Matrix Norm

Jiaxin Hu

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1 Frobenius Norm

Lemma 1 (Frobenius norm of product of matrices)**.** *For arbitrary two matrices, $\boldsymbol{A} \in \mathbb{R}^{m \times r}$ and $\boldsymbol{B} \in \mathbb{R}^{r \times n}$, we have*

$$\|\boldsymbol{AB}\|_F \leq \|\boldsymbol{A}\|_2 \|\boldsymbol{B}\|_F ,$$

where $\|\cdot\|_2$ is the spectral norm of matrix and $\|\cdot\|_F$ is Frobenius norm of matrix.

Proof. First, let $\|\cdot\|$ denote the l_2 norm of vector. The spectral norm of matrix $\boldsymbol{A} \in \mathbb{R}^{m \times r}$ is defined as:

$$\|\boldsymbol{A}\|_2 = \max_{x \in \mathbb{R}^r, \|x\| \leq 1} \|\boldsymbol{A}x\| .$$

Therefore, we have $\|\boldsymbol{A}x\| \leq \|\boldsymbol{A}\|_2 \|x\|$ for $\forall x \in \mathbb{R}^r$. Let $\boldsymbol{B} = [b_1, \ldots, b_n] \in \mathbb{R}^{r \times n}$, where $b_j \in \mathbb{R}^r, j \in [n]$ are the columns of $\boldsymbol{B}$. Then we have

$$\|\boldsymbol{AB}\|_F^2 = \sum_{j=1}^n \|\boldsymbol{A}b_j\|^2 \leq \|\boldsymbol{A}\|_2^2 \sum_{j=1}^n \|b_j\|^2 = \|\boldsymbol{A}\|_2^2 \|\boldsymbol{B}\|_F^2 .$$

That implies

$$\|\boldsymbol{AB}\|_F \leq \|\boldsymbol{A}\|_2 \|\boldsymbol{B}\|_F .$$

□