Graphic Lasso: Variation of Single-layer model

Jiaxin Hu

December 31, 2020

1 Variation of Single-layer model

Let $Q(\Omega) = \text{tr}(S\Omega) - \log |\Omega|$. Consider the primal minimization problem

$$\begin{aligned} &\min_{\Omega = [\![\omega_{j,j'}]\!]} Q(\Omega), \\ &s.t. \quad \sum_{j \neq j'} |\omega_{j,j'}|^{1/2} \leq C. \end{aligned} \tag{1}$$

Also consider the Lagrangian problem

$$\min_{\Omega = [\![\omega_{j,j'}]\!]} Q(\Omega) + \lambda \sum_{j \neq j'} |\omega_{j,j'}|^{1/2}, \tag{2}$$

where $\lambda \geq 0$.

My statement is that the two problems (1) and (2) are equivalent in the sense that for given constrain C there always exists a λ such that two problems share the same minimizer. The equivalence also holds for given λ. Below is a simple verification.

Proof. For given C, let Ω^* be the minimizer of problem (1). Note that $\text{tr}(S\Omega)$ and $-\log |\Omega|$ are convex functions of Ω. Let λ be the solution satisfies the first order condition

$$\nabla Q(\Omega^*) + \lambda \nabla \sum_{j \neq j'} |\omega^*_{j,j'}|^{1/2} = 0.$$

Then, Ω^* is also the minimizer of problem (2).

For given λ, let Ω^* be the minimizer of problem (2). Let $C = \sum_{j \neq j'} |\omega^*_{j,j'}|^{1/2}$. Suppose there exists a solution of problem (1), Ω' such that $Q(\Omega') \leq Q(\Omega^*)$ and $\sum_{j \neq j'} |\omega'_{j,j'}|^{1/2} \leq C$. Then, we have

$$Q(\Omega') + \lambda \sum_{j \neq j'} |\omega'_{j,j'}|^{1/2} \leq Q(\Omega^*) + \lambda \sum_{j \neq j'} |\omega^*_{j,j'}|^{1/2},$$

which contradicts to the fact that Ω^* is the minimizer of problem (2). Therefore, Ω^* is also the minimizer of the problem (1). $\square$

Therefore, the estimate $\hat{\Omega}$, which is the minimizer of problem (1), shares the same accuracy property of the minimizer of problem (2). The condition for λ would reflect on the condition of C.