Linear Algebra

A summary for MIT 18.06SC

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1 Matrices & Spaces

1.1 Basic concepts

- Given vectors $v_1, ..., v_n$ and scalars $c_1, ..., c_n$, the sum $c_1 v_1 + \cdots + c_n v_n$ is called a *linear combination* of $v_1, ..., v_n$.
- The vectors $v_1, ..., v_n$ are *linearly independent* (or just *indepedent*) if $c_1 v_1 + \cdots + c_n v_n = 0$ holds only when $c_1 = ... = c_n = 0$. If the vectors $v_1, ..., v_n$ are *dependent*, there exist scalars $c_1, ..., c_n$ which are not all equal to 0 such that $c_1 v_1 + \cdots + c_n v_n = 0$.
- Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$, the multiplication $\boldsymbol{A}x$ is a linear combination of the columns of $\boldsymbol{A}$, and $x^T \boldsymbol{A}$ is a linear combination of the rows of $\boldsymbol{A}$.
- Matrix multiplication is typically not communicative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$. Lemma 1 describes a special case where matrix multiplication is communicative.
- Suppose $\boldsymbol{A}$ is a square matrix. The matrix $\boldsymbol{A}$ is *invertible* or *non-singular* if there exists an $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I$. Otherwise, the matrix $\boldsymbol{A}$ is *singular*, and the determinant of $\boldsymbol{A}$ is 0.
- The inverse of a matrix product $\boldsymbol{AB}$ is $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. The product of invertible matrices is still invertible.
- The transpose of a matrix product $\boldsymbol{AB}$ is $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$. For any invertible matrix $\boldsymbol{A}$, $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$.
- A matrix $\boldsymbol{Q}$ is orthogonal if $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$. A matrix $\boldsymbol{Q}$ is unitary if $\boldsymbol{Q}^* = \boldsymbol{Q}^{-1}$, where $\boldsymbol{Q}^*$ is the conjugate transpose of $\boldsymbol{Q}$.

Lemma 1 (Communicative matrix multiplication)**.** *For matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times n}$, the matrix multiplication of $\boldsymbol{A}$ and $\boldsymbol{B}$ is communicative, i.e. $\boldsymbol{AB} = \boldsymbol{BA}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same set of eigenvectors corresponding to their non-zero eigenvalues.*

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same set of eigenvectors corresponding to their non-zero eigenvalues, there exists a matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ such that

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}_A\boldsymbol{Q}^{-1}, \quad \boldsymbol{B} = \boldsymbol{Q}\boldsymbol{D}_B\boldsymbol{Q}^{-1},$$

where columns of $\boldsymbol{Q}$ are eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\boldsymbol{D}_A \in \mathbb{R}^{n\times n}, \boldsymbol{D}_B \in \mathbb{R}^{n\times n}$ are diagonal matrices whose diagonal elements are eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Because matrix multiplication is communicative for two diagonal matrices with same dimensions, we have

$$\boldsymbol{AB} = \boldsymbol{QD}_A\boldsymbol{Q}^{-1}\boldsymbol{QD}_B\boldsymbol{Q}^{-1} = \boldsymbol{QD}_A\boldsymbol{D}_B\boldsymbol{Q}^{-1} = \boldsymbol{QD}_B\boldsymbol{D}_A\boldsymbol{Q}^{-1} = \boldsymbol{QD}_B\boldsymbol{Q}^{-1}\boldsymbol{QD}_A\boldsymbol{Q}^{-1} = \boldsymbol{BA}.$$

Therefore, the matrix multiplication of $\boldsymbol{A}$ and $\boldsymbol{B}$ is communicative. □

1.2 Permutation of matrices

Let $\boldsymbol{A}$ be a matrix, we swap the rows of $\boldsymbol{A}$ by multiplying a *permutation matrix* $\boldsymbol{P}$ on the left of $\boldsymbol{A}$. For example,

$$\boldsymbol{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix}$$

where a_k refers to the k-th row of $\boldsymbol{A}$. The inverse of the permutation matrix $\boldsymbol{P}$ is $\boldsymbol{P}^{-1} = \boldsymbol{P}^T$, which implies the orthogonality of permutation matrix. For an $n \times m$ matrix, there are $n!$ different row permutation matrices, which form a *multiplicative group*.

Similarly, we also swap the columns of the matrix $\boldsymbol{A}$ by multiplying a permutation matrix on the right of $\boldsymbol{A}$.

1.3 Elimination of matrices

Elimination is an important technique in linear algebra. We eliminate the matrix by multiplications and subtractions. Take a 3-by-3 matrix $\boldsymbol{A}$ as an example.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 2}} \boldsymbol{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

In step 1, we choose the number 1 in row 1 column 1 as a *pivot*. Then we recopy the first row, multiply an appropriate number (in this case, 3) and subtract these values from the numbers in the second row. We have thus eliminated 3 in row 2 column 1. Similarly, in step 2, we choose 2 in row 2 column 2 as a pivot and eliminate the number 4 in row 3 column 2. The number 5 in row 3 column 3 is also a pivot. The matrix $\boldsymbol{U}$ is an upper traingular matrix.

The *elimination matrix* used to eliminate the entry in row m column n is denoted as $\boldsymbol{E}_{mn}$. In the previous example,

$$\boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}; \quad \boldsymbol{E}_{32}(\boldsymbol{E}_{21}\boldsymbol{A}) = \boldsymbol{U}.$$

Pivots are non-zero. If there exists a 0 in the pivot position, we exchange the row with one below to get a non-zero value in pivot position. If all numbers below the pivots are 0, we skip this column and find a pivot in the next column.

We write $\boldsymbol{E}_{32}(\boldsymbol{E}_{21}\boldsymbol{A}) = (\boldsymbol{E}_{32}\boldsymbol{E}_{21})\boldsymbol{A} = \boldsymbol{U}$, because matrix multiplication is associative. Let $\boldsymbol{E}$ denote the product of all elimination matrices. If we need to permute the rows during the process, we multiply a permutation matrix on the left of $\boldsymbol{A}$. Therefore, the elimination process of $\boldsymbol{A}$ is

$$\boldsymbol{EPA} = \boldsymbol{U}, \tag{1}$$

where $\boldsymbol{U}$ is an upper triangular matrix.

Next, we prove the invertibility of the elimination matrix.

Lemma 2 (Invertiblity of elimination matrix)**.** *Suppose there is an elimination matrix $\boldsymbol{E}_{ij} \in \mathbb{R}^{n\times n}$ that multiplies a scalar $-c$ to the j-th row and subtracts the row from i-th row, where $i \neq j$. The matrix $\boldsymbol{E}_{ij}$ is invertible.*

Proof. We write the elimination matrix as

$$\boldsymbol{E}_{ij} = \boldsymbol{I}_n + ce_ie_j^T,$$

where $e_i \in \mathbb{R}^n$ denotes the vector with 1 in the i-th entry and 0 elsewhere. Because $i \neq j$, $e_i^T e_j = 0$. We have

$$(\boldsymbol{I}_n + ce_ie_j^T)(\boldsymbol{I}_n - ce_ie_j^T) = \boldsymbol{I}_n - c^2 e_ie_j^T e_ie_j^T = \boldsymbol{I}_n; \quad (\boldsymbol{I}_n - ce_ie_j^T)(\boldsymbol{I}_n + ce_ie_j^T) = \boldsymbol{I}_n.$$

Therefore, $\boldsymbol{I}_n - ce_ie_j^T$ is the inverse of $\boldsymbol{E}_{ij}$. The elimination matrix $\boldsymbol{E}_{ij}$ is invertible. $\square$

Corollary 1 (Inverse of elimination matrix)**.** Suppose the elimination matrix $\boldsymbol{E}_{ij}$ in lemma 2 is a lower/upper-triangular matrix. The inverse $\boldsymbol{E}_{ij}^{-1}$ is also a lower/upper-triangular matrix.

Proof. By the proof of lemma 2, the matrix $\boldsymbol{E}_{ij}$ and its inverse are written as

$$\boldsymbol{E}_{ij} = \boldsymbol{I}_n + ce_ie_j^T, \quad \boldsymbol{E}_{ij}^{-1} = \boldsymbol{I}_n - ce_ie_j^T.$$

Without the loss of generality (WLOG), we assume that $\boldsymbol{E}_{ij}$ is a lower-triangular matrix. Then $ce_ie_j^T$ and $-ce_ie_j^T$ are also lower-triangular matrices. Therefore, $\boldsymbol{E}_{ij}^{-1}$ is a lower-triangular matrix. $\square$

1.4 Gauss-Jordan Elimination

We also use elimination to find the inverse of an invertible matrix.

Suppose $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is an invertible matrix. The inverse of $\boldsymbol{A}$, denoted $\boldsymbol{A}^{-1}$, satisfies

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n. \tag{2}$$

Suppose there exists an elimination matrix $\boldsymbol{E}$ such that $\boldsymbol{E}\boldsymbol{A} = \boldsymbol{I}_n$. Multiplying $\boldsymbol{E}$ on the both sides of the equation (2), we have $\boldsymbol{E}\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} = \boldsymbol{E}$. To obtain a such $\boldsymbol{E}$, we eliminate the *augmented matrix* $[\boldsymbol{A}|\boldsymbol{I}_n]$ until $\boldsymbol{A}$ becomes $\boldsymbol{I}_n$. Then, the augmented matrix becomes $\boldsymbol{E}[\boldsymbol{A}|\boldsymbol{I}_n] = [\boldsymbol{I}_n|\boldsymbol{E}]$, where $\boldsymbol{E}$ is the inverse of $\boldsymbol{A}$.

We call this elimination process of finding $\boldsymbol{E}$ as *Gauss-Jordan Elimination*.

1.5 Factorization of matrices

By elimination, any square matrix $\boldsymbol{A}$ has the factorization (1). By lemma 2, $\boldsymbol{E}$ is invertible. Multiplying $\boldsymbol{E}^{-1}$ on both sides of equation (1) yields,

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{E}^{-1}\boldsymbol{U}.$$

Note that $\boldsymbol{E}$ is a lower-triangular matrix. By corollary 1, $\boldsymbol{E}^{-1}$ is also a lower-triangular matrix. Let $\boldsymbol{L}$ denote $\boldsymbol{E}^{-1}$, where the letter L refers to "lower triangular". Therefore, any square matrix $\boldsymbol{A}$ has a factorization:

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}, \tag{3}$$

where $\boldsymbol{U}$ is an upper triangular matrix with pivots on the diagonal, $\boldsymbol{L}$ is a lower triangular matrix with ones on the diagonal, and $\boldsymbol{P}$ is a permutation matrix. However, the equation (3) is not the unique factorization of $\boldsymbol{A}$. For example, $c\boldsymbol{L}$ and $c^{-1}\boldsymbol{U}$ also factorize $\boldsymbol{A}$, where c is a non-zero scalar.

1.6 Time complexity of elimination

For an n-by-n matrix, a single elimination step multiplies a selected row and subtracts the selected row from another row. A single elimination step requires $\mathcal{O}(n)$ operations. To eliminate the elements below the first diagonal element, we need to repeat single elimination $(n-1)$ times and thus require $\mathcal{O}(n^2)$ operations. Similarly, we require $\mathcal{O}((n-1)^2)$ operations to eliminate the elements below the second diagonal element. Repeat the elimination until we meet the n-th diagonal element. Therefore, we require in total $\mathcal{O}(n^3)$ operations to obtain an upper-triangular matrix by elimination:

$$1^2 + 2^2 + \cdots + (n)^2 = \sum_i^n i^2 \approx \int_0^n x^2 dx = \frac{1}{3}n^3 = \mathcal{O}(n^3).$$

1.7 Reduced row echelon form of matrices

In previous sections, we convert a matrix $\boldsymbol{A}$ to an upper triangular matrix $\boldsymbol{U}$. Next, we convert $\boldsymbol{U}$ into the *reduced row echelon form* (RREF), which is a simpler form than upper triangle. We use $\boldsymbol{R} = RREF(\boldsymbol{A})$ to denote the reduced row echelon form of $\boldsymbol{A}$. In the matrix $\boldsymbol{R}$, the pivots are equal to 1, and the elements above and below the pivots are eliminated to 0. In the previous example,

$$\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is another example,

$$\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \boldsymbol{R} = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that there are r pivots in $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. With proper permutation, the matrix $\boldsymbol{R}$ is in the form of $\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$, where $\boldsymbol{F} \in \mathbb{R}^{r\times(n-r)}$ is an arbitrary matrix. The columns in $\boldsymbol{A}$ which correspond to the identity matrix $\boldsymbol{I}_r$ are called *pivot columns*. The other columns are *free columns*.

1.8 Vector space, Subspace, and Column space

- *Vector space* is a collection of vectors that is closed under linear combination (addition and multiplication by any real number); i.e. for any vectors in the collection, all the combinations of these vectors are still in the collection.

- *Subspaces of the vector space* is a vector space that is contained inside of another vector space.

Note that any vector space or subspace must include an origin. For a vector space $\mathcal{A}$, the subspace of $\mathcal{A}$ can be $\mathcal{A}$ itself or a set that contains only a zero vector.

- Vectors $v_1, ..., v_n$ *span* a space that consists of all the linear combination of these vectors.

- *Column space* of a matrix $\boldsymbol{A}$ is the space spanned by the columns of $\boldsymbol{A}$. Let $C(\boldsymbol{A})$ denote the column space of $\boldsymbol{A}$.

If $v_1, ..., v_n$ span a space $\mathcal{S}$, then $\mathcal{S}$ is the smallest space that contains these vectors.

- *Basis* of a vector space is a set of vectors $v_1, ..., v_n$ that satisfy: (1) $v_1, ..., v_n$ are independent; (2) $v_1, ..., v_n$ span the space.
- *Dimension* of a space is the number of vectors in a basis of this space. Let $dim(\mathcal{A})$ denote the dimension of space $\mathcal{A}$.

1.9 Matrix rank

Let $rank(\boldsymbol{A})$ denote the *rank* of matrix $\boldsymbol{A}$. We have

$$rank(\boldsymbol{A}) \triangleq dim(C(\boldsymbol{A})) = \#\text{ of pivot columns of }\boldsymbol{A}. \tag{4}$$

If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $rank(\boldsymbol{A}) = r$, we have $r \le \min\{m, n\}$. The matrix is *full rank* if $r = \min\{m, n\}$.

The rank of a square matrix is closely related to the invertibility.

Lemma 3 (Full rankness and invertibility)**.** *A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is full rank, if and only if $\boldsymbol{A}$ is an invertible matrix.*

Proof. ($\Rightarrow$) Suppose $\boldsymbol{A}$ is full rank.
Let $\boldsymbol{R}$ denote the RREF form of $\boldsymbol{A}$. There exist an elimination matrix $\boldsymbol{E}$ and a permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{EPA} = \boldsymbol{R}.$$

By the full rankness of $\boldsymbol{A}$, $\boldsymbol{A}$ has n pivot columns, and thus $\boldsymbol{R} = \boldsymbol{I}_n$. By lemma 2, $\boldsymbol{E}$ is invertible. The permutation matrix $\boldsymbol{P}$ is also invertible. Then, the matrix product $\boldsymbol{EP}$ is invertible, and $\boldsymbol{A}$ is the inverse of $\boldsymbol{EP}$. Therefore, $\boldsymbol{A}$ is invertible.
($\Leftarrow$) Suppose $\boldsymbol{A}$ is invertible. We prove the full rankness of $\boldsymbol{A}$ by contradiction.
Assume that $rank(\boldsymbol{A}) < n$. By equation (4), $dim(C(\boldsymbol{A})) = rank(\boldsymbol{A}) < n$, which implies that the columns of $\boldsymbol{A}$ are linearly dependent. Then, there exists a non-zero vector v such that

$$\boldsymbol{A}v = 0. \tag{5}$$

By assumption, the inverse $\boldsymbol{A}^{-1}$ exists. Multiplying $\boldsymbol{A}^{-1}$ on both sides of equation (5), we have

$$v = \boldsymbol{A}^{-1}0 = 0. \tag{6}$$

However, equation (6) contradicts the fact that v is a non-zero vector. Therefore, $\boldsymbol{A}$ is full rank. $\square$

The rank of $\boldsymbol{A}$ also affects the number of solutions to the system $\boldsymbol{A}x = b$. We will discuss the relationship between matrix rank and the solutions in next section.

2 Solving $Ax = b$

In this section, we discuss the solutions to the linear system $\boldsymbol{A}x = b$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are vectors.

2.1 Solving $Ax = 0$: nullspace

The *nullspace* of matrix $\boldsymbol{A}$ is the collection of all solutions x to the system $\boldsymbol{A}x = 0$. Let $N(\boldsymbol{A})$ denote the nullspace of $\boldsymbol{A}$.

Lemma 4 (Nullspace)**.** *The nullspace of matrix $\boldsymbol{A}$ is a vector space.*

Proof. We only need to prove that $N(\boldsymbol{A})$ is closed under linear combination. For any $v_1, v_2 \in N(\boldsymbol{A})$, we have,

$$\boldsymbol{A}(c_1v_1 + c_2v_2) = c_1\boldsymbol{A}v_1 + c_2\boldsymbol{A}v_2 = 0, \quad \forall c_1, c_2 \in \mathbb{R}. \tag{7}$$

The equation (7) implies that $N(\boldsymbol{A})$ is closed under linear combination. Therefore, $N(\boldsymbol{A})$ is a vector space. ☐

Lemma 5 (The rank of nullspace)**.** *If $rank(\boldsymbol{A}) = r$, the rank of nullspace $rank(N(\boldsymbol{A})) = n - r$.*

Proof. Let $\boldsymbol{R}$ denote the RREF($\boldsymbol{A}$). We write $\boldsymbol{R}$ as $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$, where $\boldsymbol{F} \in \mathbb{R}^{r\times(n-r)}$ is an arbitrary matrix. Let $\boldsymbol{X} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix}$. We have

$$\boldsymbol{R}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix} = 0.$$

Therefore, the columns of $\boldsymbol{X}$ are independent solutions to the system $\boldsymbol{A}x = 0$ Next, we show that any vector $x \in N(\boldsymbol{A})$ is a linear combination of the columns of $\boldsymbol{X}$.
Suppose there is a solution $x = (x_1, x_2) \in N(\boldsymbol{A})$, where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$. We have

$$\boldsymbol{R}x = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \boldsymbol{F}x_2 \\ 0 \end{bmatrix} = 0.$$

This implies that $x_1 = -\boldsymbol{F}x_2$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix} x_2 = \boldsymbol{X}x_2$. Any vector in $N(\boldsymbol{A})$ is a linear combination of the columns of $\boldsymbol{X}$, i.e. $C(\boldsymbol{X}) = N(\boldsymbol{A})$. Therefore, the rank of nullspace $rank(N(\boldsymbol{A})) = dim(C(\boldsymbol{X})) = n - r$. ☐

Recall the definitions of pivot columns and free columns. In $\boldsymbol{A}x = b$, the variables in x that correspond to pivot columns are called *pivot variables*, and others are *free variables*. If $rank(\boldsymbol{A}) = r$, there are $n - r$ free variables.
In the proof of lemma 5, the columns of $\boldsymbol{X} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix}$ compose the basis of $N(\boldsymbol{A})$. Practically, we calculate the columns by assigning 1 to a free variable and 0 to other free variables, and then we solve the system $\boldsymbol{A}x = 0$.

2.2 Solving $Ax = b$: complete solutions

Lemma 6 (Solvability of $\boldsymbol{A}x = b$)**.** *The system $\boldsymbol{A}x = b$ is solvable only when $b \in C(\boldsymbol{A})$.*

Proof. If $\boldsymbol{A}x = b$ is solvable, there exists a vector x such that $\boldsymbol{A}x = b$. For any x, $\boldsymbol{A}x \in C(\boldsymbol{A})$. Therefore, $b \in C(\boldsymbol{A})$. ☐

Lemma 7 (Complete solution)**.** *The complete solution of $\boldsymbol{A}x = b$ is given by $x_{comp} = x_p + x_n$, where x_p is a solution such that $\boldsymbol{A}x_p = b$, and $x_n \in N(\boldsymbol{A})$.*

Proof. Suppose $x = x_p + x_0$ is a solution to $\boldsymbol{A}x = b$. We have

$$\boldsymbol{A}x - \boldsymbol{A}x_p = \boldsymbol{A}(x - x_p) = \boldsymbol{A}x_0 = 0.$$

Therefore, $x_0 \in N(\boldsymbol{A})$. $\square$

Usually, we find a solution x_p by assigning 0 to free variables, and we solve the system $\boldsymbol{A}x = b$.

The following table discusses the rank of $\boldsymbol{A}$, the form of $\boldsymbol{R}$, the dimension of nullspace $N(\boldsymbol{A})$, and the number of solutions to $\boldsymbol{A}x = b$.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
$\boldsymbol{R}$	$\boldsymbol{I}$	$\begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$
$dim(N(\boldsymbol{A}))$	0	0	$n - r$	$n - r$
# solutions to $\boldsymbol{A}X = b$	1	0 or 1	infinitely many	0 or infinitely many