Proofs

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To do list:

- Proof of Conjecture 1.
- Accuracy for the post-processing.
- Derivation of $Q(\xi)$.

1 Problem Formulation and Model

Consider two random tensors $\mathcal{A}, \mathcal{B}' \in \mathbb{R}^{d^{\otimes m}}$, where $\mathcal{A}(\omega)$ and $\mathcal{B}'(\omega)$ denote the tensor entry indexed by $\omega = (i_1, \ldots, i_m) \in [n]^m$. Suppose $\mathcal{A}$ and $\mathcal{B}'$ are super-symmetric; i.e., $\mathcal{A}(\omega) = \mathcal{A}(f(\omega)), \mathcal{B}(\omega) = \mathcal{B}'(f(\omega))$ for any function f permutes the indices in ω for all $\omega \in [n]^m$. Consider the bivariate generative model for the entries $\{\omega : 1 \leq i_1 \leq \cdots \leq i_m \leq n\}$

$$(\mathcal{A}(\omega), \mathcal{B}'(\omega)) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right), \quad \text{and} \quad (\mathcal{A}(\omega), \mathcal{B}'(\omega)) \perp (\mathcal{A}(\omega'), \mathcal{B}'(\omega')), \text{ for all } \omega \neq \omega',$$

where the correlation $\rho \in (0,1)$ and $\perp$ denote the statistical independence. We call $\mathcal{A}$ and $\mathcal{B}'$ as two correlated Wigner tensors.

Suppose we observe the tensor pair $\mathcal{A}$ and $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}' \circ \pi^*$, where $\pi^* : [n] \mapsto [n]$ denotes a permutation on $[d]$, and by definition $\mathcal{B}(i_1, \ldots, i_m) = \mathcal{B}'(\pi(i_1), \ldots, \pi(i_m))$ for all $(i_1, \ldots, i_m) \in [n]^m$.

This work aims to recover the true matching π given the noisy observations $\mathcal{A}, \mathcal{B}$.

2 Gaussian Tensor Matching

Notations.

1. L_p norm for function $f : \mathbb{R} \mapsto \mathbb{R}$ with $p \in [1, \infty)$:

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p \, dt \right)^{1/p}.$$

2. $[n]^m$: denote the dimensional-m space with elements $\{(i_1, \ldots, i_m) : i_k \in [n] \text{ for all } k \in [m]\}$.

2.1 Matching via Empirical Distributions

We construct the L_p distance statistics, $d_p(\mu_i, \nu_k)$, to evaluate the similarity between the pairs (i, k),

$$d_p(\mu_i, \nu_k) = \left(\int_{\mathbb{R}} |F_n^i(t) - G_n^k(t)|^p dt \right)^{1/p}, \tag{1}$$

where

$$F_n^i(t) = \frac{1}{n^{m-1}} \sum_{(i_2,\ldots,i_m)\in[n]^{m-1}} \mathbb{1}\{\mathcal{A}_{i,i_2,\ldots,i_m} \le t\}, \text{ and } G_n^k(t) = \frac{1}{n^{m-1}} \sum_{(i_2,\ldots,i_m)\in[n]^{m-1}} \mathbb{1}\{\mathcal{B}_{k,i_2,\ldots,i_m} \le t\}.$$

The Gaussian tensor matching algorithm using $d_p(\mu_i, \nu_k)$ is in Algorithm 1, where the p should be given in practice.

Algorithm 1 Gaussian tensor matching via empirical distribution

Input: Gaussian tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n^{\otimes m}}$.

1: Calculate the distance statistics $d_p(\mu_i, \nu_k)$ in (1) for each pair of $(i, k) \in [n]^2$.
2: Sort $\{d_p(\mu_i, \nu_k) : (i, k) \in [n]^2\}$ and let S be the set of indices of the smallest d elements.
3: **if** there exists a permutation $\hat{\pi}$ such that $S = \{(i, \hat{\pi}(i)) : i \in [n]\}$ **then**
4: Output $\hat{\pi}$.
5: **else**
6: Output error.
7: **end if**

Output: Estimated permutations $\hat{\pi}$ or error.

The theoretical guarantee for the success of Algorithm 1 is below.

Theorem 2.1 (**Conjecture.** Guarantee of Algorithm 1)**.** *Let $\rho = \sqrt{1-\sigma^2}$. Suppose $\sigma \le c/\log n$ for sufficiently small constant $c \in (0, 1/2)$. Algorithm 1 recover the true permutation π^* with probability tends to 1.*

Where is m in the Theorem 2.1?

The order m is not reflected in Theorem 2.1 due to the inequalities in Conjecture 1. The right hand sides of the tail bounds are independent on m. Therefore, we need to check Conjecture 1 first.

Conjecture 1 (Tail bounds for empirical process)**.** Consider the correlated pairs of normal variables (X_i, Y_i) for $i \in [n]$, where $X_i, Y_i \sim N(0, 1)$ and $\mathrm{cov}(X_i, Y_i) = \rho$. Let $\rho = \sqrt{1-\sigma^2}$, and F_n, G_n denote the empirical CDF of $\{X_i\}$ and $\{Y_i\}$. Then, the L_p norm between F_n and G_n satisfies:

1. if $\rho > 0$,
$$\mathbb{P}(\|F_n - G_n\|_p \ge \sqrt{\frac{\sigma}{n}}) \le C_1 \exp\left(-\frac{1}{\sigma}\right); \tag{2}$$

2. if $\rho = 0$,
$$\mathbb{P}(\|F_n - G_n\|_p \le \sqrt{\frac{\sigma}{n}}) \le C_2 \exp\left(-\frac{1}{\sigma}\right), \tag{3}$$

for $p \in [1, \infty)$ with universal positive constants C_1 and C_2.

Proof of Theorem 2.1. Without loss of generality, we assume the true permutation π^* is the identity mapping; i.e., $\pi^*(i) = i$ for all $i \in [n]$. For simplicity, let d_{ik} denote the distance statistics $d_p(\mu_i, \nu_j)$ in (1) with general $p \in [1, \infty)$. To guarantee the Algorithm 1 outputs the true permutation with probability, it suffices to show

$$\min_{i \neq k \in [n]^2} d_{ik} > \max_{i \in [n]} d_{ii}$$

with probability tends to 1.

Note that

$$\mathbb{P}\left(\min_{i \neq k \in [n]^2} d_{ik} < \sqrt{\frac{\sigma}{n^{m-1}}}\right) \leq n^2 C_2 \exp\left(-\frac{1}{\sigma}\right)$$

where the inequality follows by the inequality (3) in Conjecture 1.

Also, note that

$$\mathbb{P}\left(\max_{i \in [n]} d_{ii} \geq \sqrt{\frac{\sigma}{n^{m-1}}}\right) \leq n C_1 \exp\left(-\frac{1}{\sigma}\right),$$

where the inequality follows by the inequality (2) in Conjecture 1.

Take $\sigma \leq \frac{c}{\log n}$ for $c < 1/2$. Therefore, we have

$$\mathbb{P}\left(\min_{i \neq k \in [n]^2} d_{ik} > \sqrt{\frac{\sigma}{n^{m-1}}} > \max_{i \in [n]} d_{ii}\right) \geq 1 - \left(\mathbb{P}\left(\min_{i \neq k \in [n]^2} d_{ik} < \sqrt{\frac{\sigma}{n^{m-1}}}\right) + \mathbb{P}\left(\max_{i \in [n]} d_{ii} \geq \sqrt{\frac{\sigma}{n^{m-1}}}\right)\right)$$

$$\geq 1 - n^{2-\frac{1}{c}} - n^{1-\frac{1}{c}} \to 1,$$

when n goes to infinity.

We then finish the proof of Theorem 2.1. $\square$

2.2 Seeded matching

We consider the high-degree seed set

$$\mathcal{S} = \{(i, k) \in [n]^2 \colon a_i, b_k \geq \xi, d_p(\mu_i, \nu_k) \leq \zeta\}, \tag{4}$$

where

$$a_i = \frac{1}{\sqrt{n^{m-1}}} \sum_{\omega \in [n]^{m-1}} \mathcal{A}_{i,\omega}, \quad b_k = \frac{1}{\sqrt{n^{m-1}}} \sum_{\omega \in [n]^{m-1}} \mathcal{B}_{k,\omega},$$

are the counterparts of "degrees" for Gaussian tensors.

Let $\pi_0 : S \mapsto T$ denotes the mapping corresponding to the seeds, where $S, T \subset [n]$ and $\pi_0(j) = \pi(j)$ for all $j \in S$.

Define the neighbourhood

$$\mathcal{N} = \{(i_2, \ldots, i_m) : i_l \in S, \text{ for all } l = 2, \ldots, m\}$$

with $|\mathcal{N}| = |\mathcal{S}|^{m-1}$, and define $\pi_0(\mathcal{N})$ by replacing i_l to $\pi_0(i_l)$ in the definition of $\mathcal{N}$ for all $l = 2, \ldots, m$. Then, we define the similarity between the node i in $\mathcal{A}$ and node k in $\mathcal{B}$ as

$$H_{ik} = \sum_{\omega \in \mathcal{N}} \mathcal{A}_{i,\omega} \mathcal{B}_{k,\pi_0(\omega)}. \tag{5}$$

We find the rest of the mapping via the matrix H.

See the improved matching strategy in Algorithm 2 with seeded matching as a subroutine in Sub-algorithm 1.

Algorithm 2 Gaussian tensor matching with seed improvement

Input: Gaussian tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n^{\otimes m}}$, threshold ξ, ζ.

1: Calculate the distance statistics $d_p(\mu_i, \nu_k)$ in (1) for each pair of $(i,k) \in [n]^2$.
2: Obtain the high-degree set $\mathcal{S}$ in (4).
3: **if** there exists a permutation π_0 such that $\mathcal{S} = \{(i, \pi_0(i)) : i \in [n]\}$ **then**
4: Run Sub-Algorithm 1 with seed π_0 and output $\hat{\pi}$.
5: **else**
6: Output error.
7: **end if**

Output: Estimated permutations $\hat{\pi}$ or error.

Sub-Algorithm 1: seeded matching

Input: Gaussian tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n^{\otimes m}}$, seed $\pi_0 : S \mapsto T$.

8: For $i \in S^c$ and $k \in T^c$, obtain the similarity matrix $H = [\![H_{ik}]\!]$ as (5).
9: Find the optimal bipartite permutation $\tilde{\pi}_1$ such that

$$\tilde{\pi}_1 = \arg\max_{\pi : S^c \mapsto T^c} \sum_{i \in S^c} H_{i,\pi(i)}.$$

Let π_1 denote the matching on $[n]$ such that $\pi_1|_S = \pi_0$ and $\pi_1|_{S^c} = \tilde{\pi}_1$.

10: For each pair $(i,k) \in [n]^2$, calculate $W_{ik} = \sum_{\omega \in [n]^{m-1}} \mathcal{A}_{i,\omega} \mathcal{B}_{k,\pi_1(\omega)}$.
11: Sort $\{W_{ik} : (i,k) \in [n]^2\}$ and let $\hat{S}$ denote the set of indices of largest d elements.
12: **if** three exists a permutation $\hat{\pi}$ such that $\hat{S} = \{(i, \hat{\pi}(i)) : i \in [n]\}$. **then**
13: Output $\hat{\pi}$.
14: **else**
15: Output error.
16: **end if**

Output: Estimated permutations $\hat{\pi}$ or error.

The theoretical guarantee for Algorithm 2 is below.

Note that the purple parts (lines 3-4) in Sub-algorithm 1 can be considered as a non-iterative clean up for the π_1. Without the post-processing, let the estimate $\hat{\pi} = \pi_1$. In the following theorems, we develop the guarantees **without** this clean up.

Theorem 2.2 (**Conjecture**: Guarantee for Algorithm 2)**.** *Let $\rho = \sqrt{1-\sigma^2}$. Suppose $\sigma \leq c/\log^{1/3(m-1)} n$ for sufficiently small constant c. Choose thresholds $\xi \geq c_1\sqrt{\log^{1/(m-1)} n}$ with universal positive constant c_1 and $\zeta \leq \sqrt{\sigma/n^{m-1}}$. Algorithm 2 recover the true permutation π^* with probability tends to 1.*

Remark 1 (From matrix matching to tensor matching)**.** When $m = 2$, our results coincide with the results of matrix matching in Ding et al. (2021). The improvement of tensor matching with increasing order m is mainly comes from decreasing number of necessary seeds. Intuitively, in tensor cases, we need less seeds to obtain the description of the unseeded pairs with the same accuracy, which results in a looser upper bound of σ. Note that a larger σ indicates a smaller correlation between two tensors and thereof a weaker "signal" in the matching problem. Therefore, we allow a weaker signal assumption $\sigma = \mathcal{O}(\frac{1}{\log^{1/3(m-1)} n})$ as m increases.

Proof of Theorem 2.2. The proof of Theorem 2.2 separates into two parts: (1) accuracy for the seeded Sub-algorithm 1; (2) high-degree seed set $\mathcal{S}$ generates a desirable seed for seeded algorithm to succeed.

For (1), Theorem 2.3 indicates the seeded Sub-algorithm 1 successfully recovers the true matching when the seed set $\mathcal{S}$ includes $c_0 \log^{1/(m-1)} n$ true pairs for some constant $c_0 \gtrsim \mathcal{O}(1/\rho)$ and no fake pairs. For simplicity, let $s = c_0 \log^{1/(m-1)} n$ denote the number of necessary true pairs in the seed. Hence, we only need to show the set $\mathcal{S}$ (4) with proper thresholds ξ and ζ satisfies the conditions for Theorem 2.3 under $\sigma \leq c/\log^{1/3(m-1)} n$ with small c.

Note that for $(i,k) \in [n]^2$ (Need derivation here.)

$$\mathbb{P}(a_i \geq \xi, b_k \geq \xi) = \begin{cases} Q^2(\xi) & \text{if } (i,k) \text{ is a fake pair, i.e., } i \neq \pi^*(k) \\ Q(\xi)\exp(-C\sigma^2\xi^2) & \text{if } (i,k) \text{ is a true pair, i.e., } i = \pi^*(k), \end{cases}$$

where Q is the complementary CDF of normal distribution and C is a positive constant. Also, by the Conjecture 1, we have

$$\mathbb{P}\left(d_{ik}(\mu_i,\nu_k) \leq \sqrt{\frac{\sigma}{n^{m-1}}}\right) \begin{cases} \leq C_2 \exp\left(-\frac{1}{\sigma}\right) & \text{if } (i,k) \text{ is a fake pair, i.e., } i \neq \pi^*(k) \\ \geq 1 - C_1 \exp\left(-\frac{1}{\sigma}\right) & \text{if } (i,k) \text{ is a true pair, i.e., } i = \pi^*(k). \end{cases}$$

Take $\zeta \leq \sqrt{\sigma/n^{m-1}}$. Then, for $\mathcal{S}$ satisfying the conditions for Lemma 2.3, we have

1. $\mathcal{S}$ has s true pairs with high probability (the expectation of the true pairs in $\mathcal{S}$ is larger than s)
$$nQ(\xi)\exp(-C\sigma^2\xi^2) \geq s; \tag{6}$$
2. $\mathcal{S}$ has no fake pairs (the expectation of the fake pairs in $\mathcal{S}$ converges to 0 as $n \to \infty$)
$$n^2Q^2(\xi)C_2\exp\left(-\frac{1}{\sigma}\right) = o(1). \tag{7}$$

Take $\xi \geq c_1\sqrt{s}$. By inequality (6), we have $Q(\xi) \geq \frac{s}{n}\exp\left(Cc_1^2\sigma^2 s\right)$. Pluging the inequality for $Q(\xi)$ into the inequality (7), we have

$$C_2 s^2 \exp\left(2Cc_1^2\sigma^2 s - \frac{1}{\sigma}\right) = o(1),$$

which implies $\sigma \leq \frac{c}{s^{1/3}}$ with small constant c such that $2Cc_1^2c^2 - \frac{1}{c^2} < 0$.

Note that $s = c_0 \log^{1/(m-1)} n$. We finish the proof of Theorem 2.2.

$\square$

Theorem 2.3 (Accuracy for seeded Sub-algorithm 1)**.** *Suppose the seed π_0 corresponds to $s = |\mathcal{S}| = c_0 \log^{1/(m-1)} n$ true pairs for some constant $c_0 \gtrsim \mathcal{O}(1/\rho)$ and no fake pairs. The Sub-algorithm 1 recovers the true permutation π^* with probability tends to 1.*

Proof for Theorem 2.3. Without loss of generality, we assume the true permutation π^* is the identity mapping; i.e., $\pi^*(i) = i$ for all $i \in [n]$. Without post-processing, it suffices to show the $\tilde{\pi}_1$ recovers all the true pairs out of the seed set $\mathcal{S}$; i.e.,

$$\pi^*/\pi_0 = \underset{\pi: S^c \mapsto T^c}{\arg\max} \sum_{i \in S^c} H_{i,\pi(i)},$$

where π^*/π_0 is the mapping excluding the pairs in the seed π_0. It suffices to show that

$$\min_{i \in S^c} H_{ii} > \max_{i \neq j \in S^c} H_{ij} \tag{8}$$

holds with high probability tends to 1. By the inequality (12) in Lemma 1, we have

$$\mathbb{P}\left(\min_{i \in S^c} \frac{1}{s^{m-1}} H_{ii} \leq \rho - t_1\right) \leq 2(n-s)\exp\left(-\min\left\{\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right\} s^{m-1} t_1^2\right) \tag{9}$$

for $t_1 \in [0, \min\{2\rho, 2\sqrt{2}\sqrt{1-\rho^2}\}]$ and by the inequality (13) in Lemma 1

$$\mathbb{P}\left(\max_{i \neq j \in S^c} H_{ij} \geq t_2\right) \leq 2(n-s)^2 \exp\left(-\frac{1}{4} s^{m-1} t_2^2\right), \tag{10}$$

for $t_2 \in [0, \sqrt{2}]$. To let event (8) holds with probability tends to 1, we need the probabilities (9) and (10) goes to 0 as $n \to \infty$ with proper t_1 and t_2, which implies

$$\rho - t_1 > t_2 \quad \text{and} \quad t_1^2 > \frac{\log n}{\min\left\{\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right\} s^{m-1}}, \quad t_2^2 > \frac{8 \log n}{s^{m-1}}. \tag{11}$$

Take $s^{m-1} = c \log n$ such that $c \leq \frac{n^{m-1}}{\log n}$. Now, to finish the proof of Lemma 2.3, we only need to verify that there exist c, t_1, t_2 that satisfy all the inequalities in (11).

Consider $t_1 = \sqrt{\frac{2 \log n}{\min\left\{\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right\} s^{m-1}}}$ and $t_2 = \sqrt{\frac{16 \log n}{s^{m-1}}}$. Note that $\min\left\{\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right\} \geq \frac{1}{32}$. Then, we need

$$\rho - \sqrt{\frac{64}{c}} > \sqrt{\frac{16}{c}}, \quad \text{and thus} \quad c \geq \frac{12}{\rho},$$

which can be satisfied when sufficiently large n under the constraint $c \leq \frac{n^{m-1}}{\log n}$.

We then finish the proof of Lemma 1.

$\square$

Lemma 1 (Tail bounds for the product of normal variables)**.** *Consider the correlated pairs of normal variables* (X_i, Y_i) *for* $i \in [n]$*, where* $X_i, Y_i \sim N(0,1)$*. Let* $H = \frac{1}{n}\sum_{i\in[n]} X_i Y_i$*. If* $cov(X_i, Y_i) = \rho > 0$*, then we have*

$$\mathbb{P}\left(|H-\rho| \geq t\right) \leq 4\exp\left(-\min\left\{\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right\} nt^2\right), \tag{12}$$

for constant $t \in [0, \min\{2\rho, 2\sqrt{2}\sqrt{1-\rho^2}\}]$*. If* $cov(X_i, Y_i) = 0$*, then, we have*

$$\mathbb{P}\left(|H| \geq t\right) \leq 2\exp\left(-\frac{nt^2}{4}\right), \tag{13}$$

for constant $t \in [0, \sqrt{2}]$*.*

Proof of Lemma 1. Consider the case that $\rho > 0$. The proof of inequality (13) is involved as an intermediate step under the case $\rho > 0$. Note that $Y_i = \rho X_i + \sqrt{1-\rho^2} Z_i$, where Z_i is independent with X_i. Then it is equivalent to develop the tail bound for the sum $\frac{1}{n}\sum_{i=1}^n(\rho X_i^2 + \sqrt{1-\rho^2} X_i Z_i)$. We consider the tail probabilities for X_i^2 and $X_i Z_i$ separately.

Tail probability of X_i^2. Note that X_i^2s are sub-exponential variables with parameters (2,4) and expectation 1, and with Bernstein-type bound, we have

$$\mathbb{P}\left(|\frac{1}{n}\sum_{i=1}^n (X_i^2 - 1)| \geq t\right) \leq 2\exp\left(-\frac{nt^2}{8}\right),$$

when $t \in [0,1]$.

Tail probability of $X_i Z_i$. Note that for $\lambda^2 \leq \frac{1}{2}$

$$\mathbb{E}[\exp(\lambda X_i Z_i)] = \mathbb{E}_{X_i}[\mathbb{E}_{Z_i}[\exp(\lambda X_i Z_i)|X_i]] = \mathbb{E}_{X_i}[\exp(\lambda^2 X_i^2/2)] \leq \frac{1}{\sqrt{1-\lambda^2}} \leq \exp(2\lambda^2/2),$$

where the second and third inequalities follow by the properties of sub-Gaussian variables, and the last inequality follows by the inequality $\frac{1}{\sqrt{1-x}} \leq \exp(x)$ for $|x| \leq 1/2$. Hence, $X_i Z_i$ is also sub-exponential with parameters $(\sqrt{2}, \sqrt{2})$ with expectation 0. By Bernstein-type bound, we have

$$\mathbb{P}\left(|\frac{1}{n}\sum_{i=1}^n X_i Z_i| \geq t\right) \leq 2\exp\left(-\frac{nt^2}{4}\right),$$

for $t \in [0, \sqrt{2}]$. Then, we finish the proof of inequality (13).

Therefore, we have

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n(\rho X_i^2 + \sqrt{1-\rho^2} X_i Z_i) - \rho \geq t\right) &= \mathbb{P}\left(\rho\frac{1}{n}\sum_{i=1}^n (X_i^2 - 1) + \sqrt{1-\rho^2}\frac{1}{n}\sum_{i=1}^n X_i Z_i \geq t\right) \\
&\leq \mathbb{P}\left(\rho\frac{1}{n}\sum_{i=1}^n (X_i^2 - 1) \geq \frac{t}{2}\right) + \mathbb{P}\left(\sqrt{1-\rho^2}\frac{1}{n}\sum_{i=1}^n X_i Z_i \geq \frac{t}{2}\right) \\
&\leq \exp\left(-\frac{nt^2}{32\rho^2}\right) + \exp\left(-\frac{nt^2}{16(1-\rho^2)}\right) \\
&\leq 2\exp\left(-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right) nt^2\right),
\end{aligned}$$

for $t \in [0, \min\{2\rho, 2\sqrt{2}\sqrt{1-\rho^2}\}]$. Similarly, we also have

$$\mathbb{P}\left(\rho - \frac{1}{n}\sum_{i=1}^{n}(\rho X_i^2 + \sqrt{1-\rho^2}X_i Z_i) \geq t\right) \leq 2\exp\left(-\min\left(\frac{1}{32\rho^2}, \frac{1}{16(1-\rho^2)}\right)nt^2\right),$$

with $t \in [0, \min\{2\rho, 2\sqrt{2}\sqrt{1-\rho^2}\}]$.

Then, we finish the proof of Lemma 1. $\square$

References

Ding, J., Ma, Z., Wu, Y., and Xu, J. (2021). Efficient random graph matching via degree profiles. *Probability Theory and Related Fields*, 179(1):29–115.