

MLE for Gaussian Matching

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1 MLE for Gaussian Tensor matching

Recall that we have two random tensors $\mathcal{A}, \mathcal{B}' \in \mathbb{R}^{n^{\otimes m}}$, where $\mathcal{A}(\omega)$ and $\mathcal{B}'(\omega)$ denote the tensor entry indexed by $\omega = (i_1, \dots, i_m) \in [n]^m$. Suppose \mathcal{A} and \mathcal{B}' are super-symmetric; i.e., $\mathcal{A}(\omega) = \mathcal{A}(f(\omega))$, $\mathcal{B}'(\omega) = \mathcal{B}'(f(\omega))$ for any function f permutes the indices in ω for all $\omega \in [n]^m$. Consider the bivariate generative model for the entries $\{\omega : 1 \leq i_1 \leq \dots \leq i_m \leq n\}$

$$(\mathcal{A}(\omega), \mathcal{B}'(\omega)) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right), \quad \text{and} \quad (\mathcal{A}(\omega), \mathcal{B}'(\omega)) \perp (\mathcal{A}(\omega'), \mathcal{B}'(\omega')), \quad \text{for all } \omega \neq \omega',$$

where the correlation $\rho \in (0, 1)$ and \perp denote the statistical independence.

Suppose we observe the tensor pair \mathcal{A} and $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}' \circ \pi^*$, where $\pi^* : [n] \mapsto [n]$ denotes a permutation on $[d]$, and $\mathcal{B}(i_1, \dots, i_m) = \mathcal{B}'(\pi(i_1), \dots, \pi(i_m))$ for all $(i_1, \dots, i_m) \in [n]^m$. Equivalently, let $\Pi^* \in \{0, 1\}^{n \times n}$ denote the permutation matrix, where

$$\Pi_{ij}^* = \begin{cases} 1 & \pi^*(i) = j \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$\mathcal{B}' = \mathcal{B} \times_1 \Pi^* \times_2 \dots \times_m \Pi^*.$$

Let $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$. The likelihood function of $\pi : [n] \mapsto [n]$ (or corresponding $\Pi \in \{0, 1\}^{n \times n}$) with given observations \mathcal{A}, \mathcal{B} is

$$\begin{aligned} \mathcal{L}(\pi, \rho | \mathcal{A}, \mathcal{B}) &= \frac{1}{[\det(2\pi\Sigma)]^{n^m}} \exp\left(-\frac{1}{2} \sum_{\omega \in [n]^m} (\mathcal{A}(\omega), \mathcal{B}(\pi \circ \omega)) \Sigma^{-1} (\mathcal{A}(\omega), \mathcal{B}(\pi \circ \omega))^T\right) \\ &= \frac{1}{[\det(2\pi\Sigma)]^{n^m}} \exp\left(-\frac{1}{2(1-\rho^2)} \sum_{\omega \in [n]^m} \mathcal{A}(\omega)^2 + \mathcal{B}(\pi \circ \omega)^2 - 2\rho \mathcal{A}(\omega) \mathcal{B}(\pi \circ \omega)\right). \end{aligned}$$

Note that $\sum_{\omega \in [n]^m} \mathcal{B}(\pi \circ \omega)^2 = \sum_{\omega \in [n]^m} \mathcal{B}(\omega)^2$. The only term dependent to π in \mathcal{L} is $\mathcal{A}(\omega) \mathcal{B}(\pi \circ \omega)$. For any $\rho \in (0, 1)$, the MLE for π^* , denoted $\hat{\pi}$, satisfies

$$\hat{\pi} = \arg \max_{\pi : [n] \mapsto [n]} \mathcal{L}(\pi | \mathcal{A}, \mathcal{B}, \rho) = \arg \max_{\pi : [n] \mapsto [n]} \sum_{\omega \in [n]^m} \mathcal{A}(\omega) \mathcal{B}(\pi \circ \omega) = \arg \max_{\Pi \in \{0, 1\}^{n \times n}} \langle \mathcal{A}, \mathcal{B} \times_1 \Pi \times_2 \dots \times_m \Pi \rangle.$$

Solving $\hat{\pi}$ can be viewed as a special case of *multiway assignment problem* which is a higher-order generalization of *quadratic assignment problem*.

For the MLE of ρ , consider the log-likelihood ignoring constant terms

$$\begin{aligned}\ell(\pi, \rho | \mathcal{A}, \mathcal{B}) &= -\frac{n^m}{2} \log(1 - \rho^2) - \frac{\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2}{2(1 - \rho^2)} + \frac{\rho}{1 - \rho^2} \sum_{\omega \in [n]^m} \mathcal{A}(\omega) \mathcal{B}(\pi \circ \omega) \\ &= -\frac{n^m}{2} \log \sigma^2 - \frac{\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2}{2\sigma^2} + \frac{\sqrt{1 - \sigma^2}}{\sigma^2} \sum_{\omega \in [n]^m} \mathcal{A}(\omega) \mathcal{B}(\pi \circ \omega),\end{aligned}$$

where $\sigma^2 = 1 - \rho^2$. Take the first derivative over σ^2 and plug in the MLE of π^* . The MLE of σ^2 , denoted $\hat{\sigma}^2$, satisfies

$$-\frac{n^m}{2} + \frac{\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2}{2\hat{\sigma}^2} - \left(\frac{\sqrt{1 - \hat{\sigma}^2}}{\hat{\sigma}^2} + \frac{1}{\sqrt{1 - \hat{\sigma}^2}} \right) \sum_{\omega \in [n]^m} \mathcal{A}(\omega) \mathcal{B}(\hat{\pi} \circ \omega) = 0.$$

By the invariance of MLE, the MLE of ρ is equal to $\hat{\rho} = \sqrt{1 - \hat{\sigma}^2}$.

References