Graphic Lasso: Membership Constrain for Multi-layer model

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Suppose we have a dataset with p variables and K categories. In multi-layer model, we assume the rank of decomposition r is known, and the precision matrices are of form

$$\Omega^k = \Theta_0 + \sum_{l=1}^{r} u_{lk}\Theta_l, \quad \text{for} \quad k = 1, ..., K. \tag{1}$$

Let $\mathbf{u}_l = (u_{1l}, ..., u_{Kl}) \in \mathbb{R}^K, l \in [r]$, and $\boldsymbol{U} = [\mathbf{u}_1, ..., \mathbf{u}_r] \in \mathbb{R}^{K\times r}$.

1 Option 2 (Without Intercept)

Let $\boldsymbol{U}$ be positive membership matrix, i.e., for each of $\boldsymbol{U}$ there is only 1 positive elements and others remain 0.

Proposition 1. *Assume $\boldsymbol{U}$ is a positive membership matrix, and $\|\Theta_l\|_{max} = \alpha_l, l \in [r]$, where α_l are fixed constants. Further, assume $\{\Theta_l\}$ are "irreducible" in the sense that $\Theta_k \neq C\Theta_l$, for all $k \neq l$ and any $C \in \mathbb{R}$. Then, the membership matrix $\boldsymbol{U}$ and $\{\Theta_l\}$ are identifiable up to permutation.*

Proof. Since $\boldsymbol{U}$ is a positive membership matrix, the model (1) becomes

$$\Omega^k = u_{k,l(k)}\Theta_{l(k)}, \quad k = 1, ..., K, \tag{2}$$

where $l(k) \in [r]$ is the group assignment of the k-th category. Let $I_l = \{k|u_{kl} > 0, k \in [K]\}, k \in [r]$. Suppose there exist parameters $(\tilde{\boldsymbol{U}}, \{\tilde{\Theta}_l\})$ also satisfy the equation (2) with corresponding $\tilde{I}_l$. Now, we prove $(\tilde{\boldsymbol{U}}, \{\tilde{\Theta}_l\}) = (\boldsymbol{U}, \{\Theta_l\})$. We prove by cases.

1. For all $l \in [r]$, suppose $I_l = \tilde{I}_l$. Without the loss of generality, we consider a category k with $l(k) = 1$. Then, we have
$$u_{k1}\Theta_1 = \tilde{u}_{k1}\tilde{\Theta}_1.$$
Since $\|\Theta_1\|_{\max} = \left\|\tilde{\Theta}_1\right\|_{\max} = \alpha_1$, we must have $u_{k1} = \tilde{u}_{k1}$. Same results apply to all k with $l(k) = 1$ and other groups. Therefore, we have $(\tilde{\boldsymbol{U}}, \{\tilde{\Theta}_l\}) = (\boldsymbol{U}, \{\Theta_l\})$.

2. Suppose not all nonzero set $I_l = \tilde{I}_l$. To avoid the permutation case, we assume $\tilde{I}_l \neq I_k, k \neq l, k \in [r]$. Without the loss of generality, we assume there exist $k_0 \in \tilde{I}_1 \cap I_1$ and $k_1 \in \tilde{I}_1 \cap I_2$. Then, we have
$$u_{k_0,1}\Theta_1 = \tilde{u}_{k_0,1}\tilde{\Theta}_1, \quad \text{and} \quad u_{k_1,1}\Theta_1 = \tilde{u}_{k_1,2}\tilde{\Theta}_2.$$
The above equations imply that there exists a constant C such that $\tilde{\Theta}_1 = C\tilde{\Theta}_2$, which contradicts to the "irreducible" condition of $\{\Theta_l\}$. Therefore, such k_1 does note exists, and thus for all l we have $I_l = \tilde{I}_l$. Then, we go back to case 1, and we have $(\tilde{\boldsymbol{U}}, \{\tilde{\Theta}_l\}) = (\boldsymbol{U}, \{\Theta_l\})$.

Therefore, the parameters $(\tilde{\boldsymbol{U}}, \{\tilde{\Theta}_l\}) = (\boldsymbol{U}, \{\Theta_l\})$ are identifiable.

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