# Linear Algebra

A summary for MIT 18.06SC

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## 1 Matrices & Spaces

### 1.1 Basic concepts

- Given vectors $v_1, ..., v_n$ and scalars $c_1, ..., c_n$, the sum $c_1v_1 + \cdots + c_nv_n$ is called a *linear combination* of $v_1, ..., v_n$.
- The vectors $v_1, ..., v_n$ are *linearly independent* (or just *indepedent*) if $c_1v_1 + \cdots + c_nv_n = 0$ holds only when all $c_1, ..., c_n = 0$. If the vectors $v_1, ..., v_n$ are *dependent*, there exist scalars $c_1, ..., c_n$ which are not all equal to 0 such that $c_1v_1 + \cdots + c_nv_n = 0$.
- Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and a vector $x \in \mathbb{R}^n$ the multiplication $\boldsymbol{A}x$ is a linear combination of the columns of $\boldsymbol{A}$, and $x^T\boldsymbol{A}$ is a linear combination of the rows of $\boldsymbol{A}$.
- Matrix multiplication is typically not communicative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$. Lemma 1 describes a special case where matrix multiplication is communicative.
- Suppose $\boldsymbol{A}$ is a square matrix. The matrix $\boldsymbol{A}$ is *invertible* or *non-singular* if there exists a $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I$. Otherwise, the matrix $\boldsymbol{A}$ is *singular*, and the determinant of $\boldsymbol{A}$ is 0.
- The inverse of a matrix product $\boldsymbol{AB}$ is $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. The product of invertible matrices is still invertible.
- The transpose of a matrix product $\boldsymbol{AB}$ is $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$. For any invertible matrix $\boldsymbol{A}$, $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$.
- A matrix $\boldsymbol{Q}$ is orthogonal if $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$. A matrix $\boldsymbol{Q}$ is unitary if $\boldsymbol{Q}^* = \boldsymbol{Q}^{-1}$, where $\boldsymbol{Q}^*$ is the conjugate transpose of $\boldsymbol{Q}$.

**Lemma 1** (Communicative matrix multiplication)**.** *For matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$, the matrix multiplication of $\boldsymbol{A}$ and $\boldsymbol{B}$ is communicative, i.e. $\boldsymbol{AB} = \boldsymbol{BA}$, if all the normalized eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical.*

*Proof.* If all the normalized eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical, there exists a matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}_A\boldsymbol{Q}^{-1}, \quad \boldsymbol{B} = \boldsymbol{Q}\boldsymbol{D}_B\boldsymbol{Q}^{-1},$$

where columns of $\boldsymbol{Q}$ are normalized eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\boldsymbol{D}_A \in \mathbb{R}^{n\times n}, \boldsymbol{D}_B \in \mathbb{R}^{n\times n}$ are diagonal matrices whose diagonal elements are corresponding eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$. Because matrix multiplication is communicative for two diagonal matrices with same dimensions, we have

$$\boldsymbol{AB} = \boldsymbol{QD}_A\boldsymbol{Q}^{-1}\boldsymbol{QD}_B\boldsymbol{Q}^{-1} = \boldsymbol{QD}_A\boldsymbol{D}_B\boldsymbol{Q}^{-1} = \boldsymbol{QD}_B\boldsymbol{D}_A\boldsymbol{Q}^{-1} = \boldsymbol{QD}_B\boldsymbol{Q}^{-1}\boldsymbol{QD}_A\boldsymbol{Q}^{-1} = \boldsymbol{BA}.$$

Therefore, the matrix multiplication of $\boldsymbol{A}$ and $\boldsymbol{B}$ is communicative. □

### 1.2 Permutation of matrices

For any matrix $\boldsymbol{A}$, we swap its rows by multiplying a *permutation matrix* $\boldsymbol{P}$ on the left of $\boldsymbol{A}$. For example,

$$\boldsymbol{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix}$$

where $a_k$ refers to the $k$-th row of $\boldsymbol{A}$. The inverse of permutation matrix $\boldsymbol{P}$ is $\boldsymbol{P}^{-1} = \boldsymbol{P}^T$, which implies the orthogonality of permutation matrix. For an $n \times m$ matrix, there are $n!$ different row permutation matrices, which form a *multiplicative group*.

Similarly, we also swap the columns of the matrix $\boldsymbol{A}$ by multiplying a permutation matrix on the right of $\boldsymbol{A}$.

### 1.3 Elimination of matrices

Elimination is an important technique in linear algebra. We eliminate the matrix by multiplications and subtractions. Take a 3-by-3 matrix $\boldsymbol{A}$ as an example.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 2}} \boldsymbol{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

In step 1, we choose the number 1 in row 1 column 1 as a *pivot*, then we recopy the first row, multiply an appropriate number (in this case, 3), and subtract those values from the numbers in the second row. We have thus eliminated 3 in row 2 column 1. Similarly, in step 2, we choose 2 in row 2 column 2 as a pivot, and eliminate the number 4 in row 3 column 2. The number 5 in row 3 column 3 is also a pivot. The matrix $\boldsymbol{U}$ is an upper traingular matrix.

The *elimination matrix* used to eliminate the entry in row $m$ column $n$ is denoted $\boldsymbol{E}_{mn}$. In the previous example,

$$\boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}; \quad \boldsymbol{E}_{32}(\boldsymbol{E}_{21}\boldsymbol{A}) = \boldsymbol{U}.$$

Pivots are non-zero. If there is a 0 in the pivot position, we must exchange the row with one below to get a non-zero value in pivot position. If there is not non-zero value below the 0 pivot, we skip this column, and find a pivot in next column.

We write $\boldsymbol{E}_{32}(\boldsymbol{E}_{21}\boldsymbol{A}) = (\boldsymbol{E}_{32}\boldsymbol{E}_{21})\boldsymbol{A} = \boldsymbol{U}$ because matrix multiplication is associative. Let $\boldsymbol{E}$ denote the product of all elimination matrices. If we need to permute the rows during the process, we multiply a permutation matrix on the left of $\boldsymbol{A}$. Therefore, the elimination process of $\boldsymbol{A}$ is

$$\boldsymbol{EPA} = \boldsymbol{U}, \tag{1}$$

where $\boldsymbol{U}$ is an upper triangular matrix.

Next, we prove the invertibility of the elimination matrix.

**Lemma 2** (Invertiblity of elimination matrix)**.** *Suppose there is an elimination matrix $\boldsymbol{E}_{ij} \in \mathbb{R}^{n\times n}$ that multiplies a scalar $-c$ to the $j$-th row, and subtracts the row from $i$-th row, where $i \neq j$. The matrix $\boldsymbol{E}_{ij}$ is invertible.*

*Proof.* We write the elimination matrix as

$$\boldsymbol{E}_{ij} = \boldsymbol{I}_n + ce_ie_j^T,$$

where $e_i \in \mathbb{R}^n$ denotes the vector with value 1 in the $i$-th entry and value 0 elsewhere. Note that $e_i^T e_j = 0$ because $i \neq j$. We have

$$(\boldsymbol{I}_n + ce_ie_j^T)(\boldsymbol{I}_n - ce_ie_j^T) = \boldsymbol{I}_n - c^2 e_ie_j^T e_ie_j^T = \boldsymbol{I}_n; \quad (\boldsymbol{I}_n - ce_ie_j^T)(\boldsymbol{I}_n + ce_ie_j^T) = \boldsymbol{I}_n.$$

Therefore, $\boldsymbol{I}_n - ce_ie_j^T$ is the inverse of $\boldsymbol{E}_{ij}$. The elimination matrix $\boldsymbol{E}_{ij}$ is invertible. □

**Corollary 1** (Inverse of elimination matrix)**.** Suppose the elimination matrix $\boldsymbol{E}_{ij}$ in lemma 2 is a lower/upper-triangular matrix. The inverse $\boldsymbol{E}_{ij}^{-1}$ is also a lower/upper-triangular matrix.

*Proof.* By the proof of lemma 2, the matrix $\boldsymbol{E}_{ij}$ and its inverse are written as

$$\boldsymbol{E}_{ij} = \boldsymbol{I}_n + ce_ie_j^T, \quad \boldsymbol{E}_{ij}^{-1} = \boldsymbol{I}_n - ce_ie_j^T.$$

WLOG, we assume $\boldsymbol{E}_{ij}$ is a lower-triangular matrix. Then $ce_ie_j^T$ and $-ce_ie_j^T$ are also lower-triangular matrices. Therefore, $\boldsymbol{E}_{ij}^{-1}$ is a lower-triangular matrix. □

### 1.4 Gauss-Jordan Elimination

We also use elimination to find the inverse of any invertible matrix.

Suppose $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is an invertible matrix. The inverse of $\boldsymbol{A}$, $\boldsymbol{A}^{-1}$ , satisfies

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n. \tag{2}$$

Suppose there is an elimination $\boldsymbol{E}$ such that $\boldsymbol{E}\boldsymbol{A} = \boldsymbol{I}_n$. Multiplying $\boldsymbol{E}$ on the both side of the equation (2), we have $\boldsymbol{E}\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} = \boldsymbol{E}$. To obtain an such $\boldsymbol{E}$, we eliminate the *augmented matrix* $[\boldsymbol{A}|\boldsymbol{I}_n]$ until $\boldsymbol{A}$ becomes $\boldsymbol{I}_n$. Then, the augmented matrix becomes $\boldsymbol{E}[\boldsymbol{A}|\boldsymbol{I}_n] = [\boldsymbol{I}_n|\boldsymbol{E}]$, where $\boldsymbol{E}$ is the inverse of $\boldsymbol{A}$.

We call this elimination process of finding $\boldsymbol{E}$ as *Gauss-Jordan Elimination.*

### 1.5 Factorization of matrices

By elimination, for any square matrix $\boldsymbol{A}$, we have equation (1). By lemma 2, $\boldsymbol{E}$ is invertible. We multiply $\boldsymbol{E}^{-1}$ on both sides of equation (1). We have,

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{E}^{-1}\boldsymbol{U}.$$

Note that $\boldsymbol{E}$ is a lower-triangular matrix. By corollary 1, $\boldsymbol{E}^{-1}$ is also a lower-triangular matrix. Let $\boldsymbol{L}$ denote $\boldsymbol{E}^{-1}$, where the letter $L$ refers to "lower triangular". Therefore, any square matrix $\boldsymbol{A}$ has a factorization:

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}, \tag{3}$$

where $\boldsymbol{U}$ is an upper triangular matrix with pivots on the diagonal, $\boldsymbol{L}$ is lower triangular matrix with ones on the diagonal, and $\boldsymbol{P}$ is a permutation matrix. However, the equation (3) is not the unique factorization of $\boldsymbol{A}$. For example, $c\boldsymbol{L}$ and $c^{-1}\boldsymbol{U}$ also factorize $\boldsymbol{A}$, where $c$ is a non-zero scalar.

## 1.6 Time complexity of elimination

For an $n$-by-$n$ matrix, a single elimination multiplies a selected row and subtracts the selected row from another row. A single elimination requires $\mathcal{O}(n)$ operations. To eliminate the elements below the first diagonal element, we need repeat single eliminations $(n-1)$ times, and thus require $\mathcal{O}(n^2)$ operations. Similarly, we require $\mathcal{O}((n-1)^2)$ to eliminate the elements below the second diagonal element. Repeat the elimination until we meet the $n$-th diagonal element. Therefore, we require $\mathcal{O}(n^3)$ operations to obtain an upper-triangular matrix by elimination:

$$1^2 + 2^2 + \cdots + (n)^2 = \sum_i^n i^2 \approx \int_0^n x^2 dx = \frac{1}{3}n^3 = \mathcal{O}(n^3).$$

## 1.7 Reduced row echelon form of matrices

In previous sections, we convert any matrix $\boldsymbol{A}$ to an upper triangular matrix $\boldsymbol{U}$. Next, we convert $\boldsymbol{U}$ into *reduced row echelon form* (RREF), which is a simpler form than upper triangle. We use $\boldsymbol{R} = RREF(\boldsymbol{A})$ to denote the reduced row echelon form of $\boldsymbol{A}$. In $\boldsymbol{R}$, the pivots are equal to 1, and the elements above and below the pivots are eliminated to 0. In the previous example,

$$\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is another example,

$$\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{0 above and below pivots}} \boldsymbol{R} = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume there are $r$ pivots in $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. With proper permutation, the matrix $\boldsymbol{R}$ is in form $\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$, where $\boldsymbol{F} \in \mathbb{R}^{r \times (n-r)}$ is an arbitrary matrix. The columns in $\boldsymbol{A}$ which correspond to the identity matrix $\boldsymbol{I}_r$ are called *pivot columns*. The other columns are *free columns*.

## 1.8 Vector space, Subspace and Column space

- *Vector space* is a collection of vectors that is closed under linear combination (addition and multiplication by any real number); i.e. for any vectors in the collection, all the combinations of these vectors are still in the collection.

- *Subspaces of the vector space* is a vector space that is contained inside of another vector space.

Note that any vector space or subspace must include an origin. For a vector space $\mathcal{A}$, the subspace of $\mathcal{A}$ can be $\mathcal{A}$ itself or a set that contains only a zero vector.

- Vectors $v_1, ..., v_n$ *span* a space that consists all the combination of those vectors.

- *Column space* of matrix $\boldsymbol{A}$ is the space spanned by the columns of $\boldsymbol{A}$. Let $C(\boldsymbol{A})$ denote the column space of $\boldsymbol{A}$.

If $v_1, ..., v_n$ span a space $\mathcal{S}$, then $\mathcal{S}$ is the smallest space that contain those vectors.

- *Basis* of a vector space is a sequence of vectors $v_1, ..., v_n$ that satisfy: (1) $v_1, ..., v_n$ are independent; (2) $v_1, ..., v_n$ span the space.
- *Dimension* of the space is the number of vectors in a basis of the space. Let $dim(\mathcal{A})$ denote the dimension of space $\mathcal{A}$.

### 1.9 Matrix rank

The *rank* of a matrix $\boldsymbol{A}$ is defined as the dimension of the columns space of $\boldsymbol{A}$. Rank of matrix $\boldsymbol{A}$ is also equal to the number of pivot columns of $\boldsymbol{A}$. Let $rank(\boldsymbol{A})$ denote the rank of matrix $\boldsymbol{A}$. We have

$$rank(\boldsymbol{A}) = \# \text{ of pivot columns of } \boldsymbol{A} = dim(C(\boldsymbol{A})). \tag{4}$$

If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $rank(\boldsymbol{A}) = r$ , we have $r \le \min\{m, n\}$. We say the matrix is *full rank* if $r = \min\{m, n\}$.

The rank of a square matrix is closely related to its invertibility.

**Lemma 3** (Full rankness and invertibility)**.** *A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is full rank, if and only if $\boldsymbol{A}$ is an invertible matrix.*

*Proof.* First, we assume $\boldsymbol{A}$ is full rank and prove the invertibility of $\boldsymbol{A}$.
Consider the RREF form of $\boldsymbol{A}$, $\boldsymbol{R} = RREF(\boldsymbol{A})$. There are elimination matrix $\boldsymbol{E}$ and permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{EPA} = \boldsymbol{R}.$$

By the full rankness of $\boldsymbol{A}$, $\boldsymbol{A}$ has $n$ pivot columns, and thus $\boldsymbol{R} = \boldsymbol{I}_n$. By lemma 2, $\boldsymbol{E}$ is invertible. The permutation matrix $\boldsymbol{P}$ is also invertible. Then, the matrix product $\boldsymbol{EP}$ is invertible, and $\boldsymbol{A}$ is the inverse of $\boldsymbol{EP}$. Therefore, $\boldsymbol{A}$ is invertible.
Second, we assume $\boldsymbol{A}$ is invertible and prove the full rankness of $\boldsymbol{A}$ by contradiction.
Assume $\boldsymbol{A}$ has an inverse $\boldsymbol{A}^{-1}$ and $rank(\boldsymbol{A}) < n$. By equation (4), $dim(C(\boldsymbol{A})) = rank(\boldsymbol{A}) < n$, which implies that the columns of $\boldsymbol{A}$ are linearly dependent. Then, there exists a non-zero vector $v$ such that

$$\boldsymbol{A}v = 0. \tag{5}$$

Multiplying $\boldsymbol{A}^{-1}$ on both sides of equation (5), we have

$$v = \boldsymbol{A}^{-1}0 = 0.$$

However, it contradicts the fact that $v$ is non-zero. Therefore, $\boldsymbol{A}$ is full rank. □

The rank of $\boldsymbol{A}$ also effects the number of solutions to the system $\boldsymbol{A}x = b$. We will discuss it in next section.

## 2 Solving $Ax = b$

Here we discuss the solutions of the linear system $Ax = b$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are vectors.

## 2.1 Solving $Ax = 0$: Nullspace

The *nullspace* of matrix $\boldsymbol{A}$ is the collection of all solutions $x$ to the system $\boldsymbol{A}x = 0$. Let $N(\boldsymbol{A})$ denote the nullspace of $\boldsymbol{A}$.

**Lemma 4** (Nullspace)**.** *The nullspace of matrix $\boldsymbol{A}$ is a vector space.*

*Proof.* We need to show $N(\boldsymbol{A})$ is closed to linear combination to prove that $N(\boldsymbol{A})$ is a vector space. For $\forall v_1, v_2 \in N(\boldsymbol{A})$, we have,

$$\boldsymbol{A}(c_1v_1 + c_2v_2) = c_1\boldsymbol{A}v_1 + c_2\boldsymbol{A}v_2 = 0, \quad \forall c_1, c_2 \in \mathbb{R}. \tag{6}$$

The equation (6) implies that any linear combination of vectors in $N(\boldsymbol{A})$ is also a vector in $N(\boldsymbol{A})$. Therefore, $N(\boldsymbol{A})$ is closed to linear combination. □

**Lemma 5** (The rank of nullspace)**.** *If $rank(\boldsymbol{A}) = r$, the rank of nullspace $rank(N(\boldsymbol{A})) = n - r$ .*

*Proof.* Let $\boldsymbol{R}$ denote the RREF($\boldsymbol{A}$). We write $\boldsymbol{R}$ in form $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$, where $\boldsymbol{F} \in \mathbb{R}^{r\times(n-r)}$ is arbitrary matrix. Let $\boldsymbol{X} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix}$. We have

$$\boldsymbol{R}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix} = 0.$$

Therefore, each column of $\boldsymbol{X}$ is a special solution to the system $\boldsymbol{A}x = 0$. Next, we show other solutions to $\boldsymbol{A}x = 0$ are linear combinations of those special solutions.
Suppose there is a solution $x = (x_1, x_2) \in N(\boldsymbol{A})$, where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$. We have

$$\boldsymbol{R}x = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{F} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \boldsymbol{F}x_2 \\ 0 \end{bmatrix} = 0.$$

That implies $x_1 = -\boldsymbol{F}x_2$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix} x_2 = \boldsymbol{X}x_2$. Any vector in $N(\boldsymbol{A})$ is a linear combination of special solutions, i.e. $C(\boldsymbol{X}) = N(\boldsymbol{A})$. Therefore, the rank of nullspace is $rank(N(\boldsymbol{A})) = dim(C(\boldsymbol{X})) = n - r$. □

Recall the definitions of pivot column and free column. In $\boldsymbol{A}x = b$, the variables in $x$ that correspond to pivot columns are called *pivot variables*, and others are *free variables*. If $rank(\boldsymbol{A}) = r$, there are $n - r$ free variables, which coincides with $rank(N(\boldsymbol{A}))$.
In the proof of lemma 5, the columns of $\boldsymbol{X} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I}_{n-r} \end{bmatrix}$ are special solutions that compose the basis of $N(\boldsymbol{A})$. Practically, we find those special solutions by assigning 1 to a free variable and 0 to other free variables, and then solving the system $\boldsymbol{A}x = 0$.

## 2.2 Solving $Ax = b$: complete solutions

**Lemma 6** (Solvability of $\boldsymbol{A}x = b$)**.** *The system $\boldsymbol{A}x = b$ is solvable only when $b \in C(\boldsymbol{A})$.*

*Proof.* If $\boldsymbol{A}x = b$ is solvable, there exists a $x$ such that $\boldsymbol{A}x = b$. For any $x$, $\boldsymbol{A}x \in C(\boldsymbol{A})$. Therefore, $b \in C(\boldsymbol{A})$. □

**Lemma 7** (Complete solution)**.** *The complete solution of* $\boldsymbol{A}x = b$ *is given by* $x_{comp} = x_p + x_n$*, where* $x_p$ *is a particular solution such that* $\boldsymbol{A}x_p = b$*, and* $x_n \in N(\boldsymbol{A})$*.*

*Proof.* Suppose $x = x_p + x_0$ is an arbitrary solution to $\boldsymbol{A}x = b$. We have

$$\boldsymbol{A}x - \boldsymbol{A}x_p = \boldsymbol{A}(x - x_p) = \boldsymbol{A}x_0 = 0.$$

Therefore, $x_0 \in N(\boldsymbol{A})$. □

Usually, we find a particular solution by assigning 0 to free variables, and solving the system $\boldsymbol{A}x = b$.

The following table discusses the rank of $\boldsymbol{A}$, the form of $\boldsymbol{R}$, dimension of nullspace $N(\boldsymbol{A})$, and solutions of $\boldsymbol{A}x = b$.

| | $r = m = n$ | $r = n < m$ | $r = m < n$ | $r < m, r < n$ |
|---|---|---|---|---|
| $\boldsymbol{R}$ | $\boldsymbol{I}$ | $\begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix}$ | $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$ | $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & 0 \end{bmatrix}$ |
| $dim(N(\boldsymbol{A}))$ | 0 | 0 | $n - r$ | $n - r$ |
| # solutions to $\boldsymbol{A}X = b$ | 1 | 0 or 1 | infinitely many | 0 or infinitely many |