Thought about STD

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May 17, 2021

1 Equivalent model of STD

Claim: the new noise tensor consists of i.i.d. (sub-)Gaussian entries if and only if all $\boldsymbol{X}_k, k \in [3]$ has orthogonal columns and the diagonal elements of $(\boldsymbol{X}_k^T \boldsymbol{X}_k)^{-1}$ keep the same for $k \in [3]$.

Proof. Let $\mathcal{E}' = \mathcal{E} \times_1 \boldsymbol{M}_1 \times_2 \boldsymbol{M}_2 \times_3 \boldsymbol{M}_3$, where $\mathcal{E}$ has independent sub-Gaussian-σ entries. By the Prop 1 in the supplementary of STD paper, $\mathcal{E}'$ is a sub-Gaussian tensor with a different parameter σ', where σ' depends on $\boldsymbol{M}_k$. Now, consider the case that $\mathcal{E}$ has i.i.d. Gaussian entries with variance 1. We only need to show that $\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i',j',k'}) = 0$ and $\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i,j,k}) = \sigma^2$, for arbitrary (i,j,k), $(i',j',k') \neq (i,j,k)$, and for a constant σ^2.

Note that for arbitrary (i,j,k) we have

$$\mathcal{E}'_{i,j,k} = \sum_{a \in [d_1], b \in [d_2], c \in [d_3]} \mathcal{E}_{abc} \boldsymbol{M}_{1,ia} \boldsymbol{M}_{2,jb} \boldsymbol{M}_{3,kc}.$$

Then, the covariance is equal to

$$\begin{aligned}
\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i',j',k'}) &= \sum_{a \in [d_1], b \in [d_2], c \in [d_3]} \boldsymbol{M}_{1,ia} \boldsymbol{M}_{2,jb} \boldsymbol{M}_{3,kc} \boldsymbol{M}_{1,i'a} \boldsymbol{M}_{2,j'b} \boldsymbol{M}_{3,k'c} \\
&= \sum_{b \in [d_2], c \in [d_3]} \boldsymbol{e}_{1,i}^T \boldsymbol{M}_1 \boldsymbol{M}_1^T \boldsymbol{e}_{1,i'} \boldsymbol{M}_{2,jb} \boldsymbol{M}_{3,kc} \boldsymbol{M}_{2,j'b} \boldsymbol{M}_{3,k'c} \\
&= [\boldsymbol{e}_{1,i}^T (\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1} \boldsymbol{e}_{1,i'}][\boldsymbol{e}_{2,j}^T (\boldsymbol{X}_2^T \boldsymbol{X}_2)^{-1} \boldsymbol{e}_{2,j'}][\boldsymbol{e}_{3,k}^T (\boldsymbol{X}_3^T \boldsymbol{X}_3)^{-1} \boldsymbol{e}_{3,k'}], \qquad (1)
\end{aligned}$$

where $\boldsymbol{e}_{k,i} \in \mathbb{R}^{p_k}$ whose i-th elements is 1 and the other entries remain 0, for $k \in [3], i \in [p_k]$. The equation (1) implies that the covariance equal to 0 for all pairs $(i',j',k') \neq (i,j,k)$ if and only if all $\boldsymbol{X}_k, k \in [3]$ should have orthogonal columns.

Next, note the variance of $\mathcal{E}_{i,j,k}$ is

$$\begin{aligned}
\text{cov}(\mathcal{E}'_{i,j,k}, \mathcal{E}'_{i,j,k}) &= [\boldsymbol{e}_{1,i}^T (\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1} \boldsymbol{e}_{1,i}][\boldsymbol{e}_{2,j}^T (\boldsymbol{X}_2^T \boldsymbol{X}_2)^{-1} \boldsymbol{e}_{2,j}][\boldsymbol{e}_{3,k}^T (\boldsymbol{X}_3^T \boldsymbol{X}_3)^{-1} \boldsymbol{e}_{3,k}] \\
&= (\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1}_{ii} (\boldsymbol{X}_2^T \boldsymbol{X}_2)^{-1}_{jj} (\boldsymbol{X}_3^T \boldsymbol{X}_3)^{-1}_{kk}. \qquad (2)
\end{aligned}$$

The equation (2) implies that all entries of $\mathcal{E}'$ have equal variance if and only if the diagonal elements of $(\boldsymbol{X}_k^T \boldsymbol{X}_k)^{-1}$ should be the same for $k \in [3]$.

□

Alternative Way

Sub-Claim: Suppose $\mathcal{E} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ has i.i.d. Gaussian entries, the product $\mathcal{E}_1 = \mathcal{E} \times_1 \boldsymbol{M}_1$ has i.i.d. entries if and only if $\boldsymbol{X}_1 \in \mathbb{R}^{d_1 \times p_1}$ has orthogonal columns, where $\boldsymbol{M}_1 = (\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1} \boldsymbol{X}_1^T \in \mathbb{R}^{p_1 \times d_1}$.

Proof of Sub-Claim: It is equivalent to show the vectorized tensor $\text{vec}(\mathcal{E}_1) = (\boldsymbol{M}_1 \otimes \boldsymbol{I}_{d_2 d_3})\text{vec}(\mathcal{E})$ has a diagonal covariance matrix, where $\otimes$ is the matrix Kronecker product and $\boldsymbol{I}_{d_2 d_3}$ is the identity matrix with dimension $d_2 d_3$. Note that $\mathcal{E}$ has i.i.d. Gaussian entries. Then, the covariance matrix

$$\text{cov}(\text{vec}(\mathcal{E}_1)) = \boldsymbol{M}_1 \boldsymbol{M}_1^T \otimes \boldsymbol{I}_{d_2 d_3} = (\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1} \otimes \boldsymbol{I}_{d_2 d_3}$$

is diagonal if and only if $(\boldsymbol{X}_1^T \boldsymbol{X}_1)^{-1}$ is diagonal, i.e., the columns of $\boldsymbol{X}_1$ are orthogonal. $\square$

Proof of Claim: ($\Rightarrow$) Suppose all the matrix $\boldsymbol{X}_k, k \in [3]$ have orthogonal columns. Then by the Sub-Claim, the tensor $\mathcal{E}_1 = \mathcal{E} \times_1 \boldsymbol{M}_1$ has i.i.d. Gaussian entries. Similarly, the tensor $\mathcal{E}_2 = \mathcal{E}_1 \times_2 \boldsymbol{M}_2$ and $\mathcal{E}' = \mathcal{E}_2 \times_3 \boldsymbol{M}_3$ have i.i.d. Gaussian entries.

($\Leftarrow$) Suppose the tensor $\mathcal{E}' = \mathcal{E} \times \boldsymbol{M}_1 \times_2 \boldsymbol{M}_2 \times_3 \boldsymbol{M}_3$ has i.i.d. Gaussian entries. Let $\boldsymbol{N}_k = (\boldsymbol{M}_k^T \boldsymbol{M}_k)^{-1} \boldsymbol{M}_k^T$ for $k \in [3]$. Note that

$$\mathcal{E}' \times \boldsymbol{N}_1 \times_2 \boldsymbol{N}_2 \times_3 \boldsymbol{N}_3 = \mathcal{E}.$$

$\square$