

Graphic Lasso: Scaled membership (Simple Case)

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1 Thoughts

1. In Note 0323, I decomposed the original difference of the likelihood into 5 terms H_1, \dots, H_5 , and I tried to use the following inequality to show the MLE estimate is near to the true parameters.

$$0 \geq G(\hat{u}, \hat{\Theta}) \geq H_1 + H_5 - H_2 - |H_3| + H_4.$$

However, from G to H_1, \dots, H_5 , there are a lot of inequalities. I think this may be the reason why I can not show $\hat{u} \rightarrow u$ and $\hat{\Theta} \rightarrow \Theta$.

Therefore, in the following new proof, I would like to use the original G and show that $\hat{u}\hat{\Theta} \rightarrow u\Theta$ and further $\hat{u} \rightarrow u, \hat{\Theta} \rightarrow \Theta$.

In the discrete case, we have $\sum_{al} D_{al} \|\Delta_{al}\|_F \rightarrow 0$, where D_{al} is the entries of confusion matrix and $\Delta_{al} = \hat{\Theta}^l - \Theta^a$. Then, we know that

$$D_{al} \|\Delta_{al}\| + D_{a'l} \|\Delta_{a'l}\| \geq \min\{D_{al}, D_{a'l}\} \|\Theta^a - \Theta^{a'}\| \geq \min\{D_{al}, D_{a'l}\} \delta,$$

where δ is the minimal gap between Θ^l . Thus, for each a , there is only one l such that D_{al} does not tend to 0, i.e., with proper permutation, all the off-diagonal elements in the confusion matrix tends to 0.

In our case, $\sum_{k=1}^K \|\hat{u}_k \hat{\Theta} - u_k \Theta\|$ is an analogy of $\sum_{al} D_{al} \|\Delta_{al}\|_F$ in the continuous case. Since we do not have minimal gap here and $\hat{\Theta}, \Theta$ are positive definite, I think similar techniques can be applied to our case from the angle of u_k . See Step I for details.

2. The constraint $\|u\|_F^2 = K$ is crucial since we need $u_k \geq a > 0$ and the norm of u grows along with K .
3. **new.** In previous proof, I used $\sum_{k=1}^K \|\Delta_k\|_F = \sum_{k=1}^K u_k \|\Delta_k/u_k\| \leq \sum_{k=1}^K \max_k u_k \|\Delta_k/u_k\|$ to get the accuracy rate and the term $\max_k u_k$ brought an extra term factor \sqrt{K} . In our meeting, we think there are only finite number of u_k s achieve the rate \sqrt{K} . This idea makes sense but is hard to prove. However, noticed that $\|\Delta_k/u_k\|_F \leq \|\Delta\|_F + \max_k (\hat{u}_k/u_k - 1) \|\hat{\Theta}\|_F$, we only need to consider the sum $\sum_{k=1}^K u_k$, which is easily bounded by Cauchy Schwartz. Since $\sum_k u_k \leq K \max_k u_k$, we obtain a shaper bound and finally the accuracy rate is the same as hard membership case.

2 Simple case

Consider the model in which K categories share the same precision matrix structure with different magnitude. The optimization problem is stated below:

$$\begin{aligned} \min_{\{u, \Theta\}} \quad & \mathcal{L}(u, \Theta) = \sum_{k=1}^K \langle S^k, \Omega^k \rangle - \log \det(\Omega^k), \\ \text{s.t.} \quad & \Omega^k = u_k \Theta, \quad k = 1, \dots, K, \\ & u_k \geq a, \|u\|_F^2 = K, \quad a > 0, \\ & \Theta \text{ is positive definite with, and } \tau_1 < \varphi_{\min}(\Theta) \leq \varphi_{\max}(\Theta) < \tau_2, \tau_1, \tau_2 > 0 \end{aligned}$$

Lemma 1 (Precision matrix Accuracy). *Let $\{u, \Theta\}$ denote the true parameters. Consider a estimation $\{\hat{u}, \hat{\Theta}\}$ such that $\mathcal{L}(\hat{u}, \hat{\Theta}) \geq \mathcal{L}(u, \Theta)$. With probability tends to 1 as $n \rightarrow \infty$, we have the accuracy rates*

$$K \left\| \hat{\Theta} - \Theta \right\|_F + \sum_{k=1}^K |\hat{u}_k - u_k| \tau_2 \leq 16 \tau_2^2 \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}},$$

and

$$\sum_{k=1}^K \left\| \hat{\Omega}^k - \Omega^k \right\|_F = \sum_{k=1}^K \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F \leq 16 \tau_2^2 \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}}.$$

Proof. We prove the accuracy rate by two steps.

Step I: Show that $\hat{u} \rightarrow u$ and $\hat{\Theta} \rightarrow \Theta$.

First, we define

$$\begin{aligned} G(\hat{u}, \hat{\Theta}) &= \mathcal{L}(\hat{u}, \hat{\Theta}) - \mathcal{L}(u, \Theta) \\ &= \sum_{k=1}^K \langle S^k, \hat{u}_k \hat{\Theta} \rangle - \langle S^k, u_k \Theta \rangle - \log \det(\hat{u}_k \hat{\Theta}) + \log \det(u_k \Theta). \end{aligned}$$

Let $\Delta_k = \hat{u}_k \hat{\Theta} - u_k \Theta$. By Taylor expansion, we have

$$\begin{aligned} -\log \det(\hat{u}_k \hat{\Theta}) + \log \det(u_k \Theta) &\geq -\langle (u_k \Theta)^{-1}, \Delta_k \rangle + \frac{1}{2u_k^2 \tau_2^2 + \|\Delta_k\|_F^2} \|\Delta_k\|_F^2, \\ &\geq -\langle u_k^{-1} \Sigma^{-1}, \Delta_k \rangle + \frac{1}{2u_k^2 \tau_2^2 + \|\Delta_k\|_F^2} \|\Delta_k\|_F^2. \end{aligned} \quad (1)$$

Plugging the inequality (1) into G , we have

$$G(\hat{u}, \hat{\Theta}) \geq \sum_{k=1}^K \langle S^k - u_k^{-1} \Sigma, \Delta_k \rangle + \frac{1}{2K \tau_2^2 + (\sum_{k=1}^K \|\Delta_k\|_F)^2} \sum_{k=1}^K \|\Delta_k\|_F^2. \quad (2)$$

Let $X_1^k, \dots, X_n^k \sim_{i.i.d.} \mathcal{N}(0, \Sigma/u_k)$. We know that

$$S_{jl}^k = \frac{1}{n} \sum_{i=1}^n \left[X_{ij}^k X_{jl}^k - X_{.j}^k X_{.l}^k \right].$$

Since $X_{\cdot j}^k, X_{\cdot l}^k \rightarrow 0$ almost sure when $n \rightarrow \infty$, we have

$$|S_{jl}^k - \Sigma_{jl}/u_k| = |\frac{1}{n} X_{ij}^k X_{jl}^k - \Sigma_{jl}/u_k| \leq C \sqrt{\frac{\log p}{n}}, \quad (3)$$

with high probability. Therefore, by the assumption $\mathcal{L}(\hat{u}, \hat{\Theta}) \geq \mathcal{L}(u, \Theta)$, we have

$$0 \geq G(\hat{u}, \hat{\Theta}) \geq \frac{1}{2K\tau_2^2 + (\sum_{k=1}^K \|\Delta_k\|_F)^2} \sum_{k=1}^K \|\Delta_k\|_F^2 - C \sqrt{\frac{\log p}{n}} \sum_{k=1}^K \|\Delta_k\|, \quad (4)$$

which implies that

$$C \sqrt{\frac{\log p}{n}} K \left[2K\tau_2^2 + (\sum_{k=1}^K \|\Delta_k\|_F)^2 \right] - \sum_{k=1}^K \|\Delta_k\|_F \geq 0.$$

Note that $\sqrt{\frac{\log p}{n}} \rightarrow 0$ as $n \rightarrow \infty$. We need

$$\sum_{k=1}^K \|\Delta_k\|_F = \sum_{k=1}^K \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F \rightarrow 0, \quad n \rightarrow \infty.$$

Since $\|\Delta_k\|_F \geq 0$, we also have

$$\|\Delta_k\|_F = \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F \rightarrow 0, \quad n \rightarrow \infty, \quad \text{for all } k \in [K]$$

and thus

$$\left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F / u_k \rightarrow 0, \quad \text{for all } k \in [K], \quad \text{and} \quad \sum_{k=1}^K \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F / u_k \rightarrow 0.$$

For arbitrary $k, k' \in [K]$, note that

$$\left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F / u_k + \left\| \hat{u}_{k'} \hat{\Theta} - u_{k'} \Theta \right\|_F / u_{k'} \geq \left\| (\hat{u}_k / u_k - \hat{u}_{k'} / u_{k'}) \hat{\Theta} \right\|_F \rightarrow 0,$$

which implies for any pair (k, k') , we need

$$\frac{\hat{u}_k}{u_k} - \frac{\hat{u}_{k'}}{u_{k'}} \rightarrow 0, \quad \text{and thus} \quad \hat{u} \rightarrow cu,$$

for some constant c . By the assumption that $\|\hat{u}\|_F = \|u\|_F = K$, the constant $c = 1$ and therefore we obtain that $\hat{u} \rightarrow u$ as $n \rightarrow \infty$. On the other hand, given $\hat{u} \rightarrow u$, we also have

$$\|\Delta_k\|_F = \left\| u_k(\hat{\Theta} - \Theta) + (\hat{u}_k - u_k)\hat{\Theta} \right\|_F \rightarrow 0, \quad \text{for all } k \in [K],$$

which implies that $\left\| \hat{\Theta} - \Theta \right\|_F \rightarrow 0$.

Sanity Check: Let $S^k = u_k^{-1} \Sigma$.

The inequality (2) becomes,

$$0 \geq G(\hat{u}, \hat{\Theta}) \geq \frac{1}{2K\tau_2^2 + (\sum_{k=1}^K \|\Delta_k\|_F)^2} \sum_{k=1}^K \|\Delta_k\|_F^2,$$

which requires $\sum_{k=1}^K \|\Delta_k\|_F^2 = 0$, otherwise, the right hand side tends to a positive constant as $n \rightarrow \infty$. Therefore, from $\sum_{k=1}^K \|\Delta_k\|_F^2 = 0$, we have $\hat{u}_k = u_k$ and $\hat{\Theta} = \Theta$, and thus we obtain the conclusion that MLE is near the true parameters.

Step II: Sharpen the accuracy rate.

Note that accuracy rate bound from inequality (4) is sub-optimal since it does not use the common structure of the precision matrix. Therefore, back to the inequality (2) of G .

$$\begin{aligned} G(\hat{u}, \hat{\Theta}) &\geq \sum_{k=1}^K \langle S^k - u_k^{-1}\Sigma, \Delta_k \rangle + \sum_{k=1}^K \frac{1}{2u_k^2\tau_2^2 + (\sum_{k=1}^K \|\Delta_k\|_F)^2} \|\Delta_k\|_F^2, \\ &\geq \sum_{k=1}^K \langle [u_k S^k - \Sigma], \Delta_k/u_k \rangle + \frac{1}{4\tau_2^2} \sum_{k=1}^K \|\Delta_k/u_k\|_F^2, \\ &= I_1 + I_2. \end{aligned}$$

where the second inequality follows by the conclusion in Step I, and I_1, I_2 denote the two terms respectively. Let $\Delta = \hat{\Theta} - \Theta$. Note that

$$\Delta_k/u_k = \hat{u}_k/u_k \hat{\Theta} - \Theta = \Delta + (\hat{u}_k/u_k - 1) \hat{\Theta}. \quad (5)$$

For I_1 , by the decomposition (5), we have

$$\begin{aligned} I_1 &= \sum_{k=1}^K \langle [u_k S^k - \Sigma], \Delta \rangle + \sum_{k=1}^K (\hat{u}_k/u_k - 1) \langle [u_k S^k - \Sigma], \hat{\Theta} \rangle \\ &\leq \sum_{k=1}^K \langle [u_k S^k - \Sigma], \Delta \rangle + \max_{k \in [K]} |(\hat{u}_k/u_k - 1)| \sum_{k=1}^K |\langle [u_k S^k - \Sigma], \hat{\Theta} \rangle|, \end{aligned}$$

By similar process to obtain the inequality (3), we have

$$\max_{(i,j)} \left| \sum_{k=1}^K [u_k S_{jl}^k - \Sigma_{jl}] \right| \leq \sqrt{KC} \sqrt{\frac{\log p}{n}},$$

with high probability. Therefore, we have

$$|I_1| \leq \sqrt{KC} \sqrt{\frac{p^2 \log p}{n}} \left[\|\Delta\|_F + \max_{k \in [K]} |(\hat{u}_k/u_k - 1)| \|\hat{\Theta}\|_F \right]. \quad (6)$$

For I_2 , note that for n large enough,

$$\begin{aligned} \|\Delta_k/u_k\|_F &= \|\Delta\|_F + \max_{k \in [K]} |(\hat{u}_k/u_k - 1)| \|\hat{\Theta}\|_F \\ &\quad + \left\| \Delta + |(\hat{u}_k/u_k - 1)| \hat{\Theta} \right\|_F - \left(\|\Delta\|_F + \max_{k \in [K]} |(\hat{u}_k/u_k - 1)| \|\hat{\Theta}\|_F \right) \\ &\geq \frac{1}{2} \left[\|\Delta\|_F + \max_{k \in [K]} |(\hat{u}_k/u_k - 1)| \|\hat{\Theta}\|_F \right], \end{aligned}$$

where the inequality follows the fact that both $\|\Delta\|_F, \max_{k \in [K]} |\hat{u}_k/u_k - 1| \rightarrow 0$ as $n \rightarrow \infty$. This inequality makes sense since $\|A + B\|_F^2$ are near to $\|A\|_F^2 + \|B\|_F^2$ when all the entries in A, B are close to 0. Therefore, we have

$$\begin{aligned} I_2 &\geq \frac{1}{16\tau_2^2} \sum_{k=1}^K \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right]^2 \\ &= \frac{1}{16\tau_2^2} K \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right]^2. \end{aligned} \quad (7)$$

Combining the inequality (6), (7) with the assumption that $G(\hat{u}, \hat{\Theta}) \leq 0$, we have

$$\begin{aligned} 0 &\geq I_2 - |I_1| \\ &\geq \frac{1}{16\tau_2^2} K \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right]^2 \\ &\quad - \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}} \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right], \end{aligned}$$

which implies that

$$K \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right] \leq 16\tau_2^2 \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}}. \quad (8)$$

Note that we have $\sum_{k=1}^K u_k \leq \sqrt{K \sum_{k=1}^K u_k^2} = K$ by Cauchy Schwartz and $\tau_2 = \|\hat{\Theta}\|_2 \leq \|\hat{\Theta}\|_F$. Hence, we obtain the accuracy for the additive error

$$K \|\Delta\|_F + \sum_{k=1}^K |\hat{u}_k - u_k| \tau_2 \leq K \|\Delta\|_F + \sum_{k=1}^K u_k \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \leq 16\tau_2^2 \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}},$$

where the last inequality follows the inequality (8). Last, note that

$$\begin{aligned} \sum_{k=1}^K \|\Delta_k\|_F &= \sum_{k=1}^K u_k \|\Delta_k/u_k\|_F \\ &\leq \sum_{k=1}^K u_k \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right] \\ &\leq K \left[\|\Delta\|_F + \max_{k \in [K]} |\hat{u}_k/u_k - 1| \|\hat{\Theta}\|_F \right] \\ &\leq 16\tau_2^2 \sqrt{K} C \sqrt{\frac{p^2 \log p}{n}} \end{aligned}$$

□