Changes in the final manuscript

This file summarizes the changes in the final manuscript compared with the submission after the second-round revision. We list the changes by types and code the modifications by red color. See the color-coded manuscript "ieee_final_colorcode.pdf" for all detailed modifications.

Explanation and statement modifications.

1. We have revised the explanations for the boundedness constraints c_3, c_4 in Page 4:

   "... Third, the constant c_3 requires that all slides in $\mathcal{S}$ have non-degenerate norm. Particularly, the lower bound c_3 excludes the no purely zero slide case to avoid trivial non-identifiability of model (2); see Example 2 below. The upper bound c_4 is a technical constraint to avoid the slides with unbounded norm as dimension grows; in practice, the constraint $\max_{a:} \|\mathrm{Mat}(\mathcal{S})_{a:}\| \leq c_4$ would likely never be active with a large $c_4 \geq \|\mathcal{Y}\|_F$. ..."

2. We have corrected the conclusions related to $\Delta^2_{\boldsymbol{X}}$ in Theorem 2:

   "... Further, we define the parameter space $\mathcal{P}'(\gamma') := \mathcal{P} \cap \{\Delta^2_{\boldsymbol{X}} = p^{\gamma'}\}$, where $\Delta^2_{\boldsymbol{X}}$ is the mean tensor minimal gap in (9). When $\gamma' < -(K-1)$, we have

$$\liminf_{p\to\infty} \inf_{\hat{z}_{\text{stat}}} \sup_{(z,\boldsymbol{\theta},\mathcal{S})\in\mathcal{P}'(\gamma')} \mathbb{E}\left[p\ell(\hat{z}_{\text{stat}}, z)\right] \geq 1."$$

Related discussion in Page 6 and the Proof of Theorem 2 in Appendix D are also revised correspondly.

Minor technical condition modifications.

1. We have added the ranges of the number of communities $r \geq 2$ (or $r \geq 1$), order $K \geq 2$, and dimension $p \to \infty$ in the statements of Theorems 1, 2, 3, 4, 5, Lemma 1, Corollary 1, Proposition 1 in the main text, and the Proofs of Theorem 1, 2, 3, 4, 5, Lemmas 8, 11, 12, 13 in the Appendices. We take the modification in Theorem 4 as a typical example here:

   "Consider the general sub-Gaussian dTBM with fixed $r \geq 1$, $K \geq 2$, i.i.d. noise ..."

   "...With probability going to 1 as $p \to \infty$, we have ..."

   "...We have ... with probability going to 1 as $p \to \infty$. ..."

2. We have clarified the technical assumptions in Lemma 1 and Theorems 3, 5.

   In Lemma 1, we have added the lower bound of degree $\boldsymbol{\theta}$ and removed the Assumption 1:

   "Consider the dTBM model (2) under the parameter space $\mathcal{P}$ in (3) with $r \geq 2$. Suppose $\boldsymbol{\theta}$ is balanced satisfying (7) and $\min_{i\in[p]} \theta(i) \geq c$ from some constant $c > 0$. Then, as $p \to \infty$, for all i, j such that $z(i) \neq z(j)$, we have ..."

   In Theorems 3 and 5, we have clarified the linear local stability condition:

"... Assume the local linear stability of degree holds in the neighborhood $\mathcal{N}(z, \varepsilon)$ for all $\varepsilon \leq E_0$ and some $E_0 \geq \check{C} \log^{-1} p$ with some positive constant $\check{C}$. ..."

Proof modifications.

1. We have added discussions of extreme cases with $r = 1$ in the Proofs of Theorem 1, 4,and 5.

   In the Proof of Theorem 1, we have added following statements:

   > "... if the model (27) violates Assumption 2. Note that $\Delta^2_{\text{min}} = 1$ when there exists a $k \in [K]$ such that $r_k = 1$. Hence, we consider the case that $r_k \geq 2$ for all $k \in [K]$. Without loss of generality, ...
   >
   > First, we show the uniqueness of $\boldsymbol{M}_k$ for all $k \in [K]$. When $r_k = 1$, all possible $\boldsymbol{M}_k$ is equal to the vector $\mathbf{1}_{p_k}$, and the uniqueness holds trivially. Hence, we consider the case that $r_k \geq 2$. Without loss of generality, we consider $k = 1$ with $r_1 \geq 2$ and show the uniqueness of the first mode membership matrix; ..."

   In the Proofs of Theorem 4 and 5, we have added following statement:

   > "For the case $r = 1$, we have $\ell(z^{(t)}, z) = 0$ trivially for all $t \geq 0$. Hence, we focus on the proof of the first mode clustering $z_1^{(t+1)}$ with $r \geq 2$; ..."

2. We have revised the Proofs of Lemma 1 and 9 for better presentations.

   In Proof of Lemma 1, we showed the equivalence between mean tensor and core tensor minimal gaps via the cosine terms:

   > ..." Then, we have
   >
   > $$\cos(\boldsymbol{S}_{z_1(i):}, \boldsymbol{S}_{z_1(j):}) = \frac{\langle \boldsymbol{S}_{z_1(i):}, \boldsymbol{S}_{z_1(j):} \rangle}{\|\boldsymbol{S}_{z_1(i):}\| \|\boldsymbol{S}_{z_1(j):}\|} = (1+o(1)) \frac{\langle \boldsymbol{X}_{i:}, \boldsymbol{X}_{j:} \rangle}{\|\boldsymbol{X}_{i:}\| \|\boldsymbol{X}_{j:}\|} = (1+o(1)) \cos(\boldsymbol{X}_{i:}, \boldsymbol{X}_{j:}),$$
   >
   > where the second inequality follows by the balance assumption on $\boldsymbol{\theta}$. ..."

   In Proof of Lemma 9, we used a more classical textbook result to upper bound maximal inner product between low-rank tensor and random noise tensor:

   > "... Consider the SVD for matrix $\boldsymbol{T} = \boldsymbol{U} \Sigma \boldsymbol{V}^T$ with orthogonal matrices $\boldsymbol{U} \in \mathbb{R}^{m \times 2r}, \boldsymbol{V} \in \mathbb{R}^{n \times 2r}$ and diagonal matrix $\Sigma \in \mathbb{R}^{2r \times 2r}$. We have
   >
   > $$\begin{aligned} \sup_{\boldsymbol{T} \in \mathbb{R}^{m \times n}, \text{rank}(\boldsymbol{T}) \leq 2r, \|\boldsymbol{T}\|_F = 1} \langle \boldsymbol{T}, \boldsymbol{Y} - \boldsymbol{X} \rangle &= \sup_{\boldsymbol{T} \in \mathbb{R}^{m \times n}, \text{rank}(\boldsymbol{T}) \leq 2r, \|\boldsymbol{T}\|_F = 1} \langle \boldsymbol{U} \Sigma, \boldsymbol{E} \boldsymbol{V} \rangle \\ &= \sup_{\boldsymbol{v} \in \mathbb{R}^{2nr}} \boldsymbol{v}^T \boldsymbol{e} \leq C \sigma \sqrt{nr}, \end{aligned}$$
   >
   > with probability $1 - \exp(-C_2 nr)$, where C, C_2 are two positive constants, the vectorization $\boldsymbol{e} = \text{Vec}(\boldsymbol{E}\boldsymbol{V}) \in \mathbb{R}^{2nr}$ has independent mean-zero sub-Gaussian entries with bounded variance σ^2 due to the orthogonality of $\boldsymbol{V}$, and the last inequality follows from Rigollet and Hütter (2015, Theorem 1.19). ..."