

Initialization convergence

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1 Model&Algorithm

Suppose we have p nodes from r communities and observe the adjacent tensor $\mathcal{Y} \in \{0, 1\}^{p \times p \times p}$ whose entry \mathcal{Y}_{ijk} refers to the connection of the triplet (i, j, k) . Let $\theta = (\theta_1, \dots, \theta_p) \in \mathbb{R}^p$ denote the degree-corrected parameters and $z = (z_1, \dots, z_p) \in [r]^p$ denote the clustering assignment. Consider the hDCBM model

$$\mathbb{E}[\mathcal{Y}] = \mathcal{X} = \mathcal{S} \times_1 \Theta \mathbf{M} \times_2 \Theta \mathbf{M} \times_3 \Theta \mathbf{M},$$

where $\mathcal{S} \in \mathbb{R}^{r \times r \times r}$ is the symmetric core tensor, $\Theta = \text{diag}(\theta) \in \mathbb{R}^{p \times p}$ and $\mathbf{M} \in \mathbb{R}^{p \times r}$ is the hard membership matrix based on z . Consider the parameter space $\mathcal{P}_z(r, \beta)$ for $\beta > 1$ where

$$\mathcal{P}_z(r, \beta) = \left\{ z : \frac{p}{\beta r} \leq \sum_{j=1}^p \mathbf{1}\{z_j = a\} = p_a \leq \frac{\beta p}{r}, \quad \text{for all } a \in [r] \right\},$$

and the parameter space $\mathcal{P}(\delta, \Delta_{\min}, \alpha)$ for (Θ, \mathcal{S}) , where

$$\mathcal{P}(\delta, \Delta_{\min}, \alpha) = \left\{ (\mathcal{S}, \Theta) : \begin{aligned} & \min_{u \neq u' \in [r]} \min_{v, w \in [p]} (\mathcal{S}_{uvw} - \mathcal{S}_{u'vw})^2 = \Delta_{\min}^2, \\ & \frac{1}{p_a} \sum_{i: z_i = a} \theta_i^2 \in [1 - \delta, 1 + \delta], \text{ for all } a \in [r], \\ & \max_{u, v, w \in [r]} \mathcal{S}_{uvw} \leq \alpha, \mathcal{S}_{uvw} > 0, \text{ for all } u, v, w \in [p] \end{aligned} \right\}.$$

Notations

1. Let $\bar{\theta} = \max_{j \in [p]} \theta_j$.
2. Let $\mathbf{X}_k = \mathcal{M}_k(\mathcal{X})$ and \mathbf{X}_{kj} denote the j -th row of \mathbf{X}_k .

See Algorithm 1 for detailed algorithm. The convergence of the initialization is stated below:

Theorem 1.1 (Initialization convergence(conjecture)). Suppose $\delta = o(1)$, $\Delta_{\min} > 0$, $\|\theta\|_{\max} = o(p/r)$. Let \hat{z} denote the output of Algorithm 1. With probability at least $1 - C \exp(-cp)$, we have

$$\sum_{j: \hat{z}_j \neq z_j^*} \theta_j^2 \leq C \frac{(2\beta^2 + 1)\beta r}{\Delta_{\min}^2 p(1 - \delta)} \left[\frac{16\alpha^2(1 + M)}{\Delta_{\min}^2} + 1 \right] \bar{\theta}^3 \alpha (p^{3/2}r + pr + r^3).$$

Corollary 1. Suppose $\min_{j \in [p]} \theta_j > a$. Then, we have

$$h(\hat{z}, z^*) \leq C \frac{(2\beta^2 + 1)\beta r}{\Delta_{\min}^2 p(1 - \delta)} \left[\frac{16\alpha^2(1 + M)}{\Delta_{\min}^2} + 1 \right] \bar{\theta}^3 \alpha (p^{3/2}r + pr + r^3).$$

Remark 1. To let the misclassification error decreases along with p , we need Δ_{\min}^2 increases at least as $\mathcal{O}(p^{1/2})$.

2 Proof of Theorem 1

For simplicity, we ignore the permutation of assignment here.

2.1 Gap-free estimation error bound

Note that our model is equal to

$$\mathcal{Y} = \mathcal{X} + \mathcal{Z}.$$

The \mathcal{Z} has independent entries following $\text{subG}(\bar{\theta}^3 \alpha)$, which follows by the fact that $\|\mathcal{X}\|_{\max} = \bar{\theta}^3 \alpha$. By Proposition 1 in Han et al. (2020), we have

$$\left\| \hat{\mathcal{Y}} - \mathcal{X} \right\|_F^2 \leq C \bar{\theta}^3 \alpha (p^{3/2}r + pr + r^3),$$

with probability at least $1 - C \exp(-cp)$ and C is a constant.

2.2 Measurement of misclassification

For any set $S \subset [p]$, note that

$$\begin{aligned} \sum_{j \in S} \|\mathbf{X}_{kj}\|_F^2 &= \sum_{j \in S} \theta_j^2 \left(\sum_{k, l \in [p]} \left[\theta_k \theta_l \mathcal{S}_{z_j^* z_k^* z_l^*} \right]^2 \right) \\ &\geq \sum_{j \in S} \theta_j^2 \frac{p(1 - \delta)}{\beta r} \Delta_{\min}^2, \end{aligned}$$

where the second inequality follows by the fact that at most 1 $\mathcal{S}_{uvw} \leq \Delta_{\min}$ (otherwise, the minimal gap would smaller than Δ_{\min}) and the minimal size of the cluster is $\frac{p}{\beta r}$. Hence, to bound $\sum_{j \in S} \theta_j^2$, it is sufficient to bound $\sum_{j \in S} \|\mathbf{X}_{kj}\|_F^2$.

2.3 Weighted k -means

Further, let $\mathbf{X}_{kj}^s = \mathbf{X}_{kj} / \|\mathbf{X}_{kj}\|_F$. Note that $\mathbf{X}_{kj}^s = \mathbf{X}_{ki}^s$ is $z_i^* = z_j^*$. Note that the weighted k -means implies that

$$\sum_{j=1}^p \left\| \hat{\mathbf{Y}}_{kj} \right\|_F^2 \left\| (\hat{\mathbf{Y}}_{kj}^s) - \hat{x}_{\hat{z}_j} \right\|_F^2 \leq M \sum_{j=1}^p \left\| \hat{\mathbf{Y}}_{kj} \right\|_F^2 \left\| (\hat{\mathbf{Y}}_{kj}^s) - \mathbf{X}_{kj}^s \right\|_F^2,$$

where \hat{z}, \hat{x} are the estimated assignment and centroids. Hence, we may bound the term $\left\| (\hat{\mathbf{Y}}_{kj}^s)^T - \hat{x}_{\hat{z}_j} \right\|_F^2$ by the easier term $\left\| (\hat{\mathbf{Y}}_{kj}^s)^T - \mathbf{X}_{kj}^s \right\|_F^2$. Particularly, we have

$$\sum_{j=1}^p \left\| \hat{\mathbf{Y}}_{kj} \right\|_F^2 \left\| (\hat{\mathbf{Y}}_{kj}^s) - \mathbf{X}_{kj}^s \right\|_F^2 \leq 2 \sum_{j=1}^p \left\| \mathbf{Y}_{kj} - \mathbf{X}_{kj} \right\|_F^2 = 2 \left\| \hat{\mathcal{Y}} - \mathcal{X} \right\|_F^2,$$

where the inequality follows by the triangle inequality

$$\left\| \frac{x}{\|x\|_F} - \frac{y}{\|y\|_F} \right\|_F \leq \frac{2\|x - y\|_F}{\|x\| \vee \|y\|}.$$

2.4 Quantify the number of misclassification

Similarly with Lemma 6 in [Gao et al. \(2018\)](#), let $S = \{j \in [p] : \left\| \hat{x}_{\hat{z}_j} - \mathbf{X}_{kj}^s \right\|_F \geq \frac{1}{2\alpha} \Delta_{\min}\}$. First, for $z_j^* \neq z_i^*$, we know that

$$\left\| \mathbf{X}_{kj}^s - \mathbf{X}_{ki}^s \right\|_F^2 \geq \frac{\sum_{k,l \in [p]} \Delta_{\min}^2}{\sum_{k,l \in [p]} \alpha^2} = \frac{\Delta_{\min}^2}{\alpha^2}.$$

Next, we partition the clusters in three subsets based on the sets C_u , where

$$C_u = \left\{ j \in [p] : z_j^* = u, \left\| \hat{x}_{\hat{z}_j} - \mathbf{X}_{kj}^s \right\|_F \leq \frac{1}{2\alpha} \Delta_{\min} \right\}$$

and

$$\begin{aligned} R_1 &= \{u \in [r] : C_u = \emptyset\} \\ R_2 &= \{u \in [r] : C_u \neq \emptyset, \text{ for all } i, j \in C_u, \hat{z}_i = \hat{z}_j\} \\ R_3 &= \{u \in [r] : C_u \neq \emptyset, \text{ exist } i, j \in C_u, \hat{z}_i \neq \hat{z}_j\}. \end{aligned}$$

Note that

$$\sum_{j: \hat{z}_j \neq z_j^*} \theta_j^2 \leq \sum_{j \in S} \theta_j^2 + \sum_{j \in \bigcup_{u \in R_3} C_u} \theta_j^2.$$

Also, note that $|R_2| + 2|R_3| \leq r = |R_1| + |R_2| + |R_3|$, which implies that $|R_3| \leq |R_1|$. Then, we have

$$\begin{aligned}
\sum_{j \in \cup_{u \in R_3} C_u} \theta_j^2 &\leq |R_3|(1 + \delta) \frac{\beta p}{r} \\
&\leq |R_1|(1 + \delta) \frac{\beta p}{r} \\
&\leq \frac{1 + \delta}{1 - \delta} \beta^2 \sum_{j \in \cup_{u \in R_1} C_u} \theta_j^2 \\
&\leq 2\beta^2 \sum_{j \in S} \theta_j^2,
\end{aligned}$$

where the third inequality follows by the fact that $\sum_{j \in \cup_{u \in R_1} C_u} \theta_j^2 \geq |R_1|(1 - \delta) \frac{p}{\beta r}$, and the last inequality follows by the fact that $\cup_{u \in R_1} C_u \subset \{j \in S\}$ and $\delta = o(1)$. Therefore, we obtain

$$\sum_{j: \hat{z}_j \neq z_j^*} \theta_j^2 \leq (2\beta^2 + 1) \sum_{j \in S} \theta_j^2.$$

2.5 Assemble

Now, to bound the desire misclassification rate $\sum_{j: \hat{z}_j \neq z_j^*} \theta_j^2$, we only need to bound the $\sum_{j \in S} \|(\mathcal{M}_k(\mathcal{X}))_j\|_F^2$.

Note that

$$\begin{aligned}
\sum_{j \in S} \|\mathbf{X}_{kj}\|_F^2 &\leq 2 \sum_{j \in S} \|\hat{\mathbf{Y}}_{kj}\|_F^2 + 2 \sum_{j \in S} \|\hat{\mathbf{Y}}_{kj} - \mathbf{X}_{kj}\|_F^2 \\
&\leq 2 \sum_{j \in S} \|\hat{\mathbf{Y}}_{kj}\|_F^2 + 2 \|\hat{\mathbf{Y}}_k - \mathbf{X}_k\|_F^2,
\end{aligned}$$

where the second term is bounded by first part. For the first term, note that

$$\begin{aligned}
\sum_{j \in S} \|\hat{\mathbf{Y}}_{kj}\|_F^2 &\leq \frac{4\alpha^2}{\Delta_{\min}^2} \sum_{j \in S} \|\hat{\mathbf{Y}}_{kj}\|_F^2 \|\hat{x}_{\hat{z}_j} - \mathbf{X}_{kj}^s\|_F^2 \\
&\leq \frac{8\alpha^2}{\Delta_{\min}^2} \sum_{j \in S} \|\hat{\mathbf{Y}}_{kj}\|_F^2 \left[\left\| (\hat{\mathbf{Y}}_{kj}^s) - \hat{x}_{\hat{z}_j} \right\|_F^2 + \left\| (\hat{\mathbf{Y}}_{kj}^s) - \mathbf{X}_{kj}^s \right\|_F^2 \right] \\
&\leq \frac{8\alpha^2(1 + M)}{\Delta_{\min}^2} \sum_{i=1}^n \|\hat{\mathbf{Y}}_{kj}\|_F^2 \left\| (\hat{\mathbf{Y}}_{kj}^s) - \mathbf{X}_{kj}^s \right\|_F^2 \\
&\leq \frac{16\alpha^2(1 + M)}{\Delta_{\min}^2} \left\| \hat{\mathbf{Y}}_k - (\mathcal{M}_k(\mathcal{X})) \right\|_F^2,
\end{aligned}$$

where the first inequality follows by the definition of S , third and fourth inequality follow by the statements of k -mean in Section 2.3. Then, we know that the misclassification is bounded by the

estimation error with polynomial rate,

$$\begin{aligned}
\sum_{j: \hat{z}_j \neq z_j^*} \theta_j^2 &\leq (2\beta^2 + 1) \sum_{j \in S} \theta_j^2 \\
&\leq \frac{(2\beta^2 + 1)\beta r}{\Delta_{\min}^2 p(1 - \delta)} \sum_{j \in S} \|\mathbf{X}_{kj}\|_F^2 \\
&\leq \frac{2(2\beta^2 + 1)\beta r}{\Delta_{\min}^2 p(1 - \delta)} \left[\frac{16\alpha^2(1 + M)}{\Delta_{\min}^2} + 1 \right] \left\| \hat{\mathbf{Y}}_k - (\mathcal{M}_k(\mathcal{X})) \right\|_F^2 \\
&\leq C \frac{(2\beta^2 + 1)\beta r}{\Delta_{\min}^2 p(1 - \delta)} \left[\frac{16\alpha^2(1 + M)}{\Delta_{\min}^2} + 1 \right] \bar{\theta}^3 \alpha (p^{3/2} r + pr + r^3)
\end{aligned}$$

with probability at least $1 - C \exp(-cp)$.

Algorithm 1 High-order weighted k -means clustering

Input: Observation $\mathcal{Y} \in \{0, 1\}^{p \times \dots \times p}$, r , relaxation factor in k -means $M > 1$, SCORE normalization function h .

- 1: Compute $\hat{U}_k = \text{SVD}_{r_k}(\mathcal{M}_k(\mathcal{Y}))$ for $k \in [d]$
- 2: **for** $k \in [3]$ **do**
- 3: Estimate the singular space \hat{U}_k via

$$\hat{U}_k = \text{SVD}_r(\mathcal{M}_k(\mathcal{Y} \times_1 \tilde{U}_1^T \times \dots \times_{k-1} \tilde{U}_{k-1}^T \times_{k+1} \tilde{U}_{k+1}^T \times \dots \times_3 \tilde{U}_3^T))$$

- 4: **end for**
- 5: Obtain

$$\hat{\mathcal{Y}} = \mathcal{Y} \times_1 \hat{U}_1 \hat{U}_1^T \times_2 \hat{U}_2 \hat{U}_2^T \times_3 \hat{U}_3 \hat{U}_3^T$$

- 6: **for** $k \in [3]$ **do**
- 7: Let $\mathbf{Y}_k = \mathcal{M}_k(\hat{\mathcal{Y}})$ and let $\hat{\mathbf{Y}}_{kj}$ denote the rows of $\hat{\mathbf{Y}}_k$ for $j \in [p]$. Obtain the SCORE normalized $\hat{\mathbf{Y}}_k^s$ via $\hat{\mathbf{Y}}_{kj}^s = \frac{\hat{\mathbf{Y}}_{kj}}{h(\hat{\mathbf{Y}}_{kj})}$ for $j \in [p]$.
- 8: Find the initial assignment $z_k^{(0)} \in [r]^p$ and centroids $\hat{x}_1, \dots, \hat{x}_r \in \mathbb{R}^{p^2}$ such that

$$\sum_{j=1}^p h(\hat{\mathbf{Y}}_{kj})^2 \left\| (\hat{\mathbf{Y}}_{kj}^s)^T - \hat{x}_{(z_k^{(0)})_j} \right\|_F^2 \leq M \min_{x_1, \dots, x_{r_k}, z_k} \sum_{j=1}^p h(\hat{\mathbf{Y}}_{kj})^2 \left\| (\hat{\mathbf{Y}}_{kj}^s)^T - \hat{x}_{(z_k^{(0)})_j} \right\|_F^2$$

- 9: **end for**
 - 10: Find the average of $z_k^{(0)}, k \in [3], z^{(0)}$.
- Output:** $\{z^{(0)} \in [r]^p\}$
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References

- Gao, C., Ma, Z., Zhang, A. Y., and Zhou, H. H. (2018). Community detection in degree-corrected block models. The Annals of Statistics, 46(5):2153–2185.
- Han, R., Luo, Y., Wang, M., and Zhang, A. R. (2020). Exact clustering in tensor block model: Statistical optimality and computational limit. arXiv preprint arXiv:2012.09996.