

# Thought about STD

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## 1 Equivalent model of STD

Claim: the new noise tensor consists of i.i.d. (sub-)Gaussian entries if and only if all  $\mathbf{X}_k, k \in [3]$  has orthogonal columns and the diagonal elements of  $(\mathbf{X}_k^T \mathbf{X}_k)^{-1}$  keep the same for  $k \in [3]$ .

*Proof.* Let  $\mathcal{E}' = \mathcal{E} \times_1 \mathbf{M}_1 \times_2 \mathbf{M}_2 \times_3 \mathbf{M}_3$ , where  $\mathcal{E}$  has independent sub-Gaussian- $\sigma$  entries. By the Prop 1 in the supplementary of STD paper,  $\mathcal{E}'$  is a sub-Gaussian tensor with a different parameter  $\sigma'$ , where  $\sigma'$  depends on  $\mathbf{M}_k$ . Now, consider the case that  $\mathcal{E}$  has i.i.d. Gaussian entries with variance 1. We only need to show that  $\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i',j',k'}) = 0$  and  $\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i,j,k}) = \sigma^2$ , for arbitrary  $(i, j, k)$ ,  $(i', j', k') \neq (i, j, k)$ , and for a constant  $\sigma^2$ .

Note that for arbitrary  $(i, j, k)$  we have

$$\mathcal{E}'_{i,j,k} = \sum_{a \in [d_1], b \in [d_2], c \in [d_3]} \mathcal{E}_{abc} \mathbf{M}_{1,ia} \mathbf{M}_{2,jb} \mathbf{M}_{3,kc}.$$

Then, the covariance is equal to

$$\begin{aligned} \text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i',j',k'}) &= \sum_{a \in [d_1], b \in [d_2], c \in [d_3]} \mathbf{M}_{1,ia} \mathbf{M}_{2,jb} \mathbf{M}_{3,kc} \mathbf{M}_{1,i'a} \mathbf{M}_{2,j'b} \mathbf{M}_{3,k'c} \\ &= \sum_{b \in [d_2], c \in [d_3]} \mathbf{e}_{1,i}^T \mathbf{M}_1 \mathbf{M}_1^T \mathbf{e}_{1,i'} \mathbf{M}_{2,jb} \mathbf{M}_{3,kc} \mathbf{M}_{2,j'b} \mathbf{M}_{3,k'c} \\ &= [\mathbf{e}_{1,i}^T (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{e}_{1,i'}] [\mathbf{e}_{2,j}^T (\mathbf{X}_2^T \mathbf{X}_2)^{-1} \mathbf{e}_{2,j'}] [\mathbf{e}_{3,k}^T (\mathbf{X}_3^T \mathbf{X}_3)^{-1} \mathbf{e}_{3,k'}], \end{aligned} \quad (1)$$

where  $\mathbf{e}_{k,i} \in \mathbb{R}^{p_k}$  whose  $i$ -th elements is 1 and the other entries remain 0, for  $k \in [3], i \in [p_k]$ . The equation (1) implies that the covariance equal to 0 for all pairs  $(i', j', k') \neq (i, j, k)$  if and only if all  $\mathbf{X}_k, k \in [3]$  should have orthogonal columns.

Next, note the variance of  $\mathcal{E}'_{i,j,k}$  is

$$\begin{aligned} \text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i,j,k}) &= [\mathbf{e}_{1,i}^T (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{e}_{1,i}] [\mathbf{e}_{2,j}^T (\mathbf{X}_2^T \mathbf{X}_2)^{-1} \mathbf{e}_{2,j}] [\mathbf{e}_{3,k}^T (\mathbf{X}_3^T \mathbf{X}_3)^{-1} \mathbf{e}_{3,k}] \\ &= (\mathbf{X}_1^T \mathbf{X}_1)^{-1}_{ii} (\mathbf{X}_2^T \mathbf{X}_2)^{-1}_{jj} (\mathbf{X}_3^T \mathbf{X}_3)^{-1}_{kk}. \end{aligned} \quad (2)$$

The equation (2) implies that all entries of  $\mathcal{E}'$  have equal variance if and only if the diagonal elements of  $(\mathbf{X}_k^T \mathbf{X}_k)^{-1}$  should be the same for  $k \in [3]$ .

□