Solution to "Chapter 2: Basic tail and concentration bounds"

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1 Summary

Theorem 1.1 (Markov's inequality)**.** *Suppose $X \geq 0$ is a random variable with finite mean, we have*

$$\mathbb{P}(X \geq t) \leq \frac{E[X]}{t}, \quad \forall\ t > 0.$$

Theorem 1.2 (Chebyshev's inequality)**.** *Suppose $X \geq 0$ is a random variable with finite mean μ and finite variance, we have*

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{var(X)}{t^2}, \quad \forall\ t > 0.$$

Theorem 1.3 (Markov's inequality for polynomial moments)**.** *Suppose the random variable X has a central moment of order k. Applying Markov's inequality to the random variable $|X - \mu|^k$ yields*

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\mathbb{E}[|X - \mu|^k]}{t^k}, \quad \forall\ t > 0.$$

Theorem 1.4 (Chernoff bound)**.** *Suppose the random variable X has a moment generating function in the neighborhood of 0, i.e. $\varphi_X(\lambda) = \mathbb{E}[e^{\lambda X}] < +\infty, \forall \lambda \in (-b, b), b > 0$. Applying Markov's inequality to the random variable $Y = e^{\lambda(X-\mu)}$ yields*

$$\mathbb{P}((X - \mu) \geq t) \leq \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda t}}.$$

Optimizing the choice of λ for the tightest bound yields the Chernoff bound

$$\mathbb{P}((X - \mu) \geq t) \leq \inf_{\lambda \in [0,b)} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda t}}.$$

2 Exercises

Exercise 2.1 (Tightness of inequalities.) The Markov and Chebyshev's inequalities can not be improved in general.

(a) Provide a random variable $X \geq 0$ for which Markov's inequality (1.1) is met with equality.

(b) Provide a random variable Y for which Chebyshev's inequality (1.2) is met with equality.

Solution:

(a) Recall the proof of Markov's inequality. For any $t > 0$,

$$\mathbb{E}[X] = \int_0^t x f_X(x)dx + \int_t^{+\infty} x f_X(x)dx \geq \int_t^{+\infty} x f_X(x) \geq t \int_t^{+\infty} f_X(x) = t\mathbb{P}(X \geq t).$$

If Markov's inequality meets the equality, the inequalities above should meet equality.

Consider a variable X with distribution $P(X = 0) = 1$. For any $t > 0$, the variable X satisfies

$$\int_0^t x f_X(x)dx = 0 \text{ and } \int_t^{+\infty} x f_X(x)dx = \int_t^{+\infty} t f_X(x)dx.$$

Therefore, for variable X, the Markov's inequality is met with equality.

(b) Chebyshev's inequality follows by applying Markov's inequality to the non-negative random variable $Y = \mathbb{E}(X - \mathbb{E}[X])^2$. Let the distribution of Y be $\mathbb{P}(Y = 0) = 1$. Then the Markov's inequality for Y and the Chebyshev's inequality for X meet the equalities. By transformation, the distribution of random variable X is $\mathbb{P}(X = \mathbb{E}[X]) = 1$. Therefore, for any random variable X with distribution $\mathbb{P}(X = c) = 1, c \in \mathbb{R}$, the Chebyshev's inequality is met with equality.

Exercise 2.2

Lemma 1 (Standard normal distribution)**.** *Let $\phi(z)$ be the density function of a standard normal $Z \sim N(0,1)$ variable. Then,*

$$\phi'(z) + z\phi(z) = 0, \tag{1}$$

and

$$\phi(z)(\frac{1}{z} - \frac{1}{z^3}) \leq \mathbb{P}(Z \geq z) \leq \phi(z)(\frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5}), \quad \textit{for all } z > 0. \tag{2}$$

Proof. First, we prove the equation (1). The pdf of standard normal distribution $\phi(z)$ satisfies

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}); \quad \phi'(z) = -z\frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) = -z\phi(z).$$

Next, we prove the equation (2). Using equation (1), we have

$$\begin{aligned}
\mathbb{P}(Z \geq z) &= \int_z^{+\infty} \phi(t)dt = \int_z^{+\infty} -\frac{1}{t}\phi'(t)dt = \frac{1}{z}\phi(z) - \int_z^{+\infty} \frac{1}{t^2}\phi(t)dt \\
&= \frac{1}{z}\phi(z) + \int_z^{+\infty} \frac{1}{t^3}\phi'(t)dt = \frac{1}{z}\phi(z) - \frac{1}{z^3}\phi(z) + \int_z^{+\infty} \frac{3}{t^4}\phi(t)dt
\end{aligned}$$

Since $\frac{3}{t^4}\phi(t) \geq 0$, therefore $\mathbb{P}(Z \geq z) \geq \phi(z)(\frac{1}{z} - \frac{1}{z^3})$. On the other hand,

$$\int_z^{+\infty} \frac{3}{t^4}\phi(t)dt = \int_z^{+\infty} -\frac{3}{t^5}\phi'(t)dt = \frac{3}{z^5}\phi(z) - \int_z^{+\infty} \frac{15}{t^6}\phi(t)dt \leq \frac{3}{z^5}\phi(z).$$

Therefore, $\mathbb{P}(Z \geq z) \leq \phi(z)(\frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5})$. $\square$

Exercise 2.3

Lemma 2 (Polynomial bound and Chernoff bound)**.** *Suppose $X \geq 0$, and that the moment generating function of X exists in the neighborhood of 0. Given some $\delta > 0$ and integer $k \in \mathbb{Z}_+$, we have*

$$\inf_{k \in \mathbb{Z}_+} \frac{\mathbb{E}[|X|^k]}{\delta^k} \leq \inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}}.$$

Consequently, an optimized bound based on polynomial moments is always at least as good as the Chernoff upper bound.

Proof. By power series, we have

$$e^{\lambda X} = \sum_{k=0}^{+\infty} \frac{X^k \lambda^k}{k!}, \quad \forall \lambda \in \mathbb{R} \tag{3}$$

Since the moment generating function $\varphi_X(\lambda)$ exists in the neighbor hood of 0, there exists a constant $b > 0$ such that

$$\inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}} = \inf_{\lambda \in (0, b)} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}} < +\infty.$$

Taking the expectation on both sides of the power series (3) yields

$$\mathbb{E}[e^{\lambda X}] = \sum_{k=0}^{+\infty} \frac{\mathbb{E}[|X|^k] \lambda^k}{k!} < +\infty, \quad \forall \lambda \in (0, b).$$

Therefore, the moment $\mathbb{E}[|X|^k] < +\infty$, $\forall k \in \mathbb{Z}_+$ exists. Applying the power serious to $e^{\lambda \delta}$, we obtain the result

$$\inf_{k \in \mathbb{Z}_+} \frac{\mathbb{E}[|X|^k]}{\delta^k} \leq \sum_{k=0}^{+\infty} \frac{\mathbb{E}[|X|^k]}{\delta^k} = \sum_{k=0}^{+\infty} \frac{\frac{\mathbb{E}[|X|^k] \lambda^k}{k!}}{\frac{\lambda^k \delta^k}{k!}} = \inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \delta}}.$$

$\square$