

Graphic Lasso: Clustering accuracy

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1 Accuracy

Consider the model

$$\mathbb{E}[\mathcal{Y}] = f(\mathcal{C} \times \mathbf{M}_1 \times_2 \cdots \times_K \mathbf{M}_K),$$

where $\mathcal{Y} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$, $\mathcal{C} = \llbracket c_{r_1, \dots, r_K} \rrbracket \in \mathbb{R}^{R_1 \times \cdots \times R_K}$, $\mathbf{M}_k \in \{0, 1\}^{d_k \times r_k}$ for all $k \in [K]$ are membership matrices, and $f(\cdot)$ is the link function. Define the misclassification rate on the k -th mode as

$$MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) = \max_{r \in [R_k], a \neq a' \in [R_k]} \min\{D_{ar}^{(k)}, D_{a'r}^{(k)}\}$$

where $D^{(k)} \in \mathbb{R}^{R_k \times R_k}$ is the confusion matrix on the k -th, and $D_{rr'}^{(k)} = \frac{1}{d_k} \sum_{i=1}^{d_k} \mathbf{I}\{\mathbf{M}_{k,ir_k} = \hat{\mathbf{M}}_{k,ir_k} = 1\}$. Define the minimal gap between blocks as $\delta = \min_k \delta^{(k)}$, where

$$\delta^{(k)} = \min_{r_k \neq r'_k} \max_{r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_K} (f(c_{r_1, \dots, r_k, \dots, r_K}) - f(c_{r_1, \dots, r'_k, \dots, r_K}))^2.$$

Theorem 1.1. *Let $\{\mathcal{C}, \mathbf{M}_k\}$ denote the true parameters, and $\Theta = \llbracket \Theta_{i_1, \dots, i_K} \rrbracket = \mathcal{C} \times_1 \mathbf{M}_1 \times_2 \cdots \times_K \mathbf{M}_K$. Suppose $0 < a_1 < \text{Var}(\mathcal{Y}_{i_1, \dots, i_K} | \Theta_{i_1, \dots, i_K}) < a_2 < \infty$. Let σ denote the sub-Gaussian parameter of \mathcal{Y} . For any $\epsilon \in [0, 1]$, the MLE estimator $\{\hat{\mathbf{M}}_k\}$ satisfies the following bound*

$$\mathbb{P}(MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon) \leq 2^{1+\sum_k d_k} \exp\left(-\frac{C\epsilon^2 \tau^{3K-2} \delta^2 \prod_k d_k}{\sigma^2 a_2^2 \|\mathcal{C}\|_{\max}^2}\right),$$

where $\tau > 0$ is the lower bound the cluster proportion.

Proof. Recall the objective function in our model is

$$\mathcal{L}_{\mathcal{Y}}(\mathcal{C}, \{\mathbf{M}_k\}) = \langle \mathcal{Y}, \Theta \rangle + \sum_{i_1, \dots, i_K} b(\Theta_{i_1, \dots, i_K}), \quad (1)$$

where $\Theta = \mathcal{C} \times_1 \mathbf{M}_1 \times_2 \cdots \times_K \mathbf{M}_K$, and $b'(\cdot) = f(\cdot)$. The deviation between the MLE $\{\hat{\mathcal{C}}, \hat{\mathbf{M}}_k\}$ and the true parameters $\{\mathcal{C}, \mathbf{M}_k\}$ comes from two aspects: the label assignment and the estimation of the core tensor. We tease apart these two parts.

1. First, we suppose the membership $\{\mathbf{M}_k\}$ are given. We now assess the stochastic error due to the estimation of \mathcal{C} , conditional on $\{\mathbf{M}_k\}$. Noted that the objective function is a convex function, the MLE of \mathcal{C} satisfies the first-order condition. Then, for each $(r_1, \dots, r_K), r_k \in [R_k], k = 1, \dots, K$ we have

$$\hat{c}_{r_1, \dots, r_K} = (b')^{-1}\left(\frac{1}{d_1 \cdots d_K p_{r_1}^{(1)} \cdots p_{r_K}^{(K)}} [\mathcal{Y} \times_1 \mathbf{M}_1^T \times_2 \cdots \times_K \mathbf{M}_K^T]_{r_1, \dots, r_K}\right), \quad (2)$$

where $p_{r_k}^{(k)} = \frac{1}{d_k} \sum_{i=1}^{d_k} \mathbf{I}\{M_{k,ir_k} = 1\}$ is the portion of the r_k -th cluster on the k -th mode.

Consider the function $F(\mathbf{M}_k) = \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{C}}, \{\mathbf{M}_k\})$, where $\hat{\mathcal{C}} = \llbracket \hat{c}_{r_1, \dots, r_K} \rrbracket$ is the estimation (2). The function $F(\mathbf{M}_k)$ is of form

$$F(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} [b'(\hat{c}_{r_1, \dots, r_K}) \hat{c}_{r_1, \dots, r_K} - b(\hat{c}_{r_1, \dots, r_K})] = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} h(b'(\hat{c}_{r_1, \dots, r_K})),$$

where $h(x) = x(b')^{-1}(x) - b((b')^{-1}(x))$. Let $G(\mathbf{M}_k) = \mathbb{E}[F(\mathbf{M}_k)]$ denote the expectation of $F(\mathbf{M}_k)$ with respect to $\hat{\mathcal{C}}$. We have that

$$G(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} h(\mu_{r_1, \dots, r_K}),$$

where

$$\mu_{r_1, \dots, r_K} = \mathbb{E}[b'(\hat{c}_{r_1, \dots, r_K})] = \frac{1}{\prod_k p_{r_k}^{(k)}} [b'(\mathcal{C}) \times_1 \mathbf{D}^{(1),T} \times_2 \dots \times_K \mathbf{D}^{(K),T}]_{r_1, \dots, r_K}. \quad (3)$$

Therefore, the deviation $F(\mathbf{M}_k) - G(\mathbf{M}_k)$ quantifies the stochastic error due to the estimation of \mathcal{C} . Further, we define the residual tensor for block means, $\mathcal{R}(\mathbf{M}_k) = \llbracket R_{r_1, \dots, r_K} \rrbracket$, where

$$R_{r_1, \dots, r_K} = b'(\hat{c}_{r_1, \dots, r_K}) - \mathbb{E}[b'(\hat{c}_{r_1, \dots, r_K})].$$

2. Next, we free $\{\mathbf{M}_k\}$ and quantify the total deviation. Considering the MLE $\{\hat{\mathbf{M}}_k\}$, we have

$$(\hat{\mathbf{M}}_1, \dots, \hat{\mathbf{M}}_K) = \arg \max_{\{\mathbf{M}_k\}} F(\mathbf{M}_k).$$

The expectation with respect to \mathcal{C} of the objective function at the true parameter is

$$G(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} h(b'(c_{r_1, \dots, r_K})).$$

Correspondingly, the expected objective function at the MLE is

$$G(\hat{\mathbf{M}}_k) = \sum_{r_1, \dots, r_K} \prod_k \hat{p}_{r_k}^{(k)} h(\mu_{r_1, \dots, r_K}),$$

where μ_{r_1, \dots, r_K} is defined in (3) and $\hat{p}_{r_k}^{(k)}$ are obtained by $\hat{\mathbf{M}}_k$. We use $G(\hat{\mathbf{M}}_k) - G(\mathbf{M}_k)$ to measure the stochastic deviation caused by mismatch in label assignment.

The Lemma 1 indicates that, if there is non-negligible mismatch between \mathbf{M}_k and $\hat{\mathbf{M}}_k$, the estimate $\hat{\mathbf{M}}_k$ can not be the global optimizer to the objective function (1).

Back to probability of misclassification rate. By Lemma 1, we have

$$\mathbb{P}(MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon) \leq \mathbb{P}\left(G(\hat{\mathbf{M}}_k) - G(\mathbf{M}_k) \leq -\frac{\epsilon}{4a_2} \tau^{K-1} \delta'\right). \quad (4)$$

Notice that total deviation between $\{\mathbf{M}_k\}$ and $\hat{\mathbf{M}}_k$ is decomposed in three parts.

$$\begin{aligned} F(\hat{\mathbf{M}}_k) - F(\mathbf{M}_k) &= F(\hat{\mathbf{M}}_k) - G(\hat{\mathbf{M}}_k) + G(\hat{\mathbf{M}}_k) - G(\mathbf{M}_k) + G(\mathbf{M}_k) - F(\mathbf{M}_k) \\ &\leq 2r - \frac{\epsilon}{4a_2} \tau^{K-1} \delta, \end{aligned} \quad (5)$$

where the last inequality follows the triangle inequality, and $r = \sup_{\{\mathbf{M}_k\}} |F(\mathbf{M}_k) - G(\mathbf{M}_k)|$. Since $\{\hat{\mathbf{M}}_k\}$ is MLE, the left hand side of the inequality (5) is larger or equal than 0. Plugging the decomposition (5) in to the probability (4), we obtain that

$$\begin{aligned} \mathbb{P}(MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon) &\leq \mathbb{P}\left(F(\hat{\mathbf{M}}_k) - F(\mathbf{M}_k) \leq 2r - \frac{\epsilon}{4\alpha W} \tau^{K-1} \delta\right) \\ &\leq \mathbb{P}\left(r \geq \frac{\epsilon}{8\alpha} \tau^{K-1} \delta\right) \end{aligned} \quad (6)$$

Now, the problem transfers to a find a probability of r . Consider the term r , we have

$$\begin{aligned} |F(\mathbf{M}_k) - G(\mathbf{M}_k)| &\leq \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} |h(b'(\hat{c}_{r_1, \dots, r_K})) - h(\mu_{r_1, \dots, r_K})| \\ &\leq \|\mathcal{C}\|_{\max} \|\mathcal{R}(\mathbf{M}_k)\|_{\max}, \end{aligned} \quad (7)$$

where the last inequality follows by the Taylor Expansion

$$|h(b'(\hat{c}_{r_1, \dots, r_K})) - h(\mu_{r_1, \dots, r_K})| \leq \sup_{x=b'(c_{r_1, \dots, r_K})} |h'(x)| \|\mathcal{R}(\mathbf{M}_k)\|_{\max}$$

, and $\sup_{x=b'(c_{r_1, \dots, r_K})} |h'(x)| = \sup_{x=b'(c_{r_1, \dots, r_K})} |(b')^{-1}(x)| = \sup_{c_{r_1, \dots, r_K}} |c_{r_1, \dots, r_K}| \leq \|\mathcal{C}\|_{\max}$.

Combining the probability (6) with the upper bound (7), we obtain the accuracy of MCR

$$\begin{aligned} \mathbb{P}(MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon) &\leq \mathbb{P}\left(\sup_{\{\mathbf{M}_k\}} \|\mathcal{R}\|_{\max} \geq \frac{\epsilon}{8\alpha \|\mathcal{C}\|_{\max}} \tau^{K-1} \delta\right) \\ &\leq \mathbb{P}\left(\sup_{I_{r_1, \dots, r_K}} \frac{\sum_{(i_1, \dots, i_K) \in I_{r_1, \dots, r_K}} \mathcal{Y}_{i_1, \dots, i_K} - \mathbb{E}[\mathcal{Y}_{i_1, \dots, i_K}]}{|I_{r_1, \dots, r_K}|} \geq \frac{\epsilon}{8\alpha_2 \|\mathcal{C}\|_{\max}} \tau^{K-1} \delta\right) \\ &\leq 2^{1+\sum d_k} \exp\left(-\frac{\epsilon^2 \tau^{2K-2} \delta^2 L}{C \sigma^2 \alpha^2 \|\mathcal{C}\|_{\max}^2}\right), \end{aligned}$$

where $I_{r_1, \dots, r_K} = \{(i_1, \dots, i_K) | \mathbf{M}_{k, i_k r_k} = 1, k \in [K]\}$ is the collection of the indices of the elements belong to the cluster (r_1, \dots, r_K) , the last inequality follows by the Hoeffding's inequality, and $L = \min |I_{r_1, \dots, r_K}| \geq \tau^K \prod_k d_k$. \square

Lemma 1. For an fixed $\epsilon > 0$, suppose $MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon$ for some $k \in [K]$. We have

$$G(\hat{\mathbf{M}}_k) - G(\mathbf{M}_k) \leq -\frac{\epsilon}{4a_2} \tau^{K-1} \delta.$$

Proof. We provide the proof for $k = 1$. The proof for other $k \in [K]$ is similar. Since $MCR(\hat{\mathbf{M}}_1, \mathbf{M}_1) \geq \epsilon$, there exist some $r_1 \in [R_1]$ and $a_1 \neq a'_1$ such that $\min\{D_{a_1, r_1}^{(1)}, D_{a'_1, r_1}^{(1)}\} \geq \epsilon$. Let $\mathcal{N} = \llbracket h(b'(c_{r_1, \dots, r_K})) \rrbracket$ and $W = \prod_k \hat{p}_{r_k}^{(k)}$. Then, there exists c^* such that

$$\begin{aligned} &[\mathcal{N} \times_1 \mathbf{D}^{(1), T} \times_2 \dots \times_K \mathbf{D}^{(K), T}]_{r_1, \dots, r_K} \\ &= D_{a_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} h(b'(c_{a_1, \dots, a_K})) + D_{a'_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} h(b'(c_{a'_1, \dots, a_K})) \\ &+ (W - D_{a_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} - D_{a'_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)}) c^*. \end{aligned}$$

Recall the definition of μ_{r_1, \dots, r_K} in (3). Then, by Taylor Expansion of function $h(\cdot)$ at the point μ_{r_1, \dots, r_K} , we have

$$\begin{aligned} & \frac{1}{W} [\mathcal{N} \times_1 \mathbf{D}^{(1),T} \times_2 \dots \times_K \mathbf{D}^{(K),T}]_{r_1, \dots, r_K} - h(\mu_{r_1, \dots, r_K}) \\ & \geq \frac{1}{2W} D_{a_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} h''(\mu_{r_1, \dots, r_K}) (b'(c_{a_1, \dots, a_K}) - \mu_{r_1, \dots, r_K})^2 \\ & + \frac{1}{2W} D_{a_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} h''(\mu_{r_1, \dots, r_K}) (b'(c_{a'_1, \dots, a_K}) - \mu_{r_1, \dots, r_K})^2 \\ & + \frac{1}{2W} (W - D_{a_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} - D_{a'_1, r_1}^{(1)} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)}) h''(\mu_{r_1, \dots, r_K}) (c^* - \mu_{r_1, \dots, r_K})^2, \end{aligned}$$

where $h''(x) = \frac{1}{b''(b'^{-1}(x))}$, and $\inf_{x=b'(c_{r_1, \dots, r_K})} h''(x) = \inf_{c_{r_1, \dots, r_K}} \frac{1}{b''(c_{r_1, \dots, r_K})} \geq \frac{1}{\text{Var}(Y_{i_1, \dots, i_K})} \geq \frac{1}{a_2}$. By the inequality $a^2 + b^2 \geq \frac{(a+b)^2}{2}$, we obtain that

$$\begin{aligned} & \frac{1}{W} [\mathcal{N} \times_1 \mathbf{D}^{(1),T} \times_2 \dots \times_K \mathbf{D}^{(K),T}]_{r_1, \dots, r_K} - h(\mu_{r_1, \dots, r_K}) \\ & \geq \frac{1}{a_2 4W} \min\{D_{a_1, r_1}^{(1)}, D_{a'_1, r_1}^{(1)}\} D_{a_2, r_2}^{(2)} \dots D_{a_K, r_K}^{(K)} (b'(c_{a_1, \dots, a_K}) - b'(c_{a'_1, \dots, a_K}))^2. \end{aligned} \quad (8)$$

Noted $h(\cdot)$ is a convex function, for other $r'_1 \in [R_1]/\{r_1\}$, by Jensen's inequality, we have

$$\frac{1}{W} [\mathcal{N} \times_1 \mathbf{D}^{(1),T} \times_2 \dots \times_K \mathbf{D}^{(K),T}]_{r'_1, \dots, r_K} - h(\mu_{r'_1, \dots, r_K}) \geq 0. \quad (9)$$

Combing the inequality (8) and (9), we obtain that

$$G(\hat{\mathbf{M}}_k) - G(\mathbf{M}_k) \leq -\frac{\epsilon}{4\alpha} \tau^{K-1} \delta,$$

where the inequality follows by the fact that $\sum_{r_k} D_{a_k r_k}^{(k)} = p_{a_k}^{(k)} \geq \tau$. □

2 Discussion

Following is the discussion about the definition of $G(\mathbf{M}_k) = \mathbb{E}[F(\mathbf{M}_k)]$, which is the expectation of $F(\mathbf{M}_k)$ with respect to $\hat{\mathcal{C}} = [\hat{c}_{r_1, \dots, r_K}]$.

2.1 Least Squared model

In the least squared model, with given membership $\{\mathbf{M}_k\}$, the estimation of the core tensor is

$$\hat{c}_{r_1, \dots, r_K} = \frac{1}{d_1 \dots d_K p_{r_1}^{(1)} \dots p_{r_K}^{(K)}} [\mathcal{Y} \times_1 \mathbf{M}_1 \times_2 \dots \times_K \mathbf{M}_K]_{r_1, \dots, r_K}.$$

We define the function $F(\mathbf{M}_k) = \mathcal{L}_{\mathcal{Y}}(\mathcal{C}, \{\mathbf{M}_k\})$. A straightforward calculation shows that

$$F(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} \hat{c}_{r_1, \dots, r_K}^2.$$

Let $G(\mathbf{M}_k) = \mathbb{E}[F(\mathbf{M}_k)]$ denote the expectation of $F(\mathbf{M}_k)$ with respect to $\hat{\mathcal{C}}$. We have

$$G(\mathbf{M}_k) = \mathbb{E}[F(\mathbf{M}_k)] = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} \mathbb{E}(\hat{c}_{r_1, \dots, r_K}^2). \quad (10)$$

Notice that $\mathbb{E}[\hat{c}_{r_1, \dots, r_K}^2] = \text{Var}(\hat{c}_{r_1, \dots, r_K}) + (\mathbb{E}[\hat{c}_{r_1, \dots, r_K}])^2$, and $(\mathbb{E}[\hat{c}_{r_1, \dots, r_K}])^2 = \mu_{r_1, \dots, r_K}^2$ where μ_{r_1, \dots, r_K} is defined in (3). Since for each entry $\text{Var}(\mathcal{Y}_{i_1, \dots, i_K}) = \text{Var}(\epsilon_{i_1, \dots, i_K}) = \sigma_0^2$, and $\epsilon_{i_1, \dots, i_K}$ are i.i.d., the variance is equal to

$$\text{Var}(\hat{c}_{r_1, \dots, r_K}) = \frac{1}{\prod_k d_k^2 \prod_k [p_{r_k}^{(k)}]^2} \prod_k d_k \prod_k p_{r_k}^{(k)} \sigma_0^2 = \frac{1}{\prod_k d_k \prod_k p_{r_k}^{(k)}} \sigma_0^2. \quad (11)$$

Plugging the variance (11) into the definition (10), we have

$$G(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \frac{1}{\prod_k d_k} \sigma_0^2 + \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} \mu_{r_1, \dots, r_K}^2.$$

Since the first term is independent with the membership $\{\mathbf{M}_k\}$, we can ignore the first term, and the conclusion won't change.

2.2 Exponential Family model

In the exponential family model, with given membership $\{\mathbf{M}_k\}$, the estimation of the core tensor is

$$\hat{c}_{r_1, \dots, r_K} = (b')^{-1} \frac{1}{d_1 \dots d_K p_{r_1}^{(1)} \dots p_{r_K}^{(K)}} [\mathcal{Y} \times_1 \mathbf{M}_1 \times_2 \dots \times_K \mathbf{M}_K]_{r_1, \dots, r_K}.$$

Then, the corresponding function $F(\mathbf{M}_k)$ is of form

$$F(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} h(b'(\hat{c}_{r_1, \dots, r_K})),$$

where $h(x) = x(b')^{-1}(x) - b((b')^{-1}(x))$. The expectation $G(\mathbf{M}_k)$ is

$$G(\mathbf{M}_k) = \sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} \mathbb{E}[h(b'(\hat{c}_{r_1, \dots, r_K}))].$$

Consider the Taylor Expansion of $h(\cdot)$ at the point $\mathbb{E}[b'(\hat{c}_{r_1, \dots, r_K})]$. For simplicity, let \hat{c} denote $\hat{c}_{r_1, \dots, r_K}$. We have

$$h(b'(\hat{c})) = h(\mathbb{E}[b'(\hat{c})]) + h'(\mathbb{E}[b'(\hat{c})])(b'(\hat{c}) - \mathbb{E}[b'(\hat{c})]) + \frac{h''(\alpha b'(\hat{c}) + (1 - \alpha)\mathbb{E}(b'(\hat{c})))}{2} (b'(\hat{c}) - \mathbb{E}[b'(\hat{c})])^2,$$

for some $\alpha \in [0, 1]$. Since the expectation of the first term $\mathbb{E}[h'(\mathbb{E}[b'(\hat{c})])(b'(\hat{c}) - \mathbb{E}[b'(\hat{c})])] = 0$, we only need to prove that $\mathbb{E}\left[\frac{h''(\alpha b'(\hat{c}) + (1 - \alpha)\mathbb{E}(b'(\hat{c})))}{2} (b'(\hat{c}) - \mathbb{E}[b'(\hat{c})])^2\right]$ is not related to $\{\mathbf{M}_k\}$.

Below is just my thoughts.

Note that $h''(x) = \frac{1}{b''(b'^{-1}(x))}$, and $\alpha b'(\hat{c}) + (1 - \alpha)\mathbb{E}(b'(\hat{c}))$ is a linear combination of all entries of $b'(\mathcal{C})$ and $\mathcal{Y} = b'(\mathcal{C}) \times_1 \mathbf{M}_1 \times_2 \dots \times_K \mathbf{M}_K + \mathcal{E}$, where $\mathcal{E} = \llbracket \epsilon_{i_1, \dots, i_K} \rrbracket$ is a sub-gaussian mean-zero noise tensor. Recall the assumption that $0 < a_1 < \text{Var}(\mathcal{Y}_{i_1, \dots, i_K} | c_{r_1, \dots, r_K}) = b''(c_{r_1, \dots, r_K}) < a_2 < \infty$. We have

$$\inf_{r_1, \dots, r_K} \frac{1}{\text{Var}(\mathcal{Y}_{i_1, \dots, i_K} | c_{r_1, \dots, r_K} + \epsilon_{i_1, \dots, i_K})} \leq h''(\alpha b'(\hat{c}) + (1 - \alpha)\mathbb{E}(b'(\hat{c}))) \leq \sup_{r_1, \dots, r_K} \frac{1}{\text{Var}(\mathcal{Y}_{i_1, \dots, i_K} | c_{r_1, \dots, r_K})},$$

which implies that

$$\frac{1}{4a_2} \leq h''(\alpha b'(\hat{c}) + (1 - \alpha)\mathbb{E}(b'(\hat{c}))) \leq \frac{1}{a_1}.$$

2.3 Weakest Condition under which the clustering accuracy holds?

The following are three necessary conditions:

1. Irreducibility. This implies the minimal gap between blocks $\delta = \min_k \delta^{(k)} > 0$, where

$$\delta^{(k)} = \min_{r_k \neq r'_k} \max_{r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_K} (f(c_{r_1, \dots, r_k, \dots, r_K}) - f(c_{r_1, \dots, r'_k, \dots, r_K}))^2.$$

Otherwise, two blocks with the same mean are not able to be distinguished.

2. Convex loss function. The loss function should be convex, otherwise the estimation of \mathcal{C} would not be unique with given membership $\{\mathbf{M}_k\}$, and thus the stochastic error for estimation will be hard to measure.
3. Bounded variance. This implies there exist two positive constants a_1, a_2 such that $0 < a_1 < \text{Var}(\mathcal{Y}_{i_1, \dots, i_K} | \Theta_{i_1, \dots, i_K}) < a_2 < \infty$.

The bounded variance condition may not be sufficient. Because in $G(\mathbf{M}_k)$, the rest parts other than $\sum_{r_1, \dots, r_K} \prod_k p_{r_k}^{(k)} h(\mu_{r_1, \dots, r_K})$ are also related to the membership $\{\mathbf{M}_k\}$.