Estimation of Monge Matrix

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I was thinking of two possible definitions to Monge Tensor which may reflect the property in order-3 cumulant tensor. These two definitions imply different orderings structure. Note that order-3 moment tensor is equal to the order-3 cumulant tensor. Whereas, order-4 or higher-order moment tensors are different with cumulant tensor.

Before the definition of Monge Tensor, we first define the population and sample order-3 cumulant tensor for mean 0 random vector $X = [\![X_1, ..., X_D]\!] \in \mathbb{R}^D$ with sample $X^l \in \mathbb{R}^d, l = 1, ..., N$ according to the Definition 5.4 in (De Lathauwer, 2010).

Definition 1 (Population and sample order-3 cumulant tensor)**.** Consider a real stochastic vector $X \in \mathbb{R}^D$. The population order-3 cumulant tensor is defined by the element-wise equation,

$$\mathcal{C}_{i,j,k} = \mathbb{E}[X_i X_j X_k], \quad \text{for all} \quad i, j, k \in [D].$$

The sample order-3 cumulant tensor is defined as

$$\mathcal{S}_{ijk} = \frac{1}{N} \sum_{l=1}^{N} X_{li} X_{lj} X_{lk}, \quad \text{for all} \quad i, j, k \in [D].$$

Note that the definition can be rewritten in a compact tensor form.

Definition 2 (Tensor form of order-3 cumulant tensor)**.** Consider a real stochastic vector $X \in \mathbb{R}^D$. The population order-3 cumulant tensor is defined as

$$\mathcal{C} = \mathbb{E}\left[X^{\otimes 3}\right],$$

and sample order-3 cumulant tensor is defined as

$$\mathcal{S} = \frac{1}{N} \mathcal{D} \times_1 \boldsymbol{X} \times_2 \boldsymbol{X} \times_3 \boldsymbol{X} = \frac{1}{N} [\![\boldsymbol{X}, \boldsymbol{X}, \boldsymbol{X}]\!],$$

where $\boldsymbol{X} = [X^1, ..., X^N] \in \mathbb{R}^{D \times N}$ collects the sample vectors, $\mathcal{D} \in \mathbb{R}^{N \times N \times N}$ is a super-diagonal tensor with diagonal value 1, and $[\![\cdot, \cdot, \cdot]\!]$ denotes the CP-style tensor product.

For simplicity, we represent the definition of anti-Monge tensor.

1 One-step definition

In matrix case, a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is an anti-Monge matrix if and only if $\boldsymbol{D}_1 \boldsymbol{A} \boldsymbol{D}_2^T \geq 0$, where

$$\boldsymbol{D}_1 = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(m-1)\times m} \tag{1}$$

, and $\boldsymbol{D}_2 \in \mathbb{R}^{(n-1)\times n}$ is defined analogously. By this idea, we define the one-step definition of anti-Monge Tensor.

Definition 3 (One-step definition of anti-Monge Tensor)**.** A tensor $\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times n_3}$ is an anti-Monge Tensor if and only if

$$\mathcal{X} \times_1 \boldsymbol{D}_1 \times_2 \boldsymbol{D}_2 \times_3 \boldsymbol{D}_3 \geq 0$$

, where $\boldsymbol{D}_k \in \mathbb{R}^{(n_k-1)\times n_k}, k = 1, 2, 3$ are defined as (1), and $\geq$ refers to element-wise inequality.

Application to tensor cumulant

Consider the data matrix $\boldsymbol{X} \in \mathbb{R}^{D\times N}$. Let $\tilde{X}_d \in \mathbb{R}^N$ denote the d-th row of $\boldsymbol{X}$. Then $\tilde{X}_d$ collects N realizations of feature d. Suppose that

$$\mathbf{1_n}^T \left[\left(\tilde{X}_{i+1} - \tilde{X}_i\right) * \left(\tilde{X}_{j+1} - \tilde{X}_j\right) * \left(\tilde{X}_{k+1} - \tilde{X}_k\right) \right] \geq 0, \quad \text{for all} \quad i, j, k \in [D-1], \tag{2}$$

where $*$ denotes the Hardamard (element-wise) product. With $\boldsymbol{D} \in \mathbb{R}^{(D-1)\times D}$, we have

$$\boldsymbol{D}\boldsymbol{X} = \begin{bmatrix} \tilde{X}_2 - \tilde{X}_1 \\ \vdots \\ \tilde{X}_D - \tilde{X}_{D-1} \end{bmatrix}.$$

Then, we have

$$\mathcal{S} \times_1 \boldsymbol{D} \times_2 \boldsymbol{D} \times_3 \boldsymbol{D} = \frac{1}{N} [\![\boldsymbol{X}, \boldsymbol{X}, \boldsymbol{X}]\!] \times_1 \boldsymbol{D} \times_2 \boldsymbol{D} \times_3 \boldsymbol{D} = \frac{1}{N} [\![\boldsymbol{D}\boldsymbol{X}, \boldsymbol{D}\boldsymbol{X}, \boldsymbol{D}\boldsymbol{X}]\!] \geq 0,$$

where the last inequality follows by the assumption (2). Hence, the sample order-3 cumulant tensor $\mathcal{S}$ is an anti-Monge tensor.

2 Three-step definition

Other definition requires the slices of the tensor from every mode are anti-Monge matrix.

Definition 4 (Three-step definition of anti-Monge Tensor)**.** A tensor $\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times n_3}$ is an anti-Monge Tensor if and only if $\mathcal{X}[i, ,], \mathcal{X}[, j,], \mathcal{X}[, , k]$ are anti-Monge matrix for all $i \in [n_1], j \in [n_2], k \in [n_3]$, i.e.,

$$\mathcal{X} \times_2 \boldsymbol{D}_2 \times_3 \boldsymbol{D}_3 \geq 0, \quad \mathcal{X} \times_1 \boldsymbol{D}_1 \times_3 \boldsymbol{D}_3 \geq 0, \quad \mathcal{X} \times_1 \boldsymbol{D}_1 \times_2 \boldsymbol{D}_2 \geq 0.$$

Application to tensor cumulant

Consider the data matrix $\boldsymbol{X} \in \mathbb{R}^{D\times N}$. Let $\tilde{X}_d \in \mathbb{R}^N$ denote the d-th row of $\boldsymbol{X}$. Then $\tilde{X}_d$ collects N realizations of feature d. Suppose that

$$\left[\left(\tilde{X}_{i+1} - \tilde{X}_i\right) * \left(\tilde{X}_{j+1} - \tilde{X}_j\right)\right]^T \tilde{X}_k \geq 0, \quad \text{for all} \quad i, j \in [D-1], k \in [D]. \tag{3}$$

Then, we have

$$\mathcal{S} \times_1 \boldsymbol{D} \times_2 \boldsymbol{D} = \frac{1}{N}[\![\boldsymbol{X}, \boldsymbol{X}, \boldsymbol{X}]\!] \times_1 \boldsymbol{D} \times_2 \boldsymbol{D} = \frac{1}{N}[\![\boldsymbol{D}\boldsymbol{X}, \boldsymbol{D}\boldsymbol{X}, \boldsymbol{X}]\!] \geq 0, \tag{4}$$

where the last inequality follows by the assumption (2). The inequality (4) also holds for $\mathcal{S} \times_1 \boldsymbol{D} \times_3 \boldsymbol{D}$ and $\mathcal{S} \times_2 \boldsymbol{D} \times_3 \boldsymbol{D}$ since the inequality holds after switching the index i, j, k. Therefore, the order-3 cumulant $\mathcal{S}$ is an anti-Monge tensor.

References

De Lathauwer, L. (2010). Algebraic methods after prewhitening. In *Handbook of Blind Source Separation*, pages 155–177. Elsevier.