

Linear Algebra

A summary for MIT 18.06SC

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1 Matrices & Spaces

1.1 Basic concepts

- Given vectors v_1, \dots, v_n and scalars c_1, \dots, c_n , the sum $c_1v_1 + \dots + c_nv_n$ is called a *linear combination* of v_1, \dots, v_n .
 $c_1=c_2=\dots=c_n=0$
- The vectors v_1, \dots, v_n are *linearly independent* (or just *independent*) if $c_1v_1 + \dots + c_nv_n = 0$ holds only when all $c_1, \dots, c_n = 0$. If the vectors v_1, \dots, v_n are *dependent*, there exist scalars c_1, \dots, c_n which are not all equal to 0 such that $c_1v_1 + \dots + c_nv_n = 0$.
- Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$, the multiplication $\mathbf{A}x$ is a linear combination of the columns of \mathbf{A} , and $x^T \mathbf{A}$ is a linear combination of the rows of \mathbf{A} .
- Matrix multiplication is typically not communicative, i.e. $\mathbf{AB} \neq \mathbf{BA}$. Lemma 1 describes a special case where matrix multiplication is communicative.
- Suppose \mathbf{A} is a square matrix. The matrix \mathbf{A} is *invertible* or *non-singular* if there exists an \mathbf{A}^{-1} such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$. Otherwise, the matrix \mathbf{A} is *singular*, and the determinant of \mathbf{A} is 0.
- The inverse of a matrix product \mathbf{AB} is $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. The product of invertible matrices is still invertible.
- The transpose of a matrix product \mathbf{AB} is $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. For any invertible matrix \mathbf{A} , $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
- A matrix \mathbf{Q} is orthogonal if $\mathbf{Q}^T = \mathbf{Q}^{-1}$. A matrix \mathbf{Q} is unitary if $\mathbf{Q}^* = \mathbf{Q}^{-1}$, where \mathbf{Q}^* is the conjugate transpose of \mathbf{Q} .

Lemma 1 (Communicative matrix multiplication). *For matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times n}$, the matrix multiplication of \mathbf{A} and \mathbf{B} is communicative, i.e. $\mathbf{AB} = \mathbf{BA}$, if \mathbf{A} and \mathbf{B} have the same set of eigenvectors corresponding to their non-zero eigenvalues.*

Proof. If \mathbf{A} and \mathbf{B} have the same set of eigenvectors corresponding to their non-zero eigenvalues, there exists a matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{Q}\mathbf{D}_A\mathbf{Q}^{-1}, \quad \mathbf{B} = \mathbf{Q}\mathbf{D}_B\mathbf{Q}^{-1},$$

where columns of \mathbf{Q} are eigenvectors of \mathbf{A} and \mathbf{B} , and $\mathbf{D}_A \in \mathbb{R}^{n \times n}, \mathbf{D}_B \in \mathbb{R}^{n \times n}$ are diagonal matrices whose diagonal elements are eigenvalues of \mathbf{A} and \mathbf{B} , respectively. Because matrix multiplication is communicative for two diagonal matrices with same dimensions, we have

$$\mathbf{AB} = \mathbf{Q}\mathbf{D}_A\mathbf{Q}^{-1}\mathbf{Q}\mathbf{D}_B\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}_A\mathbf{D}_B\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}_B\mathbf{D}_A\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{D}_B\mathbf{Q}^{-1}\mathbf{Q}\mathbf{D}_A\mathbf{Q}^{-1} = \mathbf{BA}.$$

Therefore, the matrix multiplication of \mathbf{A} and \mathbf{B} is communicative. \square

1.2 Permutation of matrices

~~Let \mathbf{A} be a matrix. the rows of \mathbf{A}~~ extra whitespace?
~~For any matrix \mathbf{A} , we swap its rows by multiplying a permutation matrix \mathbf{P} on the left of \mathbf{A} . For example,~~

$$\mathbf{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix}$$

where a_k refers to the k -th row of \mathbf{A} . The inverse of the permutation matrix \mathbf{P} is $\mathbf{P}^{-1} = \mathbf{P}^T$, which implies the orthogonality of permutation matrix. For an $n \times m$ matrix, there are $n!$ different row permutation matrices, which form a *multiplicative group*.

Similarly, we also swap the columns of the matrix \mathbf{A} by multiplying a permutation matrix on the right of \mathbf{A} .

1.3 Elimination of matrices

Elimination is an important technique in linear algebra. We eliminate the matrix by multiplications and subtractions. Take a 3-by-3 matrix \mathbf{A} as an example.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 2}} \mathbf{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

In step 1, we choose the number 1 in row 1 column 1 as a *pivot*, then we recopy the first row, multiply an appropriate number (in this case, 3) and subtract those values from the numbers in the second row. We have thus eliminated 3 in row 2 column 1. Similarly, in step 2, we choose 2 in row 2 column 2 as a pivot and eliminate the number 4 in row 3 column 2. The number 5 in row 3 column 3 is also a pivot. The matrix \mathbf{U} is an upper triangular matrix.

The *elimination matrix* used to eliminate the entry in row m column n is denoted as \mathbf{E}_{mn} . In the previous example,

$$\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}; \quad \mathbf{E}_{32}(\mathbf{E}_{21}\mathbf{A}) = \mathbf{U}.$$

Pivots are non-zero. If there ~~is~~ a 0 in the pivot position, we ~~must~~ exchange the row with one below to get a non-zero value in pivot position. If ~~there is not~~ non-zero value below the 0 pivot, we skip this column and find a pivot in the next column. ~~all numbers below the pivot are zero~~

We write $\mathbf{E}_{32}(\mathbf{E}_{21}\mathbf{A}) = (\mathbf{E}_{32}\mathbf{E}_{21})\mathbf{A} = \mathbf{U}$, because matrix multiplication is associative. Let \mathbf{E} denote the product of all elimination matrices. If we need to permute the rows during the process, we multiply a permutation matrix on the left of \mathbf{A} . Therefore, the elimination process of \mathbf{A} is

$$\mathbf{E}\mathbf{PA} = \mathbf{U}, \tag{1}$$

where \mathbf{U} is an upper triangular matrix.

Next, we prove the invertibility of the elimination matrix.

Let \mathbf{E} in ... be an elimination matrix that ...

Lemma 2 (Invertibility of elimination matrix). *Suppose there is an elimination matrix $\mathbf{E}_{ij} \in \mathbb{R}^{n \times n}$ that multiplies a scalar $-c$ to the j -th row and subtracts the row from i -th row, where $i \neq j$. The matrix \mathbf{E}_{ij} is invertible.*

Proof. We write the elimination matrix as

$$\mathbf{E}_{ij} = \mathbf{I}_n + ce_i e_j^T,$$

where $e_i \in \mathbb{R}^n$ denotes the vector with value 1 in the i -th entry and value 0 elsewhere. Because $i \neq j$, $e_i^T e_j = 0$. We have

$$(\mathbf{I}_n + ce_i e_j^T)(\mathbf{I}_n - ce_i e_j^T) = \mathbf{I}_n - c^2 e_i e_j^T e_i e_j^T = \mathbf{I}_n; \quad (\mathbf{I}_n - ce_i e_j^T)(\mathbf{I}_n + ce_i e_j^T) = \mathbf{I}_n.$$

Therefore, $\mathbf{I}_n - ce_i e_j^T$ is the inverse of \mathbf{E}_{ij} . The elimination matrix \mathbf{E}_{ij} is invertible. \square

Corollary 1 (Inverse of elimination matrix). Suppose the elimination matrix \mathbf{E}_{ij} in lemma 2 is a lower/upper-triangular matrix. The inverse \mathbf{E}_{ij}^{-1} is also a lower/upper-triangular matrix.

Proof. By the proof of lemma 2, the matrix \mathbf{E}_{ij} and its inverse are written as

explain acronym in its first appearance $\mathbf{E}_{ij} = \mathbf{I}_n + ce_i e_j^T, \quad \mathbf{E}_{ij}^{-1} = \mathbf{I}_n - ce_i e_j^T.$

Without loss of generality (WLOG)

WLOG, we assume that \mathbf{E}_{ij} is a lower-triangular matrix. Then $ce_i e_j^T$ and $-ce_i e_j^T$ are also lower-triangular matrices. Therefore, \mathbf{E}_{ij}^{-1} is a lower-triangular matrix. \square

1.4 Gauss-Jordan Elimination

We also use elimination to find the inverse of an invertible matrix.

of \mathbf{A} , denoted \mathbf{A}^{-1} ,

Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ is an invertible matrix. The inverse of \mathbf{A} , \mathbf{A}^{-1} , satisfies

$$\mathbf{AA}^{-1} = \mathbf{I}_n. \quad \text{exists an elimination matrix sides} \quad (2)$$

Suppose there is an elimination \mathbf{E} such that $\mathbf{EA} = \mathbf{I}_n$. Multiplying \mathbf{E} on the both side of the equation (2), we have $\mathbf{EAA}^{-1} = \mathbf{A}^{-1} = \mathbf{E}$. To obtain a such \mathbf{E} , we eliminate the augmented matrix $[\mathbf{A}|\mathbf{I}_n]$ until \mathbf{A} becomes \mathbf{I}_n . Then, the augmented matrix becomes $\mathbf{E}[\mathbf{A}|\mathbf{I}_n] = [\mathbf{I}_n|\mathbf{E}]$, where \mathbf{E} is the inverse of \mathbf{A} .

We call this elimination process of finding \mathbf{E} as *Gauss-Jordan Elimination*.

1.5 Factorization of matrices

any square matrix \mathbf{A} has the factorization (1)

By elimination, for any square matrix \mathbf{A} , we have equation (1). By lemma 2, \mathbf{E} is invertible. We multiply \mathbf{E}^{-1} on both sides of equation (1). We have, yields

Multiplying

$$\mathbf{PA} = \mathbf{E}^{-1}\mathbf{U}.$$

Note that \mathbf{E} is a lower-triangular matrix. By corollary 1, \mathbf{E}^{-1} is also a lower-triangular matrix. Let \mathbf{L} denote \mathbf{E}^{-1} , where the letter L refers to “lower triangular”. Therefore, any square matrix \mathbf{A} has a factorization:

$$\mathbf{PA} = \mathbf{LU}, \quad (3)$$

where \mathbf{U} is an upper triangular matrix with pivots on the diagonal, \mathbf{L} is a lower triangular matrix with ones on the diagonal, and \mathbf{P} is a permutation matrix. However, the equation (3) is not the unique factorization of \mathbf{A} . For example, $c\mathbf{L}$ and $c^{-1}\mathbf{U}$ also factorize \mathbf{A} , where c is a non-zero scalar.

1.6 Time complexity of elimination

~~a single elimination → a single elimination step~~

For an n -by- n matrix, a single elimination multiplies a selected row and subtracts the selected row from another row. ~~A single elimination~~ requires $\mathcal{O}(n)$ operations. To eliminate the elements below the first diagonal element, we need to repeat single elimination $(n-1)$ times and thus require $\mathcal{O}(n^2)$ operations. Similarly, we require $\mathcal{O}((n-1)^2)$ operations to eliminate the elements below the second diagonal element. Repeat the elimination until we meet the n -th diagonal element. Therefore, we require total $\mathcal{O}(n^3)$ operations to obtain an upper-triangular matrix by elimination:

in total

$$1^2 + 2^2 + \cdots + (n)^2 = \sum_i^n i^2 \approx \int_0^n x^2 dx = \frac{1}{3}n^3 = \mathcal{O}(n^3).$$

1.7 Reduced row echelon form of matrices

In previous sections, we convert a matrix \mathbf{A} to an upper triangular matrix \mathbf{U} . Next, we convert \mathbf{U} into the *reduced row echelon form* (RREF), which is a simpler form than upper triangle. We use $\mathbf{R} = RREF(\mathbf{A})$ to denote the reduced row echelon form of \mathbf{A} . In \mathbf{R} , the pivots are equal to 1, and the elements above and below the pivots are eliminated to 0. In the previous example,

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{\text{make pivots = 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is another example,

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{make pivots = 1}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \mathbf{R} = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that there are r pivots in $\mathbf{A} \in \mathbb{R}^{m \times n}$. With proper permutation, the matrix \mathbf{R} is in form $\begin{bmatrix} \mathbf{I}_r & \mathbf{F} \\ 0 & 0 \end{bmatrix}$, where $\mathbf{F} \in \mathbb{R}^{r \times (n-r)}$ is an arbitrary matrix. The columns in \mathbf{A} which correspond to the identity matrix \mathbf{I}_r are called *pivot columns*. The other columns are *free columns*.

“,” and

1.8 Vector space, Subspace and Column space

~~closed~~

- *Vector space* is a collection of vectors that is ~~close~~ under linear combination (addition and multiplication by any real number); i.e. for any vectors in the collection, all the combinations of these vectors are still in the collection.
- *Subspaces of the vector space* is a vector space that is contained inside of another vector space.

Note that any vector space or subspace must include an origin. For a vector space \mathcal{A} , the subspace of \mathcal{A} can be \mathcal{A} itself or a set that contains only a zero vector.

~~consists of linear combinations~~

- Vectors v_1, \dots, v_n *span* a space that ~~consists~~ all the ~~combination~~ of these vectors.
- *Column space* of a matrix \mathbf{A} is the space spanned by the columns of \mathbf{A} . Let $C(\mathbf{A})$ denote the column space of \mathbf{A} .

~~contains~~

If v_1, \dots, v_n span a space \mathcal{S} , then \mathcal{S} is the smallest space that ~~contain~~ these vectors.

- *Basis* of a vector space is ~~a sequence~~^{a set} of vectors v_1, \dots, v_n that satisfy: (1) v_1, \dots, v_n are independent; (2) v_1, \dots, v_n span the space.
- *Dimension* of a space is the number of vectors in a basis of this space. Let $\dim(\mathcal{A})$ denote the dimension of space \mathcal{A} .

1.9 Matrix rank

Let $\text{rank}(\mathbf{A})$ denote the *rank* of matrix \mathbf{A} . We have

$$\text{rank}(\mathbf{A}) \stackrel{\Delta}{=} \dim(C(\mathbf{A})) = \# \text{ of pivot columns of } \mathbf{A}. \quad (4)$$

If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\mathbf{A}) = r$, we have $r \leq \min\{m, n\}$. The matrix is *full rank* if $r = \min\{m, n\}$.

The rank of a square matrix is closely related to the invertibility.

Lemma 3 (Full rankness and invertibility). *A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is full rank, if and only if \mathbf{A} is an invertible matrix.*

Proof. (\Rightarrow) Suppose \mathbf{A} is full rank. (Good. The current structure is better than your previous one)

Let \mathbf{R} denote the RREF form of \mathbf{A} . There exist an elimination matrix \mathbf{E} and a permutation matrix \mathbf{P} such that

$$\mathbf{E}\mathbf{P}\mathbf{A} = \mathbf{R}.$$

By the full rankness of \mathbf{A} , \mathbf{A} has n pivot columns, and thus $\mathbf{R} = \mathbf{I}_n$. By lemma 2, \mathbf{E} is invertible. The permutation matrix \mathbf{P} is also invertible. Then, the matrix product \mathbf{EP} is invertible, and \mathbf{A} is the inverse of \mathbf{EP} . Therefore, \mathbf{A} is invertible.

(\Leftarrow) Suppose \mathbf{A} is invertible. We prove the full rankness of \mathbf{A} by contradiction.

Assume that $\text{rank}(\mathbf{A}) < n$. By equation (4), $\dim(C(\mathbf{A})) = \text{rank}(\mathbf{A}) < n$, which implies that the columns of \mathbf{A} are linearly dependent. Then, there exists a non-zero vector v such that

$$\mathbf{A}v = 0. \quad (5)$$

By assumption, the inverse \mathbf{A}^{-1} exists. Multiplying \mathbf{A}^{-1} on both sides of equation (5), we have

$$v = \mathbf{A}^{-1}0 = 0. \quad (6)$$

However, equation (6) contradicts the fact that v is a non-zero vector. Therefore, \mathbf{A} is full rank. \square

The rank of \mathbf{A} also affects the number of solutions to the system $\mathbf{Ax} = b$. We will discuss the relationship between matrix rank and the solutions in next section.

2 Solving $\mathbf{Ax} = b$

In this section, we discuss the solutions to the linear system $\mathbf{Ax} = b$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are vectors.

2.1 Solving $Ax = 0$: Nullspace

The *nullspace* of matrix \mathbf{A} is the collection of all solutions x to the system $\mathbf{Ax} = 0$. Let $N(\mathbf{A})$ denote the nullspace of \mathbf{A} .

Lemma 4 (Nullspace). *The nullspace of matrix \mathbf{A} is a vector space.*

Proof. We only need to prove that $N(\mathbf{A})$ is closed under linear combination. For $\forall v_1, v_2 \in N(\mathbf{A})$, we have,

$$\mathbf{A}(c_1v_1 + c_2v_2) = c_1\mathbf{Av}_1 + c_2\mathbf{Av}_2 = 0, \quad \forall c_1, c_2 \in \mathbb{R}. \quad (7)$$

The equation (7) implies that $N(\mathbf{A})$ is closed under linear combination. Therefore, $N(\mathbf{A})$ is a vector space. \square

Lemma 5 (The rank of nullspace). *If $\text{rank}(\mathbf{A}) = r$, the rank of nullspace $\text{rank}(N(\mathbf{A})) = n - r$.*

Proof. Let \mathbf{R} denote the RREF(\mathbf{A}). We write \mathbf{R} in form $\mathbf{R} = \begin{bmatrix} \mathbf{I}_r & \mathbf{F} \\ 0 & 0 \end{bmatrix}$, where $\mathbf{F} \in \mathbb{R}^{r \times (n-r)}$ is

an arbitrary matrix. Let $\mathbf{X} = \begin{bmatrix} -\mathbf{F} \\ \mathbf{I}_{n-r} \end{bmatrix}$. We have

Is ``a special solution'' a terminology?

If so, then say "is called a special solution".

If not, then fine.

$$\mathbf{RX} = \begin{bmatrix} \mathbf{I}_r & \mathbf{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\mathbf{F} \\ \mathbf{I}_{n-r} \end{bmatrix} = 0.$$

Therefore, each column of \mathbf{X} is a special solution to the system $\mathbf{Ax} = 0$. Next, we show that any vector $x \in N(\mathbf{A})$ is a linear combination of the columns of \mathbf{X} .

Suppose there is a solution $x = (x_1, x_2) \in N(\mathbf{A})$, where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$. We have

$$\mathbf{Rx} = \begin{bmatrix} \mathbf{I}_r & \mathbf{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \mathbf{Fx}_2 \\ 0 \end{bmatrix} = 0.$$

This implies that $x_1 = -\mathbf{Fx}_2$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{F} \\ \mathbf{I}_{n-r} \end{bmatrix} x_2 = \mathbf{X}x_2$. Any vector in $N(\mathbf{A})$ is a linear combination of the columns of \mathbf{X} , i.e. $C(\mathbf{X}) = N(\mathbf{A})$. Therefore, the rank of nullspace $\text{rank}(N(\mathbf{A})) = \dim(C(\mathbf{X})) = n - r$. type "i.e.\ C(X)=..." in latex. Note "\\" after ":" \square

Recall the definitions of pivot columns and free columns. In $\mathbf{Ax} = b$, the variables in x that correspond to pivot columns are called *pivot variables*, and others are *free variables*. If $\text{rank}(\mathbf{A}) = r$, there are $n - r$ free variables.

In the proof of lemma 5, the columns of $\mathbf{X} = \begin{bmatrix} -\mathbf{F} \\ \mathbf{I}_{n-r} \end{bmatrix}$ are special solutions that compose the basis of $N(\mathbf{A})$. Practically, we find these special solutions by assigning 1 to a free variable and 0 to other free variables, and then we solve the system $\mathbf{Ax} = 0$.

2.2 Solving $Ax = b$: complete solutions

Lemma 6 (Solvability of $\mathbf{Ax} = b$). *The system $\mathbf{Ax} = b$ is solvable only when $b \in C(\mathbf{A})$.*

Proof. If $\mathbf{Ax} = b$ is solvable, there exists a x such that $\mathbf{Ax} = b$. For any x , $\mathbf{Ax} \in C(\mathbf{A})$. Therefore, $b \in C(\mathbf{A})$. \square

Lemma 7 (Complete solution). *The complete solution of $\mathbf{Ax} = b$ is given by $x_{\text{comp}} = x_p + x_n$, where x_p is a particular solution such that $\mathbf{Ax}_p = b$, and $x_n \in N(\mathbf{A})$.*

Proof. Suppose $x = x_p + x_0$ is ~~an arbitrary~~^a solution to $\mathbf{A}x = b$. We have

$$\mathbf{A}x - \mathbf{A}x_p = \mathbf{A}(x - x_p) = \mathbf{A}x_0 = 0.$$

Therefore, $x_0 \in N(\mathbf{A})$. □

what is the difference between “a particular solution” vs. “a special solution”? If they are the same, then stick with one phrase

Usually, we find a particular solution by assigning 0 to free variables, and we solve the system

$$\mathbf{A}x = b.$$

The following table discusses the rank of \mathbf{A} , the form of \mathbf{R} , the dimension of nullspace $N(\mathbf{A})$, and the number of solutions to $\mathbf{A}x = b$.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
\mathbf{R}	\mathbf{I}	$\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}$	$[\mathbf{I} \quad \mathbf{F}]$	$\begin{bmatrix} \mathbf{I} & \mathbf{F} \\ 0 & 0 \end{bmatrix}$
$\dim(N(\mathbf{A}))$	0	0	$n - r$	$n - r$
# solutions to $\mathbf{A}X = b$	1	0 or 1	infinitely many	0 or infinitely many