

Graphic Lasso: Clustering accuracy for precision matrix model

Jiixin Hu

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1 With convex penalty L_1 norm

The precision model is stated as

$$\mathbb{E}[S^k] = \Omega^k = \sum_{l=1}^r u_{kl} \Theta^l, \quad k \in [K].$$

Consider the following penalized optimization problem

$$\max_{\mathbf{U}, \Theta^l} \mathcal{L}_S(\mathbf{U}, \Theta^l) = - \sum_{k=1}^K \text{tr}(S^k \Omega^k) + \log \det(\Omega^k) + \lambda \left\| \Omega^k \right\|,$$

where \mathbf{U} is a membership matrix, and $\{\Theta^l\}$ are irreducible and invertible.

Proposition 1. *The loss function \mathcal{L}_S satisfies the conditions for Theorem 3.1, and thus the clustering accuracy for precision matrix model is guaranteed.*

Proof. First, we introduce some useful notations.

Given the membership \mathbf{U}' , let $\hat{\Theta}^l(\mathbf{U}') = \arg \max_{\mathbf{U}', \Theta^l}$. Particularly, for each $l \in [r]$, we have

$$\hat{\Theta}^l(\mathbf{U}') = \arg \max_{\Theta} - \sum_{k \in I'_l} \langle S^k, \Theta \rangle + |I'_l| \log \det(\Theta) + \lambda |I'_l| \|\Theta\|_1,$$

index l

where $I'_l = \{k : u'_{kl} \neq 0\}$ is the index set for the l -th group based on the membership \mathbf{U}' . The sample-based loss is defined as

$$F(\mathbf{U}') = \mathcal{L}_S(\mathbf{U}', \hat{\Theta}^l(\mathbf{U}')).$$

Correspondingly, define the population-based loss function as

$$l(\mathbf{U}, \Theta^l) = \mathbb{E}_S[\mathcal{L}_S(\mathbf{U}, \Theta^l)] = - \sum_{k=1}^K \text{tr}(\Sigma^k \Omega^k) + \log \det(\Omega^k) + \lambda \sum_{k=1}^K \left\| \Omega^k \right\|_1.$$

Given the membership \mathbf{U}' , let $\tilde{\Theta}^l(\mathbf{U}') = \arg \max_{\mathbf{U}', \Theta^l}$. Particularly, for each $l \in [r]$, we have

$$\tilde{\Theta}^l(\mathbf{U}') = \arg \max_{\Theta} - \sum_{k \in I'_l} \langle \Sigma^k, \Theta \rangle + |I'_l| \log \det(\Theta) + \lambda |I'_l| \|\Theta\|_1. \quad (1)$$

Then, the population-based loss is defined as

$$G(\mathbf{U}') = l(\mathbf{U}', \tilde{\Theta}^l(\mathbf{U}')).$$

Note that $\hat{\Theta}^l(\mathbf{U}')$ and $\tilde{\Theta}^l(\mathbf{U}')$ do not have closed forms. But both of them only utilize $|I_l'|$ sample covariance(true covariance) matrices based on the membership.

Next, we verify the functions $F(\cdot)$ and $G(\cdot)$ satisfy the conditions in the Theorem 3.1. Let $\{\mathbf{U}, \Theta^l\}$ denote the true membership and precision matrices, and define $\hat{\mathbf{U}} = \arg \max_{\mathbf{U}} F(\mathbf{U})$. We also define the confusion matrix $D = \llbracket D_{ij} \rrbracket \in \mathbb{R}^{r \times r}$, where $D_{ij} = \sum_{k=1}^K \mathbf{I}\{u_{ki} = \hat{u}_{kj} = 1\}$.

1. (Self-consistency) First, we consider the explicit formulas for $G(\hat{\mathbf{U}})$ and $G(\mathbf{U})$.

$$\begin{aligned} G(\hat{\mathbf{U}}) &= l(\hat{\mathbf{U}}, \tilde{\Theta}^l(\hat{\mathbf{U}})) \\ &= \sum_{l=1}^r \left[\sum_{k \in \hat{I}_l} -\langle \Sigma^k, \tilde{\Theta}^l(\hat{\mathbf{U}}) \rangle + |\hat{I}_l| \log \det(\tilde{\Theta}^l(\hat{\mathbf{U}})) - \lambda |\hat{I}_l| \left\| \tilde{\Theta}^l(\hat{\mathbf{U}}) \right\|_1 \right] \\ &= \sum_{l=1}^r \left[\sum_{a=1}^r D_{al} \left(-\langle \Sigma^a, \tilde{\Theta}^l(\hat{\mathbf{U}}) \rangle + \log \det(\tilde{\Theta}^l(\hat{\mathbf{U}})) - \lambda \left\| \tilde{\Theta}^l(\hat{\mathbf{U}}) \right\|_1 \right) \right], \end{aligned}$$

and

$$\begin{aligned} G(\mathbf{U}) &= l(\mathbf{U}, \tilde{\Theta}^l(\mathbf{U})) \\ &= \sum_{l=1}^r \left[-|I_l| \langle \Sigma^k, \tilde{\Theta}^l(\mathbf{U}) \rangle + |I_l| \log \det(\tilde{\Theta}^l(\mathbf{U})) - \lambda |I_l| \left\| \tilde{\Theta}^l(\mathbf{U}) \right\|_1 \right] \\ &= \sum_{l=1}^r \left[\sum_{a=1}^r D_{al} \left(-\langle \Sigma^a, \tilde{\Theta}^l(\mathbf{U}) \rangle + \log \det(\tilde{\Theta}^l(\mathbf{U})) - \lambda \left\| \tilde{\Theta}^l(\mathbf{U}) \right\|_1 \right) \right]. \end{aligned}$$

Define the function

$$h^k(\Theta) = -\langle \Sigma^k, \Theta \rangle + \log \det(\Theta) - \lambda \|\Theta\|_1.$$

By the definition (1), we know that

$$\tilde{\Theta}^k(\mathbf{U}) = \arg \max_{\Theta} h^k(\Theta), k = 1, \dots, r.$$

Therefore, we have the self-consistency of \mathbf{U} , i.e., $G(\hat{\mathbf{U}}) \leq G(\mathbf{U})$.

Next, we want to find the function which links the subtraction $G(\hat{\mathbf{U}}) - G(\mathbf{U})$ with the misclassification rate $MCR(\hat{\mathbf{U}}, \mathbf{U})$, where $MCR(\hat{\mathbf{U}}, \mathbf{U}) = \max_{l, a \neq a' \in [r]} \min\{D_{al}, D_{a'l}\}$.

Suppose $MCR(\hat{\mathbf{U}}, \mathbf{U}) \geq \epsilon$. There exist $l, k \neq k' \in [r]$ such that $\min\{D_{kl}, D_{k'l}\} \geq \epsilon$.

Then, we have

$$\begin{aligned} G(\hat{\mathbf{U}}) - G(\mathbf{U}) &\leq D_{kl} \left(h^k(\tilde{\Theta}^l(\hat{\mathbf{U}})) - h^k(\tilde{\Theta}^k(\mathbf{U})) \right) + D_{k'l} \left(h^k(\tilde{\Theta}^l(\hat{\mathbf{U}})) - h^k(\tilde{\Theta}^{k'}(\mathbf{U})) \right) \\ &\leq \epsilon C(\mathbf{U}, \Theta^l, \lambda), \end{aligned}$$

where C is a function of the true parameters $\{\mathbf{U}, \Theta^l\}$. **Need to figure out the explicit form of C in next step.**

2. (Bounded difference between sample- and population-based loss) For arbitrary \mathbf{U} , consider the absolute subtraction

$$\begin{aligned} |F(\mathbf{U}) - G(\mathbf{U})| &= |\mathcal{L}_S(\mathbf{U}, \hat{\Theta}^l(\mathbf{U})) - l(\mathbf{U}, \tilde{\Theta}^l(\mathbf{U}))| \\ &\leq |\mathcal{L}_S(\mathbf{U}, \hat{\Theta}^l(\mathbf{U})) - l(\mathbf{U}, \hat{\Theta}^l(\mathbf{U}))| + |l(\mathbf{U}, \hat{\Theta}^l(\mathbf{U})) - l(\mathbf{U}, \tilde{\Theta}^l(\mathbf{U}))| \\ &= M_1 + M_2. \end{aligned}$$

Conjecture:

For M_1 ,

$$M_1 = \left| \sum_{l=1}^r \sum_{k \in I_l} \langle \Sigma^k - S^k, \hat{\Theta}^l(\mathbf{U}) \rangle \right| = \max_{k, (ij)} |\Sigma_{ij}^k - S_{ij}^k| C_1(\mathbf{U}, \Theta^l, p),$$

use Cauchy-Schwarz inequality.
multiple ways to bound by, ℓ_1 , ℓ_2 , ℓ_∞ norm, etc.

where C_1 is a function of the true parameters $\{\mathbf{U}, \Theta^l\}$ and the dimension p .

For M_2 , note that $l(\mathbf{U}, \Theta)$ is a convex function of Θ and thus l is local Lipschitz. We may have

$$M_2 \leq \max_{l \in [r]} \sup_{\Theta^l} \left| \frac{\partial}{\partial \Theta^l} l(\mathbf{U}, \Theta^l) \right| \left\| \hat{\Theta}^l(\mathbf{U}) - \tilde{\Theta}^l(\mathbf{U}) \right\|_{\max}$$

Also, we can consider $\max_{l \in [r]} \sup_{\Theta^l} \left| \frac{\partial}{\partial \Theta^l} l(\mathbf{U}, \Theta^l) \right| = C_2(\mathbf{U}, \Theta^l, \lambda)$, where C_2 is the function of the true parameters $\{\mathbf{U}, \Theta^l\}$ and tuning parameter λ . Since $\hat{\Theta}^l$ is the sample-based estimation and $\tilde{\Theta}^l$ is the population-based estimation, my conjecture is that $\left\| \hat{\Theta}^l(\mathbf{U}) - \tilde{\Theta}^l(\mathbf{U}) \right\|_{\max} = C_3(\max_{k, (ij)} |\Sigma_{ij}^k - S_{ij}^k|)$.

Therefore, we bound the difference as

$$|F(\mathbf{U}) - G(\mathbf{U})| \leq C'(\mathbf{U}, \Theta^l, p, \lambda) C''(\max_{k, (ij)} |\Sigma_{ij}^k - S_{ij}^k|),$$

and then we can utilize of residual to find a $p(t) = \mathbb{P}(|F(\mathbf{U}) - G(\mathbf{U})| \geq t) \rightarrow 0$ as $t \rightarrow \infty$.

□

2 Misclassification error

We explore the perturbed version of the self-consistency in this section.

Lemma 1 (Self-consistency of \mathbf{U}). *Suppose $MCR(\hat{\mathbf{U}}, \mathbf{U}) \geq \epsilon$ and the minimal gap between $\{\Theta^l\}$ denoted δ is positive. For $\lambda \leq C' \left(\frac{\log p}{n} \right)^{1/2}$ with some constant C' , we have the perturbation version of the self-consistency.*

$$G(\hat{\mathbf{U}}) - G(\mathbf{U}) \leq -\frac{\epsilon}{4\tau^2} \delta^2 + \epsilon \lambda \sqrt{p} C \left(\frac{p \log p}{n} \right)^{1/2} < 0.$$

Proof. Suppose $MCR(\hat{\mathbf{U}}, \mathbf{U}) \geq \epsilon$. Let $\{\mathbf{U}, \Theta^l\}$ denote the true parameters, and $\Theta^l = (\Sigma^l)^{-1}$. Define the function

$$h^k(\Theta) = -\langle \Sigma^k, \Theta \rangle + \log \det(\Theta) - \lambda \|\Theta\|_1.$$

There exist $l, k \neq k' \in [r]$ such that $\min\{D_{kl}, D_{k'l}\} \geq \epsilon$. Then, we have

$$\begin{aligned} G(\hat{U}) - G(U) &\leq D_{kl} \left(h^k(\tilde{\Theta}^l(\hat{U})) - h^k(\tilde{\Theta}^k(U)) \right) + D_{k'l} \left(h^{k'}(\tilde{\Theta}^l(\hat{U})) - h^{k'}(\tilde{\Theta}^{k'}(U)) \right) \\ &\leq D_{kl} \left(h^k(\tilde{\Theta}^l(\hat{U})) - h^k(\Theta^k) \right) + D_{k'l} \left(h^{k'}(\tilde{\Theta}^l(\hat{U})) - h^{k'}(\Theta^k) \right), \end{aligned} \quad (2)$$

where the second inequality follows the fact that $h^k(\Theta^k) \leq h^k(\tilde{\Theta}^k(U))$ since $h^k(\tilde{\Theta}^k(U))$ is the maximizer of $h^k(\Theta)$ by the definition. For simplicity, let $\hat{\Theta}$ denote $\tilde{\Theta}^l(\hat{U})$. Define $\Delta^k = \hat{\Theta} - \Theta^k$.

(0228) Try to prove Δ^k is bounded by delta, the minimal gap between true Θ_k and Θ_l . Intuitively, this should hold provided (a) true Θ_k are quite distinct from each other, and (b) penalization lambda is small; (c) MCR is not too large.

Consider the function $f^k(t) = \log \det(\Theta^k + t\Delta^k)$, and by Taylor expansion we have

(Think about why. What happens if the two groups (k,l) have one member in each, and lambda=0)

$$f^k(1) - f^k(0) = \langle \Sigma^k, \Delta^k \rangle - \text{vec}(\Delta^k)^T \int_0^1 (1-v)(\Theta^k + v\Delta^k)^{-1} \otimes (\Theta^k + v\Delta^k)^{-1} d\text{vec}(\Delta^k).$$

Then, we have

$$\begin{aligned} h^k(\tilde{\Theta}^k) - h^k(\hat{\Theta}^k) &= \langle \Sigma^k, \Delta^k \rangle - f^k(1) + f^k(0) - \lambda \left(\|\Theta^k\|_1 - \|\hat{\Theta}\|_1 \right) \\ &\geq A_1 - |A_2|, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \text{vec}(\Delta^k)^T \int_0^1 (1-v)(\Theta^k + v\Delta^k)^{-1} \otimes (\Theta^k + v\Delta^k)^{-1} d\text{vec}(\Delta^k), \\ A_2 &= \lambda \left(\|\Theta^k\|_1 - \|\hat{\Theta}\|_1 \right). \end{aligned}$$

p1: Δ^k here is not the delta in precision matrix accuracy

By Guo's paper, we know that

$$A_1 \geq \frac{1}{4\tau^2} \|\Delta^k\|_F^2,$$

p2: we can not find the lower bound for $\min\|\Delta^k\|$. For example, if the Δ in pink has $\min\|\Delta^k\|$, and we plug the upper bound of $\min\|\Delta^k\|$ in the constrain for lambda (green parts), then the pink part may not smaller than 0.

where $\max_{k \in [r]} \varphi_{\max}(\Theta^k) \leq \tau < \infty$. Also note that

$$|A_2| \leq \lambda \|\Theta^k - \hat{\Theta}\|_1 \leq \lambda \sqrt{p} \|\Delta^k\|_F.$$

Plug the inequalities (3) and (4) in to the inequality (2), we obtain that

$$G(\hat{U}) - G(U) \leq D_{kl} \left(-\frac{1}{4\tau^2} \|\Delta^k\|_F^2 + \lambda \sqrt{p} \|\Delta^k\|_F \right) + D_{k'l} \left(-\frac{1}{4\tau^2} \|\Delta^{k'}\|_F^2 + \lambda \sqrt{p} \|\Delta^{k'}\|_F \right).$$

Intuitively, if we have λ very small, then we obtain the perturbation version of self-consistency. By a straightforward calculation, if we have

$$\lambda \leq \frac{1}{4\tau^2 \sqrt{p}} \min_{k \in [r]} \|\Delta^k\|_F, \quad (5)$$

then the perturbation version of self-consistency holds. Recall our previous conclusion for the Ω

estimation. If $\lambda = \mathcal{O} \left(\left(\frac{\log p}{n} \right)^{1/2} \right)$, we have

not directly applicable.

Your earlier result (0115.pdf) for Ω is under

constrained optimization, but not under

penalized optimization. (?)

$$\min_{k \in [r]} \|\Delta^k\|_F \leq C \left(\frac{p \log p}{n} \right)^{1/2}$$

Extension should be easy though.

with high probability. This implies that when $\lambda \leq C' \left(\frac{\log p}{n} \right)^{1/2}$, the λ satisfies the condition (5) with high probability. Finally, we obtain the perturbation version of self-consistency,

$$\begin{aligned} G(\hat{U}) - G(U) &\leq -\frac{\epsilon}{4\tau^2} \left\| \Theta^k - \Theta^{k'} \right\|_F^2 + \epsilon \lambda \sqrt{p} C \left(\frac{p \log p}{n} \right)^{1/2} \\ &\leq -\frac{\epsilon}{4\tau^2} \delta^2 + \epsilon \lambda \sqrt{p} C \left(\frac{p \log p}{n} \right)^{1/2}, \end{aligned}$$

red part is $\lambda * \sqrt{p} * (\|\Delta_k\| + \|\Delta_{k'}\|)$
 (try to bound the red part in terms of δ .
 The final bound is a function of δ .)

where δ is the minimal gap between Θ^l . □

Remark 1. When $\lambda = 0$, the subtraction $G(\hat{U}) - G(U) \leq -\frac{1}{4\tau^2} \delta^2$ agrees with the result under the case without penalty.

Remark 2. The difficulty of proving the consistency of $\hat{\Theta}^l(U)$ does not have a closed form. In other literatures, they usually consider the true Θ^l rather than $\hat{\Theta}^l(U)$ under the true membership. The possible reason is that the properties (such as singular value, minimal gap) of Θ^l are easy to describe while it is hard to tell the properties of $\hat{\Theta}^l(U)$ (except it is an optimizer). Therefore, I introduce the true precision matrices in the proof in step (2). As a result, the upper bound becomes related with the precision matrices estimation $\|\Delta\|_F = \|\hat{\Theta} - \Theta^k\|_F$, and thus the control for λ is required.

3 Others

Theorem 3.1 (General property for loss function to guarantee the clustering accuracy). *Let $\{\mathcal{C}, \mathbf{M}_k\}$ denote the true parameters, and $\mathcal{L}_Y(\mathcal{C}', \mathbf{M}'_k)$ denote the sample-based loss function. Define the sample-based loss function with respect to \mathbf{M}'_k as*

$$F(\mathbf{M}'_k) = \mathcal{L}_Y(\hat{\mathcal{C}}(\mathbf{M}'_k), \mathbf{M}'_k),$$

where

$$\hat{\mathcal{C}}(\mathbf{M}'_k) = \arg \max_{\mathcal{C}} \mathcal{L}_Y(\mathcal{C}, \mathbf{M}'_k).$$

Correspondingly, define the population-based loss function with respect to \mathbf{M}'_k as

$$G(\mathbf{M}'_k) = l(\tilde{\mathcal{C}}(\mathbf{M}'_k), \mathbf{M}'_k),$$

where

$$l(\mathcal{C}, \mathbf{M}_k) = \mathbb{E}_Y[\mathcal{L}_Y(\mathcal{C}, \mathbf{M}_k)], \quad \text{and} \quad \tilde{\mathcal{C}}(\mathbf{M}'_k) = \arg \max_{\mathcal{C}} l(\mathcal{C}, \mathbf{M}'_k).$$

Suppose the loss function satisfies the following properties

1. (Self-consistency to \mathbf{M}_k) Suppose $MCR(\mathbf{M}'_k, \mathbf{M}_k) \geq \epsilon$ for $\epsilon > 0$. We have

$$G(\mathbf{M}'_k) - G(\mathbf{M}_k) \leq -C(\epsilon),$$

where $C(\cdot)$ takes positive values.

2. (Bounded difference between sample- and population-based loss) The difference between sample-based and population-based loss function is bounded in probability, i.e.,

$$p(t) = \mathbb{P}(|F(\mathbf{M}'_k) - G(\mathbf{M}'_k)| \geq t) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Let $\{\hat{\mathbf{M}}_k\}$ be the maximizer of $F(\mathbf{M}_k)$. Then, we have the following clustering accuracy, for any $\epsilon > 0$,

$$\mathbb{P}(MCR(\hat{\mathbf{M}}_k, \mathbf{M}_k) \geq \epsilon) \leq p \left(\frac{C(\epsilon)}{2} \right).$$