Referee's comments: Stationary probability vectors of higher-order two-dimensional transition probability tensors

In this paper, the authors studied the stationary probability vectors for a special class of transition probability tensors that satisfy (1) symmetry and (2) dimension $n = 2$. These two assumptions makes the problem easy to handle because such a tensor is completely determined by one number (either $a = p_{1\ldots1}$ or $b = p_{21\ldots1}$, since $a + b = 1$). The authors explicitly solved the fixed-point equations and obtained the stationary probability vectors. The results generalize the earlier work from order-4 tensors [Culp, Pearson and Zhang] to order-m tensors.

Major Points:

1. Because a symmetric and dimensional-2 transition tensor is determined by only one number, it is not surprising that its eigenvectors can be fully characterized. The mathematical techniques involved are quite straightforward, and to me, the main novelty is to study the tensor eigenvectors in the context of higher-order Markov chain. The authors obtained neat results on the stationary probability vectors, but I think further discussion on the probabilistic interpretation is imperative.

   For example, what is the local dynamics of these stationary vectors? If we start the Markov chain with a certain initial distribution, will the system always converge to the uniform stationary distribution? Any oscillation behaviors, or local attraction? When the transition tensor has two stationary probability vectors, which one will the Markov chain converge to? In short, the implication of the tensor results in the context of Markov chain would be needed.

2. The authors excluded the matrix $m = 2$ case from the very beginning. However, for most readers who are familiar with stochastic transition matrices, it will be much easier to understand the unique challenges (if any) for transition tensors by making a contrast. It should be remarked explicitly which lemmas/results fail in the matrix case and any fundamental distinctions between $m = 2$ vs. general even-order tensors. In particular, when $m = 2$, the symmetric transition tensor is a doubly stochastic matrix. We know that the stationary distribution is uniform if and only if the transition matrix is doubly stochastic. Does the similar property hold for higher-order transition tensors?

3. The Lemma 2.3 was opaque to me. When $m = 3$, $f_1(0, 1) = a$ by Lemma 2.1 (ii). However, if we plug $x = 0$ in Lemma 2.3 (ii), $g_1(0) = (a - b)/2 + a$, which appears different than a. In fact, there seems to be quite a lot typos/mistakes in the proof of Lemma 2.3.

   In equation (2.7), t starts from 1, 3, $\ldots$, up to $m - 2$. However, m is assumed to be even, so is $m - 2$. Should $m - 2$ be replaced by $m - 1$?

   Similar issue on page 7 lines 17-23.

   On page 8 line 12, the second line is not equal to the first line. The index t should run from 1, 2, $\ldots$, to $m - 1$ (not $m - 2$). Again, if we plug $x = 0$ in the first line, then $g_1(0) = a$.

However, plugging $x = 0$ in the third line leads to $(a - b)/2 + a$.

4. I believe the general conclusions (Theorem 2.1) are still true, since the derivative does not depend on the constants. However, the presentation is made needlessly complicated by describing the similar derivations twice: once in Part 2 and once in Part 3 for odd m. I think it should be made more concise; the author can describe one scenario in detail, and then simply comment on the difference for the other scenario.

   Similarly, in Lemma 2.3, the proof for odd m case is almost the same as the even m case. I do not think it is necessary to include the detailed calculation twice; it would be helpful though to have a simple numerical example.

Minor Points:

1. The title should include "symmetric" as key word. Symmetry is key assumption that makes the problems easy to solve.

2. page 2 line 18: "stachastic" should read "stochastic".