Conjecture 1

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Conjecture 1 (Tail bounds for empirical process)**.** Consider the correlated pairs of normal variables (X_i, Y_i) for $i \in [n]$, where $X_i, Y_i \sim N(0,1)$ and $\text{cov}(X_i, Y_i) = \rho$. Let $\rho = \sqrt{1-\sigma^2}$, and F_n, G_n denote the empirical CDF of $\{X_i\}$ and $\{Y_i\}$. Then, the L_p norm between F_n and G_n satisfies:

1. if $\rho > 0$,
$$\mathbb{P}(\|F_n - G_n\|_p \geq \sqrt{\frac{\sigma}{n}}) \leq C_1 \exp\left(-\frac{1}{\sigma}\right); \tag{1}$$

2. if $\rho = 0$,
$$\mathbb{P}(\|F_n - G_n\|_p \leq \sqrt{\frac{\sigma}{n}}) \leq C_2 \exp\left(-\frac{1}{\sigma}\right), \tag{2}$$

for $p \in [1, \infty)$ with universal positive constants C_1 and C_2.

Proof Sketches: We prove the inequalities (1) and (2) separately.

For inequality (1), by Ding et al. (2021) (Need to figure out the specific derivation), we have
$$\|F_n - G_n\|_p = \mathcal{O}\left(\sqrt{\frac{\sigma}{n}}\right),$$
for general $p \geq 1$ when n is large enough. Therefore, for a fixed $q \geq 1$ and fixed $\lambda > 0$, we have
$$\lim_{n\to\infty} \mathbb{E}[\exp(\lambda\sqrt{n}\|F_n - G_n\|_q)] \leq C\exp(\lambda\sigma), \tag{3}$$
for some positive constant C. The inequality (3) then indicates that $\|F_n - G_n\|_q$ is a sub-Gaussian when n is large enough, i.e.,
$$\mathbb{E}[\exp(\lambda\|F_n - G_n\|_q)] = C'\exp(\lambda\sqrt{\sigma/n} + \lambda^2\xi^2/2).$$
for some positive constant C' with mean $\mathcal{O}(\sqrt{\sigma/n})$ and sub-Gaussianity parameter $\xi^2 \leq \mathcal{O}(\sqrt{\sigma/n})$. Otherwise, the inequality (3) dose not hold when $\xi^2 \gtrsim \sqrt{\sigma/n}$. Take $\xi^2 = \sigma^2/n$. By Chernoff bound, we have
$$\mathbb{P}(\|F_n - G_n\|_q \geq \mathbb{E}[\|F_n - G_n\|_q] + t) \leq \exp\left(-\frac{t^2}{2\xi^2}\right) \lesssim \exp\left(-\frac{nt^2}{\sigma^2}\right).$$
Take $t \asymp \sqrt{\sigma/n}$. We have
$$\mathbb{P}(\|F_n - G_n\|_q \geq C''\sqrt{\frac{\sigma}{n}}) \leq C_1\exp\left(-\frac{1}{\sigma}\right).$$

For inequality (2), we follow the inequality (14) in Ding et al. (2021) (Also need to figure out the specific derivation).

Therefore, when n is large enough, we have Conjecture 1.

References

Ding, J., Ma, Z., Wu, Y., and Xu, J. (2021). Efficient random graph matching via degree profiles. *Probability Theory and Related Fields*, 179(1):29–115.