Complete proof for Precision clustering

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July 15, 2021

1 Definitions

1.1 Model

Suppose we have K categories in R groups. Let $z(k) \in [R]^K$ denote the group assignment, and $X_{z(k)} \sim \mathcal{N}_p(0, \Sigma_{z(k)})$, where

$$\Sigma_{z(k)}^{-1} = \Omega_{z(k)} = \Theta_0 + u_k \Theta_{z(k)},$$

where Σ_r, Ω_r are the true covariance and precision matrices, respectively, Θ_0 is denoted as intercept matrix and Θ_r for $r \in [R]$ are denoted as factor matrices. For simplicity, we let $\Theta = \{\Theta_r\}_{r=0}^R$ denote the intercept and the sequence of factor matrices. Let $I_r = \{k \in [K] : z(k) = r\}$, and S_k denote the sample covariance matrix for k-th category with n independent sample $X_{z(k),1}, ..., X_{z(k),n}$.

1.2 Parameter space

Define the the parameter space for the assignment, $\mathcal{P}_z$ as following

$$\mathcal{P}_z(R, \beta) = \left\{ z \in [R]^K : \quad \frac{K}{\beta R} \leq |I_r| \leq \frac{K\beta}{R}, r \in [R] \right\}.$$

With given assignment $z \in \mathcal{P}(R, \beta)$, define the true parameter space $\mathcal{P}^*(z, \tau_1, \tau_2, \delta, m, M)$ and the estimator search space $\mathcal{P}(z, \tau_1, \tau_2, \delta, m, M))$ as following

$$\begin{aligned}\mathcal{P}^*(z, \tau_1, \tau_2, \delta, m, M) = \Bigg\{ (u, \Theta) : \quad & \Theta_0, \Theta_r \text{ are positive definite for all } r \in [R]; \\ & 0 < \tau_1 < \min_{r \in \{0\} \cup [R]} \varphi_{\min}(\Theta_r) \leq \max_{r \in \{0\} \cup [R]} \varphi_{\max}(\Theta_r) < \tau_2; \\ & \max_{r, r' \in [R]} \cos(\Theta_r, \Theta_{r'}) < \delta < 1; \quad \langle \Theta_0^{-1}, \Theta_r \rangle = 0, r \in [R]; \\ & m < \min_{k \in [K]} |u_k| \leq \max_{k \in [K]} |u_k| < M; 0 < a < \frac{m^4 \tau_1^2}{M^4 \tau_2^2} - \delta, \\ & \sum_{k \in I_r} u_k^2 = K, \sum_{k \in I_r} u_k = 0, \text{ for all } r \in [R] \Bigg\},\end{aligned}$$

and

$$\mathcal{P}(z,\tau_1,\tau_2,\delta,m,M)=\Bigg\{(u,\Theta):\quad \Theta_0,\Theta_r \text{ are positive definite for all } r\in[R];$$

$$\max_{r,r'\in[R]}\cos(\Theta_r,\Theta_{r'})<\delta<1;\ \langle\Theta_0^{-1},\Theta_r\rangle=0, r\in[R];$$

$$m<\min_{k\in[K]}|u_k|\le\max_{k\in[K]}|u_k|<M; 0<a<\frac{m^4\tau_1^2}{M^4\tau_2^2}-\delta,$$

$$\sum_{k\in I_r}u_k^2=K,\sum_{k\in I_r}u_k=0,\ \text{for all } r\in[R]\Bigg\},$$

1.3 Estimators

Let (z^*,u^*,Θ^*) denote the true parameters. Let $\mathcal{Q}(z,u,\Theta)$ denote the negative log-likelihood function, where

$$\mathcal{Q}(z,u,\Theta)=\sum_{k\in[K]}\mathcal{Q}_k(z(k),u,\Theta)=\sum_{k\in[K]}\langle S_k,\Theta_0+u_k\Theta_{z(k)}\rangle-\log\det(\Theta_0+u_k\Theta_{z(k)}).$$

Then, we let $(\hat{z},\hat{u},\hat{\Theta})$ denote the MLE where,

$$(\hat{z},\hat{u},\hat{\Theta})=\operatorname*{arg\,min}_{z\in\mathcal{P}_z(R,\beta),\ (u,\Theta)\in\mathcal{P}(z,\delta,m,M)}\mathcal{Q}(z,u,\Theta),$$

and let $(\tilde{u},\tilde{\Theta})$ denote the oracle estimator with given z^*, where

$$(\tilde{u},\tilde{\Theta})=\operatorname*{arg\,min}_{(u,\Theta)\in\mathcal{P}(z^*,\tau_1,\tau_2,,\delta,m,M)}\mathcal{Q}(z,u,\Theta).$$

For simplicity, in the following proof, the notation $\hat{\theta}$ implies θ some parameter derived from the MLE, $\tilde{\theta}$ implies some parameters derived from the oracle estimator, and θ^* implies some parameters derived from the true parameters.

1.4 Misclassification loss

Define the function

$$\hat{\Omega}_k(a)=\hat{\Theta}_0+\hat{u}_k\hat{\Theta}_a.$$

Similar definitions $\tilde{\Omega}_k(a)$ and $\Omega_k^*(a)$ are proposed with $(z^*,\tilde{u},\tilde{\Theta})$ and (z^*,u^*,Θ^*). Then, we define the misclassification loss

$$\begin{aligned}\ell(z,z^*)&=\sum_{k\in[K]}\|\Omega_k^*(z(k))-\Omega_k^*(z^*(k))\|_F^2\\&=\sum_{k\in[K]}\sum_{b\in[R]/z^*(k)}\|\Omega_k^*(b)-\Omega_k^*(z^*(k))\|_F^2\,\mathbf{1}\left\{z(k)=b\right\}.\end{aligned}$$

Also define the minimal gap between different groups

$$\begin{aligned}
\Delta_{\min}^2(p,m,\tau_1,\tau_2,\delta) &= \min_{k,k'\in[K]}\min_{a\neq b\in[R]} \|\Omega_k^*(a)-\Omega_{k'}^*(b)\|_F^2 \\
&\geq \min_{a\neq b\in[R]} \|u_k^*\Theta_a^* - u_{k'}^*\Theta_b^*\|_F^2 \\
&= \min_{a\neq b\in[R]} \left[(u_k^*)^2\|\Theta_a^*\|_F^2 + (u_{k'}^*)^2\|\Theta_a^*\|_F^2 - 2u_k^*u_{k'}^*\langle\Theta_a^*,\Theta_b^*\rangle\right] \\
&\geq 2p\left[m^2\tau_1^2 - M^2\tau_2^2\delta\right]
\end{aligned}$$

where the last inequality follow by the fact that

$$\|A-B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 - 2\langle A,B\rangle \geq p\left[\varphi_{\min}^2(A)+\varphi_{\min}^2(B)\right] - 2\|A\|_F\|B\|_F\cos(A,B),$$

for $A,B\in\mathbb{R}^{p\times p}$. Note that $\Delta_{\min}$ is a increasing function in p,m and a decreasing function in δ. For simplicity, we use $\Delta_{\min}^2$ to denote the minimal gap.

Last, we consider the Hamming loss $h(z,z^*) = \sum_{k\in[K]}\mathbf{1}\{z(k)\neq z^*(k)\}$, where

$$\ell(z,z^*) \geq \Delta_{\min}^2 h(z,z^*).$$

1.5 Error decomposition

Suppose $z^*(k)=a$. We need to analyze the following event to study the misclassification of MLE $\hat{z}$ where $\hat{z}(k)=b$.

$$\mathcal{Q}_k(b,\hat{u},\hat{\Theta}) \leq \mathcal{Q}_k(a,\hat{u},\hat{\Theta}). \tag{1}$$

Define the errors

$$\hat{\Delta}(a,b) = \hat{\Omega}_k(a) - \Omega_k^*(b); \quad \tilde{\Delta}(a,b) = \tilde{\Omega}_k(a) - \Omega_k^*(b); \quad \Delta(a,b) = \hat{\Omega}_k(a) - \tilde{\Omega}_k(b).$$

By the Taylor Expansion, we have

$$\mathcal{Q}_k(b,\hat{u},\hat{\Theta}) - \mathcal{Q}_k(a,u^*,\Theta^*) = \langle S_k-\Sigma_k, \hat{\Delta}(b,a)\rangle + T_2(b,a), \tag{2}$$

where

$$\begin{aligned}
T_2(b,a) &= \mathrm{vec}(\hat{\Delta}(b,a))^T\int_0^1 (1-v)(\Omega_k^*+\hat{\Delta}(b,a))^{-1}\otimes(\Omega_k^*+\hat{\Delta}(b,a))^{-1}dv\,\mathrm{vec}(\hat{\Delta}(b,a)) \\
&= c\left\|\hat{\Delta}(b,a)\right\|_F^2,
\end{aligned}$$

with a constant c related to the τ_1,τ_2 and the second equation follows by the Lemma 3 that $\left\|\hat{\Delta}(b,a)\right\|_F$ is bounded for n large enough.

Plugging the Taylor Expansion (2) into the event (1), the event is upper bounded by the event

$$\langle S_k-\Sigma_k, \hat{\Delta}(b,a)-\hat{\Delta}(a,a)\rangle \leq c\left[\left\|\hat{\Delta}(a,a)\right\|_F^2 - \left\|\hat{\Delta}(b,a)\right\|_F^2\right].$$

Rearranging the inequality, we have

$$\langle S_k - \Sigma_k, \tilde{\Omega}_k(b) - \tilde{\Omega}_k(a)\rangle \le -c\bar{\Delta}_k(a,b)^2 + cG_k(a,b,\hat{z}) + cH_k(a,b) + F_k(a,b,\hat{z}),$$

where

$$\begin{aligned}
\bar{\Delta}_k(a,b)^2 &= \|\Omega_k^*(a) - \Omega_k^*(b)\|_F^2 . \\
F_k(a,b,\hat{z}) &= \langle S_k - \Sigma_k, \Delta(a,a) - \Delta(b,b)\rangle. \\
G_k(a,b,\hat{z}) &= \left(\left\|\hat{\Delta}(a,a)\right\|_F^2 - \left\|\tilde{\Delta}(a,a)\right\|_F^2\right) - \left(\left\|\hat{\Delta}(b,a)\right\|_F^2 - \left\|\tilde{\Delta}(b,a)\right\|_F^2\right). \\
H_k(a,b) &= \left\|\tilde{\Delta}(a,a)\right\|_F^2 - \left(\left\|\tilde{\Delta}(b,a)\right\|_F^2 - \bar{\Delta}_k(a,b)^2\right).
\end{aligned}$$

Last, we define oracle misclassification loss as

$$\xi_{\text{ideal}}(\epsilon) = \sum_{k\in[K]} \sum_{b\in[R]/z^*(k)} \|\Omega_k^*(z^*(k)) - \Omega_k^*(b)\|_F^2 \cdot \mathbf{1}\left\{\langle S_k - \Sigma_k, \tilde{\Omega}_k(b) - \tilde{\Omega}_k(z^*(k))\rangle \le -c(1-\epsilon)\bar{\Delta}_k(a,b)^2\right\}.$$

2 Useful Lemmas

Lemma 1 (Intercept estimation)**.** *The oracle estimator and MLE of the intercept are equivalent, i.e.,* $\hat{\Theta}_0 = \tilde{\Theta}_0$.

Lemma 2 (Oracle estimation error)**.** *The oracle estimator* $(\tilde{u}, \tilde{\Theta})$ *satisfy the following inequalities simultaneously with probability at least* $1 - \mathcal{O}(1/n)$*,*

$$\left\|\tilde{\Theta}_0 - \Theta_0^*\right\|_F \le C_0 p\sqrt{\frac{\log p \log n}{nK}}, \quad \left\|\tilde{\Theta}_r - \Theta_r^*\right\|_F \le C_r p\sqrt{\frac{\log p \log n}{n|I_r^*|}}, \quad |\tilde{u}_k - u_k^*| \le C_k p\sqrt{\frac{\log p \log n}{n}},$$

for some large positive constants C_0, C_r, C_k *and* $\min_{r\in[R]} |I_r^*| \ge \frac{K}{\beta R}$.

Lemma 3 (MLE estimation error)**.** *The MLE* $(\hat{z}, \hat{u}, \hat{\Theta})$ *satisfy the following inequalities simultaneously with probability at least* $1 - \mathcal{O}(1/n)$*,*

$$\sum_{k\in[K]} \left\|\hat{\Omega}_k(\hat{z}(k)) - \Omega_k^*(z^*(k))\right\|_F \le CK\sqrt{\frac{\log n \log p}{n}},$$

and that $\hat{z} \to z^*$*, i.e.,* $\ell(\hat{z}, z^*) \to 0$ *as* $n \to \infty$.

Lemma 4 (Conditions check)**.** *For* n *large enough such that* $\ell(\hat{z}, z^*) \le \tau = \frac{K}{2\beta R}$*, we have*

1.

$$\sum_{k\in[K]} \max_{b\in[K]/z^*(k)} \frac{F_k(z^*(k), b, \hat{z})^2 \left\|\Omega_k^*(z^*(k)) - \Omega_k^*(b)\right\|_F^2}{\bar{\Delta}_k(z^*(k), b)^4 \ell(\hat{z}, z^*)} \le C_1\epsilon^2,$$

hols with probability at least $1 - \eta_1$ *for small positive constant* C_1*, and* $\epsilon, \eta_1 > 0$*;*

2.

$$\max_{T\subset[K]} \frac{\tau}{4\Delta_{\min}^2|T|+\tau} \sum_{k\in[K]} \max_{b\in[K]/z^*(k)} \frac{G_k(z^*(k),b,\hat{z})^2 \left\|\Omega_k(u^*,\Theta^*,z^*(k))-\Omega_k(u^*,\Theta^*,b)\right\|_F^2}{\bar{\Delta}_k(z^*(k),b)^4\ell(\hat{z},z^*)} \leq C_2\epsilon^2,$$

holds with probability at least $1-\eta_2$ for small positive constant C_2, and $\epsilon,\eta_2>0$;

3.

$$\max_{k\in[K]} \max_{b\in[K]/z^*(k)} \frac{|H_k(z^*(k),b)|}{\bar{\Delta}_k(z^*(k),b)^2} \leq C_3\epsilon,$$

holds with probability at least $1-\eta_3$ for small positive constant C_3, and $\epsilon,\eta_3>0$.

3 Main theorems

Theorem 3.1 (Error decomposition)**.** *For n large enough such that $\ell(\hat{z},z^*)\leq\tau=\frac{K}{2\beta R}$, the MLE $\hat{z}$ satisfies following inequality*

$$\ell(\hat{z},z^*)\leq C\xi_{ideal}(\epsilon),$$

with probability at least $1-\eta_1-\eta_2-\eta_3$ for some positive constant C.

Remark 1. The parameter ϵ in $\xi_{\text{ideal}}(\epsilon)$ is the same as the ϵ in Lemma 4.

Theorem 3.2 (Oracle misclassification rate)**.** *Assume $\Delta_{\min}=\mathcal{O}(K^\gamma)$ for some $\gamma>0$, $n=C_n\exp(\Delta_{\min}^2)$ for C_n large enough, and the ϵ in Lemma 4 small enough. With probability $1-\eta_1-\eta_2-\eta_3-\exp(-\Delta_{\min})$ as $K\to\infty$*

$$\xi_{ideal}(\epsilon)\leq K\exp\left(-(1-c\epsilon)^2C\Delta_{\min}^2\right),$$

where c,C are two positive constants.

Remark 2. In fact, if Lemma 4 holds for the decreasing sequence $\epsilon_K\to 0$ and $\eta_1,\eta_2,\eta_3\to\mathcal{O}\left(\exp(-\Delta_{\min})\right)$ as $K\to 0$, we have

$$\xi_{\text{ideal}}(\epsilon)\leq K\exp\left(-(1+o(1))C\Delta_{\min}^2\right),$$

with $1-\mathcal{O}\left(\exp(-\Delta_{\min})\right)$.

4 Proofs

Proof of Lemma 1. To show the equivalence $\hat{\Theta}_0=\tilde{\Theta}_0$, we only need to show that

$$\frac{\partial\mathcal{Q}(\hat{z},u,\Theta)}{\partial\Theta_0}=\frac{\partial\mathcal{Q}(z^*,u,\Theta)}{\partial\Theta_0}.$$

For any $z \in \mathcal{P}_z(R,\beta)$ and $(u,\Theta) \in \mathcal{P}(z,\tau_1,\tau_2,\delta,m,M)$, we have

$$\begin{aligned}
\frac{\partial \mathcal{Q}(z,u,\Theta)}{\partial \Theta_0} &= \frac{\partial}{\partial \Theta_0}\sum_{k\in[K]}\langle S_k,\Theta_0\rangle + \frac{\partial}{\partial \Theta_0}\sum_{k\in[K]}\log\det(\Theta_0 + u_k\Theta_{z(k)}) \\
&= \frac{\partial}{\partial \Theta_0}\sum_{k\in[K]}\langle S_k,\Theta_0\rangle + \sum_{k\in[K]}(\Theta_0 + u_k\Theta_{z(k)})^{-1}.
\end{aligned}$$

Note that by the matrix inverse lemma and the fact that $(u,\Theta) \in \mathcal{P}(z,\tau_1,\tau_2,\delta,m,M)$, we have

$$\begin{aligned}
(\Theta_0 + u_k\Theta_{z(k)})^{-1} &= \Theta_0^{-1} + \frac{u_k}{1+u_k\langle\Theta_0^{-1},\Theta_{z(k)}\rangle}\Theta_0^{-1}\Theta_{z(k)}\Theta_0^{-1} \\
&= \Theta_0^{-1} + u_k\Theta_0^{-1}\Theta_{z(k)}\Theta_0^{-1},
\end{aligned}$$

and

$$\sum_{k\in[K]}(\Theta_0 + u_k\Theta_{z(k)})^{-1} = K\Theta_0^{-1}.$$

Hence, the partial derivative with respect to Θ_0 is independent with the assignment z, and thus $\hat{\Theta}_0 = \tilde{\Theta}_0$. $\square$

Proof of Lemma 2. Very similar to Note 0626. $\square$

Proof of Lemma 3. First part is very similar to Note 0411.

For the second part, suppose we already have

$$\sum_{k\in[K]}\left\|\hat{\Omega}_k(\hat{z}(k)) - \Omega_k^*(z^*(k))\right\|_F \leq CK\sqrt{\frac{\log n\log p}{n}},$$

with high probability. We show $\ell(\hat{z},z^*) \to 0$ as $n\to\infty$ by contradiction.

If $\ell(\hat{z},z^*) \to c$, then there exist $k,k' \in [K]$ such that $\hat{z}(k) = \hat{z}(k') = a, a\in[R]$ and $z^*(k) = b, z^*(k') = c, b \neq c \in [R]$. Then, we have

$$\begin{aligned}
\sum_{k\in[K]}\left\|\hat{\Omega}_k(\hat{z}(k)) - \Omega_k^*(z^*(k))\right\|_F &\geq \left\|\hat{\Omega}_k(a) - \Omega_k^*(b)\right\|_F + \left\|\hat{\Omega}_{k'}(a) - \Omega_{k'}^*(c)\right\|_F \\
&\geq \left\|\hat{u}_k\hat{\Theta}_a - u_k^*\Theta_b^*\right\|_F + \left\|\hat{u}_{k'}\hat{\Theta}_a - u_{k'}^*\Theta_c^*\right\|_F - 2\left\|\tilde{\Theta}_0 - \Theta^*\right\|_F \\
&\geq m\left\|\frac{u_k^*}{\hat{u}_k}\Theta_b^* - \frac{u_{k'}^*}{\hat{u}_{k'}}\Theta_c^*\right\|_F - 2\left\|\tilde{\Theta}_0 - \Theta^*\right\|_F.
\end{aligned}$$

Note that by Lemma 2, we have $\left\|\tilde{\Theta}_0 - \Theta_0^*\right\|_F \leq C_0 p\sqrt{\frac{\log p\log n}{nK}}$ with probability $1-\mathcal{O}(1/n)$, and

$$\left\|\frac{u_k^*}{\hat{u}_k}\Theta_b^* - \frac{u_{k'}^*}{\hat{u}_{k'}}\Theta_c^*\right\|_F^2 \geq 2p\left[\frac{m^2}{M^2}\tau_1^2 - \frac{M^2}{m^2}\tau_2^2\delta\right] > 2p\frac{M^2}{m^2}\tau_2^2 a > 0,$$

by the fact that $(u^*,\Theta^*) \in \mathcal{P}^*(z^*,\tau_1,\tau_2,\delta,m,M)$. This contradicts to the conclusion in Lemma 3. $\square$

Proof of Lemma 4. **For first and second parts,** □

Proof of Lemma 3.1. Very similar to the Proof of Theorem 3.1 in (Gao and Zhang, 2019). □

References

Gao, C. and Zhang, A. Y. (2019). Iterative algorithm for discrete structure recovery. arXiv preprint arXiv:1911.01018.