Graphic Lasso: Possible Accuracy

Jiaxin Hu

January 4, 2021

Let $Q(\Omega) = \mathrm{tr}(S\Omega) - \log|\Omega|$. Consider the primal minimization problem

$$\min_{\Omega = [\![\omega_{j,j'}]\!]} Q(\Omega),$$
$$s.t. \quad \sum_{j \neq j'} |\omega_{j,j'}|^{1/2} \leq C.$$

For simplicity, let $|\Omega|^{1/2} = \sum_{j \neq j'} |\omega_{j,j'}|^{1/2}$, T denote the set of indices of non-zero off-diagonal elements, and $q = |T|$. We assume following assumptions.

1. There exist two constants τ_1, τ_2 such that $0 < \tau_1 < \phi_{\min}(\Omega_0) \leq \phi_{\max}(\Omega_0) < \tau_2 < \infty$, for all $p \geq 1, k = 1, ..., K$, where $\phi_{\min}(\cdot), \phi_{\max(\cdot)}$ denote the minimal and maximal eigenvalues, respectively.

2. There exists a constant $\tau_3 > 0$ such that $\min_{(j,j') \in T} |\omega_{0,j,j'}| \geq \tau_3$.

Theorem 0.1 (Consistency (Preliminary)). *Suppose* $\frac{(p+q)\log p}{n} = o(1)$*, two assumptions hold, and* $C = \mathcal{O}\left[(p+q)\left(\frac{\log p}{n}\right)^{1/2}\right]$. *There exists a local minimizer such that*

$$\left\|\hat{\Omega} - \Omega\right\|_F = O_p\left[\left\{\frac{(p+q)\log p}{n}\right\}^{1/2}\right].$$

Proof. Let Ω, Σ denote the true precision matrix and covariance matrix, $G(\Delta) = Q(\Omega + \Delta) - Q(\Omega)$, and $\mathcal{A} = \left\{\|\Delta\|_F \leq Mr_n, |\Omega + \Delta|^{1/2} \leq C\right\}$, where $r_n = \left[\frac{(p+q)\log p}{n}\right]^{1/2}$. Note that $G(0) = 0$. To prove the existence of the local minimizer inside $\mathcal{A}$, we only need to show that $G(\Delta) > 0$ for all $\Delta \in \partial\mathcal{A} = \left\{\|\Delta\|_F = Mr_n, |\Omega + \Delta|^{1/2} \leq C\right\}$.

By Guo et al., we have the following decomposition.

$$G(\Delta) = \mathrm{tr}(S(\Omega + \Delta)) - \mathrm{tr}(\Omega) - \log|\Omega + \Delta| + \log|\Omega| = I_1 + I_2,$$

where

$$I_1 = \mathrm{tr}((S - \Sigma)\Delta), \quad I_2 = (\tilde{\Delta})^T \int_0^1 (1 - v)(\Omega + v\Delta)^{-1} \otimes (\Omega + v\Delta)^{-1} dv \tilde{\Delta}.$$

Further, with probability tending to 1, there exist two constants C_1, C_2 s.t.,

$$I_1 \leq C_1 \left(\frac{\log p}{n}\right)^{1/2} \left(|\Delta_T^-|_1 + |\Delta_{T^c}^-|_1\right) + C_2 \left(\frac{p\log p}{n}\right)^{1/2} \left\|\Delta^+\right\|_F, \quad I_2 \geq \frac{1}{4\tau_2^2} \|\Delta\|_F^2.$$

By Guo et al, we have

$$I_{1,1} = C_1 \left(\frac{\log p}{n}\right)^{1/2} |\Delta_T^-|_1 + C_2 \left(\frac{p \log p}{n}\right)^{1/2} \left\|\Delta^+\right\|_F \leq M(C_1 + C_2)\frac{(p+q)\log p}{n}.$$

Note that $|\Delta_{T^c}^-|_1 = \sum_{(j,j')\in T^c} |\delta_{j,j'}| \leq \sum_{(j,j')\in T^c} |\delta_{j,j'}|^{1/2}$ for r_n small enough. Also, note that $|\Omega + \Delta|^{1/2} = \sum_{(j,j')\in T} |\omega_{j,j'} + \delta_{j,j'}|^{1/2} + \sum_{(j,j')\in T^c} |\delta_{j,j'}|^{1/2} \leq C$. Then, we have

$$|\Delta_{T^c}^-|_1 \leq C - \sum_{(j,j')\in T} |\omega_{j,j'} + \delta_{j,j'}|^{1/2} \leq C, \tag{1}$$

and thus

$$I_{1,2} = C_1 \left(\frac{\log p}{n}\right)^{1/2} |\Delta_{T^c}^-|_1 \leq C_1 C \left(\frac{\log p}{n}\right)^{1/2}.$$

(I guess the inequality (1) can be improved.) Therefore, we have

$$\begin{aligned} G(\Delta) &\geq I_2 - I_{1,1} - I_{1,2} \\ &\geq \frac{(p+q)\log p}{n}\left(\frac{M^2}{4\tau_2^2} - M(C_1 + C_2) - C_1 C \left(\frac{(p+q)^2 \log p}{n}\right)^{-1/2}\right). \end{aligned}$$

Since $C = \mathcal{O}\left[(p+q)\left(\frac{\log p}{n}\right)^{1/2}\right]$, we have $G(\Delta) > \epsilon$ for some $\epsilon > 0$ and M sufficiently large.

$\square$