

Graphic Lasso: Possible Accuracy for Multi-Layer Model

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1 Discussion about Identifiability

Suppose we have a dataset with p variables and K categories. In multi-layer model, we assume the rank of decomposition r is known, and the precision matrices are of form

$$\Omega^k = \Theta_0 + \sum_{l=1}^r u_{lk} \Theta_l, \quad \text{for } k = 1, \dots, K. \quad (1)$$

The identifiability problem for $\{\Theta_0, \Theta_1, \dots, \Theta_r, \mathbf{u}_1, \dots, \mathbf{u}_r\}$ is actually an identifiability problem for tensor decomposition.

Let $\mathcal{Y} \in \mathbb{R}^{p \times p \times K}$ denote the collection of K networks, where $\mathcal{Y}[:, k] = \Omega^k, k \in [K]$. Let $\mathcal{C} \in \mathbb{R}^{p \times p \times (r+1)}$ denote the collection of “core” networks, where $\mathcal{C}[:, 1] = \sqrt{K} \Theta_0, \mathcal{C}[:, l] = \Theta_{l-1}, l = 2, \dots, (r+1)$. Let $\mathbf{U} \in \mathbb{R}^{K \times (r+1)} = (\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_r)$ denote the factor matrix, where $\mathbf{u}_0 = \mathbf{1}_K / \sqrt{K}$. Rewrite the model (1) in tensor form.

$$\mathcal{Y} = \mathcal{C} \times_3 \mathbf{U}. \quad (2)$$

Therefore, the identifiability problem for $\{\Theta_l, \mathbf{u}_l\}$ becomes the identifiability problem for $\{\mathcal{C}, \mathbf{U}\}$. Before we discuss the identifiable condition case by case, we first assume \mathcal{C} is full rank on mode 3.

1. No sparsity constrain on \mathbf{U} .

Proposition 1. *The decomposition \mathcal{C} and \mathbf{U} are identifiable if \mathbf{U} is an orthonormal matrix, i.e., $\mathbf{U}^T \mathbf{U} = \mathbf{I}_{r+1}$.*

Proof. Let $\text{Unfold}(\cdot)$ denote the unfold representation of a tensor on mode 3. The model (2) is equal to

$$\text{Unfold}(\mathcal{Y}) = \mathbf{U} \text{Unfold}(\mathcal{C}).$$

By matrix SVD, we have $\text{Unfold}(\mathcal{Y}) = \tilde{\mathbf{U}} \Sigma \mathbf{V}^T$, where $\tilde{\mathbf{U}}$ is an orthonormal matrix. The SVD decomposition is unique up to orthogonal rotation (ignore row permutation).

Note that $\mathbf{u}_0 = \mathbf{1}_K / \sqrt{K}$. There always has a unique orthonormal matrix \mathbf{R} such that the first column of $\tilde{\mathbf{U}} \mathbf{R}$ is equal to $\mathbf{1}_K / \sqrt{K}$. Let $\mathbf{U} = \tilde{\mathbf{U}} \mathbf{R}$ and $\text{Unfold}(\mathcal{C}) = \mathbf{R}^T \Sigma \mathbf{V}$. Then, \mathbf{U} and \mathcal{C} are identifiable. \square

2. Membership constrain on \mathbf{U} . (Without intercept Θ_0)

If \mathbf{U} is a membership matrix, we are clustering K categories into r groups. Then, the model (1) becomes

$$\Omega^k = \Theta_{i_k}, \quad \text{for } k = 1, \dots, K,$$

where $i_k \in [r]$ is the group for the k -th category. Then, let $\mathcal{C} \in \mathbb{R}^{p \times p \times r}$, where $\mathcal{C}[:, l] = \Theta_l, l = 1, \dots, r$, and $\mathbf{U} \in \mathbb{R}^{K \times r} = (\mathbf{u}_1, \dots, \mathbf{u}_r)$.

Proposition 2. *The decomposition \mathcal{C} and \mathbf{U} are identifiable up to permutation if \mathbf{U} is a membership matrix, i.e., in each row of \mathbf{U} there is only 1 copy of 1 and massive 0.*

Proof. If \mathbf{U} is a membership matrix, the model (2) is a special case of tensor block model. By Proposition 1 in Wang, the matrix \mathbf{U} is identifiable if \mathcal{C} is irreducible on mode 3. In our case, we assume \mathcal{C} is full rank on mode 3, and thus $\{\mathbf{U}, \mathcal{C}\}$ are identifiable. \square

Remark 1. The sparsity of Θ_l won't affect the identifiability in these two cases under the assumption that \mathcal{C} is full rank on mode 3. In no sparsity constrain case, we only need the full rankness of $\text{Unfold}(\mathcal{C})$, and the sparsity on the first and second mode of \mathcal{C} does not affect the rank of $\text{Unfold}(\mathcal{C})$. In membership constrain, we only need the mode 3 irreducibility of \mathcal{C} .

Remark 2. The two cases above are two extreme cases. Intermediate cases include the fuzzy clustering, where $\sum_{l=1}^r u_{lk} = 1, k \in [K]$, and the sparsity constrain for the column, where $|\mathbf{u}_l|_0 < a, l \in [r]$.

2 A simple extension

Let $Q^k(\Omega) = \text{tr}(S^k \Omega) - \log |\Omega|$. Assume the rank of decomposition r is known. Consider the constrained optimization problem

$$\begin{aligned} \min_{\mathcal{C}} \quad & \sum_{k=1}^K [Q^k(\Omega^k)] \\ \text{s.t.} \quad & \Omega^k = \Theta_0 + \sum_{l=1}^r u_{lk} \Theta_l, \quad \text{for } k = 1, \dots, K, \\ & \|\Theta_l\|_0 \leq b, \quad \text{for } l = 1, \dots, r, \\ & \|\Theta_0\|_0 \leq b_0, \\ & \text{with more identifiability conditions,} \end{aligned}$$

where a, b, b_0 are fixed positive constants, $|\cdot|_0$ refers to the vector L_0 norm, and $\|\cdot\|_0$ refers to the matrix L_0 norm. For simplicity, let $\hat{\mathcal{C}} = \{\hat{\Theta}_0, \hat{\Theta}_1, \dots, \hat{\Theta}_r, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_r\}$ denote the estimation, and $\hat{\Omega}^k = \hat{\Theta}_0 + \sum_{l=1}^r \hat{u}_{lk} \hat{\Theta}_l$ for $k = 1, \dots, K$.

For true precision matrices Ω^k , let $T^k = \{(j, j') | \omega_{j, j'}^k \neq 0\}$ and $q^k = |T^k|$. Assume $0 < \tau_1 < \phi_{\min}(\Omega^k) \leq \phi_{\max}(\Omega^k) < \tau_2, k = 1, \dots, K$, for some positive constant τ_1, τ_2 . **This condition can be transferred as some conditions for $\{\Theta_l, \mathbf{u}_l\}$. I will specify the conditions later.**

Theorem 2.1. *Suppose two assumptions hold. Let $\{\Omega^k\}$ denote the true precision matrices. For the estimation $\hat{\mathcal{C}}$ such that $\sum_{k=1}^K [Q^k(\hat{\Omega}^k)] \leq \sum_{k=1}^K [Q^k(\Omega^k)]$ and satisfies the constrains, the following accuracy bound holds with probability tending to 1.*

$$\sum_{k=1}^K \left\| \hat{\Omega}^k - \Omega^k \right\|_F \leq CK \left(C_1(b_0 + rb) \left(\frac{\log p}{n} \right)^{1/2} + C_2 \left(\frac{p \log p}{n} \right)^{1/2} \right),$$

where n is the sample size for each category, and C, C_1, C_2 are positive constants independent with p, n .

Proof. Let Ω^k denote the true precision matrices for $k = 1, \dots, K$. Consider the estimation \hat{C} such that $\sum_{k=1}^K [Q^k(\hat{\Omega}^k)] \leq \sum_{k=1}^K [Q^k(\Omega^k)]$. Let $\Delta^k = \hat{\Omega}^k - \Omega^k$. For a matrix M , let M_T denote the matrix M with all elements outside the index set T replaced by 0, and $\tilde{M} = \text{vec}(M)$ be the vectorization of M . Define the function

$$G(\{\Delta^k\}) = \sum_{k=1}^K \text{tr}(S(\Omega^k + \Delta^k)) - \text{tr}(\Omega^k) - \log |\Omega^k + \Delta^k| + \log |\Omega^k| = I_1 + I_2, \quad (3)$$

where

$$I_1 = \sum_{k=1}^K \text{tr}((S^k - \Sigma^k)\Delta^k), \quad I_2 = \sum_{k=1}^K (\tilde{\Delta}^k)^T \int_0^1 (1-v)(\Omega^k + v\Delta^k)^{-1} \otimes (\Omega^k + v\Delta^k)^{-1} dv \tilde{\Delta}^k.$$

With probability tending to 1, we have

$$|I_1| \leq C_1 \left(\frac{\log p}{n} \right)^{1/2} \sum_{k=1}^K (|\Delta_{T^k}^k|_1 + |\Delta_{T^{k,c}}^k|_1) + C_2 \left(\frac{p \log p}{n} \right)^{1/2} \sum_{k=1}^K \|\Delta^k\|_F, \quad I_2 \geq \frac{1}{4\tau_2^2} \sum_{k=1}^K \|\Delta^k\|_F^2.$$

Note that $|\Delta_{T^k}^k|_1 \leq \sqrt{q^k} \|\Delta^k\|_F$. Then, we only need to deal with $|\Delta_{T^{k,c}}^k|_1$. Rewrite the term, we have

$$|\Delta_{T^{k,c}}^k|_1 = |\hat{\Theta}_{0,T^{k,c}} + \hat{u}_{1k}\hat{\Theta}_{1,T^{k,c}} + \dots + \hat{u}_{rk}\hat{\Theta}_{r,T^{k,c}}|_1 \leq (b_0 + rb) \|\Delta^k\|_{\max} \leq (b_0 + rb) \|\Delta^k\|_F.$$

To let the equation (3) smaller than 0, we have

$$I_2 \leq -I_1 \leq |I_1|. \quad (4)$$

Plugging the upper bound of $|I_1|$ and the lower bound of I_2 into the inequality (4), we have

$$\frac{1}{4\tau_2^2} \sum_{k=1}^K \|\Delta^k\|_F^2 \leq C_1 \left(\frac{\log p}{n} \right)^{1/2} \sum_{k=1}^K (\sqrt{q^k} \|\Delta^k\|_F + (b_0 + rb) \|\Delta^k\|_F) + C_2 \left(\frac{p \log p}{n} \right)^{1/2} \sum_{k=1}^K \|\Delta^k\|_F.$$

By Cauchy Schwartz inequality, we know that $\sum_{k=1}^K \|\Delta^k\|_F^2 \geq \frac{1}{K} (\sum_{k=1}^K \|\Delta^k\|_F)^2$. Also, note that $q^k \leq (b_0 + rb), k = 1, \dots, K$. Dividing by $\sum_{k=1}^K \|\Delta^k\|_F$ on both sides of the inequality, we obtain the accuracy rate

$$\sum_{k=1}^K \|\Delta^k\|_F = \sum_{k=1}^K \|\hat{\Omega}^k - \Omega^k\|_F \leq 4\tau_2^2 K \left(C_1(b_0 + rb) \left(\frac{\log p}{n} \right)^{1/2} + C_2 \left(\frac{p \log p}{n} \right)^{1/2} \right). \quad (5)$$

□

Remark 3. The accuracy (5) holds when q^k are fixed. Otherwise, the accuracy is of order $\mathcal{O}_p \left[q \left\{ \frac{\log p}{n} \right\}^{1/2} \right]$.

Remark 4. This proof does not utilize the special structure of Ω^k . We can go through the proof with the constrain $|\Omega^k| < s$.

Remark 5. Both accuracy results of our constrained estimator and penalized estimator are of order $F(p, q) \left(\frac{\log p}{n} \right)^{1/2}$, where $F(p, q) = (p + q)^{1/2}$ for penalized estimator and $F(p, q) = (p + q^2)^{1/2}$ with $q = b_0 + rb$ in our estimator. In case of growing (p, n) and fixed q , the two estimators share the same accuracy rate.

3 Others

Can the factor K be improved?

First, consider the case $r = 0, K > 1$. Then, we have $\Theta_0 = \Omega^k, k = 1, \dots, K$. Let $\hat{\Theta}_0$ be the estimator of $\Omega^k, k = 1, \dots, K$, and thus $\Delta^k = \hat{\Omega}^k - \Omega^k = \Delta, k = 1, \dots, K$. Define the function

$$G(\Delta) = \frac{1}{K} \sum_{k=1}^K \text{tr}(S^k(\Theta_0 + \Delta)) - \text{tr}(S^k \Theta_0) - \log |\Omega_0 + \Delta| + \log |\Omega_0| = I_1 + I_2,$$

where

$$I_1 = \text{tr}\left(\left(\frac{1}{K} \sum_{k=1}^K S^k - \Sigma\right)\Delta\right), \quad I_2 = (\tilde{\Delta})^T \int_0^1 (1-v)(\Theta_0 + v\Delta)^{-1} \otimes (\Theta_0 + v\Delta)^{-1} dv \tilde{\Delta}.$$

Note that $\frac{1}{K} \sum_{k=1}^K S^k$ can be considered as the sample covariance matrix with sample size nK . Then, the upper bound for $|I_1|$ is

$$|I_1| \leq C_1 \left(\frac{\log p}{nK}\right)^{1/2} (|\Delta|_1) + C_2 \left(\frac{p \log p}{nK}\right)^{1/2} \|\Delta\|_F.$$

Since $I_2 \geq \frac{1}{4\tau_2^2} \|\Delta\|_F^2$, $|\Delta|_1 \leq \sqrt{q} \|\Delta\|_F + (b_0 + rb) \|\Delta\|_F$, and we need $I_2 \leq |I_1|$, we obtain the error bound

$$\|\Delta\|_F^2 \leq \frac{4\tau_2^2}{K^{1/2}} F(p, q, n) \|\Delta\|_F,$$

and thus $\|\Delta\|_F = \left\| \hat{\Theta}_0 - \Theta_0 \right\|_F$ decreases in K of order $\mathcal{O}(K^{-1/2})$.

This result agrees with the intuition. As K growing, the sample size for estimating the Θ_0 becomes larger. Then, the error for the estimation goes smaller.

Comparison Table.

4 Next

- Think about the identifiability of the intermediate cases (sparse matrix factorization).
- Think about the proof which utilizes the special structure of the Ω^k .

	Penalized	
	L_0	L_1
Ground Truth	For $k \in [K]$, $ \Omega^k _0 < s$, where $ \cdot _0$ denote the number of nonzero elements in the matrix, and $s > 1$ is a positive constant.	For $k \in [K]$, $ \Omega^k _1 < c$, where norm $ \Omega _1 = \sum_{(i,j)} \omega_{ij} $.
Fitting techniques	The estimator is the solution to the optimization problem $\min_{\{\Omega^k\}} Q^k(\Omega^k) + \lambda \sum_{k=1}^K \Omega^k _0.$	The estimator is the solution to the optimization problem $\min_{\{\Omega^k\}} Q^k(\Omega^k) + \lambda \sum_{k=1}^K \Omega^k _1.$
Accuracy	For $\lambda \geq 0$, $\sum_k \ \Delta_k\ _F^2 \leq 4\tau_2^2 \left(F(p, q, n) \sum_k \ \Delta^k\ _F + K\lambda q \right) \sum_k \ \Delta_k\ _F^2 \leq 4\tau_2^2 (\lambda\sqrt{q} + F(p, q, n)) \sum_k \ \Delta^k\ _F,$ where $F(p, q, n) = C_1 p \left(\frac{\log p}{n} \right)^{1/2} + C_2 \left(\frac{p \log p}{n} \right)^{1/2}$. Then, the error $\sum_k \ \Delta_k\ _F = \mathcal{O}\left(\frac{K\sqrt{\lambda}}{n^{1/4}}\right).$ If $\lambda \geq \Lambda_1 \left(\frac{\log p}{n} \right)^{1/2}$, the error becomes of order $\mathcal{O}(K/n^{1/2})$.	For $\lambda \geq \Lambda_1 \left(\frac{\log p}{n} \right)^{1/2}$, we have $\sum_k \ \Delta_k\ _F^2 \leq 4\tau_2^2 (\lambda\sqrt{q} + F(p, q, n)) \sum_k \ \Delta^k\ _F,$ where $F(p, q, n) = C_1 \left(\frac{q \log p}{n} \right)^{1/2} + C_2 \left(\frac{p \log p}{n} \right)^{1/2}$. Then, we have $\sum_k \ \Delta^k\ _F \leq 4\tau_2^2 K (\lambda\sqrt{q} + F(p, q, n)).$

	Constrained	
	L_0	L_1
Ground Truth	<p>For $k \in [K]$,</p> $ \Omega^k _0 < s$ <p>, where $\cdot _0$ denote the number of nonzero elements in the matrix, and $s > 1$ is a positive constant.</p>	<p>For $k \in [K]$,</p> $ \Omega^k _1 < c,$ <p>where norm $\Omega _1 = \sum_{(i,j)} \omega_{ij}$.</p>
Fitting techniques	<p>The estimator is the solution to the optimization problem</p> $\min_{\{\Omega^k\}} Q^k(\Omega^k)$ $s.t. \quad \Omega^k _0 < s, \quad k \in [K]$	<p>The estimator is the solution to the optimization problem</p> $\min_{\{\Omega^k\}} Q^k(\Omega^k)$ $s.t. \quad \Omega^k _1 < c, \quad k \in [K]$
Accuracy	<p>We have</p> $\sum_k \ \Delta_k\ _F^2 \leq 4\tau_2^2 F(p, s, n) \sum_k \ \Delta^k\ _F,$ <p>where $F(p, s, n) = C_1 s \left(\frac{\log p}{n}\right)^{1/2} + C_2 \left(\frac{p \log p}{n}\right)^{1/2}$. Then</p> $\sum_k \ \Delta_k\ _F \leq 4\tau_2^2 K F(p, s, n).$	<p>We have</p> $\sum_k \ \Delta_k\ _F^2 \leq 4\tau_2^2 F(p, q, n) \sum_k \ \Delta^k\ _F,$ <p>where $F(p, q, n) = C_1 p \left(\frac{\log p}{n}\right)^{1/2} + C_2 \left(\frac{p \log p}{n}\right)^{1/2}$. Then,</p> $\sum_k \ \Delta_k\ _F \leq 4\tau_2^2 K F(p, q, n).$