Graphic Lasso: Possible Accuracy

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Let $Q(\Omega) = \text{tr}(S\Omega) - \log|\Omega|$. Consider the primal minimization problem

$$\min_{\Omega = [\![\omega_{j,j'}]\!]} Q(\Omega),$$
$$s.t. \quad \sum_{j \neq j'} |\omega_{j,j'}|^{1/2} \leq C.$$

For simplicity, let $|\Omega|^{1/2} = \sum_{j\neq j'} |\omega_{j,j'}|^{1/2}$, T denote the set of indices of non-zero off-diagonal elements, and $q = |T|$. We assume following assumptions.

1. There exist two constants τ_1, τ_2 such that $0 < \tau_1 < \phi_{\min}(\Omega_0) \leq \phi_{\max}(\Omega_0) < \tau_2 < \infty$, for all $p \geq 1, k = 1, ..., K$, where $\phi_{\min}(\cdot), \phi_{\max(\cdot)}$ denote the minimal and maximal eigenvalues, respectively.

2. There exists a constant $\tau_3 > 0$ such that $\min_{(j,j')\in T} |\omega_{0,j,j'}| \geq \tau_3$.

Theorem 0.1 (Consistency (Preliminary))**.** *Suppose two assumptions hold and C is a positive constant. For the estimation $\hat{\Omega}$ such that $Q(\hat{\Omega}) \leq Q(\Omega)$ and $|\hat{\Omega}|^{1/2} \leq C$, we have the following accuracy bound with probability tending to 1.*

$$\left\| \hat{\Omega} - \Omega \right\|_F = O_p \left[\left\{ \frac{(p+q)\log p}{n} \right\}^{1/2} \right].$$

Proof. Consider the following decomposition

$$G(\Delta) = \text{tr}(S(\Omega + \Delta)) - \text{tr}(\Omega) - \log|\Omega + \Delta| + \log|\Omega| = I_1 + I_2,$$

where

$$I_1 = \text{tr}((S - \Sigma)\Delta), \quad I_2 = (\tilde{\Delta})^T \int_0^1 (1-v)(\Omega + v\Delta)^{-1} \otimes (\Omega + v\Delta)^{-1} dv \tilde{\Delta}.$$

Suppose $\hat{\Omega} = \Omega + \Delta$ has larger or equal likelihood value than the true precision matrix Ω. Then, we have $G(\Delta) \leq 0$, i.e.,

$$I_2 \leq -I_1 \leq |I_1|. \tag{1}$$

Note that

$$|I_1| \leq C_1 \left(\frac{\log p}{n} \right)^{1/2} \left(|\Delta_T^-|_1 + |\Delta_{T^c}^-|_1 \right) + C_2 \left(\frac{p \log p}{n} \right)^{1/2} \left\| \Delta^+ \right\|_F, \quad I_2 \geq \frac{1}{4\tau_2^2} \left\| \Delta \right\|_F^2,$$

$|\Delta^-_T|_1 \le q^{1/2}\left\|\Delta\right\|_F$, and $|\Delta^-_{T^c}|_1 \le C$. To satisfy the inequality (1), we have

$$\frac{1}{4\tau_2^2}\left\|\Delta\right\|_F^2 \le (C_1+C_2)\left(\frac{(p+q)\log p}{n}\right)^{1/2}\left\|\Delta\right\|_F + C_1\left(\frac{(p+q)\log p}{n}\right)^{1/2} C. \tag{2}$$

Consider the equation

$$0 = -\frac{1}{4\tau_2^2}x^2 + (C_1+C_2)\left(\frac{(p+q)\log p}{n}\right)^{1/2} x + C_1\left(\frac{(p+q)\log p}{n}\right)^{1/2} C. \tag{3}$$

The solutions to the equation (3) are

$$x^* = 2\tau_2^2\left\{(C_1+C_2)\left(\frac{(p+q)\log p}{n}\right)^{1/2} \pm \sqrt{(C_1+C_2)^2\left(\frac{(p+q)\log p}{n}\right) + C_1C\left(\frac{(p+q)\log p}{n}\right)^{1/2}/\tau_2^2}\right\}$$

$$= \mathcal{O}\left[\left(\frac{(p+q)\log p}{n}\right)^{1/2}\right].$$

Therefore, to satisfy the inequality (2), we have

$$\left\|\hat{\Omega}-\Omega\right\|_F = \left\|\Delta\right\|_F = \mathcal{O}\left[\left(\frac{(p+q)\log p}{n}\right)^{1/2}\right].$$

$\square$