# (Polynomial) MLE error in dTBM and TBM

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This note shows the polynomial misclassification errors of the MLE under TBM and dTBM. We show the detailed proof for dTBM and the revision for TBM in Wang and Zeng (2019) can be obtained easily from dTBM results.

## 1 Preliminary

We consider the general Gaussian dTBM

$$\mathcal{Y} = \mathcal{S} \times_1 \boldsymbol{\Theta}_1 \boldsymbol{M}_1 \times_2 \cdots \times \boldsymbol{\Theta}_K \boldsymbol{M}_K + \mathcal{E},$$

where $\mathcal{S} \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ is the core tensor, $\boldsymbol{M}_k \in \{0,1\}^{p_k \times r_k}$ are the membership matrices corresponding to the assignment $z_k \in [p_k] \mapsto [r_k]$, $\boldsymbol{\theta}_k \in \mathbb{R}_+^{p_k}$ are heterogeneity, and $\mathcal{E} \in \mathbb{R}^{p_1 \times \cdots \times p_K}$ is noise tensor with i.i.d. standard Gaussian entries.

We consider the estimation space

$$E = \{(\hat{z}_k, \hat{\boldsymbol{\theta}}_k, \hat{\mathcal{S}}) : \hat{z}_k \text{ is a function } [p_k] \to [r_k],\ \hat{\theta}_k(i) > 0, i \in [p_k], k \in [K]\}.$$

We have the MLE of $(z_k, \boldsymbol{\theta}_k, \mathcal{S})$ that minimizes the least square error over $E$

$$(\hat{z}_{k,\text{MLE}}, \hat{\boldsymbol{\theta}}_{k,\text{MLE}}, \hat{\mathcal{S}}_{\text{MLE}}) = \underset{(z_k, \boldsymbol{\theta}_k, \mathcal{S}) \in E}{\arg\min} \|\mathcal{Y} - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2,$$

where

$$\mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S}) = \mathcal{S} \times_1 \boldsymbol{\Theta}_1 \boldsymbol{M}_1 \times_2 \cdots \times \boldsymbol{\Theta}_K \boldsymbol{M}_K.$$

Let $(z_k^*, \boldsymbol{\theta}^*, \mathcal{S}^*)$ denote the true parameters. We consider the misclassification error

$$\ell(\hat{z}_k, z_k^*) = \frac{1}{p_k} \min_{\pi \in \Pi_{p_k}} \sum_{i \in [p_k]} \mathbb{1}\{\pi \circ \hat{z}_k(i) \neq z_k^*(i)\},$$

where $\Pi_{p_k}$ is the collection of all possible permutation on $[p_k]$. Without loss of generality, we assume the identity mappings $\pi(i) = i, i \in [p_k]$ minimizes the misclassification errors.

## 2 MLE error under dTBM

We show the MLE $\hat{z}_k$ achieves polynomial misclassification error.

**Lemma 1** (Signal perturbation by misclassification)**.** Consider the general dTBM with true parameters $(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*)$, dimensions $p_k \asymp p$, and number of clusters $r_k \asymp r$, for all $k \in [K]$. Suppose $\boldsymbol{\theta}_k$ are balanced and $\min_{i\in[p_k],k\in[K]} \theta(i) \geq c$ for some constant $c$. Consider the assignments $z_k$ such that $\max_{k\in[K]} \ell(z_k, z_k^*) = \epsilon$. Then,

$$\|\mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2 \gtrsim p^K \Delta_{\min}^2 \epsilon.$$

*Proof of Lemma 1.* Without loss of generality, let $\ell(z_1, z_1^*) = \epsilon$. Rewrite the function $\mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})$ as $\mathcal{X}(z_1, \{z_k\}_{k>1}, \boldsymbol{\theta}, \mathcal{S})$. Let

$$\boldsymbol{C} = \mathrm{Mat}_1(\mathcal{C}), \quad \text{where} \quad \mathcal{C} = \mathcal{S} \times_2 \boldsymbol{\Theta}_2 \boldsymbol{M}_2 \times_3 \cdots \times_K \boldsymbol{\Theta}_K \boldsymbol{M}_K,$$

denote the extended signal tensor with parameter $(\{z_k\}_{k>1}, \{\boldsymbol{\theta}_k\}_{k>1}, \mathcal{S})$ and $\boldsymbol{C}^*$ denote the extended signal with true parameters. We rewrite the function $\mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})$ as $\mathcal{X}(z_1, \boldsymbol{\theta}_1, \boldsymbol{C})$. Then, we have

$$\|\mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2 \geq \min_{\boldsymbol{C}} \|\mathcal{X}(z_1^*, \boldsymbol{\theta}_1^*, \boldsymbol{C}^*) - \mathcal{X}(z_1, \boldsymbol{\theta}_1, \boldsymbol{C})\|_F^2.$$

Next, we lower bound $\|\mathcal{X}(z_1^*, \boldsymbol{\theta}_1^*, \boldsymbol{C}^*) - \mathcal{X}(z_1, \boldsymbol{\theta}_1, \boldsymbol{C})\|_F^2$ with an arbitrarily fixed $\boldsymbol{C}$. For an arbitrary $a \in [r]$, consider the $i \in [p_1]$ such that $z_1^*(i) = z_1(i) = a$ and $j \in [p_1]$ such that $z^*(j) = b, z_1(j) = a$ where $b \neq a$. Without loss of generality, assume $d \coloneqq \boldsymbol{\theta}_1(j)/\boldsymbol{\theta}_1(i) > 1$. We have

$$\begin{aligned}
&\|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{a:}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 + \|\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{b:}^* - \boldsymbol{\theta}_1(j)\boldsymbol{C}_{a:}\|_F^2 \\
&\geq (1-d^2)\|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{a:}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 + d^2\|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{a:}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 + d^2\|s^{-1}\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{b:}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 \\
&\geq d^2\|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{a:}^* - d^{-1}\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{b:}^*\|_F^2 \\
&\geq \|\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{b:}^*\|_F^2 \|[\boldsymbol{C}_{a:}^*]^s - [\boldsymbol{C}_{b:}^*]^s\|_F^2, \qquad (1)
\end{aligned}$$

where the second inequality follows from the triangle inequality, and the last inequality follows from Lemma 4 in the manuscript. By the assumption that $\min_{i\in[p_1]} \boldsymbol{\theta}_1(i) \geq c$, we have

$$\|\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{b:}^*\|_F^2 \geq c^2\|\boldsymbol{S}_{b:}^*\|_F^2 \prod_{k>1} \lambda^2(\boldsymbol{\Theta}_k^*\boldsymbol{M}_k^*) \gtrsim p^{K-1}, \qquad (2)$$

where the second inequality follows from Lemma 6 in the manuscript and the assumption that $\|\boldsymbol{S}_{b:}^*\|_F^2 \geq c_4^2$. Also, by the balance assumption on $\boldsymbol{\theta}_k$, we have

$$\|[\boldsymbol{C}_{a:}^*]^s - [\boldsymbol{C}_{b:}^*]^s\|_F^2 \asymp \|[\boldsymbol{S}_{a:}^*]^s - [\boldsymbol{S}_{b:}^*]^s\| \gtrsim \Delta_{\min}^2. \qquad (3)$$

Therefore, we have

$$\begin{aligned}
&\|\mathcal{X}(z_1^*, \boldsymbol{\theta}_1^*, \boldsymbol{C}^*) - \mathcal{X}(z_1, \boldsymbol{\theta}_1, \boldsymbol{C})\|_F^2 \\
&= \sum_{a\in[r]} \sum_{i:z_1(i)=a} \|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{z^*(i):}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 \\
&= \sum_{a\in[r]} \left[ \sum_{i:z_1^*(i)=z_1(i)=a} \|\boldsymbol{\theta}_1^*(i)\boldsymbol{C}_{a:}^* - \boldsymbol{\theta}_1(i)\boldsymbol{C}_{a:}\|_F^2 + \sum_{j:z_1^*(j)\neq a, z_1(j)=a} \|\boldsymbol{\theta}_1^*(j)\boldsymbol{C}_{z^*(j):}^* - \boldsymbol{\theta}_1(j)\boldsymbol{C}_{a:}\|_F^2 \right] \\
&\gtrsim p^K \Delta_{\min}^2 \epsilon,
\end{aligned}$$

where the last inequality follows from inequalities (1), (2), (3), and the assumption that $\sum_{a\in[r]} |j : z_1^*(j) \neq a, z_1(j) = a| = p_1\epsilon$. We obtain the desired lower bound by arbitrariness of $\boldsymbol{C}$.

□

**Lemma 2** (Polynomial MLE error via union bound)**.** Consider the setup in Lemma 1. With probability tends to 1 as $p \to \infty$, we have

$$\ell(\hat{z}_{k,\text{MLE}}, z_k^*) \lesssim \frac{\sigma^2}{p^{K-1}\Delta_{\min}^2}, \quad \text{for all } k \in [K].$$

*Proof of Lemma 2.* Without loss of generality, we consider the case where $k = 1$. We consider the probability

$$\begin{aligned}
&\mathbb{P}(\ell(\hat{z}_{1,\text{MLE}}, z_1^*) \geq \epsilon) \\
&\leq \mathbb{P}(\text{there exists a } (z_k, \boldsymbol{\theta}_k, \mathcal{S}) \text{ such that } \ell(z_1, z_1^*) \geq \epsilon \text{ and } \|\mathcal{Y} - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2 \leq \|\mathcal{Y} - \mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*)\|_F^2) \\
&\leq \sum_{z_1, \ell(z_1, z_1^*) \geq \epsilon} \mathbb{P}(\inf_{\{z_k\}_{k>1}, \boldsymbol{\theta}_k, \mathcal{S}} \|\mathcal{Y} - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2 \leq \|\mathcal{Y} - \mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*)\|_F^2) \\
&\leq \sum_{z_1, \ell(z_1, z_1^*) \geq \epsilon} \mathbb{P}\left( \sup_{\{z_k\}_{k>1}, \boldsymbol{\theta}_k, \mathcal{S}} \frac{\langle \mathcal{E}, \mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S}) \rangle}{\|\mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F} \gtrsim \inf_{\{z_k\}_{k>1}, \boldsymbol{\theta}_k, \mathcal{S}} \|\mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})\|_F \right) \\
&\leq \sum_{z_1, \ell(z_1, z_1^*) \geq \epsilon} \mathbb{P}\left( \sup_{\mathcal{T} \in \mathcal{Q}(2r_1, \ldots, 2r_K) \cap \{\|\mathcal{T}\|_F = 1\}} \langle \mathcal{T}, \mathcal{E} \rangle \geq \sqrt{p^K \Delta_{\min}^2 \epsilon} \right), \qquad (4)
\end{aligned}$$

where $\mathcal{Q}(\boldsymbol{r})$ is the collection the all tensors with tucker rank smaller $\boldsymbol{r}$, the last inequality follows from Lemma 1 and the fact that $\mathcal{X}(z_k^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*) - \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})$ has rank smaller than $(2r_1, \ldots, 2r_K)$ . Note that $|z_1, \ell(z_1, z_1^*) \geq \epsilon| \lesssim r^p$. Hence, by Han et al. (2022, Lemma E5), the probability (4) is of order $\exp(-p)$ when $\sqrt{p^K \Delta_{\min}^2 \epsilon} \geq C\sigma\sqrt{Kpr + r^K}$ for some constant $C$ large enough. Therefore, we have $\ell(\hat{z}_{k,\text{MLE}}, z_k^*) \lesssim \frac{\sigma^2}{p^{K-1}\Delta_{\min}^2}$ for all $k \in [K]$ with high probability tends to 0 as $p \to \infty$. $\square$

**Remark 1.** The result in Lemma 2 agrees with the misclassification results in Lemma 12 in the manuscript. Hence, we still can not improve the accuracy in $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ based on current polynomial MLE results, and thus the $K \geq 3$ condition can not be removed.

## 3 MLE error under TBM

We show the MLE $\hat{z}_k$ achieves polynomial misclassification error. Let $\mathcal{X}(\boldsymbol{M}_k, \mathcal{S}) \coloneqq \mathcal{X}(z_k, \boldsymbol{\theta}_k, \mathcal{S})$ with $\boldsymbol{\theta}_k = \mathbf{1}$.

**Lemma 3** (Signal perturbation by misclassification)**.** Consider the general TBM with true parameters $(\boldsymbol{M}_k^*, \mathcal{S}^*)$, dimensions $p_k \asymp p$, and number of clusters $r_k \asymp r$, for all $k \in [K]$. Assume the minimal cluster proportion $\tau > c$ for some constant $c$. Consider the membership matrices $\boldsymbol{M}_k$ such that $\max_{k \in [K]} \text{MCR}(\boldsymbol{M}_k, \boldsymbol{M}_k^*) = \epsilon$. Then,

$$\|\mathcal{X}(\boldsymbol{M}_k^*, \mathcal{S}^*) - \mathcal{X}(\boldsymbol{M}_k, \mathcal{S})\|_F^2 \gtrsim p^K \delta_{\min}^2 \epsilon,$$

where $\delta_{\min}^2 \leq \min_{k \in [K]} \min_{a \neq b} \|\text{Mat}_k(\mathcal{S}^*)_{a:} - \text{Mat}_k(\mathcal{S}^*)_{b:}\|_F^2$ denote the minimal gap in the core tensor.

*Proof of Lemma 3.* Without loss of generality, let $\text{MCR}(\boldsymbol{M}_1, \boldsymbol{M}_1^*) = \epsilon$. Similar as the proof in Lemma 1, we define

$$\boldsymbol{C} = \text{Mat}_1(\mathcal{C}), \quad \text{where} \quad \mathcal{C} = \mathcal{S} \times_2 \boldsymbol{M}_2 \times_3 \cdots \times_K \boldsymbol{M}_K$$

and $\boldsymbol{C}^*$ with true parameters. Note that

$$\min_{a \neq b \in [r]} \|\boldsymbol{C}_{a:}^* - \boldsymbol{C}_{b:}^*\|_F^2 \geq \min_{a \neq b} \|\boldsymbol{S}_{a:}^* - \boldsymbol{S}_{b:}^*\|_F^2 \prod_{k>1} \lambda^2(\boldsymbol{M}_k^*) \gtrsim p^{K-1}\delta_{\min}^2, \tag{5}$$

where the last inequality follows from the fact that $\lambda(\boldsymbol{M}_k) \gtrsim \sqrt{p_k \tau} \asymp \sqrt{p}$ for all $k \in [K]$. Rewrite the function $\mathcal{X}(\boldsymbol{M}_k, \mathcal{S})$ as $\mathcal{X}(\boldsymbol{M}_1, \boldsymbol{C})$. Then, for an arbitrary fixed $\boldsymbol{C}$, we have

$$\begin{aligned}
&\|\mathcal{X}(\boldsymbol{M}_1^*, \boldsymbol{C}^*) - \mathcal{X}(\boldsymbol{M}_1, \boldsymbol{C})\|_F^2 \\
&= \sum_{a \in [r]} \left[ \sum_{i: \boldsymbol{M}_{1,ia}^* = \boldsymbol{M}_{1,ia} = 1} \|\boldsymbol{C}_{a:}^* - \boldsymbol{C}_{a:}\|_F^2 + \sum_{b \neq a} \sum_{j: \boldsymbol{M}_{1,ib}^* = 1, \boldsymbol{M}_{1,ia} = 1} \|\boldsymbol{C}_{b:}^* - \boldsymbol{C}_{a:}\|_F^2 \right] \\
&\geq p_1 \epsilon \min_{b \neq a} \|\boldsymbol{C}_{a:}^* - \boldsymbol{C}_{b:}^*\|_F^2 \\
&\geq p^K \delta_{\min}^2 \epsilon,
\end{aligned}$$

where the second inequality follows from the triangle inequality and the last inequality follows from (5). We obtain the desired lower bound by arbitrariness of $\boldsymbol{C}$. □

**Lemma 4** (Polynomial MLE error via union bound)**.** Consider the setup in Lemma 3. With probability tends to 1 as $p \to \infty$, we have

$$\text{MCR}(\hat{\boldsymbol{M}}_{k,\text{MLE}}, \boldsymbol{M}_k^*) \lesssim \frac{\sigma^2}{p^{K-1}\delta_{\min}^2}, \quad \text{for all } k \in [K].$$

*Proof of Lemma 4.* Follow the proof of Lemma 2 with $\boldsymbol{\theta}_k^* = \boldsymbol{\theta}_k = \mathbf{1}$ and Lemma 3. We obtain the desired results. □

**Remark 2.** In Wang and Zeng (2019), we have $\text{MCR}(\hat{\boldsymbol{M}}_{k,\text{MLE}}, \boldsymbol{M}_k^*)$ of order $\mathcal{O}\left(\frac{p^{-(K-1)/2}\|\mathcal{S}\|_{\max}\sigma}{\delta_{\min}}\right)$ when $\tau \geq c$ for some constant $c$. Now, by Lemma 4, we have improve the TBM MLE accuracy to $\mathcal{O}(\sigma^2 p^{-(K-1)}/\delta_{\min}^2)$.

## References


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