Matching problems

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Here is a series of graph matching problem.

1. **Correlated graph matching.** (Ding et al., 2021a).

   Consider two correlated Erdős-Rényi graphs $A, B \sim G(n, q)$ on the same vertex sets $[n]$. Let $\pi : [n] \mapsto [n]$ denote the latent permutation. Assume that conditional on A, for all $i < j$, $B_{\pi(i)\pi(j)}$ are independent and distributed as

   $$B_{\pi(i)\pi(j)} \sim \begin{cases} Ber(s) & \text{if} \quad A_{ij} = 1 \\ Ber\left(\frac{q(1-s)}{1-q}\right) & \text{if} \quad A_{ij} = 0. \end{cases}$$

   Note that A, B are symmetric matrices, and A_{ii}, B_{ii} for all $i \in [n]$ are not well-defined.

   Problem: recover π given A, B.

2. **Testing correlation of graphs.**(Wu et al., 2020)

   Consider two weighted random graphs $G_1([n], W)$ and $G_2([n], W')$, where the edge weights $\{(W_{ij}, W'_{ij}) : 1 \leq i < j \leq n\}$ are i.i.d. pairs or random variables and W_{ij}, W'_{ij} have the same marginal distribution. Now, consider the hypothesis testing problem:

   $$H_0 : W_{ij}, W'_{ij} \text{ are independent} \quad \leftrightarrow \quad H_1 : W_{ij}, W'_{ij} \text{ are correlated.}$$

   Problem: Given the unlabeled version of G_1, G_2, i.e., $\tilde{G}_1 = \pi_1 \circ G_1$ and $\tilde{G}_2 = \pi_2 \circ G_2$, test H_0 versus H_1.

3. **Planted matching.**(Ding et al., 2021b)

   Consider a bipartite weighted graph G with left vertex set $[n]$ and right vertex set $[n]'$. Let $W_{ij'} = W_e$ denote the weight on the edge $e = (i, j') \in E$. Let $\pi : [n] \mapsto [n]'$ denote a perfect mapping between $[n]$ and $[n]'$, and $M = \{e = (i, j') : i = \pi(j')\}$ denote the edges corresponding to the perfect match π.

   Suppose the bipartite weighted graph G is generated as

   $$(i, j') \begin{cases} \text{connected} & \text{if } (i, j') \in M \\ \sim P((i, j') \text{ connected}) = \frac{d}{n} & \text{if } (i, j') \notin M \end{cases}, \quad \text{with weight} \quad W_{ij'} \sim \begin{cases} \mathcal{P} & \text{if } (i, j') \in M \\ \mathcal{Q} & \text{if } (i, j') \notin M \end{cases},$$

   where d is some positive constant, and $\mathcal{P}, \mathcal{Q}$ are two distributions.

   Problem: recover M based on G.

4. **Recover networks from unlabeled noisy samples.** (Josephs et al., 2021)

   Consider an underlying network A with noisy observations $\tilde{A}^{(i)}$ for $i = 1, \ldots, m$, where

$$\mathbb{P}(\tilde{A}^{(i)}_{uv} = 1 | A_{uv}) = \begin{cases} 1 - \beta & A_{uv} = 1 \\ \alpha & A_{uv} = 0. \end{cases}$$

   Then, each pair of observations $\tilde{A}^{(i)}, \tilde{A}^{(j)}$ can be viewed as correlated Erdős-Rényi graphs.

   Problem: recover A from $\tilde{A}^{(i)}, i = 1, \ldots, m$.

References

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