Graphic Lasso: Scaled membership

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1 Simple case

Now we consider a simple case that K categories share the same structure of the precision matrix but with different scalar coefficients. To make the parameters identifiable, we need the assumptions mentioned in the Proposition 2 of the Note 010921. The problem is stated as following.

$$\begin{aligned}
\min_{\{u,\Theta\}} \quad & \mathcal{L}(u,\Theta) = \sum_{k=1}^{K} \langle S^k, \Omega^k \rangle - \log\det(\Omega^k), \\
s.t. \quad & \Omega^k = u_k \Theta, \quad k = 1,...,K, \\
& u_k \geq 0, \|u\|_F = 1,
\end{aligned}$$

1.1 Trivial Accuracy

There is a trivial accuracy rate if we simply consider $u_k\Theta$ as K different precision matrix.

Lemma 1 (Trivial Accuracy)**.** *Let $\{u,\Theta\}$ denote the true parameters. Consider a estimation $\{\hat{u},\hat{\Theta}\}$ such that $\mathcal{L}(\hat{u},\hat{\Theta}) \geq \mathcal{L}(u,\Theta)$. With probability tends to 1 as $n \to \infty$, we have the accuracy*

$$\sum_{k=1}^{K} \left\| \hat{\Omega}^k - \Omega^k \right\|_F = \sum_{k=1}^{K} \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F^2 \leq CK\sqrt{\frac{p^2 \log p}{n}}.$$

Remark 1. The Trivial Accuracy does not take the advantage of the same structure. So, the accuracy is of order $\mathcal{O}(K)$ which should not be optimal.

1.2 Sharp Accuracy

If we consider the common structure of precision matrix, my conjecture is that the accuracy should be of order $\mathcal{O}(\sqrt{K})$.

Lemma 2 (Sharp Accuracy(conjecture))**.** *Let $\{u,\Theta\}$ denote the true parameters. Consider a estimation $\{\hat{u},\hat{\Theta}\}$ such that $\mathcal{L}(\hat{u},\hat{\Theta}) \geq \mathcal{L}(u,\Theta)$. With probability tends to 1 as $n \to \infty$, we have the accuracy*

$$\sum_{k=1}^{K} \left\| \hat{\Omega}^k - \Omega^k \right\|_F = \sum_{k=1}^{K} \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F \leq C\sqrt{K}\sqrt{\frac{p^2 \log p}{n}}.$$

Proof. Note that

$$\sum_{k=1}^{K} \left\| \hat{u}_k \hat{\Theta} - u_k \Theta \right\|_F^2 \le \sum_{k=1}^{K} u_k^2 \left\| \hat{\Theta} - \Theta \right\|_F^2 + \sum_{k=1}^{K} \left\| (\hat{u}_k - u_k) \hat{\Theta} \right\|_F^2,$$

where $\sum_{k=1}^{K} u_k^2 = 1$, the first term represents the estimation error from Θ, and the second term represents the estimation error from u. Consider the function

$$\begin{aligned} G(\hat{u}, \hat{\Theta}) &= \mathcal{L}(\hat{u}, \hat{\Theta}) - \mathcal{L}(u, \Theta) \\ &= \sum_{k=1}^{K} \langle S^k, \hat{u}_k \hat{\Theta} \rangle - \langle S^k, u_k \Theta \rangle - \log\det(\hat{u}_k \hat{\Theta}) + \log\det(u_k \Theta) \\ &= G_1(\hat{u}, \hat{\Theta}) + G_2(\hat{u}, \hat{\Theta}), \end{aligned}$$

where

$$\begin{aligned} G_1(\hat{u}, \hat{\Theta}) &= \sum_{k=1}^{K} \langle S^k, u_k(\hat{\Theta} - \Theta) \rangle - \log\det(u_k \hat{\Theta}) + \log\det(u_k \Theta), \\ G_2(\hat{u}, \hat{\Theta}) &= \sum_{k=1}^{K} \langle S^k, (\hat{u}_k - u_k) \hat{\Theta} \rangle - \log \hat{u}_k / u_k \end{aligned}$$

Consider G_1. Let $\Delta = \hat{\Theta} - \Theta$. By Taylor Expansion, we have

$$\begin{aligned} -\log\det(u_k \hat{\Theta}) + \log\det(u_k \Theta) &\ge -\langle (u_k \Theta)^{-1}, u_k \Delta \rangle + \frac{1}{4\tau^2} \left\| u_k \Delta \right\|_F^2, \\ &\ge -\langle u_k^{-1} \Sigma, u_k \Delta \rangle + \frac{1}{4\tau^2} u_k^2 \left\| \Delta \right\|_F^2 \end{aligned}$$

where τ is the max singular value of Θ. Then, we have

$$\begin{aligned} G_1(\hat{u}, \hat{\Theta}) &\ge \sum_{k=1}^{K} \langle S^k - u_k^{-1} \Sigma, u_k \Delta \rangle + \frac{1}{4\tau^2} u_k^2 \left\| \Delta \right\|_F^2 \\ &= \langle \sum_{k=1}^{K} u_k S^k - K\Sigma, \Delta \rangle + \frac{1}{4\tau^2} \left\| \Delta \right\|_F^2. \end{aligned}$$

Let $X_1^k, ..., X_n^k \sim_{i.i.d.} \mathcal{N}_p(0, \Sigma/u_k), k = 1, ..., K$. Note that

$$\frac{1}{K} \sum_{k=1}^{K} u_k S_{jl}^k = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n} \sum_{i=1}^{n} \Big((\sqrt{u_k} X_{ij}^k)(\sqrt{u_k} X_{il}^k) - (\sqrt{u_k} X_{.j}^k)(\sqrt{u_k} X_{.l}^k) \Big).$$

Since $\sqrt{u_k} X_i^k \sim \mathcal{N}(0, \Sigma)$, we have

$$\left| \frac{1}{nK} \sum_{k=1}^{K} \sum_{i=1}^{n} (\sqrt{u_k} X_{ij}^k)(\sqrt{u_k} X_{il}^k) - \Sigma_{jl} \right| \le C \sqrt{\frac{\log p}{nK}},$$

with hight probability. Thus, we have

$$\begin{aligned}
G_1(\hat{u}, \hat{\Theta}) &\geq \frac{1}{4\tau^2} \|\Delta\|_F^2 - C\sqrt{K}\sqrt{\frac{\log p}{n}} \|\Delta\|_1 \\
&\geq \frac{1}{4\tau^2} \|\Delta\|_F^2 - C\sqrt{K}\sqrt{\frac{p^2 \log p}{n}} \|\Delta\|_F \qquad (1)
\end{aligned}$$

Consider G_2. Note that

$$\begin{aligned}
-\log \hat{u}_k/u_k &= -\log\det(\hat{u}_k\Theta) + \log\det(u_k\Theta) \\
&\geq -\langle u_k^{-1}\Sigma, (\hat{u}_k - u_k)\Theta\rangle + \frac{1}{4\tau^2}(\hat{u}_k - u_k)^2 \|\Theta\|_F^2 .
\end{aligned}$$

Plug into the G_2. We have

$$G_2(\hat{u}, \hat{\Theta}) = \sum_{k=1}^{K} \langle S^k - u_k^{-1}\Sigma, (\hat{u}_k - u_k)\hat{\Theta}\rangle + \langle u_k^{-1}\Sigma, (\hat{u}_k - u_k)(\hat{\Theta} - \Theta)\rangle + \frac{1}{4\tau^2}(\hat{u}_k - u_k)^2 \|\Theta\|_F^2$$

Conjecture.

$$\begin{aligned}
G_2(\hat{u}, \hat{\Theta}) \geq \frac{1}{4\tau^2} &\left(\sum_{k=1}^{K} \left\|(\hat{u}_k - u_k)\hat{\Theta}\right\|_F\right)^2 - C\sqrt{K}\sqrt{\frac{p^2 \log p}{n}} \sum_{k=1}^{K} \left\|(\hat{u}_k - u_k)\hat{\Theta}\right\|_F \\
&+ \frac{1}{2\tau^2} \|\Delta\| \sum_{k=1}^{K} \left\|(\hat{u}_k - u_k)\hat{\Theta}\right\|_F . \qquad (2)
\end{aligned}$$

Combining the upper bounds (1) with (2), we have

$$\begin{aligned}
0 &\geq G_1(\hat{u}, \hat{\Theta}) + G_2(\hat{u}, \hat{\Theta}) \\
&\geq \left(\frac{1}{2\tau} \|\Delta\|_F + \frac{1}{2\tau} \sum_{k=1}^{K} \left\|(\hat{u}_k - u_k)\hat{\Theta}\right\|_F\right)^2 - C\sqrt{K}\sqrt{\frac{p^2 \log p}{n}} \left(\|\Delta\|_F + \sum_{k=1}^{K} \left\|(\hat{u}_k - u_k)\hat{\Theta}\right\|_F\right),
\end{aligned}$$

which implies the accuracy rate. □