Algorithm: Multiway spherical clustering for degree-corrected tensor block model

Sub-algorithm 1: Weighted higher-order initialization

Input: Observation $\mathcal{Y} \in \mathbb{R}^{p\times\cdots\times p}$, cluster number r, relaxation factor $\eta > 1$ in k-means clustering.

Compute factor matrix $\boldsymbol{U}_{\text{pre}} = \text{SVD}_r(\text{Mat}(\mathcal{Y}))$ and the $(K-1)$-mode projection $\mathcal{X}_{\text{pre}} = \mathcal{Y} \times_1 \boldsymbol{U}_{\text{pre}}\boldsymbol{U}_{\text{pre}}^T \times_2 \cdots \times_{K-1} \boldsymbol{U}_{\text{pre}}\boldsymbol{U}_{\text{pre}}^T$.

Compute factor matrix $\hat{\boldsymbol{U}} = \text{SVD}_r(\text{Mat}(\mathcal{X}_{\text{pre}}))$ and denoised tensor $\hat{\mathcal{X}} = \mathcal{Y} \times_1 \hat{\boldsymbol{U}}\hat{\boldsymbol{U}}^T \times_2 \cdots \times_K \hat{\boldsymbol{U}}\hat{\boldsymbol{U}}^T$.

Let $\hat{\boldsymbol{X}} = \text{Mat}(\hat{\mathcal{X}})$ and $S_0 = \{i \in [p] : \|\hat{\boldsymbol{X}}_{i:}\| = 0\}$. Set $\hat{z}(i)$ randomly in $[r]$ for $i \in S_0$.

For all $i \in S_0^c$, compute normalized rows $\hat{\boldsymbol{X}}_{i:}^s := \|\hat{\boldsymbol{X}}_{i:}\|^{-1}\hat{\boldsymbol{X}}_{i:}$.

Solve the clustering $\hat{z}\colon [p] \to [r]$ and centroids $(\hat{\boldsymbol{x}}_j)_{j\in[r_k]}$ using weighted k-means, such that

$$\sum_{i\in S_0^c} \|\hat{\boldsymbol{X}}_{i:}\|^2 \|\hat{\boldsymbol{X}}_{i:}^s - \hat{\boldsymbol{x}}_{\hat{z}(i)}\|^2 \leq \eta \min_{\bar{\boldsymbol{x}}_j, j\in[r], \bar{z}(i), i\in S_0^c} \sum_{i\in S^c} \|\hat{\boldsymbol{X}}_{i:}\|^2 \|\hat{\boldsymbol{X}}_{i:}^s - \bar{\boldsymbol{x}}_{\bar{z}(i)}\|^2.$$

Output: Initial clustering $z^{(0)} \leftarrow \hat{z}$.

Sub-algorithm 2: Angle-based iteration

Input: Observation $\mathcal{Y} \in \mathbb{R}^{p\times\cdots\times p}$, initialization $z^{(0)}\colon [p] \to [r]$ from Sub-algorithm 1, iteration number T.

for $t = 0$ to $T-1$ **do**

  Update the block tensor $\mathcal{S}^{(t)}$ via $\mathcal{S}^{(t)}(a_1, ..., a_K) = \text{Ave}\{\mathcal{Y}(i_1, \ldots, i_K) : z^{(t)}(i_k) = a_k, k \in [K]\}$.

  Calculate reduced tensor $\mathcal{Y}^{\text{d}} \in \mathbb{R}^{p\times r\times\cdots\times r}$ via

$$\mathcal{Y}^{\text{d}}(i, a_2, \ldots, a_K) = \text{Ave}\{\mathcal{Y}(i, i_2, \ldots, i_K) : z^{(t)}(i_k) = a_k, k \neq 1\}.$$

  Let $\boldsymbol{Y}^{\text{d}} = \text{Mat}(\mathcal{Y}^{\text{d}})$ and $J_0 = \{i \in [p] : \left\|\boldsymbol{Y}_{i:}^{\text{d}}\right\| = 0\}$. Set $z^{(t+1)}(i)$ randomly in $[r]$ for $i \in J_0$.

  Let $\boldsymbol{S}^{(t)} = \text{Mat}(\mathcal{S}^{(t)})$. For all $i \in J_0^c$ update the cluster assignment by

$$z(i)^{(t+1)} = \arg\max_{a\in[r]} \cos\left(\boldsymbol{Y}_{i:}^{\text{d}},\ \boldsymbol{S}_{a:}^{(t)}\right).$$

end for

Output: Estimated clustering $z^{(T)} \in [r]^p$.