

Linear Algebra – Part II

A summary for MIT 18.06SC

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1 Orthogonality

1.1 Orthogonal vectors and subspaces

Definition 1 (*Orthogonal vectors*). Let $x, y \in \mathbb{R}^n$ be two vectors. The vectors x and y are orthogonal, denoted $x \perp y$, if and only if $x^T y = y^T x = 0$.

Definition 2 (*Orthogonal subspaces*). Let S, T be two subspaces S, T . The subspaces S and T are orthogonal, denoted $S \perp T$, if and only if for any $s \in S$ and $t \in T$, $s^T t = t^T s = 0$.

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix. There are four subspaces related to \mathbf{A} : column space $C(\mathbf{A})$, row space $C(\mathbf{A}^T)$, nullspace $N(\mathbf{A})$, and left nullspace $N(\mathbf{A}^T)$. Suppose the rank of \mathbf{A} is $\text{rank}(\mathbf{A}) = r$, the dimensions of these subspaces are:

$$\dim(C(\mathbf{A})) = \dim(C(\mathbf{A}^T)) = r, \quad \dim(N(\mathbf{A})) = n - r, \quad \dim(N(\mathbf{A}^T)) = m - r.$$

Theorem 1.1 (Orthogonality of matrix subspaces). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix. The row space $C(\mathbf{A}^T)$ and the nullspace $N(\mathbf{A})$ are orthogonal. The column space $C(\mathbf{A})$ and the left nullspace $N(\mathbf{A}^T)$ are orthogonal; i.e.,

$$C(\mathbf{A}^T) \perp N(\mathbf{A}) \quad \text{and} \quad C(\mathbf{A}) \perp N(\mathbf{A}^T).$$

Proof. For any vector $x \in N(\mathbf{A})$, we have $\mathbf{A}x = 0$. Specifically,

$$\mathbf{A}x = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $a_i \in \mathbb{R}^n$ is the i -th row of \mathbf{A} , for all $i \in [m]$. By Definition 1, x is orthogonal to the rows in matrix \mathbf{A} . For any vector $v \in C(\mathbf{A}^T)$, v is a linear combination of the rows $v = c_1 a_1 + \dots + c_m a_m$, where $c_i \in \mathbb{R}$, for all $i \in [m]$. Taking inner product between vectors v and x , we have

$$v^T x = c_1 a_1^T x + \dots + c_m a_m^T x = 0.$$

Therefore, $v \perp x$, and $N(\mathbf{A}) \perp C(\mathbf{A}^T)$.

Similarly, for any $x \in N(\mathbf{A}^T)$, we have $\mathbf{A}^T x = 0$, which implies $N(\mathbf{A}^T) \perp C(\mathbf{A})$. □

Theorem 1.2 (Relationship between $\mathbf{A}^T \mathbf{A}$ and \mathbf{A}). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix. We have

$$N(\mathbf{A}^T \mathbf{A}) = N(\mathbf{A}); \quad \text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A}).$$

Proof. First, we prove that $N(\mathbf{A}^T \mathbf{A}) = N(\mathbf{A})$.

On one hand, if $x \in N(\mathbf{A})$, then $\mathbf{A}x = 0$, which implies that $\mathbf{A}^T \mathbf{A}x = \mathbf{A}^T 0 = 0$. Therefore, for any $x \in N(\mathbf{A})$, the vector $x \in N(\mathbf{A}^T \mathbf{A})$.

On the other hand, we prove by contradiction that for any $x \in N(\mathbf{A}^T \mathbf{A})$ the vector $x \in N(\mathbf{A})$.

Suppose there is a vector $x \in N(\mathbf{A}^T \mathbf{A})$ but $x \notin N(\mathbf{A})$. We have

$$\mathbf{A}x = b \neq 0, \quad \mathbf{A}^T \mathbf{A}x = 0 \quad \Rightarrow \quad \mathbf{A}^T b = 0. \quad (1)$$

By the first equality in (1), $b \in C(\mathbf{A})$, and by the third equation in (1), $b \in N(\mathbf{A}^T)$. This contradicts the Theorem 1.1 that $C(\mathbf{A}) \perp N(\mathbf{A}^T)$. Therefore, for any $x \in N(\mathbf{A}^T \mathbf{A})$, the vector $x \in N(\mathbf{A})$.

Next, given $N(\mathbf{A}^T \mathbf{A}) = N(\mathbf{A})$, we have $\text{rank}(\mathbf{A}^T \mathbf{A}) = n - \dim(N(\mathbf{A}^T \mathbf{A})) = n - \dim(N(\mathbf{A})) = \text{rank}(\mathbf{A})$. \square

Corollary 1 (Invertibility of $\mathbf{A}^T \mathbf{A}$). If \mathbf{A} has independent columns, then $\mathbf{A}^T \mathbf{A}$ is invertible.

Proof. Suppose $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix with independent columns; i.e., $\text{rank}(\mathbf{A}) = n$. By Theorem 1.2, $\text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A}) = n$. Since $\mathbf{A}^T \mathbf{A} \in \mathbb{R}^{n \times n}$ is a square matrix, $\mathbf{A}^T \mathbf{A}$ is invertible. \square

1.2 Projections onto subspaces

Definition 3 (*Projection and projection matrix*). Let $x \in \mathbb{R}^n$ be a vector and $\mathbf{A}^{n \times m}$ be a matrix with independent columns. The vector $p \in C(\mathbf{A})$ that satisfies

$$(x - p) \perp C(\mathbf{A}), \quad (2)$$

is called the projection of vector x onto the column space $C(\mathbf{A})$. For all $x \in \mathbb{R}^n$ and the corresponding projection p , the matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ that satisfies

$$p = \mathbf{P}x,$$

is called the projection matrix of \mathbf{A} from \mathbb{R}^n onto the column space $C(\mathbf{A})$.

Proposition 1 (Projection matrix of $C(\mathbf{A})$). Let $x \in \mathbb{R}^n$ be a vector and $\mathbf{A}^{m \times n}$ be a matrix with independent columns. The projection matrix of \mathbf{A} from \mathbb{R}^n onto the column space $C(\mathbf{A})$ is

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T.$$

Proof. By Definition 3, the projection $p \in C(\mathbf{A})$. Then, there exists a vector $\hat{x} \in \mathbb{R}^m$ such that $p = \mathbf{A}\hat{x}$. By equation (2), for all $x \in \mathbb{R}^n$, we have

$$\mathbf{A}^T(x - \mathbf{A}\hat{x}) = 0 \quad \Rightarrow \quad \hat{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T x \quad \Rightarrow \quad p = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T x.$$

Therefore, the matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is the projection matrix of \mathbf{A} from \mathbb{R}^n onto the column space $C(\mathbf{A})$. \square

Theorem 1.3 (Properties of projection matrix). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix, and \mathbf{P} be the projection matrix of \mathbf{A} onto the column space. Then,

$$\mathbf{P}^T = \mathbf{P}; \quad \mathbf{P}^2 = \mathbf{P}.$$

Proof. By Proposition 1, the projection matrix is $\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$. Then,

$$\mathbf{P}^T = (\mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T)^T = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T = \mathbf{P}.$$

$$\mathbf{P}^2 = \mathbf{P}^T \mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T = \mathbf{P}.$$

□

Corollary 2 (Projection onto $N(\mathbf{A}^T)$). Suppose \mathbf{P} is a projection matrix in Theorem 1.3. Then $\mathbf{I} - \mathbf{P}$ is also a projection matrix of \mathbf{A} from \mathbb{R}^n onto the left nullspace $N(\mathbf{A}^T)$.

Proof. By Definition 3, for any $x \in \mathbb{R}^n$, we have

$$x - \mathbf{P}x \perp C(\mathbf{A}) \quad \Rightarrow \quad (\mathbf{I} - \mathbf{P})x \perp C(\mathbf{A}).$$

By Theorem 1.1, the column space of \mathbf{A} is orthogonal to the left null space of \mathbf{A} . Then,

$$(\mathbf{I} - \mathbf{P})x \in N(\mathbf{A}^T) \quad \text{and} \quad (x - (\mathbf{I} - \mathbf{P})x) \perp N(\mathbf{A}^T).$$

Therefore, $\mathbf{I} - \mathbf{P}$ is a projection matrix of \mathbf{A} from \mathbb{R}^n onto the left nullspace $N(\mathbf{A}^T)$. □

1.3 Projection matrices and least squares

Let $y \in \mathbb{R}^n$ be a vector and $\mathbf{X} \in \mathbb{R}^{n \times (k+1)}$ be a design matrix. We propose the linear regression model as

$$y = \mathbf{X}\beta + \epsilon,$$

where $\beta = (\beta_0, \beta_1, \dots, \beta_k)$ are regression coefficients and ϵ is the noise. The least square estimate of β is the minimizer of the loss; i.e.,

$$\hat{\beta}_{LS} = \arg \min_{\beta \in \mathbb{R}^{k+1}} \|\mathbf{y} - \mathbf{X}\beta\|^2,$$

where $\|\cdot\|$ is the euclidean norm. We consider the vector $\mathbf{X}\hat{\beta}_{LS}$ as the projection of y onto the column space of \mathbf{X} , which minimizes the distance from y to the column space $C(\mathbf{X})$. Therefore, we may use projection to solve the minimization problem.

By Definition 3, the projection $\mathbf{X}\hat{\beta}_{LS}$ satisfies

$$\mathbf{X}^T(\mathbf{y} - \mathbf{X}\hat{\beta}_{LS}) = 0 \quad \Rightarrow \quad \hat{\beta}_{LS} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}.$$

The estimate $\hat{\beta}_{LS}$ is consistent with the estimates solved by using the derivative.

1.4 Orthogonal matrices and Gram-Schmidt

Definition 4 (*Orthonormal vectors*). The vectors q_1, \dots, q_n are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

Orthonormal vectors are always independent.

Definition 5 (*Orthonormal matrix and orthogonal matrix*). Let $\mathbf{Q} \in \mathbb{R}^{m \times n}$ be a matrix. The matrix \mathbf{Q} is an orthonormal matrix, if the columns of \mathbf{Q} are orthonormal. When $m = n$, the square matrix \mathbf{Q} is an orthogonal matrix.

Suppose $\mathbf{Q} \in \mathbb{R}^{m \times n}$ is an orthonormal matrix. Then, we have $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}_n$. When $m = n$, \mathbf{Q} is an orthogonal matrix and $\mathbf{Q}^T = \mathbf{Q}^{-1}$. The projection matrix of \mathbf{Q} onto the column space $C(\mathbf{Q})$, denoted \mathbf{P} , becomes $\mathbf{P} = \mathbf{I}_m$.

Definition 6 (*Gram-Schmidt Process and QR decomposition*). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix with $\text{rank}(\mathbf{A}) = n$, and $a_i \in \mathbb{R}^m$ be the column of matrix \mathbf{A} , for all $i \in [n]$. Gram-Schmidt process finds the orthonormal basis for $C(\mathbf{A})$ as following,

$$\begin{aligned} u_1 &= a_1, & e_1 &= \frac{u_1}{\|u_1\|}; \\ u_2 &= a_2 - \frac{u_1^T a_2}{u_1^T u_1} u_1, & e_2 &= \frac{u_2}{\|u_2\|}; \\ u_3 &= a_3 - \frac{u_1^T a_3}{u_1^T u_1} u_1 - \frac{u_2^T a_3}{u_2^T u_2} u_2, & e_3 &= \frac{u_3}{\|u_3\|}; \\ &\vdots & & \\ u_n &= a_n - \sum_{k=1}^{n-1} \frac{u_k^T a_n}{u_k^T u_k} u_k, & e_n &= \frac{u_n}{\|u_n\|}. \end{aligned}$$

The vectors e_1, \dots, e_n are orthonormal basis of the $C(\mathbf{A})$. Based on the basis obtained by Gram-Schmidt, we decompose the matrix \mathbf{A} as following,

$$\mathbf{A} = [a_1, \dots, a_n] = [e_1, \dots, e_n] \begin{bmatrix} e_1^T a_1 & e_1^T a_2 & \dots & e_1^T a_n \\ 0 & e_2^T a_2 & \dots & e_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n^T a_n \end{bmatrix} = \mathbf{Q}\mathbf{R}, \quad (3)$$

where $\mathbf{Q} \in \mathbb{R}^{m \times n}$ is an orthonormal matrix and $\mathbf{R} \in \mathbb{R}^{n \times n}$ is an upper triangular matrix. The matrix decomposition in equation (3) is called QR decomposition.

2 Determinants

The *determinant* is a number associated with any square matrix. For a square matrix \mathbf{A} , let $\det(\mathbf{A})$ or $|\mathbf{A}|$ denote the determinant of matrix \mathbf{A} .

2.1 Properties of determinants

We give ten properties of determinants. The last seven properties are deduced by the first three basic properties.

1. The determinant of identity matrix is equal to 1; i.e., $\det(\mathbf{I}) = 1$.
2. Exchanging two rows of a matrix reverses the sign of the matrix's determinant. Hence, an odd number of row exchanges reverse the sign of the determinant while an even number of row exchanges maintain the sign of the determinant.
3. (a) If one row of the matrix is multiplied by a constant t , the determinant of the matrix is multiplied by t .

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(b) The determinant is linearly additive on the rows of the matrix.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

4. If the matrix contains two identical rows, the determinant of the matrix is 0.

Proof. Suppose \mathbf{A} has two identical rows a_i, a_j . The matrix after exchanging a_i, a_j , denoted \mathbf{A}' , is the same as the original matrix \mathbf{A} . By property 2, we have $\det(\mathbf{A}) = -\det(\mathbf{A}') = -\det(\mathbf{A})$. Therefore, $\det(\mathbf{A}) = 0$. \square

5. Subtracting t times of the i -th row from the j -th row does not change the determinant of the matrix, where $i \neq j$.

Proof. Take a two-by-two matrix as an example. By property 3 and property 4,

$$\begin{vmatrix} a & b \\ c-ta & d-tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The proof for higher-dimension matrix is similar. \square

6. If the matrix has a row that is all zeros, the determinant of the matrix is 0.

Proof. By property 3, letting $t = 0$ leads to property 6. \square

7. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a triangular matrix with diagonal elements d_1, \dots, d_n . The determinant $\det(\mathbf{A}) = \prod_{i=1}^n d_i$.

Proof. We eliminate the matrix \mathbf{A} to a diagonal matrix, denoted \mathbf{A}' . By property 5, $\det \mathbf{A} = \det \mathbf{A}'$. Since \mathbf{A} is triangular, the diagonal elements of \mathbf{A}' are still d_1, \dots, d_n . By property 3 and 1, $\det(\mathbf{A}') = \prod_{i=1}^n d_i \det(\mathbf{I}_n) = \prod_{i=1}^n d_i$. Therefore, $\det(\mathbf{A}) = \prod_{i=1}^n d_i$. \square

8. If the square matrix is singular, the determinant of the matrix is 0.

Proof. If \mathbf{A} is a singular matrix, the reduced row echelon form of \mathbf{A} , denoted $RREF(\mathbf{A})$, has a row with all 0 entries. By and 6, $\det(RREF(\mathbf{A})) = 0$. Therefore, by property 5, we have $\det(\mathbf{A}) = \det(RREF(\mathbf{A})) = 0$. \square

9. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ be two square matrices, the determinant of the matrix product is $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.

Proof. First, if at least one of \mathbf{A}, \mathbf{B} is singular, the product \mathbf{AB} is also singular. Then, $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}) = 0$.

Second, let \mathbf{E} be an elimination matrix. The product \mathbf{EB} means implementing a linear operation on the rows of \mathbf{B} . By property 5, linear operation does not change the determinant of \mathbf{B} . Therefore, we have

$$|\mathbf{EB}| = |\mathbf{E}||\mathbf{B}| \tag{4}$$

Last, suppose that both \mathbf{A}, \mathbf{B} are invertible. By elimination, there exist a sequence of elimination matrix $\{\mathbf{E}_i\}_{i=1}^n$ such that

$$\mathbf{E}_n \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{I}.$$

By Lemma 2 in Linear Algebra-Part I, the inverse of an elimination matrix is still an elimination matrix. Let \mathbf{E}'_k denote the inverse \mathbf{E}_k^{-1} , for all $k \in [n]$. We rewrite \mathbf{A} as

$$\mathbf{A} = \mathbf{E}'_n \cdots \mathbf{E}'_1.$$

Combing equation (4), we have

$$|\mathbf{AB}| = |\mathbf{E}'_n \cdots \mathbf{E}'_1 \mathbf{B}| = |\mathbf{E}'_n| |\mathbf{E}'_{n-1} \cdots \mathbf{E}'_1 \mathbf{B}| = \cdots = |\mathbf{E}'_n| \cdots |\mathbf{E}'_1| |\mathbf{B}|. \quad (5)$$

Applying the equation (4) again to the term $|\mathbf{E}'_n| \cdots |\mathbf{E}'_1|$, we have

$$|\mathbf{E}'_n| \cdots |\mathbf{E}'_1| = |\mathbf{E}'_n| \cdots |\mathbf{E}'_3| |\mathbf{E}'_2 \mathbf{E}'_1| = \cdots = |\mathbf{E}'_n \cdots \mathbf{E}'_1|. \quad (6)$$

Hence, combining equation (5) with equation (6), we obtain the result

$$|\mathbf{AB}| = |\mathbf{E}'_n \cdots \mathbf{E}'_1| |\mathbf{B}| = |\mathbf{A}| |\mathbf{B}|.$$

□

10. Let \mathbf{A} be a square matrix, the determinant $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

Proof. By LU decomposition, we rewrite the matrix as $\mathbf{A} = \mathbf{LU}$, where \mathbf{L} is a lower-triangular matrix and \mathbf{U} is an upper-triangular matrix. Note that the transports $\mathbf{L}^T, \mathbf{U}^T$ have the same diagonal elements with \mathbf{L}, \mathbf{U} respectively. By property 7, $\det(\mathbf{L}^T) = \det(\mathbf{L})$, $\det(\mathbf{U}^T) = \det(\mathbf{U})$. Therefore, combining property 9, we have $\det(\mathbf{A}) = \det(\mathbf{L}) \det(\mathbf{U}) = \det(\mathbf{U}^T) \det(\mathbf{L}^T) = \det(\mathbf{U}^T \mathbf{L}^T) = \det(\mathbf{A}^T)$. □

2.2 Determinant computation

Proposition 2 (Big formula for computing determinant). *Let $\mathbf{A} = \llbracket a_{ij} \rrbracket \in \mathbb{R}^{n \times n}$ be a square matrix. The big formula for computing the determinant of \mathbf{A} is*

$$\det(\mathbf{A}) = \sum_{(\alpha_1, \dots, \alpha_n) \in \mathcal{P}} (-1)^{N(\alpha_1, \dots, \alpha_n)} a_{1, \alpha_1} \cdots a_{n, \alpha_n},$$

where $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is the permutation of $(1, 2, \dots, n)$, \mathcal{P} is the collection of all possible $(\alpha_1, \dots, \alpha_n)$, and $N(\alpha_1, \dots, \alpha_n)$ is the number of necessary exchanges from $(\alpha_1, \alpha_2, \dots, \alpha_n)$ to $(1, 2, \dots, n)$.

Proof. Let $\mathbf{A} = \llbracket a_{ij} \rrbracket \in \mathbb{R}^{n \times n}$ be a square matrix. We decompose the matrix \mathbf{A} by the summation of n^n matrices, denoted \mathbf{A}'_i , where $i \in [n^n]$. For any \mathbf{A}'_i , there is only one entry comes from \mathbf{A} in each row and other entries are 0. By property 3, the determinant $\det(\mathbf{A}) = \sum_{i=1}^{n^n} \det(\mathbf{A}'_i)$. Take a 3-by-3 square matrix as an example.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = \mathbf{A}'_1 + \mathbf{A}'_2 + \cdots + \mathbf{A}'_{27}.$$

There are $n!$ nonsingular matrices among matrices \mathbf{A}'_i . The number of nonsingular matrices coincides with the size of \mathcal{P} . Intuitively, there are n ways to choose an element from the first row,

after which there are only $n - 1$ ways to choose an element from the second row to avoid the zero determinant. Therefore, we have $n \times (n - 1) \times (n - 2) \times \cdots \times 2 = n!$ nonsingular matrices. By property 7 and 2, the determinant of a nonsingular \mathbf{A}'_i follows the formula

$$(-1)^{N(\alpha_1, \dots, \alpha_n)} a_{1, \alpha_1} \dots a_{n, \alpha_n},$$

where $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is the permutation of $(1, 2, \dots, n)$, \mathcal{P} is the collection of all possible $(\alpha_1, \dots, \alpha_n)$, and $N(\alpha_1, \dots, \alpha_n)$ is the number of necessary exchanges from $(\alpha_1, \alpha_2, \dots, \alpha_n)$ to $(1, 2, \dots, n)$. \square

Definition 7 (*Cofactors, cofactor matrix, and cofactor formula*). Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a square matrix, and $\mathbf{A}_{-i, -j}$ be the submatrix of \mathbf{A} after removing the i -th row and j -th column. The cofactor associated with a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} \det(\mathbf{A}_{-i, -j}).$$

The matrix $\mathbf{C} = [C_{ij}] \in \mathbb{R}^{n \times n}$ is called cofactor matrix. The cofactor formula of $\det(\mathbf{A})$ is

$$\det(\mathbf{A}) = \sum_j^n a_{ij} C_{ij} = \sum_j^n a_{ji} C_{ji}, \quad \text{for all } i \in [n].$$

Example 1 (Tridiagonal matrix). One example of using cofactor formula is computing the determinant of *tridiagonal matrix*. The tridiagonal matrix is a matrix in which only nonzero elements lie on or adjacent to the diagonal. Let $\mathbf{T}_n \in \mathbb{R}^{n \times n}$ denote the tridiagonal matrix of 1's; i.e.,

$$\mathbf{T}_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{n \times n}.$$

Let $\mathbf{T}_{n, -i, -j}$ be the submatrix of \mathbf{T}_n after removing the i -th row and the j -th column. By cofactor formula, we have

$$\det(\mathbf{T}_n) = 1 \times \det(\mathbf{T}_{n, -1, -1}) - 1 \times \det(\mathbf{T}_{n, -1, -2}).$$

By the definition of tridiagonal matrix, $\mathbf{T}_{n, -1, -1} = \mathbf{T}_{n-1}$, and $\det(\mathbf{A}_{n, -1, -2}) = 1 \times \det(\mathbf{T}_{n-2})$. Therefore,

$$\det(\mathbf{T}_n) = \det(\mathbf{T}_{n-1}) - \det(\mathbf{T}_{n-2}).$$

2.3 Inverse matrices

Previously, we use Gauss-Jordan elimination to obtain the inverse matrix of an invertible matrix. Here, we apply cofactor formula to compute the inverse matrix.

Theorem 2.1 (Inverse matrix by cofactors). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix, and \mathbf{C} be the cofactor matrix of \mathbf{A} . The inverse matrix of \mathbf{A} satisfies

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \mathbf{C}^T.$$

Proof. It is equivalent to show that $\mathbf{A}\mathbf{C}^T = \det(\mathbf{A})\mathbf{I}_n$.

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an invertible matrix, and C_{ij} be the cofactor of a_{ij} . By the definition of cofactor matrix, we have

$$\mathbf{A}\mathbf{C}^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

Therefore, the diagonal elements of $\mathbf{A}\mathbf{C}^T$ are $(\mathbf{A}\mathbf{C}^T)_{ii} = \sum_k a_{ik}C_{ik} = \det(\mathbf{A})$, for all $i \in [n]$. Consider the off-diagonal entries $(\mathbf{A}\mathbf{C}^T)_{ij} = \sum_k a_{ik}C_{jk}$, where $i \neq j$. By cofactor formula, the summation $\sum_k a_{ik}C_{jk}$ is equal to the determinant of a matrix whose i -th row and j -th row are identical. Combining property 6, the off-diagonals of $\mathbf{A}\mathbf{C}^T$ are 0. Then, we have $\mathbf{A}\mathbf{C}^T = \det(\mathbf{A})\mathbf{I}_n$. \square

Definition 8 (*Cramer's rule*). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix, $x, b \in \mathbb{R}^n$ be vectors. Applying Theorem 2.1 to the linear system $\mathbf{A}x = b$, we obtain the Cramer's rule of $x = \mathbf{A}^{-1}b$,

$$x = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T b \quad \text{and} \quad x_j = \frac{\det(\mathbf{B}_j)}{\det(\mathbf{A})},$$

where \mathbf{B}_j is the matrix \mathbf{A} after replacing the j -th column by b .

Definition 9 (*Parallelepiped*). The parallelepiped determined by n vectors $v_1, \dots, v_n \in \mathbb{R}^n$ is defined as the following subset,

$$P = \{a_1v_1 + \cdots + a_nv_n : 0 \leq a_1, \dots, a_n \leq 1\}.$$

We use $\text{vol}(P)$ to denote the volume of the parallelepiped P .

Proposition 3 (Determinants and volumes). *The absolute determinant of a square matrix \mathbf{A} is the volume of parallelepiped determined by the rows of \mathbf{A} ; i.e.,*

$$|\det(\mathbf{A})| = \text{vol}(P(\mathbf{A})),$$

where $P(\mathbf{A})$ is the parallelepiped determined by the rows of \mathbf{A} .

Example 2. The area of a triangle with vertices at $(x_1, y_1), (x_2, y_2), (x_3, y_2)$ is

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Proof. The two edges of the triangle are $v_1 = (x_2 - x_1, y_2 - y_1)$ and $v_2 = (x_3 - x_1, y_3 - y_1)$. The area of the triangle is a half of the area of the parallelepiped determined by v_1, v_2 . Therefore, the area of the triangle is

$$\frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2} \left(\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \right) = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

\square

3 Eigenvalues and eigenvectors

Definition 10 (*Eigenvalues and eigenvectors*). Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ be a square matrix. Suppose there is a nonzero vector $x \in \mathbb{R}^m$ such that

$$\mathbf{A}x = \lambda x, \quad \text{for some } \lambda \in \mathbb{C}.$$

The vector x is called the eigenvector of \mathbf{A} . The value λ is called the eigenvalue of \mathbf{A} , and x is the eigenvector associated with eigenvalue λ .

Usually, eigenvectors are normalized; i.e., $\|x\| = 1$, where $\|\cdot\|$ is the euclidean distance. For simplicity, all the eigenvectors mentioned below are normalized. Besides, the eigenvectors associated with eigenvalue 0 span the nullspace of \mathbf{A} .

Definition 11 (*Trace of square matrix*). Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a square matrix. The trace of \mathbf{A} is defined as

$$\text{tr}(\mathbf{A}) = \sum_i^n a_{ii}.$$

Definition 12 (*Characteristic polynomial*). Let λ denote an eigenvalue of the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. The determinant of $\mathbf{A} - \lambda \mathbf{I}_n$ is a polynomial of λ , denoted $P(\lambda)$. We call the polynomial $P(\lambda)$ as the characteristic polynomial of \mathbf{A} . Specifically,

$$P(\lambda) = (-1)^n \lambda^n + (-1)^n \text{tr}(\mathbf{A}) \lambda^{n-1} + \cdots + (-1)^n \det(\mathbf{A}).$$

Since the characteristic polynomial $P(\lambda)$ is of degree n , there are n solutions to the equation $P(\lambda) = 0$. The solutions to $P(\lambda) = 0$, denoted $\lambda_1, \dots, \lambda_n$, are n eigenvalues of \mathbf{A} , which may be complex. If the complex eigenvalues exist, the complex eigenvalues come in conjugate pairs, because $P(\bar{\lambda}) = \overline{P(\lambda)}$ is also equal to 0, where $\bar{\lambda}$ is the conjugate of λ . Note that n eigenvalues are not necessarily distinct with each other.

3.1 Properties for eigenvectors and eigenvalues

Theorem 3.1 (Summation and production of eigenvalues). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix, and $\lambda_1, \dots, \lambda_n$ be the n eigenvalues of \mathbf{A} . Then,

$$\sum_{i=1}^n \lambda_i = \text{tr}(\mathbf{A}); \quad \prod_{i=1}^n \lambda_i = \det(\mathbf{A}).$$

Proof. First, we re-write the characteristic polynomial $P(\lambda)$ of \mathbf{A} as

$$P(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \tag{7}$$

where λ_i are n eigenvalues of \mathbf{A} , for all $i \in [n]$. By equation (7), the coefficient for λ^{n-1} is equal to $(-1)^n \sum_{i=1}^n \lambda_i$. Compared with Definition 12, we have $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$. Similarly, the constant term in equation (7) is $(-1)^n \prod_{i=1}^n \lambda_i$ while the constant term in Definition 12 is $(-1)^n \det(\mathbf{A})$. Therefore, we have $\prod_{i=1}^n \lambda_i = \det(\mathbf{A})$. \square

Definition 13 (*Similar matrices*). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix, and $\mathbf{B} \in \mathbb{R}^{n \times n}$ be an invertible matrix. The matrix product \mathbf{BAB}^{-1} is called the similar matrix of \mathbf{A} .

Theorem 3.2 (Eigenvalues for similar matrices). *Let \mathbf{A} be a square matrix. Any similar matrices of \mathbf{A} share the same eigenvalues of \mathbf{A} .*

Proof. Let \mathbf{BAB}^{-1} be a similar matrix of \mathbf{A} , and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of \mathbf{A} . Then, for all λ_i , there is an eigenvector x_i such that

$$\lambda_i x_i = \mathbf{A}x_i \Leftrightarrow \lambda_i \mathbf{B}x_i = \mathbf{BAB}^{-1}\mathbf{B}x_i.$$

Since x_i is a nonzero vector, and \mathbf{B} is invertible, the vector $\mathbf{B}x_i$ is also a nonzero vector. Therefore, λ_i is an eigenvalue of \mathbf{BAB}^{-1} with associated eigenvector $\mathbf{B}x_i$, for all $i \in [n]$. \square

Theorem 3.3 (Eigenvalues for powers of the matrix). *Let \mathbf{A} be a matrix, λ be a eigenvalue of \mathbf{A} , and x be the eigenvector associated with λ . For any polynomial P , we have*

$$P(\mathbf{A})x = P(\lambda)x.$$

Proof. For all integer $k \geq 0$ and constant $c \in \mathbb{R}$, we have

$$c\mathbf{A}^k x = c\mathbf{A}^{k-1}\mathbf{A}x = c\lambda\mathbf{A}^{k-1}x = \dots = c\lambda^k x.$$

Therefore, $P(\mathbf{A})x = P(\lambda)x$, for any polynomial P . \square

Theorem 3.4 (Eigenvalues for the inverse). *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix, $\lambda_1, \dots, \lambda_n$ be the eigenvalues of \mathbf{A} , and x_1, \dots, x_n be the eigenvectors associated with $\lambda_1, \dots, \lambda_n$. Then, $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ are the eigenvalues of \mathbf{A}^{-1} , and x_1, \dots, x_n are eigenvectors associated with $\lambda_1^{-1}, \dots, \lambda_n^{-1}$.*

Proof. Since the nullspace of \mathbf{A} contains only a zero vector, the eigenvalue $\lambda_i > 0$, for all $i \in [n]$. For all λ_i and x_i , multiplying x_i on both sides of the equation $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$ yields

$$\mathbf{A}^{-1}\mathbf{A}x_i = x_i \Rightarrow \mathbf{A}^{-1}x_i = \lambda_i^{-1}x_i.$$

Therefore, λ_i^{-1} is the eigenvalue of \mathbf{A}^{-1} , and x_i is the eigenvector associated with λ_i^{-1} , for all $i \in [n]$. \square

Theorem 3.5 (Independence of eigenvectors). *Let \mathbf{A} be a square matrix with at least two different eigenvalues. The eigenvectors associated with different eigenvalues are independent.*

Proof. Suppose the square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ has m different eigenvalues $\lambda_1, \dots, \lambda_m$ with associated eigenvectors v_1, \dots, v_m . Consider a linear combination $c_1v_1 + \dots + c_mv_m = 0$, where $c_i \in \mathbb{R}$, for all $i \in [m]$. Then we have

$$\mathbf{A}(c_1v_1 + \dots + c_mv_m) = c_1\lambda_1v_1 + \dots + c_m\lambda_mv_m = 0, \quad (8)$$

and

$$\lambda_1(c_1v_1 + \dots + c_mv_m) = c_1\lambda_1v_1 + \dots + c_m\lambda_1v_m = 0. \quad (9)$$

Subtracting the equations (8) from (9) yields

$$c_2(\lambda_1 - \lambda_2)v_2 + \dots + c_m(\lambda_1 - \lambda_m)v_m = 0. \quad (10)$$

Consider the new linear combination $c_2(\lambda_1 - \lambda_2)v_2 + \dots + c_m(\lambda_1 - \lambda_m)v_m = 0$ and repeat the steps from equation (8) to equation (10). After m iterations, we have

$$c_m(\lambda_1 - \lambda_m)(\lambda_2 - \lambda_m) \dots (\lambda_{m-1} - \lambda_m)v_m = 0.$$

Since $\lambda_1, \dots, \lambda_m$ are distinct, and v_m is a nonzero vector, the coefficient $c_m = 0$. Plugging $c_m = 0$ back to previous steps, we have $c_1 = \dots = c_m = 0$. Therefore, v_1, \dots, v_m are independent. \square

Theorem 3.6 (Eigenvalues for triangular matrices). *The eigenvalues for a triangular matrix are the entries on the diagonal.*

Proof. Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a triangular matrix. By the property of determinant 7, we have

$$\det(\mathbf{A}) = \prod_{i=1}^n a_{ii}; \quad \det(\mathbf{A} - \lambda \mathbf{I}_n) = \prod_{i=1}^n (a_{ii} - \lambda).$$

To let $\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$, we have $\lambda = a_{ii}$, for all $i \in [n]$. Therefore, the entries on the diagonal are the eigenvalues of a triangular matrix. \square

3.2 Eigenvalues and eigenvectors for symmetric matrices

Definition 14 (Antisymmetric matrices). The matrix \mathbf{A} is an antisymmetric matrix if \mathbf{A} satisfies

$$\mathbf{A}^T = -\mathbf{A}.$$

Theorem 3.7 (Eigenvalues for symmetric and antisymmetric matrices). *All the eigenvalues of a symmetric square matrix with only real entries are real. All the eigenvalues of an antisymmetric square matrix are imaginary; i.e., $\lambda = bi$, where $b \in \mathbb{R}, i = \sqrt{-1}$.*

Proof. First, we prove by contradiction that symmetric square matrices with all real entries have real eigenvalues.

Let \mathbf{A} be a symmetric matrix with only real entries. Suppose λ is a complex eigenvalue of \mathbf{A} , and x is the eigenvector associated with λ . Since all the entries of \mathbf{A} are real, the eigenvector x is also complex. Let $\bar{\lambda}$ and \bar{x} denote the conjugate eigenvalue and eigenvector, respectively. Then we have

$$\bar{x}^T \mathbf{A} x = \bar{x}^T \lambda x, \tag{11}$$

and

$$x^T \mathbf{A} \bar{x} = x^T \bar{\lambda} \bar{x}. \tag{12}$$

By the symmetry of \mathbf{A} , we have

$$\bar{x}^T \mathbf{A} x = x^T \mathbf{A}^T \bar{x} = x^T \mathbf{A} \bar{x}.$$

Subtracting the equation (11) from equation (12) yields

$$0 = (\bar{x}^T \mathbf{A} x - x^T \mathbf{A} \bar{x}) = (\lambda - \bar{\lambda}) x^T \bar{x}.$$

Since that $x^T \bar{x}$ is a real number, the imaginary part of λ is equal to 0, which contradicts the assumption that λ is complex.

Next, we prove that the eigenvalues for antisymmetric matrices are imaginary. Let \mathbf{A} be an antisymmetric matrix, λ be a complex eigenvalue of \mathbf{A} , and x be the complex eigenvector associated with λ . By the antisymmetry of \mathbf{A} , we have

$$\bar{x}^T \mathbf{A} x = x^T \mathbf{A}^T \bar{x} = -x^T \mathbf{A} \bar{x}.$$

Then,

$$0 = (\bar{x}^T \mathbf{A} x + x^T \mathbf{A} \bar{x}) = (\lambda + \bar{\lambda}) x^T \bar{x}.$$

Since that $x^T \bar{x}$ is a real number, the real part of λ is equal to 0. Therefore, the eigenvalues for an antisymmetric matrix are imaginary. \square

Theorem 3.8 (Orthogonality of eigenvectors for symmetric matrices). *The eigenvectors of a symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ associated with different eigenvalues are orthogonal.*

Proof. Let λ_1 and λ_2 be two different eigenvalues of \mathbf{A} , and x_1, x_2 are two eigenvectors associated with λ_1 and λ_2 , respectively. By the symmetry of \mathbf{A} , we have

$$x_2^T \mathbf{A} x_1 = x_1^T \mathbf{A}^T x_2 = x_1^T \mathbf{A} x_2.$$

Then,

$$0 = (x_2^T \mathbf{A} x_1 - x_1^T \mathbf{A} x_2) = (\lambda_1 - \lambda_2) x_2^T x_1.$$

Since λ_1, λ_2 are different, we have $\lambda_1 - \lambda_2 \neq 0$ and $x_2^T x_1 = 0$. Therefore, x_1 and x_2 are orthogonal. \square

Theorem 3.9 (Eigenvectors for repeated eigenvalues of symmetric matrices). *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Suppose λ_0 is a repeated eigenvalue of \mathbf{A} with multiplicity m , where $2 \leq m \leq n$. There exist m orthonormal eigenvectors associated with λ_0 .*

Proof. First, let x_0 be a nonzero eigenvector associated with λ_0 . For any $x_0 \in \mathbb{R}^n$, there are $n - 1$ additional orthogonal vectors y_1, \dots, y_{n-1} such that $y_j \perp x_0$, for all $j \in [(n-1)]$, and $(x_0, y_1, \dots, y_{n-1})$ forms the basis of \mathbb{R}^n . Let $\mathbf{Y} = [y_1, \dots, y_{n-1}]$ and $\mathbf{X} = [x_0, \mathbf{Y}]$. Since \mathbf{A} is a symmetric matrix, we have $x_0^T \mathbf{A} = \lambda_0 x_0^T$. Consider the following matrix product,

$$\mathbf{X}^T \mathbf{A} \mathbf{X} = \begin{bmatrix} x_0^T \mathbf{A} x_0 & x_0^T \mathbf{A} \mathbf{Y} \\ \mathbf{Y}^T \mathbf{A} x_0 & \mathbf{Y}^T \mathbf{A} \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \mathbf{Y}^T \mathbf{A} \mathbf{Y} \end{bmatrix}.$$

Note that \mathbf{X} is an orthogonal matrix which satisfies $\mathbf{X}^T = \mathbf{X}^{-1}$. By Theorem 3.2, $\mathbf{X}^T \mathbf{A} \mathbf{X}$ is a similar matrix of \mathbf{A} . Hence, λ_0 is also a repeated eigenvalue of $\mathbf{X}^T \mathbf{A} \mathbf{X}$. The characteristic polynomial of $\mathbf{X}^T \mathbf{A} \mathbf{X} - \lambda \mathbf{I}_n$ is

$$P(\lambda) = \det(\mathbf{X}^T \mathbf{A} \mathbf{X} - \lambda \mathbf{I}_n) = (\lambda_0 - \lambda) \det(\mathbf{Y}^T \mathbf{A} \mathbf{Y} - \lambda \mathbf{I}_{n-1}).$$

Since λ_0 has multiplicity $m \geq 2$, the term $\det(\mathbf{Y}^T \mathbf{A} \mathbf{Y} - \lambda_0 \mathbf{I}_{n-1}) = 0$. Therefore, the dimension of the nullspace of $\mathbf{X}^T \mathbf{A} \mathbf{X} - \lambda_0 \mathbf{I}_n$ is larger than 2. Let v_1, v_2 be the two orthogonal vectors in the nullspace of $\mathbf{X}^T \mathbf{A} \mathbf{X} - \lambda_0 \mathbf{I}_n$. Then v_1, v_2 are two eigenvectors of $\mathbf{X}^T \mathbf{A} \mathbf{X}$, and $\mathbf{X} v_1, \mathbf{X} v_2$ are two orthogonal eigenvectors of \mathbf{A} associated with λ_0 .

Replace x_0 by two orthogonal vectors $x_1 = \mathbf{X} v_1, x_2 = \mathbf{X} v_2$ and repeat the above steps. Finally, we will have m orthogonal eigenvectors of \mathbf{A} associated with λ_0 . \square

3.3 Diagonalization

Definition 15 (Diagonalization). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix with n independent eigenvectors. We diagonalize the matrix \mathbf{A} as

$$\mathbf{A} = \mathbf{S}^{-1} \mathbf{\Lambda} \mathbf{S}, \tag{13}$$

where $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$ is a diagonal matrix whose entries are n eigenvalues of \mathbf{A} and $\mathbf{S} \in \mathbb{R}^{n \times n}$ is a matrix whose columns are independent eigenvectors of \mathbf{A} associated with eigenvalues in $\mathbf{\Lambda}$.

Theorem 3.10 (Diagonalization of symmetric matrices). *Let \mathbf{A} be a symmetric matrix. The matrix \mathbf{A} can be diagonalized.*

Proof. By Theorems 3.8 and Theorem 3.9, for any symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, \mathbf{A} have n orthogonal eigenvectors, even though there may have repeated eigenvalues. Therefore, the matrix \mathbf{A} can be diagonalized as equation (13). \square

Theorem 3.11 (Descent of matrix powers). *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, and n independent eigenvectors associated with the eigenvalues. The power of matrix $\mathbf{A}^k \rightarrow 0$ as $k \rightarrow +\infty$ if and only if $|\lambda_i| < 1$, for all $i \in [n]$.*

Proof. We diagonalize the matrix \mathbf{A} as $\mathbf{A} = \mathbf{S}^{-1} \mathbf{\Lambda} \mathbf{S}$. For all integer $k \geq 0$, we have $\mathbf{A}^k = \mathbf{S}^{-1} \mathbf{\Lambda}^k \mathbf{S}$. Then,

$$\lim_{k \rightarrow +\infty} \mathbf{A}^k = \lim_{k \rightarrow +\infty} \mathbf{S}^{-1} \mathbf{\Lambda}^k \mathbf{S} = \lim_{k \rightarrow +\infty} \mathbf{S}^{-1} \begin{bmatrix} |\lambda_1|^k & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & |\lambda_n|^k \end{bmatrix} \mathbf{S} = 0 \quad \Leftrightarrow \quad |\lambda_i| < 1, \text{ for all } i \in [n].$$

□