

# Solution to “Chapter 2: Basic tail and concentration bounds”

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July 22, 2020

## 1 Summary

**Theorem 1.1** (Markov’s inequality). *Let  $X \geq 0$  be a random variable with finite mean. We have*

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}, \quad \text{for all } t > 0. \quad (1)$$

**Theorem 1.2** (Chebyshev’s inequality). *Let  $X \geq 0$  be a random variable with finite mean  $\mu$  and finite variance. We have*

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\text{var}(X)}{t^2}, \quad \text{for all } t > 0. \quad (2)$$

**Theorem 1.3** (Markov’s inequality for polynomial moments). *Let  $X$  be a random variable. Suppose that the order  $k$  central moment of  $X$  exists. Applying Markov’s inequality to the random variable  $|X - \mu|^k$  yields*

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\mathbb{E}[|X - \mu|^k]}{t^k}, \quad \text{for all } t > 0.$$

**Theorem 1.4** (Chernoff bound). *Let  $X$  be a random variable. Suppose that the moment generating function of  $X$ , denoted  $\varphi_X(\lambda)$ , exists in the neighborhood of 0; i.e.,  $\varphi_X(\lambda) = \mathbb{E}[e^{\lambda X}] < +\infty$ , for all  $\lambda \in (-b, b)$  with some  $b > 0$ . Applying Markov’s inequality to the random variable  $Y = e^{\lambda(X - \mu)}$  yields*

$$\mathbb{P}((X - \mu) \geq t) \leq \frac{\mathbb{E}[e^{\lambda(X - \mu)}]}{e^{\lambda t}}, \quad \text{for all } \lambda \in (-b, b).$$

*Optimizing the choice of  $\lambda$  for the tightest bound, we obtain the Chernoff bound*

$$\mathbb{P}((X - \mu) \geq t) \leq \inf_{\lambda \in [0, b)} \frac{\mathbb{E}[e^{\lambda(X - \mu)}]}{e^{\lambda t}}.$$

**Theorem 1.5** (Hoeffding bound for bounded variable). *Let  $X$  be a random variable with mean  $\mu = \mathbb{E}(X)$ . Suppose that  $X \in [a, b]$  almost surely, where  $a \leq b \in \mathbb{R}$  are two constants. Then, we have*

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{s(b-a)^2}{8}}, \quad \text{for all } \lambda \in \mathbb{R}.$$

*Consequently, the variable  $X \sim \text{subG}\left(\frac{(b-a)^2}{4}\right)$ .*

*Proof.* See Exercise 2.4. □

**Theorem 1.6** (Moment of sub-Gaussian variable). *Let  $X \sim \text{subG}(\sigma^2)$ . For all integer  $k \geq 1$ , we have*

$$\mathbb{E}[|X|^k] \leq k2^{k/2}\sigma^k\Gamma\left(\frac{k}{2}\right), \quad (3)$$

where the Gamma function is defined as  $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$ .

**Theorem 1.7** (One-sided Bernstein's inequality). *Let  $X$  be a random variable. Suppose  $X \leq b$  almost surely. We have*

$$\mathbb{E}\left[e^{\lambda(X-\mathbb{E}[X])}\right] \leq \exp\left\{\frac{\lambda^2\mathbb{E}[X^2]/2}{1-b\lambda/3}\right\}, \quad \text{for all } \lambda \in [0, 3/b).$$

Consequently, let  $X_i$  be independent variables, and  $X_i \leq b$  almost surely, for all  $i \in [n]$ . We have

$$\mathbb{P}\left[\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \geq n\delta\right] \leq \exp\left\{-\frac{n\delta^2}{\sum_{i=1}^n \mathbb{E}[X_i^2]/n + b\delta/3}\right\}, \quad \text{for all } \delta \geq 0. \quad (4)$$

Particularly, let  $X_i$  be independent nonnegative variables, for all  $i \in [n]$ . The equation (4) becomes

$$\mathbb{P}\left[\sum_{i=1}^n (Y_i - \mathbb{E}[Y_i]) \leq n\delta\right] \leq \exp\left\{-\frac{n\delta^2}{\sum_{i=1}^n \mathbb{E}[Y_i^2]/n}\right\}, \quad \text{for all } \delta \geq 0. \quad (5)$$

**Definition 1** (Bernstein's condition). Let  $X$  be a random variable with mean  $\mu = \mathbb{E}[X]$  and variance  $\sigma^2 = \text{var}(X)$ . We say  $X$  satisfies the Bernstein's condition with parameter  $b$  if

$$\left|\mathbb{E}[(X - \mu)^k]\right| \leq \frac{1}{2}k!\sigma^2b^{k-2}, \quad \text{for } k = 3, 4, \dots$$

Note that every bounded random variable  $X$  satisfies the Bernstein's condition.

**Theorem 1.8** (Bernstein-type bound). *For any variable  $X$  satisfying the Bernstein's condition, we have*

$$\mathbb{E}\left[e^{\lambda(X-\mu)}\right] \leq \exp\left\{\frac{\lambda^2\sigma^2}{2(1-b|\lambda|)}\right\}, \quad \text{for all } |\lambda| \leq \frac{1}{b},$$

and the concentration inequality

$$\mathbb{P}[|X - \mu| \geq t] \leq 2\exp\left\{-\frac{t^2}{2(\sigma^2 + bt)}\right\}, \quad \text{for all } t \geq 0.$$

## 2 Exercises

### 2.1 Exercise 2.1

(Tightness of inequalities.) The Markov's and Chebyshev's inequalities are not able to be improved in general.

- (a) Provide a random variable  $X \geq 0$  that attains the equality in Markov's inequality (1).
- (b) Provide a random variable  $Y$  that attains the equality in Chebyshev's inequality (2).

**Solution:**

- (a) For a given constant  $t > 0$ , we define a variable  $Y_t = X - t\mathbf{1}[X \geq t]$ , where  $\mathbf{1}$  is the indicator function. Note that  $Y_t$  is a nonnegative variable. The Markov's inequality follows by taking the expectation to  $Y_t$ ,

$$\mathbb{E}[Y_t] = \mathbb{E}[X] - t\mathbb{P}[X \geq t] \geq 0.$$

Therefore, Markov's inequality meets the equality if and only if the expectation  $\mathbb{E}[Y_t] = 0$ . Since  $Y_t$  is nonnegative, we have  $\mathbb{P}(Y_t = 0) = 1$ . Note that  $Y_t = 0$  if and only if  $X = 0$  or  $X = t$ .

Hence, for the given constant  $t > 0$ , the nonnegative variable  $X$  with distribution  $\mathbb{P}(X \in \{0, t\}) = 1$  attains the equality of Markov's inequality.

- (b) Chebyshev's inequality follows by applying Markov's inequality to the nonnegative random variable  $Z = (X - \mathbb{E}[X])^2$ . ~~As in part (a),~~ given a constant  $t > 0$ , the variable  $Z = (X - \mathbb{E}[X])^2$  with distribution  $\mathbb{P}(Z \in \{0, t^2\}) = 1$  attains the equality of the Markov's inequality for  $Z$ . Consequently, the variable  $X$  attains the equality of the Chebyshev's inequality for  $X$ . By transformation, the distribution of  $X$  satisfies the followings formula,

$$\mathbb{P}(X = x) = \begin{cases} p & \text{if } x = c, \\ \frac{1-p}{2} & \text{if } x = c - t \text{ or } x = c + t, \\ 0 & \text{otherwise,} \end{cases}$$

where  $c \in \mathbb{R}$  is a constant and  $p \in [0, 1]$ .

**Remark 1** (Tightness of Markov's inequality). Only a few variables attain the equalities in Markov's and Chebyshev's inequalities. In research, we should pay attention to the concentration bounds tighter than Markov's inequality.

**2.2 Exercise 2.2**

**Lemma 1** (Standard normal distribution). *Let  $\phi(z)$  be the density function of a standard normal variable  $Z \sim N(0, 1)$ . Then,*

$$\phi'(z) + z\phi(z) = 0, \tag{6}$$

and

$$\phi(z) \left( \frac{1}{z} - \frac{1}{z^3} \right) \leq \mathbb{P}(Z \geq z) \leq \phi(z) \left( \frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5} \right), \quad \text{for all } z > 0. \tag{7}$$

*Proof.* First, we prove the equation (6).

The pdf of the standard normal distribution is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right).$$

The equation (6) follows by taking the derivative of  $\phi(z)$ . Specifically,

$$\phi'(z) = -z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) = -z\phi(z).$$

Next, we prove the equation (7).

We write the upper tail probability of the standard normal variable as

$$\mathbb{P}(Z \geq z) = \int_z^{+\infty} \phi(t) dt = \int_z^{+\infty} -\frac{1}{t} \phi'(t) dt = \frac{1}{z} \phi(z) - \int_z^{+\infty} \frac{1}{t^2} \phi(t) dt, \quad (8)$$

where the second equality follows by the equation (6). Applying the equation (6) to the last term in equation (8) yields

$$\int_z^{+\infty} \frac{1}{t^2} \phi(t) dt = \int_z^{+\infty} \frac{1}{t^3} \phi'(t) dt = -\frac{1}{z^3} \phi(z) + \int_z^{+\infty} \frac{3}{t^4} \phi(t) dt \geq -\frac{1}{z^3} \phi(z) \quad (9)$$

Plugging the equation (9) into the equation (8), we obtain  $\mathbb{P}(Z \geq z) \geq \phi(z) \left( \frac{1}{z} - \frac{1}{z^3} \right)$ . Applying the equation (6) again to the equation (9) yields

$$\int_z^{+\infty} \frac{3}{t^4} \phi(t) dt = \int_z^{+\infty} -\frac{3}{t^5} \phi'(t) dt = \frac{3}{z^5} \phi(z) - \int_z^{+\infty} \frac{15}{t^6} \phi(t) dt \leq \frac{3}{z^5} \phi(z). \quad (10)$$

Combing equations (8), (9) and (10), we obtain  $\mathbb{P}(Z \geq z) \leq \phi(z) \left( \frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5} \right)$ .  $\square$

**Remark 2.** Direct calculation of tail probability for a univariate normal variable is hard. Equation (7) provides a numerical approximation to the tail probability. Particularly, the tail probability decays at the rate of  $z^{-1} e^{-z^2/2}$  as  $z \rightarrow +\infty$ . The decay rate is faster than polynomial rate  $\mathcal{O}(z^{-\alpha})$ , for any  $\alpha \geq 1$ .

### 2.3 Exercise 2.3

**Lemma 2** (Polynomial bound and Chernoff bound). *Let  $X \geq 0$  be a nonnegative variable. Suppose that the moment generating function of  $X$ , denoted  $\varphi_X(\lambda)$ , exists in the neighborhood of  $\lambda = 0$ . Given some  $\delta > 0$ , we have*

$$\inf_{k \in \mathbb{Z}_+} \frac{\mathbb{E}[|X|^k]}{\delta^k} \leq \inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}}. \quad (11)$$

*Consequently, an optimized bound based on polynomial moments is always at least as good as the Chernoff upper bound.*

*Proof.* By power series, we have

$$e^{\lambda X} = \sum_{k=0}^{+\infty} \frac{X^k \lambda^k}{k!}, \quad \text{for all } \lambda \in \mathbb{R}. \quad (12)$$

Since the moment generating function  $\varphi_X(\lambda)$  exists in the neighborhood of  $\lambda = 0$ , there exists a constant  $b > 0$  such that

$$\mathbb{E}[e^{\lambda X}] = \sum_{k=0}^{+\infty} \frac{\mathbb{E}[|X|^k] \lambda^k}{k!} < +\infty, \quad \text{for all } \lambda \in (0, b).$$

Hence, the moment  $\mathbb{E}[|X|^k]$  exists, for all  $k \in \mathbb{Z}_+$ . Applying power series (12) to the right hand side of equation (11) yields

$$\inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}} = \frac{\sum_{k=0}^{+\infty} \frac{\mathbb{E}[|X|^k] \lambda^k}{k!}}{\sum_{k=0}^{+\infty} \frac{\lambda^k \delta^k}{k!}}. \quad (13)$$

By Cauchy's third inequality, we have

$$\frac{\sum_{k=0}^{+\infty} \frac{\mathbb{E}[|X|^k] \lambda^k}{k!}}{\sum_{k=0}^{+\infty} \frac{\lambda^k \delta^k}{k!}} \geq \inf_{k \in \mathbb{Z}_+} \frac{\mathbb{E}[|X|^k]}{\delta^k} \quad (14)$$

Therefore, we obtain the equation (11) by combining the equation (13) with equation (14).  $\square$

**Remark 3.** Applying different functions  $g(X)$  to the Markov's inequality leads to different bounds for the tail probability of variable  $X$ . Equation (11) implies that optimized polynomial bound is at least as tight as the Chernoff bound, provided that the moment generating function of  $X$  exists.

## 2.4 Exercise 2.4

In Exercise 2.4, we prove Theorem 1.5, the Hoeffding bound for a bounded variable.

*Proof.* Let  $X$  be a bounded random variable, and  $X \in [a, b]$  almost surely, where  $a \leq b \in \mathbb{R}$  are two constants. Let  $\mu = \mathbb{E}[X]$ . Define the function

$$g(\lambda) = \log \mathbb{E}[e^{\lambda X}], \quad \text{for all } \lambda \in \mathbb{R}.$$

Applying Taylor Expansion to  $g(\lambda)$  at 0, we have

$$g(\lambda) = g(0) + g'(0)\lambda + \frac{g''(\lambda_0)}{2}\lambda^2, \quad \text{where } \lambda_0 = t\lambda, \text{ for some } t \in [0, 1]. \quad (15)$$

In equation (15), the term  $g(0) = \log \mathbb{E}[e^0] = 0$ . By power series (12), we obtain the first derivative  $g'(\lambda)$  as follows,

$$\begin{aligned} g'(\lambda) &= \left( \log \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^k] \right)' \\ &= \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^{(k+1)}] \bigg/ \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^k] \\ &= \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}. \end{aligned} \quad (16)$$

Therefore,  $g'(0) = \mathbb{E}[X] = \mu$ . Taking the derivative to equation (16), we obtain the second-order derivative  $g''(\lambda)$  as follows,

$$\begin{aligned} g''(\lambda) &= \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^{(k+2)}] \bigg/ \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^k] - \left( \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^{(k+1)}] \bigg/ \sum_{k=0}^n \frac{\lambda^k}{k!} \mathbb{E}[X^k] \right)^2 \\ &= \frac{\mathbb{E}[X^2 e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} - \left( \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} \right)^2. \end{aligned}$$

We interpret the second-order derivative  $g''(\lambda)$  as the variance of  $X$  with the re-weighted distribution  $dP' = e^{\lambda X} / \mathbb{E}[e^{\lambda X}] dP_X$ , where  $P_X$  is the distribution of  $X$ . Taking integral of 1 with respect to  $dP'$ , we have

$$\int_{-\infty}^{+\infty} dP' = \int_{-\infty}^{+\infty} \frac{e^{\lambda X}}{\mathbb{E}[e^{\lambda X}]} dP_X = 1,$$

which implies that the function  $P'$  is a valid probability distribution. Under all possible re-weighted distributions, the variance of  $X$  is upper bounded as follows,

$$\text{var}(X) = \text{var}\left(X - \frac{a+b}{2}\right) \leq \mathbb{E}\left[\left(X - \frac{a+b}{2}\right)^2\right] \leq \frac{(b-a)^2}{4},$$

where the term  $\frac{(b-a)^2}{4}$  follows by letting  $X$  supported on the boundaries  $a$  and  $b$  only. Hence, the second-order derivative  $g''(\lambda) \leq \frac{(b-a)^2}{4}$ . We plug the results of  $g'$  and  $g''$  into the equation (15). Then,

$$g(\lambda) = g(0) + g'(0)\lambda + \frac{g''(\lambda_0)}{2}\lambda^2 \leq 0 + \lambda\mu + \frac{(b-a)^2}{8}\lambda^2. \quad (17)$$

Taking the exponentiation on both sides of the inequality (17), we have

$$\mathbb{E}[e^{\lambda X}] = \exp(g(\lambda)) \leq e^{\mu\lambda + \frac{(b-a)^2}{8}\lambda^2}. \quad (18)$$

The equation (18) implies that  $X$  is a sub-Gaussian variable with at most  $\sigma = \frac{(b-a)}{2}$ .  $\square$

**Remark 4.** For any bounded random variable  $X$  supported on  $[a, b]$ ,  $X$  is a sub-gaussian variable with parameter at most  $\sigma^2 = (b-a)^2/4$ . All the properties for sub-Gaussian variables apply to the bounded variables.

## 2.5 Exercise 2.5

**Lemma 3** (Sub-Gaussian bounds and means/variance). *Let  $X$  be a random variable such that*

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2 \sigma^2}{2} + \mu\lambda}, \quad \text{for all } \lambda \in \mathbb{R}. \quad (19)$$

*Then,  $\mathbb{E}[X] = \mu$  and  $\text{var}(X) \leq \sigma^2$ .*

*Proof.* By equation (19), the moment generating function of  $X$ , denoted  $\varphi_X(\lambda)$ , exists in the neighborhood of  $\lambda = 0$ . Hence, the mean and variance of  $X$  exist. For all  $\lambda$  in the neighborhood of  $\lambda = 0$ , applying power series on both sides of equation (19) yields

$$\lambda \mathbb{E}[X] + \frac{\lambda^2}{2} \mathbb{E}[X^2] + o(\lambda^2) \leq \mu\lambda + \frac{\lambda^2 \sigma^2 + \lambda^2 \mu^2}{2} + o(\lambda^2). \quad (20)$$

Dividing by  $\lambda > 0$  on both sides of equation (20) and letting  $\lambda \rightarrow 0^+$ , we have  $\mathbb{E}(X) \leq \mu$ . Dividing by  $\lambda < 0$  on both sides of equation (20) and letting  $\lambda \rightarrow 0^-$ , we have  $\mathbb{E}(X) \geq \mu$ . Therefore, we obtain the mean  $\mathbb{E}[X] = \mu$ . Then, we divide  $2/\lambda^2$  on both sides of equation (20), for  $\lambda \neq 0$ . The term  $\mathbb{E}[X]\lambda$  and  $\mu\lambda$  are cancelled. We have  $\mathbb{E}[X^2] \leq \sigma^2 + \mu^2$ , and thus the  $\text{var}(X) \leq \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \sigma^2$ .  $\square$

**Question:** Let  $\sigma_{\min}^2$  denote the smallest possible  $\sigma$  satisfying the inequality (19). Is it true that  $\text{var}(X) = \sigma_{\min}^2$ ?

**Solution:** The statement that  $\text{var}(X) = \sigma_{\min}^2$  is not necessarily true. Recall the function  $g(\lambda)$  in Exercise 2.4. By the results in Exercise 2.4, the equation (19) is equal to

$$g''(\lambda) \leq \sigma^2, \quad \text{for all } \lambda \in \mathbb{R},$$

where  $g''(\lambda)$  is the variance of  $X$  with the re-weighted distribution defined in Exercise 2.4. Therefore, we have  $\max_{\lambda} g''(\lambda) = \sigma_{\min}^2$ . Note that  $g''(0) = \text{var}(X)$ . To let the equality  $\text{var}(X) = \sigma_{\min}^2$  hold, we need to show that  $\max_{\lambda} g''(\lambda) = g''(0)$  holds for  $X$ .

However, the statement  $\max_{\lambda} g''(\lambda) = g''(0)$  is not necessarily true. A counter example is below. Consider a random variable  $Y \sim \text{Ber}(1/3)$ . The variance of  $Y$  is  $\text{var}(Y) = 2/9$ . Let  $\lambda = 1$ . The re-weighted distribution  $dP'$  is

$$P'(Y = 0) = \frac{2}{3\mathbb{E}[e^Y]} \quad \text{and} \quad P'(Y = 1) = \frac{e}{3\mathbb{E}[e^Y]}, \quad \text{where } \mathbb{E}[e^Y] = \frac{2}{3} + \frac{e}{3}.$$

The variance of  $Y$  with  $dP'$  is  $2/3\mathbb{E}[e^Y] \times e/3\mathbb{E}[e^Y] = 0.2442 > 2/9$ . Therefore, we have  $\text{var}(Y) < g''(1) \leq \max_{\lambda} g''(\lambda) = \sigma_{\min}^2$ . The statement  $\max_{\lambda} g''(\lambda) = g''(0)$  is not true for this variable  $Y$ .

**Remark 5.** Parameters of a sub-Gaussian distribution provide the exact value of the mean and an upper bound of the variance; i.e.,  $\mathbb{E}[X] = \mu$  and  $\text{var}(X) \leq \sigma^2$ . ~~Then, the tail ...~~ Suppose the moment generating function of variable  $X$  exists over the entire real interval. The tail distribution of  $X$  is ~~always~~ bounded by a sub-Gaussian distribution with a proper choice of  $\sigma^2$ .

## 2.6 Exercise 2.6

**Lemma 4** (Lower bounds on squared sub-Gaussians). *Let  $\{X_i\}_{i=1}^n$  be an i.i.d. sequence of zero-mean sub-Gaussian variables with parameter  $\sigma$ . The normalized sum  $Z_n = \frac{1}{n} \sum_{i=1}^n X_i^2$  satisfies*

$$\mathbb{P}[Z_n - \mathbb{E}[Z_n] \leq \sigma^2 \delta] \leq e^{-n\delta^2/16}, \quad \text{for all } \delta \geq 0. \quad (21)$$

The equation (21) implies that the lower tail of the sum of squared sub-Gaussian variables behaves in a sub-Gaussian way.

*Proof.* Since  $X_i^2$  are i.i.d. nonnegative variables, we apply the equation (5) to the variables  $\{X_i^2\}_{i=1}^n$ . Then, we have

$$\mathbb{P}\left[\sum_{i=1}^n (X_i^2 - \mathbb{E}[X_i^2]) \leq n\sigma^2 \delta\right] \leq \exp\left\{-\frac{n\delta^2 \sigma^4}{\mathbb{E}[X_1^4]}\right\}, \quad \text{for all } \delta \geq 0. \quad (22)$$

By equation (3), we have

$$\mathbb{E}[X_1^4] \leq 16\sigma^4. \quad (23)$$

Combing equations (22), (23) and the definition of  $Z_n$ , we obtain

$$\mathbb{P}[Z_n - \mathbb{E}[Z_n] \leq \sigma^2 \delta] \leq \exp\left\{-\frac{n\delta^2}{16}\right\}, \quad \text{for all } \delta \geq 0.$$

□

**Remark 6.** Equation (21) implies that the lower tail of the sum of squared sub-Gaussian variables behaves in a sub-Gaussian way. In following sections, we will show that the variable  $Z_n - \mathbb{E}[Z_n]$  in Lemma 4 is a sub-exponential variable.

## 2.7 Exercise 2.7

**Lemma 5** (Bennett's inequality). *Let  $X_1, \dots, X_n$  be a sequence of independent zero-mean random variables with  $|X_i| \leq b$  and  $\text{var}(X_i) = \sigma_i^2$ , for all  $i \in [n]$ . Then, we have Bennett's inequality*

$$\mathbb{P} \left[ \sum_{i=1}^n X_i \geq n\delta \right] \leq \exp \left\{ -\frac{n\sigma^2}{b^2} h \left( \frac{b\delta}{\sigma^2} \right) \right\}, \quad \text{for all } \delta \geq 0,$$

where  $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$  and  $h(t) := (1+t) \log(1+t) - t$  ~~supported on~~ <sup>for</sup>  $t \geq 0$ .

*Proof.* First, we consider the moment generating function of  $X_i$ , for all  $i \in [n]$ .

By ~~the~~ power series, for all  $i \in [n]$ , we have

~~deduction" is a special terminology;  
reserved for "deductive proof"~~

$$\mathbb{E} \left[ e^{\lambda X_i} \right] = \sum_{k=0}^{+\infty} \frac{\lambda^k \mathbb{E}[X_i^k]}{k!} = 1 + 0 + \sum_{k=2}^{+\infty} \frac{\lambda^k \mathbb{E}[X_i^k]}{k!} \leq \exp \left\{ \sum_{k=2}^{+\infty} \frac{\lambda^k \mathbb{E}[X_i^k]}{k!} \right\}, \quad (24)$$

where the 0 ~~is deduced from~~ <sup>comes from the fact that</sup>  $\mathbb{E}[X_i] = 0$ , and the last inequality follows ~~by~~ <sup>from</sup>  $1+x \leq e^x$ . By  $|X_i| < b$ , we obtain an upper bound of the sum term in equation (24) as follows

~~bound the last term~~

$$\sum_{k=2}^{+\infty} \frac{\lambda^k \mathbb{E}[X_i^k]}{k!} \leq \sum_{k=2}^{+\infty} \frac{\lambda^k \mathbb{E}[X_i^2 |X_i|^{k-2}]}{k!} \leq \sum_{k=2}^{+\infty} \frac{\lambda^k \sigma_i^2 b^{k-2}}{k!} = \sigma_i^2 \left( \frac{e^{\lambda b} - 1 - \lambda b}{b^2} \right). \quad (25)$$

Combing the equation (24) with equation (25), we ~~find an~~ <sup>obtain the following</sup> upper bound of the moment generating function of  $\sum_{i=1}^n X_i$ . That is

$$\mathbb{E} \left[ e^{\lambda \sum_{i=1}^n X_i} \right] = \prod_{i=1}^n \mathbb{E} \left[ e^{\lambda X_i} \right] \leq \exp \left\{ n\sigma^2 \left( \frac{e^{\lambda b} - 1 - \lambda b}{b^2} \right) \right\}, \quad (26)$$

where  $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$ . Combing the Chernoff bound with equation (26), the upper tail of  $\sum_{i=1}^n X_i$  follows

$$\begin{aligned} \mathbb{P} \left[ \sum_{i=1}^n X_i \geq n\delta \right] &\leq \exp \left\{ n\sigma^2 \left( \frac{e^{\lambda b} - 1 - \lambda b}{b^2} \right) - \lambda n\delta \right\} \\ &= \exp \left\{ \frac{n\sigma^2}{b^2} \left( e^{\lambda b} - \lambda b - \lambda \frac{\delta b^2}{\sigma^2} - 1 \right) \right\}, \quad \text{for all } \delta \geq 0. \end{aligned} \quad (27)$$

The upper bound (27) achieves the minimum when  $\lambda = \dots$  by the first-order condition of minimization.

By the first-order condition of minimization, the  $\lambda$  that satisfies  $\lambda b = \log \left( 1 + \frac{\delta b}{\sigma^2} \right)$  minimizes the term  $\left( e^{\lambda b} - \lambda b - \lambda \frac{\delta b^2}{\sigma^2} - 1 \right)$ . Plugging  $\lambda b = \log \left( 1 + \frac{\delta b}{\sigma^2} \right)$  into the equation (27), we obtain the Bennett's inequality

$$\mathbb{P} \left[ \sum_{i=1}^n X_i \geq n\delta \right] \leq \exp \left\{ -\frac{n\sigma^2}{b^2} h \left( \frac{b\delta}{\sigma^2} \right) \right\}, \quad \text{for all } \delta \geq 0, \quad (28)$$

where  $h(t) := (1+t) \log(1+t) - t$  ~~supported on~~ <sup>for</sup>  $t \geq 0$ .

Further, we show that the Bennett's inequality is at least as good as the Bernstein's inequality.

The Bernstein's inequality for  $\sum_{i=1}^n X_i$  is

$$\mathbb{P} \left[ \sum_{i=1}^n X_i \geq n\delta \right] \leq \exp \left\{ \frac{-3n\delta^2}{(2b\delta + 6\sigma^2)} \right\} = \exp \left\{ -\frac{n\sigma^2}{b^2} g \left( \frac{b\delta}{\sigma^2} \right) \right\}. \quad (29)$$



where  $g(t) := \frac{3t^2}{2t+6}$ , for all  $t \geq 0$ . Since  $g(t) \leq h(t)$  holds for all  $t \geq 0$ , we conclude that the Bennett's inequality (28) is at least as good as Bernstein's inequality (29).  $\square$

**Remark 7.** So far, we have three inequalities that control the tail of bounded variables: Hoeffding's inequality, Bernstein's inequality, and Bennett's inequality. Particularly, Hoeffding's inequality implies the sub-Gaussianity of bounded variables. As the proof for Lemma 5 shows, Bennett's inequality is at least as good as the Bernstein's inequality, for ~~all bounded variables.~~  
random variables