

Graphic Lasso: Scaled membership with intercept

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1 Identifiability

Consider the model

$$\Omega^k = \Theta^0 + \sum_{l=1}^r u_{kl} \Theta^l, \quad k \in [K]. \quad (1)$$

Let $U = \llbracket u_{kl} \rrbracket \in \mathbb{R}^{K \times r}$ be the membership matrix and u_l denote the l -th column of U . Let $I_l = \{k : u_{kl} \neq 0\}$ for $l \in [r]$ and $I_0 = \{k : u_{kl} = 0, l \in [r]\}$.

Lemma 1 (Identifiability of scaled membership model with intercept). *Suppose the parameter (U, Θ^l) satisfies the following condition.*

1. $\Theta^0, \Theta^1, \dots, \Theta^l$ are positive definite with bounded singular values, i.e., $0 < \tau_1 \leq \min_{l=0,1,\dots,r} \varphi_{\min}(\Theta^l) \leq \max_{l=0,1,\dots,r} \varphi_{\max}(\Theta^l) \leq \tau_2 < \infty$.
2. $\Theta^l, l = 0, 1, \dots, r$ are irreducible in the sense that $\Theta^l \neq C\Theta^{l'}$ for any pair l, l' and for any constant C .
3. The columns of U are non-overlap, with $\|u_l\|_F = 1$.
4. For all $l \in [r]$, we have $\sum_{k=1}^K u_{kl} = 0$.

Then, the parameters in model (1) are identifiable.

Intuition

Consider the linear regression model

$$\mathbb{E}[Y] = X\beta, \quad (2)$$

where $Y \in \mathbb{R}^K$, $X \in \mathbb{R}^{K \times (r+1)}$, and $\beta = (\beta_0, \beta_1, \dots, \beta_r)^T$. The linear regression model (2) is a scalar analogy of model (1). Let $Y_k = \Omega^k, k = 1, \dots, K$, $X = [1_K, U]$, and $\beta_i = \Theta^i, i = 0, 1, \dots, r$. Then, we obtain the model (1).

Hence, we may get some intuitions of the identifiability problem from the simple model (2).

1. By the textbook, when X is fixed, the necessary and sufficient condition to identify β is that X has full rank. Thus, we rule out the case that $|I_0| = 0$ and $u_{kl} = u_{k'l}$, for all $k, k' \in I_l$ and $l \in [r]$.
2. Unlike the linear regression model in textbook, X and β are both unknown, and we also want to know the identifiability of X . Suppose there exist another pair of parameters $\tilde{X}, \tilde{\beta}$ such that $X\beta = \tilde{X}\tilde{\beta}$. Then, we must have $C(X) = C(\tilde{X})$.

Though the first column of X, \tilde{X} are both 1_K , it is possible $X \neq \tilde{X}$ and $C(X) = C(\tilde{X})$. A simple example is that $X = [1_K, e_1]$ and $\tilde{X} = [1_K, 1_K + e_1]$. In previous note, the old condition 4 only guarantees the full rankness. Therefore, we need stronger condition to identify the unique X from the unique column space $C(X)$.

The new condition 4 requires the matrix X to be an orthogonal matrix. The following proof will show that the orthogonality and non-overlapping is sufficient to identify X from $C(X)$.

Proof. Suppose $\{\tilde{U}, \tilde{\Theta}^l\}$ also satisfy the model (1). By condition 4, we have

$$\begin{aligned} \sum_{k=1}^K \Omega^k &= K\Theta^0 + \sum_{k=1}^K \sum_{l=1}^r u_{kl}\Theta^l = K\Theta^0 \\ &= K\tilde{\Theta}^0 + \sum_{k=1}^K \sum_{l=1}^r \tilde{u}_{kl}\tilde{\Theta}^l = K\tilde{\Theta}^0, \end{aligned}$$

which implies that $\Theta^0 = \tilde{\Theta}^0$.

1. Suppose $I_l \neq \tilde{I}_l$ for some $l = 0, 1, \dots, r$. Then, there exist a pair $k, k' \in I_l$ but $k \in \tilde{I}_l$ and $k' \in \tilde{I}_l'$. That is

$$u_{kl}\Theta^l = \tilde{u}_{kl}\tilde{\Theta}^l, \quad u_{k'l}\Theta^l = \tilde{u}_{k'l'}\tilde{\Theta}^{l'},$$

which implies that $\tilde{\Theta}^l = C\tilde{\Theta}^{l'}$ for some constant C . This contradicts to the condition 2.

2. Suppose $I_l = \tilde{I}_l$ for all $l = 0, 1, \dots, r$. Then, for all k, l we have

$$u_{kl}\Theta^l = \tilde{u}_{kl}\tilde{\Theta}^{l'},$$

which implies $\Theta^l = C\tilde{\Theta}^{l'}$ for some constant C . Thus, we have $u_{kl}C = \tilde{u}_{kl}$. By condition 3, note that $\|u_l\|_F = C\|\tilde{u}_l\| = 1$. We have $C = 1$. Therefore, we have $U = \tilde{U}$ and $\Theta^l = \tilde{\Theta}^l$.

□

Necessity

However, condition 4 is not a necessary condition for identifiable. Here is the counterexample.

Example 1. Consider the case $K = 6, r = 2$ with following membership matrix

$$[1_K, U] = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{\sqrt{2}}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix},$$

and $\Theta^0, \Theta^1, \Theta^2$ satisfy the condition 1,2. Suppose there exist another set of parameters $\{\tilde{U}, \tilde{\Theta}^l\}$ lead to the same model.

1. Suppose $I_l \neq \tilde{I}_l$ for some $l = 0, 1, 2$.

- (a) Suppose there exist $k \in I_1$ and $k' \in I_2$ such that $k, k' \in \tilde{I}_0$. We have

$$\Theta^0 + u_{k1}\Theta^1 = \Theta^0 + u_{k'2}\Theta^2 = \tilde{\Theta}^0,$$

which contradicts to the condition 2 that $\Theta^1 \neq C\Theta^2$.

- (b) Without the loss of generality, suppose $|I_1 \cap \tilde{I}_0| > 0$ and $|I_2 \cap \tilde{I}_0| = 0$. Then, for $k \in I_1 \cap \tilde{I}_0$, we have

$$\Theta^0 - \tilde{\Theta}^0 = -u_{k1}\Theta^1. \quad (3)$$

Note that $u_{21} \neq u_{31}$. Thus, there also exists a k' such that

$$\Theta^0 - \tilde{\Theta}^0 = \tilde{u}_{k'1}\tilde{\Theta}^1 - u_{k'1}\Theta^1,$$

which implies that $\tilde{\Theta}^1 = \frac{u_{k'1} - u_{k1}}{\tilde{u}_{k'1}}\Theta^1$. If $I_2 = \tilde{I}_2$, then by $k = 5, k' = 6$, we have $\Theta^0 - \tilde{\Theta}^0 = c\Theta^2$ for some constant c , and this contradicts to (3). If $I_2 \neq \tilde{I}_2$, then there exists a $k \in I_2 \cap \tilde{I}_1$ and

$$\Theta^0 + u_{k2}\Theta^2 = \tilde{\Theta}^0 + \tilde{u}_{k1}\tilde{\Theta}^1,$$

which contradicts to the condition $\Theta^2 \neq C\Theta^1$.

- (c) Suppose $|\tilde{I}_0| = 0$. Without the loss of generality, let $|\tilde{I}_1| \geq |I_1|$ and thus there exists a $k \in I_2 \cap \tilde{I}_1$ and $k', k'' \in I_1 \cap \tilde{I}_1$ with $\tilde{u}_{k'1} \neq u_{k''1}$. Then, we have

$$\begin{aligned} \Theta^0 - \tilde{\Theta}^0 &= \tilde{u}_{k1}\tilde{\Theta}^1 - u_{k2}\tilde{\Theta}^2 \\ &= \tilde{u}_{k'1}\tilde{\Theta}^1 - u_{k'1}\Theta^1 \\ &= \tilde{u}_{k''1}\tilde{\Theta}^1 - u_{k''1}\Theta^1, \end{aligned}$$

which implies that $\tilde{\Theta}^1 = c\Theta^1$ and $\tilde{\Theta}^1 = c_1\Theta^1 - c_2\Theta^2$ for some constants c, c_1, c_2 . This contradicts to the condition 2.

2. Suppose $I_l = \tilde{I}_l, l = 1, 2$. By $k = 2, 3$ and $k = 5, 6$, we know that $\tilde{\Theta}^1 = c_1\Theta^1$ and $\tilde{\Theta}^2 = c_2\Theta^2$ for some constant c_1, c_2 . By the assumption that $\|u_l\| = \|\tilde{u}_l\| = 1$, we obtain that $c_1 = c_2 = 1$.

2 Accuracy rate

Consider a simple case of model (1) when $r = 1$. The optimization problem is stated below

$$\begin{aligned} \min_{\{u, \Theta\}} \quad & \mathcal{L}(u, \Theta) = \sum_{k=1}^K \langle S^k, \Omega^k \rangle - \log \det(\Omega^k), \\ s.t. \quad & \Omega^k = \Theta^0 + u_k \Theta, \quad k \in [K], \\ & u_k > 0, \|u\|_F^2 = 1, \\ & \Theta^0, \Theta \text{ are positive definite with, and} \\ & \tau_1 < \min\{\varphi_{\min}(\Theta^0), \varphi_{\min}(\Theta)\} \leq \max\{\varphi_{\max}(\Theta^0), \varphi_{\max}(\Theta)\} < \tau_2, \tau_1, \tau_2 > 0 \end{aligned}$$