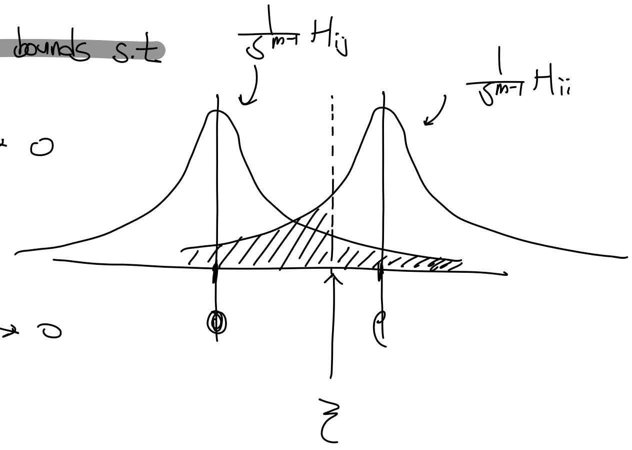
First, we need to construct tail bounds s.t

$$\begin{cases} P\left( \frac{1}{\varsigma^{m-1}} H_{ii} \leq \xi_{true} \right) \leq \longrightarrow 0 \\ P\left( \frac{1}{\varsigma^{m-1}} H_{ij} \geq \xi_{false} \right) \leq \longrightarrow 0 \end{cases}$$




$$\frac{1}{\varsigma^{m-1}} H_{ii} = \frac{1}{\varsigma^{m-1}} \sum_{\omega \in W} A_{i\omega} B_{i\pi_0(\omega)}$$

$$Z_{2i} = \rho Z_{1i} + \sqrt{1-\rho^2}\, Z'_{2i}$$

where $Z'_{2i} \perp\!\!\!\perp Z_{1i}$

$$= \frac{1}{\varsigma^{m-1}} \left( \sum_{i=1}^{\varsigma^{m-1}} \rho Z_{1i}^2 + \sqrt{1-\rho^2}\, Z_{1i} Z'_{2i} \right)$$

We need to construct tail bounds for $Z_{1i}^2$, $Z_{1i}Z_{2i}$ s.t $Z_{1i}, Z_{2i} \sim N(0,1)$ & $Z_{1i} \perp\!\!\!\perp Z_{2i}$

## $\boxed{\text{Tail Bound for } Z_{1i}^2}$

Notice $Z_{1i}^2$ is a subExp dist

$$\because E\left(e^{\lambda(X^2-1)}\right) = \frac{e^{-\lambda}}{\sqrt{1-2\lambda}} \leq e^{2\lambda^2} = e^{\frac{4\lambda^2}{2}} \quad \text{for all } |\lambda| \leq \frac{1}{4}$$

$$\Downarrow$$

$$\frac{1}{\sqrt{1-x}} \leq e^{\frac{x^2+x}{2}}$$

numerical inequality when $|x| \leq \frac{1}{2}$

$subE(2,4)$

By Bernstein type Inequality

$$P\left( \frac{1}{n} \sum_{k=1}^{n} Z_i^2 - 1 \leq t \right) \leq e^{-nt^2/8} \qquad t \in (-1, 0)$$

or $$P\left( \frac{1}{n} \sum_{k=1}^{n} Z_i^2 - 1 \geq t \right) \leq e^{-nt^2/8} \qquad t \in (0, 1)$$

$$\boxed{\text{Tail Bound for } Z_{1i} Z_{2i}'}$$

Notice $Z_{1i} Z_{2i}$ is sub Exponential

$$\because E(e^{\lambda XY}) = E_X\left(E_Y(e^{\lambda XY} \mid X)\right) = E_X\left(e^{\frac{\lambda^2 X^2}{2}}\right) = \frac{1}{\sqrt{1-\lambda^2}}$$

$$\le e^{\lambda^2} = e^{\frac{2\lambda^2}{2}}$$

for all

$|\lambda^2| \le \frac{1}{2}$

$\Leftrightarrow \lambda \le \frac{1}{\sqrt{2}}$

$\Downarrow$

$\text{SubE}(\sqrt{2}, \sqrt{2})$

Numerical Inequality $\frac{1}{1-x} \le e^{2x}$ when $|x| \le \frac{1}{2}$

$$\frac{1}{\sqrt{1-x}} \le e^{x}$$

By Bernstein type Inequality

$$P\left(\frac{1}{n}\sum_{k=1}^{n} Z_{1i} Z_{2i} \ge t\right) \le e^{-\frac{nt^2}{4}}$$

$t \in (0, \sqrt{2})$

$\therefore$ For $\frac{1}{S^{m-1}} H_{ii}$

$$P\left(\frac{1}{S^{m-1}}\left(\sum_{i=1}^{S^{m-1}} \rho Z_{1i}^2 + \sqrt{1-\rho^2}\, Z_{1i} Z_{2i}'\right) - \rho \ge t\right)$$

$$= P\left(\rho \frac{1}{S^{m-1}}\left(\sum_{i=1}^{S^{m-1}} Z_{1i}^2 - 1\right) + \sqrt{1-\rho^2}\, \frac{1}{S^{m-1}} \sum_{i=1}^{S^{m-1}} Z_{1i} Z_{2i}' \ge t\right)$$

$$\le P\left(\rho \frac{1}{S^{m-1}}\left(\sum_{i=1}^{S^{m-1}} Z_{1i}^2 - 1\right) \ge \frac{1}{2} t\right) + P\left(\sqrt{1-\rho^2}\, \frac{1}{S^{m-1}} \sum_{i=1}^{S^{m-1}} Z_{1i} Z_{2i}' \ge \frac{1}{2} t\right)$$

$$\le e^{-\frac{n}{8}\left(\frac{1}{2\rho}\right)^2 t_1^2} + e^{-\frac{n}{4}\left(\frac{1}{4(1-\rho^2)}\right) t_1^2}$$

$\rho \ge \frac{1}{\sqrt{3}}$ $\quad$ $\rho < \frac{1}{\sqrt{3}}$

$$= e^{-\frac{nt_1^2}{32\rho^2}} + e^{-\frac{nt_1^2}{16(1-\rho^2)}} \le e^{-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) nt_1^2}$$

Finally

$$P\left(\frac{1}{\delta^{m-1}} H_{ii} \le \rho - t_1\right) \le e^{-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) n t_1^2} \qquad (*)$$

for $t_1 \in (0,1)$

By the same way for $\frac{1}{\delta^{m-1}} H_{ij}$

$$P\left(\frac{1}{\delta^{m-1}} H_{ij} \ge t_2\right) \le e^{-\frac{\delta^{m-1} t_2^2}{4}} \qquad (**)$$

for $t_2 \in (0, \sqrt{2})$

We want to show $\boxed{\max_{\substack{i \neq j \\ i,j \in S^c}} \frac{1}{\delta^{m-1}} H_{ij} \le t_2 \le \rho - t_1 \le \min_{i \in S^c} \frac{1}{\delta^{m-1}} H_{ii}}$

for some $t_1, t_2$ under certain $\rho$ regime

By union bound, we have

$$(*)' \quad P\left(\min_{i \in S^c} \frac{1}{\delta^{m-1}} H_{ii} \le \rho - t_1\right) \le (n-s)\, e^{-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) n t_1^2}$$

$$\le \exp\left(-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) \delta^{m-1} t_1^2 + \log n\right)$$

To converge $\to 0$

$$t_1^2 \ge \frac{\log n}{\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) \delta^{m-1}}$$

$$(\ast\ast') \quad P\left( \max_{\substack{i \neq j \\ i,j \in S^c}} \frac{1}{S^{m-1}} H_{ij} \geq t_2 \right) \leq (n-s)^2 e^{-\frac{S^{m-1} t_2^2}{4}}$$

$$\leq \exp\left( -\frac{S^{m-1} t_2^2}{4} + 2\log n \right)$$

To converge $\longrightarrow 0$

Set $t_2^2 \geq \frac{8}{S^{m-1}} \log n$

Therefore, we have

$$\rho - t_1 \geq t_2 \qquad \text{set } t_1 = \sqrt{\frac{2\log n}{\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) S^{m-1}}} \qquad t_2 = \sqrt{\frac{9\log n}{S^{m-1}}}$$

Set $S^{m-1} = C(\log n)$ for some constant $C>0$

$$\rho - \sqrt{\frac{1}{C} \times \frac{2}{\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right)}} \geq \sqrt{\frac{9}{C}}$$

$$\rho \geq \frac{1}{\sqrt{3}} \Rightarrow \left(1 - \sqrt{\frac{64}{C}}\right)\rho \geq \sqrt{\frac{9}{C}}$$

for sufficiently large $n$.

We can always pick $C$ satisfying this Inequality

($\because$ constraints of $C$ is $n \geq (C\log n)^{\frac{1}{m}}$)

$$\rho < \frac{1}{\sqrt{3}} \Rightarrow \rho - \sqrt{\frac{32}{C}(1-\rho^2)} \geq \rho - \sqrt{\frac{32}{C} \cdot \frac{2}{3}} \geq \sqrt{\frac{9}{C}}$$

$\Rightarrow$ We can choose $C$ satisfying this Inequality

Finally we show that

$$\max_{\substack{i\neq j\\ i,j\in S^c}} \frac{1}{S^{m-1}} H_{ij} \le t_2 \le \ell - t_1 \le \min_{i\in S^c} \frac{1}{S^{m-1}} H_{ii}$$

$$= O\left(1-\frac{1}{n}\right)$$

with high pb converging to 1 under $S^{m-1} = \underbrace{C(\ell)}_{\text{constant depending on } \ell} \log n$ for sufficiently large $C$

Rmk) if we set $S^{m-1} = (C \log n)^{\varepsilon}$ where $\varepsilon > 1$
$C$ does not have to be sufficiently large