# Graphic Lasso: Clustering Accuracy

Jiaxin Hu

March 10, 2021

The precision model is stated as

$$\mathbb{E}[S^k] = \Omega^k = \sum_{l=1}^{r} u_{kl}\Theta^l, \quad k \in [K].$$

Consider the following penalized optimization problem

$$\max_{\{\boldsymbol{U},\Theta^l\}} \mathcal{L}_S(\boldsymbol{U},\Theta^l) = -\sum_{k=1}^{K} \operatorname{tr}(S^k\Omega^k) + \log\det(\Omega^k) + \lambda \left\|\Omega^k\right\|,$$

where $\boldsymbol{U}$ is a membership matrix, and $\{\Theta^l\}$ are irreducible and invertible.

## 1 Notations

**Notations.**

1. $I'_l = \{k : u'_{kl} \neq 0\}$ is the index set for the $l$-th group based on the membership $\boldsymbol{U}'$.
2. $\delta$ be the minimal gap between $\Theta^l$. That is
$$\min_{k,l\in[r]} \left\|\Theta^l - \Theta^k\right\|_F^2 = \delta^2.$$
3. Let $l(\boldsymbol{U},\Theta^l)$ be the population-based loss function. That is
$$l(\boldsymbol{U},\Theta^l) = \mathbb{E}_S[\mathcal{L}_S(\boldsymbol{U},\Theta^l)] = -\sum_{k=1}^{K} \operatorname{tr}(\Sigma^k\Omega^k) + \log\det(\Omega^k) - \lambda\sum_{k=1}^{K}\left\|\Omega^k\right\|_1.$$
4. Given the membership $\boldsymbol{U}'$, let $\hat{\Theta}^l(\boldsymbol{U}') = \arg\max_{\Theta^l\in\mathcal{P}_{\Theta,\alpha}} \mathcal{L}_S(\boldsymbol{U}',\Theta)$. Particularly, for each $l \in [r]$, we have
$$\hat{\Theta}^l(\boldsymbol{U}') = \arg\max_{\Theta} -\sum_{k\in I'_l}\langle S^k,\Theta\rangle + |I'_l|\log\det(\Theta) - \lambda|I'_l|\left\|\Theta\right\|_1,$$
5. Given the membership $\boldsymbol{U}'$, let $\tilde{\Theta}^l(\boldsymbol{U}') = \arg\max_{\boldsymbol{U}',\Theta^l}$. Particularly, for each $l \in [r]$, we have
$$\tilde{\Theta}^l(\boldsymbol{U}') = \arg\max_{\Theta} -\sum_{k\in I'_l}\langle \Sigma^k,\Theta\rangle + |I'_l|\log\det(\Theta) - \lambda|I'_l|\left\|\Theta\right\|_1.$$

6. Define functions

$$F(\boldsymbol{U}') = \mathcal{L}_S(\boldsymbol{U}', \hat{\Theta}^l(\boldsymbol{U}')), \quad G(\boldsymbol{U}') = l(\boldsymbol{U}', \tilde{\Theta}^l(\boldsymbol{U}')).$$

7. $\tau$ be the maximal singular value of the true precision matrix, i.e., $\tau = \max_{l\in[r]} \varphi_{\max}(\Theta^l)$.

8. $\tau_l$ be the minimal singular value of the true precision matrix, i.e., $\tau_l = \min_{l\in[r]} \varphi_{\min}(\Theta^l)$.

# 2 Main Result

**Theorem 2.1** (Clustering accuracy)**.** *Let $\{\boldsymbol{U}, \Theta^l\}$ denote the true parameters. Consider an estimation of membership $\hat{\boldsymbol{U}}$ such that $F(\hat{\boldsymbol{U}}) \geq F(\boldsymbol{U})$. Assume $\lambda \leq \mathcal{O}(n^{-1/2})$. Then, with high probability tends to 1 as $n \to \infty$, we have the following bound*

$$\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) \leq C_1 \exp\left\{-C_2 \frac{\epsilon\delta^2 n}{Kp^2\tau^4}\right\},$$

*where $C_1, C_2$ are two positive constants.*

**Remark 1.** Let $\epsilon = \frac{Kp^2\tau^4 \log n}{\delta^2 n}$. Then, with probability $1 - C_1 \exp\{-C_2 \log n\}$, we have

$$MCR(\hat{\boldsymbol{U}}, U) \leq \frac{Kp^2\tau^4 \log n}{\delta^2 n} \approx \mathcal{O}(n^{-1}).$$

This result implies that the MCR of MLE decays at the rate of $\mathcal{O}(n^{-1})$.

*Proof.* Since the estimate $\hat{\boldsymbol{U}}$ satisfies $F(\hat{\boldsymbol{U}}) \geq F(\boldsymbol{U})$, we have

$$F(\hat{\boldsymbol{U}}) - F(\boldsymbol{U}) = F(\hat{\boldsymbol{U}}) - G(\hat{\boldsymbol{U}}) + G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) + G(\boldsymbol{U}) - F(\boldsymbol{U}) \geq 0.$$

By Lemma 4, for any $\epsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) &= \mathbb{P}\left(G(\hat{\boldsymbol{U}}) - G(\boldsymbol{U}) \leq \epsilon\delta\left(-\frac{1}{8\tilde{\tau}}\delta + \lambda\sqrt{p}\right)\right) \\ &\leq \mathbb{P}\left(0 \leq \epsilon\delta\left(-\frac{1}{8\tilde{\tau}}\delta + \lambda\sqrt{p}\right) + 2m\right),\end{aligned}$$

where $m = \sup_{\boldsymbol{U}} |F(\boldsymbol{U}) - G(\boldsymbol{U})|$. Let $\tilde{t} = \frac{\epsilon\delta}{2}\left(\frac{1}{8\tilde{\tau}}\delta - \lambda\sqrt{p}\right)$ By Lemma 3, we obtain that

$$\begin{aligned}\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) &\leq \mathbb{P}(m \geq \tilde{t}) \\ &\leq \begin{cases} C_1 \exp\{-C_2 n a(\lambda, \tilde{t})^2\} & |a(\lambda, \tilde{t})| \leq 4\tau_l^{-1} \\ C_1 \exp\{-C_2 n a(\lambda, \tilde{t})\} & |a(\lambda, \tilde{t})| > 4\tau_l^{-1} \end{cases},\end{aligned}$$

where $a(\lambda, t) = \frac{-(2\lambda+1)+\sqrt{(2\lambda+1)^2 - 4(2\lambda^2 - t/Kp^2\tilde{\tau})}}{2}$.

Consider the Taylor Expansion of $a(\lambda, t)$ around $(0, 0)$. We have

$$\begin{aligned}a(\lambda, t) \approx \nabla a(\lambda, t)(\lambda, t)^T &= \left(-2 + \frac{-4\lambda + 2}{\sqrt{-4\lambda^2 + 4\lambda + 1 + 4t/Kp^2\tilde{\tau}}}\right)\lambda + \left(\frac{2/Kp^2\tilde{\tau}}{\sqrt{-4\lambda^2 + 4\lambda + 1 + 4t/Kp^2\tilde{\tau}}}\right)t \\ &= \mathcal{O}(-\lambda) + \mathcal{O}\left(\sqrt{\frac{t}{Kp^2\tilde{\tau}}}\right).\end{aligned}$$

Note that $\tilde{t} = \mathcal{O}(\frac{\epsilon\delta^2}{\tilde{\tau}} - \epsilon\delta\lambda\sqrt{p})$. Plug $\tilde{t}$ in $a(\lambda, t)$. We have

$$a(\lambda, \tilde{t}) = \mathcal{O}(-\lambda) + \mathcal{O}\left(\sqrt{\frac{\epsilon\delta^2}{Kp^2\tilde{\tau}^2} - \frac{\epsilon\delta\lambda}{Kp^{3/2}\tilde{\tau}}}\right).$$

Note that $\lambda = \mathcal{O}(n^{-1/2})$. Therefore, as $n \to \infty$, we have

$$a(\lambda, \tilde{t}) \to \mathcal{O}\left(\sqrt{\frac{\epsilon\delta^2}{Kp^2\tilde{\tau}^2}}\right).$$

When $K, p^2$ are sufficient large, we can consider $a(\lambda, \tilde{t}) \leq 4\tau_l^{-1}$. Then, we have

$$\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) \leq C_1 \exp\left\{-C_2 \frac{\epsilon\delta^2 n}{Kp^2\tilde{\tau}^2}\right\},$$

where $\mathbb{P}(MCR(\hat{\boldsymbol{U}}, \boldsymbol{U}) \geq \epsilon) \to 0$ as $n \to \infty$.

□

## 3 Useful Lemmas

**Lemma 1.** *Let $Z_i \sim_{i.i.d.} \mathcal{N}(0, \Sigma)$ and $\varphi_{max}(\Sigma) \leq \tau_0 < \infty$. Let $\Sigma = [\![\Sigma_{ij}]\!]$, then*

$$P\left(\left|\sum_{i=1}^n Z_{ij}Z_{ik} - n\Sigma_{jk}\right| \geq n\nu\right) \leq \begin{cases} c_1 e^{-c_2 n\nu^2}, & |\nu| \leq \delta \\ c_1 e^{-c_2 n\nu}, & |\nu| > \delta \end{cases}$$

*where $c_1, c_2$ depend on $\tau_0$ only, and $\delta = 4\tau_0$.*

*Proof of Lemma 1.* See A.J. Rothman et al. Lemma 1. □

**Lemma 2.** *Let $\tau = \max \varphi_{\max}(\Theta)$. We have*

$$vec(\Delta)^T \int_0^1 (1-v)(\Theta + v\Delta)^{-1} \otimes (Theta + v\Delta)^{-1} dv vec(\Delta) \geq \frac{1}{2(\tau^2 + \|\Delta\|_F^2)} \|\Delta\|_F^2.$$

*Proof of 2 .* Consider the integral

$$\begin{aligned} &\operatorname{vec}(\Delta)^T \int_0^1 (1-v)(\Theta + v\Delta)^{-1} \otimes (\Theta + v\Delta)^{-1} dv \operatorname{vec}(\Delta) \\ &\geq \|\Delta\|_F^2 \varphi_{\min}\left(\int_0^1 (1-v)(\Theta + v\Delta)^{-1} \otimes (\Theta + v\Delta)^{-1} dv\right) \\ &\geq \|\Delta\|_F^2 \int_0^1 (1-v)\varphi_{\min}^2\left((\Theta + v\Delta)^{-1}\right) dv \\ &\geq \frac{1}{2} \min_{\nu \in [0,1]} \varphi_{\min}^2\left((\Theta + v\Delta)^{-1}\right). \end{aligned}$$

Note that

$$\min_{\nu \in [0,1]} \varphi_{\min}^2\left((\Theta + v\Delta)^{-1}\right) \geq \min_{\nu \in [0,1]} \varphi_{\max}^{-2}(\check{\Theta} + v\Delta) \geq (\|\Theta\|_2 + \|\Delta\|_2)^{-2} \geq \frac{1}{(\tau^2 + \|\Delta\|_F^2)},$$

where the last inequality follows the fact that $\|\Delta\|_2^2 \leq \|\Delta\|_F^2$. □

**Lemma 3** (Estimation error)**.** *Given a membership $\boldsymbol{U}'$, assume $\lambda \leq \mathcal{O}(n^{-1/2})$. With high probability, we have the following probability*

$$p(t) = \mathbb{P}\left(|F(\boldsymbol{U}') - G(\boldsymbol{U}')| \geq t\right) \leq \begin{cases} C_1 \exp\left\{-C_2 n a(\lambda,t)^2\right\} & |a(\lambda,t)| \leq 4\tau_l^{-1} \\ C_1 \exp\left\{-C_2 n a(\lambda,t)\right\} & |a(\lambda,t)| > 4\tau_l^{-1} \end{cases},$$

*where $a(\lambda,t) = \frac{-(2\lambda+1)+\sqrt{(2\lambda+1)^2-4(2\lambda^2-t/Kp^2\tau^2)}}{2}$, , $C_1, C_2$ are two constants, and $p(t) \to 0$ as $t \to \infty$.*

*Proof of Lemma 3.* With given membership $\boldsymbol{U}'$, we have estimations $\hat{\Theta}^l(\boldsymbol{U}')$ and $\tilde{\Theta}^l(\boldsymbol{U}')$, which we use $\hat{\Theta}^l$ and $\tilde{\Theta}^l$ refer to them for simplicity, respectively. By the definition, we have

$$\begin{aligned} |F(\boldsymbol{U}') - G(\boldsymbol{U}')| &= |\mathcal{L}_S(\boldsymbol{U}', \hat{\Theta}^l) - l(\boldsymbol{U}', \tilde{\Theta}^l)| \\ &\leq \sum_{l=1}^{r} |f^l(\hat{\Theta}^l) - g^l(\tilde{\Theta}^l)|, \end{aligned}$$

where

$$f^l(\Theta) = -\sum_{k \in I_l'} \langle S^k, \Theta\rangle + |I_l'| \log\det(\Theta) - \lambda |I_l'| \, \|\Theta\|_1 \,,$$

and

$$g^l(\Theta) = -\sum_{k \in I_l'} \langle \Sigma^k, \Theta\rangle + |I_l'| \log\det(\Theta) - \lambda |I_l'| \, \|\Theta\|_1 \,.$$

Note that the functions $f^l(\cdot)$ and $g^l(\cdot)$ for $l \in [r]$ depends on the membership $\boldsymbol{U}'$, and $\hat{\Theta}^l, \tilde{\Theta}^l$ are unique maximizers for $f^l(\Theta), g^l(\Theta)$, respectively.

Next, for an arbitrary $l \in [r]$, we try to find the upper bound for $|f^l(\hat{\Theta}^l) - g^l(\tilde{\Theta}^l)|$. For simplicity, we use $f, g, \hat{\Theta}, \tilde{\Theta}$ denote $f^l, g^l, \hat{\Theta}^l$ and $\tilde{\Theta}^l$. Consider a new estimation $\check{\Theta}$ such that

$$\check{\Theta} = \arg\max_{\Theta} -\sum_{k \in I_l'} \langle \Sigma^k, \Theta\rangle + |I_l'| \log\det(\Theta).$$

By a straight calculation, we have the closed form of $\check{\Theta}$, which is equal to

$$\check{\Theta} = \left(\frac{\sum_{k \in I_l'} \Sigma^k}{|I_l'|}\right)^{-1}.$$

Note that $\check{\Theta}$ is the inverse of a combination of true covariance matrices. Thus $\check{\Theta}$ does not violet the singular value constrain.

Then, we have

$$\begin{aligned} |f(\hat{\Theta}) - g(\tilde{\Theta})| &\leq |f(\hat{\Theta}) - f(\check{\Theta})| + |f(\check{\Theta}) - g(\check{\Theta})| + |g(\check{\Theta}) - g(\tilde{\Theta})| \\ &= M_1 + M_2 + M_3. \end{aligned}$$

1. **For** $M_1$, we have

$$f(\hat{\Theta}) - f(\check{\Theta}) = \sum_{k \in I_l'} \langle S^k, \check{\Theta} - \hat{\Theta} \rangle + |I_l'| \left( \log\det(\hat{\Theta}) - \log\det(\check{\Theta}) \right) - \lambda |I_l'| \left( \left\| \hat{\Theta} \right\|_1 - \|\check{\Theta}\|_1 \right).$$

Define $\Delta_1 = \hat{\Theta} - \check{\Theta}$ and consider the function $m(t) = \log\det(\check{\Theta} + t\Delta_1)$. By Taylor expansion and Lemma 2, we have

$$\begin{aligned}
\log\det(\hat{\Theta}) - \log\det(\check{\Theta}) &= m(1) - m(0) \\
&= \langle \check{\Theta}^{-1}, \Delta_1 \rangle - \mathrm{vec}(\Delta_1)^T \int_0^1 (1-v)(\check{\Theta} + v\Delta_1)^{-1} \otimes (\check{\Theta} + v\Delta_1)^{-1} dv \mathrm{vec}(\Delta_1) \\
&\leq \langle \check{\Theta}^{-1}, \Delta_1 \rangle - \frac{1}{2\tau^2 + 2\|\Delta_1\|_F^2} \|\Delta_1\|_F^2,
\end{aligned}$$

Note that $f(\hat{\Theta}) - f(\check{\Theta}) \geq 0$, we have

$$\begin{aligned}
0 &\leq \sum_{k \in I_l'} \langle S^k - \Sigma^k, \Delta_1 \rangle - \frac{1}{2\tau^2 + 2\|\Delta_1\|_F^2} |I_l'| \|\Delta_1\|_F^2 + \lambda |I_l'| \|\Delta_1\|_1 \\
&\leq |I_l'| \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k| \|\Delta_1\|_1 - \frac{1}{2\tau^2 + 2\|\Delta_1\|_F^2} |I_l'| \|\Delta_1\|_F^2 + \lambda |I_l'| \|\Delta_1\|_1 \\
&\leq |I_l'| \left( -\frac{1}{2\tau^2 + 2\|\Delta_1\|_F^2} \|\Delta_1\|_F^2 + (\lambda + \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|) p \|\Delta_1\|_F \right),
\end{aligned}$$

which implies that

$$0 \leq 2(\lambda + \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|) p \|\Delta_1\|_F^2 - \|\Delta_1\|_F + 2(\lambda + \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|) p\tau^2. \tag{1}$$

By the assumption that $\lambda = \mathcal{O}(n^{-1/2})$ and the Lemma 1, we know have $\lim_{n\to\infty} \lambda = \lim_{n\to\infty} \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k| = 0$. Hence, to satisfies the condition (1), we have

$$\lim_{n\to\infty} \|\Delta_1\|_F = 0.$$

Therefore, for $n$ sufficient large, we have

$$\begin{aligned}
|f(\hat{\Theta}) - f(\check{\Theta})| &\leq |I_l'| \left( -\frac{1}{4\tau^2} \|\Delta_1\|_F^2 + (\lambda + \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|) p \|\Delta_1\|_F \right) \\
&\leq |I_l'| \tau^2 p^2 (\lambda + \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|)^2
\end{aligned}$$

where the third inequality follows by the fact the $\|\Delta\|_1 \leq p\|\Delta\|_F$, and the last inequality follows by the property of quadratic function.

2. **For** $M_2$, we have

$$\begin{aligned}
|f(\check{\Theta}) - g(\check{\Theta})| &= |\sum_{k \in I_l'} \langle S^k - \Sigma^k, \check{\Theta} \rangle| \\
&\leq |I_l'| \left\| S^k - \Sigma^k \right\|_2 \|\check{\Theta}\|_2 \\
&\leq p^2 \tau^2 |I_l'| \max_{(i,j),k \in I_l'} |S_{ij}^k - \Sigma_{ij}^k|.
\end{aligned}$$

3. **For** $M_3$, we have

$$g(\check{\Theta}) - g(\tilde{\Theta}) = \sum_{k\in I_l'} \langle \Sigma^k, \tilde{\Theta} - \check{\Theta}\rangle + |I_l'| \left(\log\det(\check{\Theta}) - \log\det(\tilde{\Theta})\right) - \lambda |I_l'| (\left\|\check{\Theta}\right\|_1 - \left\|\tilde{\Theta}\right\|_1).$$

Let $\Delta_2 = \tilde{\Theta} - \check{\Theta}$. By Taylor Expansion and the fact that $g(\check{\Theta}) - g(\tilde{\Theta}) \le 0$, with similar procedures for $M_1$, we have

$$\begin{aligned} g(\check{\Theta}) - g(\tilde{\Theta}) &\ge \sum_{k\in I_l'} \langle \Sigma^k, \Delta_2\rangle - |I_l'|(\langle \check{\Theta}^{-1}, \Delta_2\rangle - \frac{1}{4\tau^2}\left\|\Delta_2\right\|_F^2) - \lambda |I_l'| \left\|\Delta_2\right\|_1 \\ &= \frac{1}{4\tau^2}|I_l'| \left\|\Delta_2\right\|_F^2 - \lambda |I_l'| \left\|\Delta_2\right\|_1 . \end{aligned}$$

Thus, we have

$$\begin{aligned} |g(\check{\Theta}) - g(\tilde{\Theta})| &\le -\frac{1}{4\tau^2}|I_l'| \left\|\Delta_2\right\|_F^2 + \lambda |I_l'| \left\|\Delta_2\right\|_1 \\ &\le -\frac{1}{\tilde{\tau}}|I_l'| \left\|\Delta_2\right\|_F^2 + \lambda |I_l'| p \left\|\Delta_2\right\|_F \\ &\le \tau^2\lambda^2 p^2 |I_l'| \end{aligned}$$

Therefore, we have the upper bound

$$\begin{aligned} |f(\hat{\Theta}) - g(\tilde{\Theta})| &\le M_1 + M_2 + M_3 \\ &\le |I_l'| p^2\tau^2 \left[(\lambda + \max_{(i,j),k\in I_l'} |S_{ij}^k - \Sigma_{ij}^k|)^2 + \max_{(i,j),k\in I_l'} |S_{ij}^k - \Sigma_{ij}^k| + \lambda^2\right], \end{aligned}$$

and thus we have

$$\begin{aligned} |F(\boldsymbol{U}') - G(\boldsymbol{U}')| &\le \sum_{l=1}^{r} |f^l(\hat{\Theta}^l) - g^l(\tilde{\Theta}^l)| \\ &\le Kp^2\tau^2 \left[(\lambda + \max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k|)^2 + \max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k| + \lambda^2\right]. \end{aligned}$$

Intuitively, if $\lambda$ tends to 0, the error only related to the gap between population and sample $\max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k|$.

Last, we obtain the probability

$$\begin{aligned} \mathbb{P}(|F(\boldsymbol{U}') - G(\boldsymbol{U}')| \ge t) &\le \mathbb{P}\left((\lambda + \max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k|)^2 + \max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k| + \lambda^2 \ge \frac{t}{Kp^2\tau^2}\right) \\ &= \mathbb{P}\left(\max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k|^2 + (2\lambda+1)\max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k| + 2\lambda^2 - \frac{t}{Kp^2\tau^2} \ge 0\right) \\ &= \mathbb{P}\left(\max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k| \ge \frac{-(2\lambda+1) + \sqrt{(2\lambda+1)^2 - 4(2\lambda^2 - t/Kp^2\tau^2)}}{2}\right). \end{aligned}$$

Let $a(\lambda, t) = \frac{-(2\lambda+1)+\sqrt{(2\lambda+1)^2-4(2\lambda^2-t/Kp^2\tau^2)}}{2}$. Since $\lambda = \mathcal{O}(-1/n)$, $a(\lambda, t)$ is well-defined with large $n$. By the Lemma 1, we have

$$\begin{aligned} p(t) &= \mathbb{P}(|F(\boldsymbol{U}') - G(\boldsymbol{U}')| \geq t) \\ &\leq \mathbb{P}\left(\max_{(i,j),k\in K} |S_{ij}^k - \Sigma_{ij}^k| \geq a(\lambda, t)\right) \\ &\leq \begin{cases} C_1 \exp\left\{-C_2 n a(\lambda, t)^2\right\} & |a(\lambda, t)| \leq 4\tau_l^{-1} \\ C_1 \exp\left\{-C_2 n a(\lambda, t)\right\} & |a(\lambda, t)| > 4\tau_l^{-1} \end{cases} \end{aligned}$$

□

**Lemma 4** (Self-consistency of $\boldsymbol{U}$)**.** *Suppose $MCR(\boldsymbol{U}', \boldsymbol{U}) \geq \epsilon$ and the minimal gap between $\{\Theta^l\}$ denoted $\delta$ is positive. For $\lambda = \mathcal{O}(n^{-1/2})$, we have the perturbation version of the self-consistency.*

$$G(\boldsymbol{U}') - G(\boldsymbol{U}) \leq \epsilon\delta\left(-\frac{1}{8\tau^2}\delta + \lambda\sqrt{p}\right) < 0.$$

*Proof.* See note 0228. □