Graphic Lasso: Identifiability with intercept

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Consider the model

$$\Omega_k = \Theta_0 + \sum_{l=1}^{r} u_{kl}\Theta_l, \quad k \in [K]. \tag{1}$$

Let $U = [\![u_{kl}]\!] \in \mathbb{R}^{K\times r}$ be the membership matrix and u_l denote the l-th column of U. Let $I_l = \{k : u_{kl} \neq 0\}$ for $l \in [r]$ and $I_0 = \{k : u_{kl} = 0, l \in [r]\}$.

Lemma 1 (Identifiability of $r = 1$ case)**.** *Suppose $r = 1$ and the parameter (U, Θ_l) of model (1) satisfy the following conditions*

1. *Θ_0, Θ_1 are positive definite with bounded singular values.*
2. *Θ_0, Θ_1 are irreducible in the sense that $\Theta_0 \neq C\Theta_1$ for any constant C.*
3. *The matrix U reduces to a vector and $\|U\|_F = 1$.*
4. *The vector U is not align with 1_K and satisfies the linear constraint $W^T U = 0$ for a known vector $W = (w_1, ..., w_K)^T$ and $\sum_{k=1}^{K} w_k = w_0$, where $w_0 \neq 0$.*

Then, the parameter (U, Θ_l) are identifiable.

Remark 1. Geometrically, when $r = 1$, if we only have one point, we can never identify the intercept Θ_0 and the slope Θ_1. Then, we need two distinct points and thus the vector U should not be align with 1_K. Given one line, to identify the one intercept and K levels $u_k\Theta_1$, we need a linear constraint on u_k, i.e., $W^T U = 0$. Just like the set-to-0 constraint and the sum-to-0 constraint in one-way ANOVA. The extra condition $\sum_{k=1}^{K} w_k \neq 0$ guarantees the linear constraint $W^T U = 0$ does not contradict to the non-align assumption.

Proof. Suppose $(\tilde{U}, \tilde{\Theta}_l)$ also satisfy the model (1) and the conditions 1-4. By condition 4, there exist k, k' such that $u_k \neq u'_k$. Then we have

$$\Theta_0 - \tilde{\Theta}_0 = \tilde{u}_k\tilde{\Theta}_1 - u_k\Theta_1 = \tilde{u}_{k'}\tilde{\Theta}_1 - u_{k'}\Theta_1,$$

which implies that

$$(u_k - u_{k'})\Theta_1 = (\tilde{u}_k - \tilde{u}_{k'})\tilde{\Theta}_1,$$

and thus $\tilde{\Theta}_1 = c\Theta_1$ for some constant $c \neq 0$. Hence, the form of $\tilde{\Theta}_0$ is $\tilde{\Theta}_0 = \Theta_0 + a\Theta_1$ for some constant a. Therefore, we have

$$\Theta_0 + u_k\Theta_1 = \tilde{\Theta}_0 + \frac{u_k - a}{c}\tilde{\Theta}_1.$$

By the constraint $W^T\tilde{U} = 0$, we have

$$\frac{W^TU - aw_0}{c} = 0,$$

which implies that $aw_0 = 0$ and thus $a = 0$. Therefore, we have $\tilde{\Theta}_0 = \Theta_0$ and thereby $\tilde{\Theta}_1 = \Theta_1$. $\square$

Lemma 2 (Identifiability of $r \geq 2$ case)**.** *Suppose $r \geq 2$ and the parameter (U, Θ_l) of model (1) satisfy the following conditions*

1. *$\Theta^0, \Theta^1, ..., \Theta^l$ are positive definite with bounded singular values, i.e., $0 < \tau_1 \leq \min_{l=0,1,...r} \varphi_{\min}(\Theta^l) \leq \max_{l=0,1,...r} \varphi_{\max}(\Theta^l) \leq \tau_2 < \infty$.*
2. *$\Theta^l, l = 0, 1, ..., r$ are irreducible in the sense that $\Theta^l \neq C\Theta^{l'}$ for any pair l, l' and for any constant C.*
3. *The columns of U are non-overlap, with $\|u_l\|_F = 1$ for all $l \in [r]$.*
4. *There are at least two subspaces spanned by the non-zero parts of $u_l, u_{l'}$ that are not align with 1s. Particularly, u_l has at least $r+1$ different entries, and $u_{l'}$ has at least r different entries.*

Then, the parameter (U, Θ_l) are identifiable.

Remark 2. Geometrically, given the rank r, we should have r non-parallel lines in the form of $\{\Theta_0 + c\Theta_l, c \in \mathbb{R}\}$. Note that Θ_0 can be uniquely identified by the intersection of two non-parallel lines. Hence, we need constraints to make there are two determinate lines with given Ω^k.

The main idea for the condition 4 is that: Suppose $r+1$ different points are on the same line L in the ground truth, and the other points are not on L. No matter how we regroup these points, there are always two points from the $r+1$ points assigned in one group. Note that these two points are on the line L while other points outside the $r+1$ points are not on L. This implies that $r+1$ points should be assigned in one group and the other points can not be assigned in this group. Thus, we have a determinate line L. Next, we have $r-1$ groups left and r different points are on the same line L' in the ground truth. Similarly, we have another determinate line L'. The line L and L' will uniquely identify the intercept Θ_0 and thereby the $\Theta_l, l \in [r]$.

Proof. Suppose group l_1, l_2 are not align with 1s, and there are $r+1$ different u_{kl_1} and r different u_{kl_2}. Let $(\tilde{U}, \tilde{\Theta}_l)$ also satisfy the model (1) and the conditions 1-4.

Since there are $r+1$ different entries in u_{l_1}, there exist a group l such that there are at least two members $k_1, k_2 \in \tilde{I}_l$ with $u_{k_1 l_1} \neq u_{k_2 l_2}$. Then, we have

$$\Theta_0 - \tilde{\Theta}_0 = \tilde{u}_{k_1 l}\tilde{\Theta}_l - u_{k_1 l_1}\Theta_{l_1} = \tilde{u}_{k_2 l}\tilde{\Theta}_l - u_{k_2 l_2}\Theta_{l_1},$$

which implies that $\tilde{\Theta}_l = c\Theta_{l_1}$ for some constant $c \neq 0$ and $\Theta_0 - \tilde{\Theta}_0 = c_1\Theta_{l_1}$.

For members $k \in I_{l_1} \cap \tilde{I}_{l'}, l' \neq l$, we have

$$\Theta_0 + u_{kl_1}\Theta_{l_1} = \tilde{\Theta}_0 + \tilde{u}_{kl'}\tilde{\Theta}_{l'}, \tag{2}$$

which implies that $\tilde{\Theta}_{l'} = C\Theta_{l_1} = C'\tilde{\Theta}_l$ for some constants C, C'. This contradicts to the condition 2, and therefore $I_{l_1} \cap \tilde{I}_l = I_{l_1}$.

For members $k \in I_{l'} \cap \tilde{I}_l, l' \neq l_1$, we have

$$\Theta_0 + u_{kl'}\Theta_{l'} = \tilde{\Theta}_0 + \tilde{u}_{kl}\tilde{\Theta}_l,$$

which implies that $\Theta_{l'} = C\Theta_{l_1}$ for some constant C. This contradicts to the condition 2, and therefore we have $I_{l_1}^c \cap \tilde{I}_l = \emptyset$.

Hence, we have $I_{l_1} = \tilde{I}_l$. Excluding the members $k \in I_{l_1}$, we have $r-1$ groups. With similar procedures, we will have $I_{l_2} = \tilde{I}_{l'}, l' \neq l$.

By the equations (2), we will have

$$\Theta_0 - \tilde{\Theta}_0 = c_1\Theta_{l_1} = c_2\Theta_{l_2},$$

where the second equation from the proof for $I_{l_2} = \tilde{I}_{l'}$. Note that $\Theta_{l_1} \neq C\Theta_{l_2}$ for all constant C. Therefore, we have $c_1 = c_2 = 0$, and thus $\Theta_0 = \tilde{\Theta}_0$.

With $\Theta_0 = \tilde{\Theta}_0$, for all $k \in [K], l \in [r]$, we have

$$u_{kl}\Theta_l = \tilde{u}_{kl}\tilde{\Theta}_l.$$

With condition 3, we finally have $\Theta_l = \tilde{\Theta}_l, l = 0, 1, ..., r$ and $U = \tilde{U}$.

$\square$