Graphic Lasso: Self-Consistency

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1 Noiseless case

Consider the noiseless case

$$\mathcal{Y} = f(\Theta),$$

where $\Theta = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K$, and $f(\cdot)$ is an entry-wise link function. Suppose we have the following optimization problem.

$$\max_{\Theta=\mathcal{C}\times_1\boldsymbol{M}_1\times_2\cdots\times_K\boldsymbol{M}_K} \mathcal{L}_{\mathcal{Y}}(\Theta) = \langle \mathcal{Y}, \Theta \rangle - \sum_{i_1,\ldots,i_K} g(\Theta_{i_1,\ldots,i_K}). \tag{1}$$

Lemma 1 (Noiseless estimation)**.** *Let $\{\mathcal{C}, \boldsymbol{M}_k\}$ denote the true parameters and $\{\hat{\mathcal{C}}, \hat{\boldsymbol{M}}_k\}$ are the estimation which maximizes the loss function. Suppose $g(\cdot)$ is a convex function with bounded second derivative $\sup_x g''(x) \leq a$, and $\max_{r_1,\ldots,r_K} |(g')^{-1}(f(c_{r_1,\ldots,r_K}))| \leq C$, where C is a positive constant depends on $\mathcal{C}$. Assume the minimal gap between blocks is strictly larger than 0, i.e., $\delta > 0$. Then, for any $\epsilon > 0$, we have*

$$\mathbb{P}(MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon) = 0.$$

Proof. We prove the accuracy in following steps.

1. With given membership matrix $\hat{\boldsymbol{M}}_k$, the estimate $\hat{\mathcal{C}}$ is

$$\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k) = (g')^{-1}\left(\frac{1}{\prod_k d_k \prod_k \hat{p}^{(k)}_{r_k}}[f(\mathcal{C}) \times_1 \boldsymbol{M}_1\hat{\boldsymbol{M}}_1^T \times_2 \cdots \times_K \boldsymbol{M}_K\boldsymbol{M}_K^T]_{r_1,\ldots,r_K}\right).$$

   Note that the estimation $\hat{\mathcal{C}}$ depends on $\hat{\boldsymbol{M}}_k$. Therefore, we denote the estimation as $\hat{\mathcal{C}}(\hat{\boldsymbol{M}}_k) = [\![\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k)]\!]$.

2. We define some useful functions. First, we define

$$F(\hat{\boldsymbol{M}}_k) = \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{C}}(\hat{\boldsymbol{M}}_k), \hat{\boldsymbol{M}}_k) = \sum_{r_1,\ldots,r_K} \prod_k d_k \prod_k \hat{p}^{(k)}_{r_k} h(g'(\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k))),$$

   where $h(x) = x(g')^{-1}(x) - g((g')^{-1}(x))$.

   Note that $\hat{\mathcal{C}}(\hat{\boldsymbol{M}}_k)$ does not include the randomness. Thus, we have $g'(\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k)) = \mathbb{E}\left[g'(\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k))\right]$, and

$$G(\hat{\boldsymbol{M}}_k) = \sum_{r_1,\ldots,r_K} \prod_k d_k \prod_k \hat{p}^{(k)}_{r_k} h(\mathbb{E}\left[g'(\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k))\right]) = F(\hat{\boldsymbol{M}}_k),$$

which implies that there does not exist the estimation error.

Note that for true membership, we have

$$F(\boldsymbol{M}_k) = G(\boldsymbol{M}_k) = \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{C}}(\boldsymbol{M}_k), \boldsymbol{M}_k),$$

where $\hat{\mathcal{C}}(\boldsymbol{M}_k) = (g')^{-1}(f(\mathcal{C}))$ **is not equal to the true core tensor** $\mathcal{C}$.

3. We only need to consider the classification error. Under the assumptions of the positive minimal gap and the boundedness of the second derivative of g, when $MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon$ for any $\epsilon > 0$, we have
$$G(\hat{\boldsymbol{M}}_k) - G(\boldsymbol{M}_k) \leq -\frac{\epsilon}{4a}\tau^{K-1}\delta.$$

4. Since $\{\hat{\mathcal{C}}\hat{\boldsymbol{M}}_k, \hat{\boldsymbol{M}}_k\}$ is the maximizer of the loss function, we have
$$0 \leq F(\hat{\boldsymbol{M}}_k) - F(\boldsymbol{M}_k) = G(\hat{\boldsymbol{M}}_k) - G(\boldsymbol{M}_k).$$
Therefore, we obtain that
$$\mathbb{P}(MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon) = \mathbb{P}(G(\hat{\boldsymbol{M}}_k) - G(\boldsymbol{M}_k) \leq -\frac{\epsilon}{4a}\tau^{K-1}\delta) = 0.$$

□

Remark 1. The lemma 1 implies that the true membership $\boldsymbol{M}_k$ is the maximizer of the function $G(\boldsymbol{M}'_k)$. Due to the noiselessness, $G(\boldsymbol{M}'_k) = \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{C}}(\boldsymbol{M}'_k), \boldsymbol{M}'_k)$, and $\{\hat{\mathcal{C}}(\boldsymbol{M}_k), \boldsymbol{M}_k\}$ is the maximizer of the noiseless loss function. However, the true parameter $\{\mathcal{C}, \boldsymbol{M}_k\}$ is not the maximizer of the noiseless loss function, since $\hat{\mathcal{C}}(\boldsymbol{M}_k) \neq \mathcal{C}$. Therefore, we conclude that the loss function (1) is **self-consistent to** $\{\hat{\mathcal{C}}(\boldsymbol{M}_k), \boldsymbol{M}_k\}$ but not **self-consistent to** Θ.

Remark 2. Define
$$\hat{\Theta} = \hat{\mathcal{C}}(\boldsymbol{M}_k) \times_1 \boldsymbol{M}_1 \times_1 \cdots \times_K \boldsymbol{M}_K.$$
Then, $\hat{\Theta}$ is an unbiased estimate of Θ if and only if $g' = f$.

Remark 3. Which assumption in the noisy case corresponds to the self-consistency of $\boldsymbol{M}_k$?

Note that in the noisy case, we have

$$\begin{aligned}
G_{noise}(\hat{\boldsymbol{M}}_k) &= \sum_{r_1,\ldots,r_K} \prod_k d_k \prod_k \hat{p}^{(k)}_{r_k} h(\mathbb{E}\left[g'(\hat{c}_{r_1,\ldots,r_K}(\hat{\boldsymbol{M}}_k))\right]) \\
&= \langle f(\mathcal{C}) \times_1 \boldsymbol{M}_1\hat{\boldsymbol{M}}_1^T \times_2 \cdots \times_K \boldsymbol{M}_K\boldsymbol{M}_K^T, (g')^{-1}\left[f(\mathcal{C}) \times_1 \boldsymbol{M}_1\hat{\boldsymbol{M}}_1^T \times_2 \cdots \times_K \boldsymbol{M}_K\boldsymbol{M}_K^T\right]\rangle \\
&\quad - \sum_{i_1,\ldots,i_K} g\left((g')^{-1}\left[f(\mathcal{C}) \times_1 \boldsymbol{M}_1\hat{\boldsymbol{M}}_1^T \times_2 \cdots \times_K \boldsymbol{M}_K\boldsymbol{M}_K^T\right] \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K\right)_{i_1,\ldots,i_K} \\
&= F_{noiseless}(\hat{\boldsymbol{M}}_k).
\end{aligned}$$

Therefore, we use the self-consistency when we derive the misclassification error. Note that the result that when $MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon$,

$$G_{noise}(\hat{\boldsymbol{M}}_k) - G_{noise}(\boldsymbol{M}_k) \leq -\frac{\epsilon}{4a}\tau^{K-1}\delta \tag{2}$$

implies the self-consistency of $\boldsymbol{M}_k$. To obtain the result (2), we require

1. the convexity of g and $\sup_x g''(x) \geq a$;
2. minimal gap strictly larger than 0, i.e., $\delta > 0$.

2 General loss function

Consider the model

$$\mathbb{E}[\mathcal{Y}] = f(\Theta), \quad \text{where} \quad \Theta = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K.$$

Theorem 2.1 (General property for loss function to guarantee the clustering accuracy)**.** *Let $\{\mathcal{C}, \boldsymbol{M}_k\}$ denote the true parameters, and $\mathcal{L}_{\mathcal{Y}}(\mathcal{C}', \boldsymbol{M}_k')$ denote the sample-based loss function. Define the sample-based loss function with respect to $\boldsymbol{M}_k'$ as*

$$F(\boldsymbol{M}_k') = \mathcal{L}_{\mathcal{Y}}(\hat{\mathcal{C}}(\boldsymbol{M}_k'), \boldsymbol{M}_k'),$$

where

$$\hat{\mathcal{C}}(\boldsymbol{M}_k') = \arg\max_{\mathcal{C}} \mathcal{L}_{\mathcal{Y}}(\mathcal{C}, \boldsymbol{M}_k').$$

Correspondingly, define the population-based loss function with respect to $\boldsymbol{M}_k'$ as

$$G(\boldsymbol{M}_k') = l(\tilde{\mathcal{C}}(\boldsymbol{M}_k'), \boldsymbol{M}_k'),$$

where

$$l(\mathcal{C}, \boldsymbol{M}_k) = \mathbb{E}_{\mathcal{Y}}[\mathcal{L}_{\mathcal{Y}}(\mathcal{C}, \boldsymbol{M}_k)], \quad \text{and} \quad \tilde{\mathcal{C}}(\boldsymbol{M}_k') = \arg\max_{\mathcal{C}} l(\mathcal{C}, \boldsymbol{M}_k').$$

Suppose the loss function satisfies the following properties

1. *(Self-consistency to $\boldsymbol{M}_k$) Suppose $MCR(\boldsymbol{M}_k', \boldsymbol{M}_k) \geq \epsilon$ for $\epsilon > 0$. We have*

$$G(\boldsymbol{M}_k') - G(\boldsymbol{M}_k) \leq -C(\epsilon), \tag{3}$$

   where $C(\cdot)$ takes positive values.

2. *(Bounded difference between sample- and population-based loss) The difference between sample-based and population-based loss function is bounded in probability, i.e.,*

$$p(t) = \mathbb{P}(|F(\boldsymbol{M}_k') - G(\boldsymbol{M}_k')| \geq t) \to 0, \quad \text{as} \quad t \to \infty. \tag{4}$$

Let $\{\hat{\boldsymbol{M}}_k\}$ be the maximizer of $F(\boldsymbol{M}_k)$. Then, we have the following clustering accuracy, for any $\epsilon > 0$,

$$\mathbb{P}(MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon) \leq p\left(\frac{C(\epsilon)}{2}\right).$$

Proof. Since $\{\hat{\mathcal{C}}, \hat{\boldsymbol{M}}_k\}$ is the maximizer of the population-based objective function $\mathcal{L}_{\mathcal{Y}}$, we have

$$\begin{aligned} 0 &\leq F(\hat{\boldsymbol{M}}_k) - F(\boldsymbol{M}_k) \\ &= F(\hat{\boldsymbol{M}}_k) - G(\hat{\boldsymbol{M}}_k) + G(\hat{\boldsymbol{M}}_k) - G(\boldsymbol{M}_k) + G(\boldsymbol{M}_k) - F(\boldsymbol{M}_k). \end{aligned}$$

Suppose $MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon$. By the property (3), we have

$$0 \leq 2r - C(\epsilon),$$

where $r = \sup_{\boldsymbol{M}'_k} |F(\boldsymbol{M}'_k) - G(\boldsymbol{M}'_k)|$. Therefore, we have

$$\begin{aligned}\mathbb{P}(MCR(\hat{\boldsymbol{M}}_k, \boldsymbol{M}_k) \geq \epsilon) &= \mathbb{P}(G(\boldsymbol{M}'_k) - G(\boldsymbol{M}_k) \leq -C(\epsilon)) \\ &\leq \mathbb{P}(C(\epsilon) \leq 2r) \\ &= p\left(\frac{C(\epsilon)}{2}\right),\end{aligned}$$

where the last equation follows the second property (4). □

Remark 4. For the model in Tensor Block model, we have

$$C(\epsilon) = \frac{\epsilon}{4a}\tau^{K-1}\delta,$$

where a is the upper bound of $g''(x)$, τ is minimal proportion of the cluster, and δ is the minimal gap between blocks. By the sub-Gaussianity of $\mathcal{Y}$ and Hoeffding's inequality, we have

$$\begin{aligned}p(t) &\leq \mathbb{P}(C_1 \left\|g'(\hat{c}_{r_1,...,r_K}) - \mathbb{E}[g'(\hat{c}_{r_1,...,r_K})]\right\|_{\max} \geq t) \\ &\leq \mathbb{P}\left(\sup_{I_{r_1,...,r_K}} \frac{|\sum_{(i_1,...,i_K)\in I_{r_1,...,r_K}} \mathcal{Y}_{i_1,...,i_K} - \mathbb{E}[\mathcal{Y}_{i_1,...,i_K}]|}{|I_{r_1,...,r_K}|} \geq \frac{t}{C_1}\right) \\ &\leq 2^{1+\sum_k d_k} \exp\left(-\frac{t^2 L}{C_1^2}\right),\end{aligned}$$

where C_1 is a positive constant related to the true core tensor $\mathcal{C}$, $I_{r_1,...,r_K}$ is the index set of the block $(r_1, ..., r_K)$ based on the estimate membership $\hat{\boldsymbol{M}}_k$, and $L = \inf |I_{r_1,...,r_K}| \geq \tau^K \prod_k d_k$.

Remark 5. When $\tilde{\mathcal{C}}(\boldsymbol{M}_k) = \mathcal{C}$, i.e., $g' = f$ in the tensor block model, the self-consistency to $\boldsymbol{M}_k$ implies the self-consistency to $\{\mathcal{C}, \boldsymbol{M}_k\}$ or $\Theta = \mathcal{C} \times_1 \boldsymbol{M}_1 \times_2 \cdots \times_K \boldsymbol{M}_K$.

3 Precision matrix model

The precision model is stated as

$$\mathbb{E}[S^k] = \Omega^k = \sum_{l=1}^{r} u_{kl}\Theta^l, \quad k \in [K].$$

Without the sparsity penalty, we consider the optimization problem

$$\max_{\boldsymbol{U},\Theta^l} \mathcal{L}_S(\boldsymbol{U}, \Theta^l) = -\sum_{k=1}^{K} \operatorname{tr}(S^k\Omega^k) + \log\det(\Omega^k),$$

where $\boldsymbol{U}$ is a membership matrix, and $\{\Theta^l\}$ are irreducible and invertible.

Proposition 1. *The loss function $\mathcal{L}_S$ satisfies the conditions for Theorem 2.1, and thus the clustering accuracy for precision matrix model is guaranteed.*

Proof. First, we define the functions $F(\cdot)$ and $G(\cdot)$ in the Theorem 2.1 under the precision matrix context.

Given the membership matrix $\boldsymbol{U}'$, we want to find the estimate $\hat{\Theta}^l(\boldsymbol{U}') = \arg\max_{\Theta^l} \mathcal{L}_S(\boldsymbol{U}', \Theta^l)$. Note that the $\mathcal{L}_S(\boldsymbol{U}', \Theta^l)$ is concave respect to Θ^l. Then, by the first order condition, we have

$$\hat{\Theta}^l = \left(\frac{\sum_{k \in I_l'} S^k}{|I_l'|} \right)^{-1},$$

With penalty, hat Theta has no closed form. Does the subsequent calculation still go through

where $I_l' = \{k : u_{kl} = 1\}, l \in [r]$. Thus, we obtain the function $F(\boldsymbol{U}') = \mathcal{L}_S(\boldsymbol{U}', \hat{\Theta}^l(\boldsymbol{U}'))$, which is

$$F(\boldsymbol{U}') = -\sum_{l=1}^{r} |I_l'|p + |I_l'| \log\det \left(\frac{\sum_{k \in I_l'} S^k}{|I_l'|} \right)^{-1}.$$

Note that

$$l(\boldsymbol{U}', \Theta^l) = \mathbb{E}_S[\mathcal{L}_S(\boldsymbol{U}', \Theta^l] = -\sum_{k=1}^{K} \operatorname{tr}(\Sigma^k \Omega^k) + \log\det(\Omega^k).$$

Therefore, we have

$$\tilde{\Theta}^l(\boldsymbol{U}') = \left(\frac{\sum_{k \in I_l'} \Sigma^k}{|I_l'|} \right)^{-1},$$

and

$$G(\boldsymbol{U}') = l(\boldsymbol{U}', \tilde{\Theta}^l(\boldsymbol{U}')) = -\sum_{l=1}^{r} |I_l'|p + |I_l'| \log\det \left(\frac{\sum_{a=1}^{r} D_{al} \Sigma^a}{|I_l'|} \right)^{-1},$$

where D_{al} is the elements of the confusion matrix.

Next, we verify the functions $F(\cdot)$ and $G(\cdot)$ satisfy the conditions in the Theorem 2.1. Let $\{\boldsymbol{U}, \Theta^l\}$ denote the true membership and precision matrices, and $\hat{\boldsymbol{U}}$ denote the estimated $\boldsymbol{U}$ which maximizes $F(\boldsymbol{U})$.

1. (Self-consistency to $\boldsymbol{U}$)

   Consider the subtraction

   $$G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) = -\sum_{l=1}^{r} \log\det \left(\frac{\sum_{a=1}^{r} D_{al} \Sigma^a}{|\hat{I}_l|} \right) + \sum_{l=1}^{r} \left(\frac{\sum_{a=1}^{r} D_{al} \log\det(\Sigma^a)}{|\hat{I}_l|} \right).$$

   Since $MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon$, there exist $l, k \neq k' \in [r]$ such that $\min\{D_{kl}, D_{k'l}\} \geq \epsilon$. Let $\tilde{\Sigma} = \frac{\sum_{a=1}^{r} D_{al} \Sigma^a}{|\hat{I}_l|}$. Consider the function $f(t) = \log\det(\tilde{\Sigma} + t\Delta)$, where $\Delta = \Sigma - \tilde{\Sigma}$. By Taylor Expansion, we have

   $$\log\det(\Sigma) - \log\det(\tilde{\Sigma}) = f(1) - f(0) = f'(0) + \frac{f''(\xi)}{2}, \quad \text{for some} \quad \xi \in [0, 1],$$

   where

   $$f'(0) = \langle (\tilde{\Sigma})^{-1}, \Delta \rangle, \quad \text{and} \quad f''(\xi) = -\operatorname{vec}(\Delta)^T (\tilde{\Sigma} + \xi\Delta)^{-1} \otimes (\tilde{\Sigma} + \xi\Delta)^{-1} \operatorname{vec}(\Delta) \leq -s \|\Delta\|_F^2, \tag{5}$$

   where s is a positive constant which $s \leq \varphi_{\max}^{-2}(\tilde{\Sigma} + \xi\Delta)$.

   Let $\Delta^l = \Sigma^l - \tilde{\Sigma}, l \in [r]$. With the Taylor Expansion (5), we have

$$\left(\frac{\sum_{a=1}^{r} D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) - \log\det\left(\tilde{\Sigma}\right) = \sum_{a=1}^{l} \frac{D_{al}}{|\hat{I}_l|}\left[\log\det(\Sigma^a) - \log\det(\tilde{\Sigma})\right]$$
$$\leq \sum_{a=1}^{r} \frac{D_{al}}{|\hat{I}_l|}\left(\langle(\tilde{\Sigma})^{-1}, \Delta^a\rangle - \frac{1}{2}s\left\|\Delta^a\right\|_F^2\right)$$
$$\leq -\frac{D_{kl}}{2|\hat{I}_l|}s\left\|\Delta^k\right\|_F^2 - \frac{D_{k'l}}{2|\hat{I}_l|}s\left\|\Delta^{k'}\right\|_F^2,$$

where the last inequality follows by the fact that $\sum_{a=1}^{r}\frac{D_{al}}{|\hat{I}_l|}\langle\tilde{\Sigma}, \Delta^a\rangle = 0$. By the inequality $\frac{1}{2}\|A+B\|_F^2 \leq \|A\|_F^2 + \|B\|_F^2$, we obtain that

$$\left(\frac{\sum_{a=1}^{r} D_{al}\log\det(\Sigma^a)}{|\hat{I}_l|}\right) - \log\det\left(\tilde{\Sigma}\right) \leq -\frac{\min\{D_{kl}, D_{k'l}\}s}{|\hat{I}_l|}\left\|\Sigma^k - \Sigma^{k'}\right\|_F^2 \leq -\frac{\epsilon}{4s|I_l|}\delta.$$

For other $l' \in [r]/l$, since $-\log\det(\cdot)$ is a convex function, by Jensen's inequality, we have

$$\left(\frac{\sum_{a=1}^{r} D_{al'}\log\det(\Sigma^a)}{|\hat{I}_{l'}|}\right) - \log\det\left(\frac{\sum_{a=1}^{r} D_{al'}\Sigma^a}{|\hat{I}_{l'}|}\right) \leq 0.$$

Then, we have

$$G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) \leq -\frac{\epsilon}{4s}\delta,$$

which implies the self-consistency holds.

2. (Bounded difference between sample- and population-based loss)

For arbitrary $\boldsymbol{U}$, consider the absolute subtraction

$$|F(\boldsymbol{U}) - G(\boldsymbol{U})| \leq \sum_{l=1}^{r} |I_l| \left|\log\det\left(\frac{\sum_{k\in I_l} S^k}{|I_l|}\right) - \log\det\left(\mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|}\right]\right)\right|.$$

Consider the function $f(t) = -\log\det\left(\frac{\sum_{k\in I_l} S^k}{|I_l|} + t\Delta\right)$, where $\Delta = \mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|}\right] - \frac{\sum_{k\in I_l} S^k}{|I_l|}$. By the previous calculation (5), we know that $f(t)$ is a convex function. Then, the function is locally Lipschitz with $L = \sup_t |f'(t)|$. Therefore, we have

$$|F(\boldsymbol{U}) - G(\boldsymbol{U})| \leq \sum_{l=1}^{r} |I_l||f(1) - f(0)|$$
$$\leq \sum_{l=1}^{r} |I_l||f'(1)|$$
$$\leq K\sup\left|\left\langle\left(\mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|}\right]\right)^{-1}, \frac{\sum_{k\in I_l} S^k}{|I_l|} - \mathbb{E}\left[\frac{\sum_{k\in I_l} S^k}{|I_l|}\right]\right\rangle\right|$$
$$\leq Kp^2\max_{l\in[r]}\left\|\Theta^l\right\|_{\max}\max_{k,(i,j)}|S^k_{(i,j)} - \mathbb{E}[S^k_{(i,j)}]|.$$

Therefore, by Lemma 2, we have

$$
\begin{aligned}
p(t) &= \mathbb{P}(|F(\boldsymbol{U}) - G(\boldsymbol{U})| \geq t) \\
&\leq \mathbb{P}\left(Kp^2 \max_{l \in [r]} \left\| \Theta^l \right\|_{\max} \max_{k,(i,j)} |S^k_{(i,j)} - \mathbb{E}[S^k_{(i,j)}]| \geq t \right) \\
&\leq C_1 \exp\left(-C_2 \frac{\min_{k \in [K]} n_k t^2}{K^2 p^4 \max_{l \in [r]} \|\Theta^l\|^2_{\max}} \right),
\end{aligned}
$$

My conjecture: "yellow statement" holds only when penalty rho is small, say < (some function of n, d, etc). Initiatively, this is how rho in Ji Zhu's Theorem arises.

So, the key is to find this threshold for rho.

where C_1, C_2 are two constants. □

A counter example. What if the penalty dominates the log-likelihood (blue part)? Do we still have self-consistency? (my answer is no, because the population optimizer becomes hat Omega = zero.)

Remark 6. The above proof does not consider the sparsity constrains. Recall the general tensor block model. The convexity of g and boundedness of g'' (as well as irreducibility of $\mathcal{C}$) ensures the self-consistency of $\boldsymbol{M}_k$. In precision matrix model, if we add a convex sparsity penalty $R(\Theta^l)$ (e.g. L_1, L_0 norm) to the objective function, the nonlinear term $-\log\det(\Omega^k) + R(\Theta^l)$ still keeps convex, which can be considered as the function "g" in the precision matrix context. Therefore, my conjecture is that **the sparsity penalty to the objective function won't affect the self-consistency to $\boldsymbol{U}$.** Meanwhile, the difference between sample- and population-based is independent with the penalty. Thus, the loss function with sparsity penalty guarantees the clustering accuracy.

L0 is a nonconvex norm; L1 is convex.

Lemma 2. *Let $Z_i \sim_{i.i.d.} \mathcal{N}(0, \Sigma)$ and $\varphi_{max}(\Sigma) \leq \tau < \infty$. Let $\Sigma = [\![\Sigma_{ij}]\!]$, then*

$$
P\left(\left| \sum_{i=1}^n Z_{ij} Z_{ik} - n\Sigma_{jk} \right| \geq n\nu \right) \leq c_1 e^{-c_2 n \nu^2}, \quad \textit{for} \quad |\nu| \leq \delta,
$$

where c_1, c_2, δ depends on τ only.

Proof. See Lemma 1 of Rothman et.al. □