\Sigma^a}{|\hat{I}_l|}\right).
$$

Note that

$$
\sum_{l=1}^{r} |\hat{I}_l| \left(\frac{\sum_{a=1}^r D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) = \sum_{a=1}^{r}\sum_{l=1}^{r} D_{al}\log\det(\Sigma^a) = G(\boldsymbol{U}),
$$

where the second equality follows by the fact that $\sum_{l=1}^r D_{al} = |I_a|$. Since $MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon$, there exist $l, k \neq k' \in [r]$ such that $\min\{D_{kl}, D_{k'l}\} \geq \epsilon$. Let $\tilde{\Sigma} = \frac{\sum_{a=1}^r D_{al}\Sigma^a}{|\hat{I}_l|}$, and $\Delta = \Sigma - \tilde{\Sigma}$ for some matrix Σ. Consider the function $f(t) = \log\det\left(\tilde{\Sigma} + t\Delta\right)$. By Taylor Expansion, we have

$$
\log\det(\Sigma) - \log\det(\tilde{\Sigma}) = f(1) - f(0) = f'(0) + \frac{f''(\xi)}{2}, \quad \text{for some} \quad \xi \in [0,1], \tag{2}
$$

where

$$
f'(0) = \langle\tilde{\Sigma}, \Delta\rangle, \quad \text{and} \quad f''(\xi) = \text{vec}(\Delta)^T(\tilde{\Sigma} + \xi\Delta)^{-1} \otimes (\tilde{\Sigma} + \xi\Delta)^{-1}\text{vec}(\Delta). \tag{3}
$$

Particularly, by the definition of singular value, we have the lower bound of the second derivative

$$
f''(\xi) = \text{vec}(\Delta)^T(\tilde{\Sigma} + \xi\Delta)^{-1} \otimes (\tilde{\Sigma} + \xi\Delta)^{-1}\text{vec}(\Delta) \geq \|\Delta\|_F^2\, s, \tag{4}
$$

where $\|\cdot\|_F$ is the matrix Frobenius norm.

Let $\Delta^l = \Sigma^l - \tilde{\Sigma}, l \in [r]$. Combining the Taylor Expansion (1) with the lower bound (1), we have

$$\begin{aligned}\left(\frac{\sum_{a=1}^r D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) - \log\det\left(\tilde{\Sigma}\right) &= \sum_{a=1}^{l}\frac{D_{al}}{|\hat{I}_l|}\left[\log\det(\Sigma^a) - \log\det(\tilde{\Sigma})\right]\\ &\geq \sum_{a=1}^{r}\frac{D_{al}}{|\hat{I}_l|}\left(\langle\tilde{\Sigma},\Delta^a\rangle + \frac{1}{2}s\left\|\Delta^a\right\|_F^2\right)\\ &\geq \frac{D_{kl}}{2|\hat{I}_l|}s\left\|\Delta^k\right\|_F^2 + \frac{D_{k'l}}{2|\hat{I}_l|}s\left\|\Delta^{k'}\right\|_F^2,\end{aligned}$$

where the last inequality follows by the fact that $\sum_{a=1}^r \frac{D_{al}}{|\hat{I}_l|}\langle\tilde{\Sigma},\Delta^a\rangle = 0$. By the inequality $\frac{1}{2}\left\|A+B\right\|_F^2 \leq \left\|A\right\|_F^2 + \left\|B\right\|_F^2$, we obtain that

$$\left(\frac{\sum_{a=1}^r D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) - \log\det\left(\tilde{\Sigma}\right) \geq \frac{\min\{D_{kl}, D_{k'l}\}s}{|\hat{I}_l|}\left\|\Sigma^k - \Sigma^{k'}\right\|_F^2 \geq \frac{\epsilon}{4s|I_l|}\delta. \tag{5}$$

For other $l' \in [r]/l$, since $\log\det(\cdot)$ is a convex function, by Jensen's inequality, we have

$$\left(\frac{\sum_{a=1}^r D_{al'}\log\det(\Sigma^a)}{|\hat{I}_{l'}|}\right) - \log\det\left(\frac{\sum_{a=1}^r D_{al'}\Sigma^a}{|\hat{I}_{l'}|}\right) \geq 0. \tag{6}$$

Combining the the inequality (1) and (1), we obtain the misclassification error

$$G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) = \sum_{l=1}^r |\hat{I}_l|\log\det\left(\frac{\sum_{a=1}^r D_{al}\Sigma^a}{|\hat{I}_l|}\right) - \sum_{l=1}^r |\hat{I}_l|\left(\frac{\sum_{a=1}^r D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) \leq \frac{1}{4s}\epsilon\delta.$$

□

Lemma 2. *Suppose we have $|\Theta^l|_1 < b$ for all $l \in [r]$, where $|A|_1$ is the number of nonzero elements in matrix A. Then, we have*

$$|F(\boldsymbol{U}) - G(\boldsymbol{U})| \leq Krb\max_{l\in[r]}\left\|\Theta^l\right\|_{\max}\max_{k,(i,j)}|S^k_{(i,j)} - \mathbb{E}[S^k_{(i,j)}]|$$

Proof. Recall the formula of $F(\boldsymbol{U})$ and $G(\boldsymbol{U})$, where $\boldsymbol{U}$ may not be the true membership matrix. We have

$$|F(\boldsymbol{U}) - G(\boldsymbol{U})| \leq \sum_{l=1}^r |I_l|\left|\log\det\left(\frac{\sum_{k\in I_l}S^k}{|I_l|}\right) - \log\det\left(\mathbb{E}\left[\frac{\sum_{k\in I_l}S^k}{|I_l|}\right]\right)\right|.$$

Consider the function $f(t) = \log\det\left(\frac{\sum_{k\in I_l}S^k}{|I_l|} + t\Delta\right)$, where $\Delta = \mathbb{E}\left[\frac{\sum_{k\in I_l}S^k}{|I_l|}\right] - \frac{\sum_{k\in I_l}S^k}{|I_l|}$. By the previous calculation (1), we know that $f(t)$ is a convex function. Then, the function is locally Lipschitz with $L = \sup_t|f'(t)|$. Therefore, we have

$$\begin{aligned}|F(\boldsymbol{U}) - G(\boldsymbol{U})| &\leq \sum_{l=1}^r |I_l||f(1) - f(0)|\\ &\leq \sum_{l=1}^r |I_l||f'(1)|\\ &\leq K\sup\left|\left\langle\left(\mathbb{E}\left[\frac{\sum_{k\in I_l}S^k}{|I_l|}\right]\right)^{-1}, \frac{\sum_{k\in I_l}S^k}{|I_l|} - \mathbb{E}\left[\frac{\sum_{k\in I_l}S^k}{|I_l|}\right]\right\rangle\right|.\end{aligned}$$

Since $(A)^{-1}$ is convex function of A, we have

$$\left\| \left(\mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|} \right] \right)^{-1} \right\|_{\max} \leq \left\| \left(\frac{\sum_{k\in I_l} \mathbb{E}[S^k]^{-1}}{|I_l|} \right) \right\|_{\max} \leq \max_{l\in[r]} \left\| \Theta^l \right\|_{\max}.$$

We also have the sparsity

$$\left| \left(\frac{\sum_{k\in I_l} \mathbb{E}[S^k]^{-1}}{|I_l|} \right) \right|_1 \leq rb.$$

Therefore, we obtain the upper bound

$$|F(\boldsymbol{U}) - G(\boldsymbol{U})| \leq Krb \max_{l\in[r]} \left\| \Theta^l \right\|_{\max} \max_{k,(i,j)} |S^k_{(i,j)} - \mathbb{E}[S^k_{(i,j)}]|.$$

□

Lemma 3. *Let $Z_i \sim_{i.i.d.} \mathcal{N}(0, \Sigma)$ and $\varphi_{max}(\Sigma) \leq \tau < \infty$. Let $\Sigma = [\![\Sigma_{ij}]\!]$, then*

$$P\left(\left| \sum_{i=1}^{n} Z_{ij} Z_{ik} - n\Sigma_{jk} \right| \geq n\nu \right) \leq c_1 e^{-c_2 n\nu^2}, \quad \textit{for} \quad |\nu| \leq \delta,$$

where c_1, c_2, δ depends on τ only.

Proof. See Lemma 1 of Rothman et.al. □