Extension of TBSM

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ADDING BLOCKS ON FIRST MODE

Let $g \in [R]$ be the partition on second and third mode with corresponding membership matrix G. Assume there is also a block structure on the first model with partition $h \in [R_0]$ and its membership matrix H. Then the least square estimate can be obtained by maximizing the following new $f(g,h;\mathcal{Y})$.

$$f(g,h;\mathcal{Y}) = \sum_{k=1}^{R}\sum_{l=1}^{R_0} \frac{n_k(g)(n_k(g))m_l(h)}{2} \left\| \frac{\langle \mathcal{Y}, (H_l \circ G_k \circ G_k)\rangle}{n_k(g)(n_k(g)-1)m_l(h)} \right\|^2 + \sum_{1\le j<k\ne R}\sum_{l=1}^{R_0} n_j(g)n_k(g)m_l(h) \left\| \frac{\langle \mathcal{Y}, (H_l \circ G_j \circ G_k)\rangle}{n_j(g)n_k(g)m_l(h)} \right\|^2$$

Conjectures:

1. Lemma 1 is still satisfies for g, h.

Proof: Since the maximization problem for $f(g,h;\mathcal{Y})$ is equal to the minimization problem:

$$\min_{g,h,\tilde{B}} \sum_{1\le i\le m, 1\le j\ne l} (\mathcal{Y}_{ijk} - \tilde{B}_{h_i g_k g_l})^2.$$

Replace $\mathcal{Y}$ by $\mathcal{P}$ and assume (g,h) is the true partition which leads to the objective function is 0. If there are different partition (g', h') (ignoring permutation), there are at least one pairs of point s.t. $\tilde{B}_{h_i,g_k,g_l} = \tilde{B}_{h_{i1},g_{k1},g_{l1}}$ while $\tilde{B}_{h'_i,g'_k,g'_l} \ne \tilde{B}_{h'_{i1},g'_{k1},g'_{l1}}$. Then $\mathcal{P}_{ijk} - \tilde{B}_{h'_i,g'_k,g'_l}$ and $\mathcal{P}_{i_1 j_1 k_1} = \mathcal{P}_{ijk} - \tilde{B}_{h'_{i1},g'_{k1},g'_{l1}}$ can not be both are 0. The objective function for (g', h') must be strictly positive. Therefore, ignoring the permutation, the optimizer of the objective function is unique.

2. Lemma 2 is still hold on similar level. (Can be proved)

Proof:

$$2(f(g,h;\mathcal{Y}) - f(g,h;\mathcal{P})) = \sum_{k=1}^{R}\sum_{l=1}^{R_0} \frac{\|\langle \mathcal{Y}, (H_l \circ G_k \circ G_k)\rangle\|^2 - \|\langle \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle\|^2}{n_k(g)(n_k(g)-1)m_l(h)} + \sum_{1\le j\ne k\le R}\sum_{l=1}^{R_0} \frac{\|\langle \mathcal{Y}, (H_l \circ G_j \circ G_k)\rangle\|^2 - \|\langle \mathcal{P}, (H_l \circ G_j \circ G_k)\rangle\|^2}{n_j(g)n_k(g)m_l(h)}$$

For the first term, we have:

$$\frac{\|\langle \mathcal{Y}, (H_l \circ G_k \circ G_k)\rangle\|^2 - \|\langle \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle\|^2}{n_k(g)(n_k(g)-1)m_l(h)}$$
$$= \frac{\langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle^2 + |\langle 2\mathcal{P}, (H_l \circ G_k \circ G_k)\rangle \langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle|}{n_k(g)(n_k(g)-1)m_l(h)}$$

First, we can get $|\langle \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle| \lesssim m_l(h) n_k^2(g) p_{max}$. For $|\langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle|$, with Cauchy Schwarz inequality:

$$\begin{aligned}\langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle^2 &= (\sum_{i \in h_l} (\sum_{j \in G_k} \mathcal{Y}_{ijj} - \mathcal{P}_{ijj}))^2 \leq (\sum_{i \in h_l} (\sum_{j \in G_k} \mathcal{Y}_{ijj} - \mathcal{P}_{ijj})^2) m_l(h) \\ &= m_l(h) \left\| (\mathcal{Y} - \mathcal{P}) * (H_l \circ G_k \circ G_k) \right\|^2\end{aligned}$$

Then we can get:

$$\begin{aligned}\sum_{l=1}^{R_0} \frac{\langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle^2}{n_k(g)(n_k(g)-1) m_l(h)} &\leq \sum_{l=1}^{R_0} \frac{m_l(h) \left\| (\mathcal{Y} - \mathcal{P}) * (H_l \circ G_k \circ G_k) \right\|^2}{n_k(g)(n_k(g)-1) m_l(h)} \\ &= \frac{\sum_{l=1}^{R_0} \left\| (\mathcal{Y} - \mathcal{P}) * (H_l \circ G_k \circ G_k) \right\|^2}{n_k(g)(n_k(g))} \\ &= \frac{\left\| (\mathcal{Y} - \mathcal{P}) * (\omega \circ G_k \circ G_k) \right\|^2}{n_k(g)(n_k(g)-1)} \\ &\lesssim \frac{n_k^2(g) \log^2(n) \{(np_{max}) \vee \log_n\}}{n_k^2(g)}\end{aligned}$$

And:

$$\begin{aligned}&\sum_{l=1}^{R_0} \frac{|\langle 2\mathcal{P}, (H_l \circ G_k \circ G_k)\rangle \langle \mathcal{Y} - \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle|}{n_k(g)(n_k(g)-1) m_l(h)} \\ \lesssim &\sum_{l=1}^{R_0} \frac{m_l(h) n_k^2(g) p_{max} (m_l(h))^{1/2} \left\| (\mathcal{Y} - \mathcal{P}) * (H_l \circ G_k \circ G_k) \right\|}{n_k(g)(n_k(g)-1) m_l(h)} \\ \lesssim &p_{max} \sum_{l=1}^{R_0} (m_l(h))^{1/2} \left\| (\mathcal{Y} - \mathcal{P}) * (H_l \circ G_k \circ G_k) \right\| \\ \leq &\sqrt{m} \left\| (\mathcal{Y} - \mathcal{P}) * (\omega \circ G_k \circ G_k) \right\| = \sqrt{m} n_k(g) \log(n) \{(np_{max}) \vee \log_n\}^{1/2}\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{R} \sum_{l=1}^{R_0} \frac{\left\| \langle \mathcal{Y}, (H_l \circ G_k \circ G_k)\rangle \right\|^2 - \left\| \langle \mathcal{P}, (H_l \circ G_k \circ G_k)\rangle \right\|^2}{n_k(g)(n_k(g)-1) m_l(h)} \\ \lesssim &\sum_{k=1}^{R} \log^2(n) \{(np_{max}) \vee \log_n\} + n_k(g) \sqrt{m} p_{max} \log(n) \{(np_{max}) \vee \log_n\}^{1/2}\end{aligned}$$

Similar result can get for the second term in $2(f(g,h;\mathcal{Y}) - f(g,h;\mathcal{P}))$. That would lead lemma 2 still holds for the extended problem. 3. Lemma 3 will hold after changing some definition such as n_{min} and δ.