

Linear Algebra

A summary for MIT 18.06SC

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1 Matrices & Spaces

1.1 Basic concepts

- Given vectors v_1, \dots, v_n and scalars c_1, \dots, c_n , the sum $c_1v_1 + \dots + c_nv_n$ is called a *linear combination* of v_1, \dots, v_n .
- The vectors v_1, \dots, v_n are *linearly independent* (or just *independent*) if $c_1v_1 + \dots + c_nv_n = 0$ holds only when all $c_1, \dots, c_n = 0$. If the vectors v_1, \dots, v_n are *dependent*, there exist scalars c_1, \dots, c_n which are not all equal to 0 such that $c_1v_1 + \dots + c_nv_n = 0$.
- Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$ the multiplication Ax is a linear combination of the columns of A , and $x^T A$ is a linear combination of the rows of A .
- Matrix multiplication is typically not communicative, i.e. $AB \neq BA$. Lemma 1 describes a special case where matrix multiplication is communicative.
- Suppose A is a square matrix. The matrix A is *invertible* or *non-singular* if there exists a A^{-1} such that $A^{-1}A = AA^{-1} = I$. Otherwise, the matrix A is *singular*, and the determinant of A is 0.
- The inverse of a matrix product AB is $(AB)^{-1} = B^{-1}A^{-1}$. The product of invertible matrices is still invertible.
- The transpose of a matrix product AB is $(AB)^T = B^T A^T$. For any invertible matrix A , $(A^T)^{-1} = (A^{-1})^T$.
- A matrix Q is orthogonal if $Q^T = Q^{-1}$. A matrix Q is unitary if $Q^* = Q^{-1}$, where Q^* is the conjugate transpose of Q .

Lemma 1 (Communicative matrix multiplication). For matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, the matrix multiplication of A and B is communicative, i.e. $AB = BA$, if all the **normalized eigenvectors** of A and B are identical.

if A and B have the same set of eigenvectors that correspond to non-zero eigenvalues.

Proof. If all the normalized eigenvectors of A and B are identical, there exists a matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$A = QD_AQ^{-1}, \quad B = QD_BQ^{-1},$$

what is “normalized eigenvector”? Aren’t eigenvectors normalized by definition?

Your current statement of Lemma 1 fails to incorporate these two cases:

- 1. A and B are rank deficient, in which the choices of Q are infinity.**
- 2. What do you mean by “identical eigenvectors”? As a matrix, or as a set?**
The orders between eigenvectors could be different between A and B

A and B, “respectively”.

where columns of \mathbf{Q} are normalized eigenvectors of \mathbf{A} and \mathbf{B} , and $\mathbf{D}_A \in \mathbb{R}^{n \times n}$, $\mathbf{D}_B \in \mathbb{R}^{n \times n}$ are diagonal matrices whose diagonal elements are ~~corresponding~~ eigenvalues of \mathbf{A} and \mathbf{B} . Because matrix multiplication is commutative for two diagonal matrices with same dimensions, we have

$$\mathbf{AB} = \mathbf{QD}_A\mathbf{Q}^{-1}\mathbf{QD}_B\mathbf{Q}^{-1} = \mathbf{QD}_A\mathbf{D}_B\mathbf{Q}^{-1} = \mathbf{QD}_B\mathbf{D}_A\mathbf{Q}^{-1} = \mathbf{QD}_B\mathbf{Q}^{-1}\mathbf{QD}_A\mathbf{Q}^{-1} = \mathbf{BA}.$$

Therefore, the matrix multiplication of \mathbf{A} and \mathbf{B} is commutative. \square

1.2 Permutation of matrices

For any matrix \mathbf{A} , we can swap its rows by multiplying a *permutation matrix* \mathbf{P} on the left of \mathbf{A} . For example,

$$\mathbf{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix}$$

where a_k refers to the k -th row of \mathbf{A} . The inverse of permutation matrix \mathbf{P} is $\mathbf{P}^{-1} = \mathbf{P}^T$, which implies the orthogonality of permutation matrix. For an $n \times m$ matrix, there are $n!$ different row permutation matrices, which form a *multiplicative group*.

Similarly, we can also swap the columns of the matrix \mathbf{A} by multiplying a permutation matrix on the right of \mathbf{A} .

1.3 Elimination of matrices

Elimination is an important technique in linear algebra. We eliminate the matrix by multiplications and subtractions. Take a 3-by-3 matrix \mathbf{A} as an example.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\text{step 2}} \mathbf{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \begin{matrix} \text{“these”} \\ \text{avoid “those” by all means.} \end{matrix}$$

In step 1, we choose the number 1 in row 1 column 1 as a *pivot*, then we recopy the first row, multiply an appropriate number (in this case, 3), and subtract ~~those~~ values from the numbers in the second row. We have thus eliminated 3 in row 2 column 1. Similarly, in step 2, we choose 2 in row 2 column 2 as a pivot, and eliminate the number 4 in row 3 column 2. The number 5 in row 3 column 3 is also a pivot. The matrix \mathbf{U} is an upper triangular matrix.

The *elimination matrix* used to eliminate the entry in row m column n is denoted \mathbf{E}_{mn} . In the previous example,

$$\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}; \quad \mathbf{E}_{32}(\mathbf{E}_{21}\mathbf{A}) = \mathbf{U}.$$

Pivots are non-zero. If there is a 0 in the pivot position, we must exchange the row with one below to get a non-zero value in pivot position. If there is not non-zero value below the 0 pivot, we skip this column, and find a pivot in next column.

We write $\mathbf{E}_{32}(\mathbf{E}_{21}\mathbf{A}) = (\mathbf{E}_{32}\mathbf{E}_{21})\mathbf{A} = \mathbf{U}$ because matrix multiplication is associative. Let \mathbf{E} denote the product of all elimination matrices. If we need to permute the rows during the process, we multiply a permutation matrix on the left of \mathbf{A} . Therefore, the elimination process of \mathbf{A} is

$$\mathbf{EPA} = \mathbf{U}, \tag{1}$$

where U is an upper triangular matrix.

Next, we prove the invertibility of the elimination matrix.

Lemma 2 (Invertibility of elimination matrix). *Suppose there is an elimination matrix $E_{ij} \in \mathbb{R}^{n \times n}$ that multiplies a scalar $-c$ to the j -th row, and subtracts the row from i -th row, where $i \neq j$. The matrix E_{ij} is invertible.*

Proof. We write the elimination matrix as

$$E_{ij} = I_n + ce_i e_j^T,$$

where $e_i \in \mathbb{R}^n$ denotes the vector with value 1 in the i -th entry and value 0 elsewhere. Note that $e_i^T e_j = 0$ because $i \neq j$. We have

$$(I_n + ce_i e_j^T)(I_n - ce_i e_j^T) = I_n - c^2 e_i e_j^T e_i e_j^T = I_n; \quad (I_n - ce_i e_j^T)(I_n + ce_i e_j^T) = I_n.$$

Therefore, $I_n - ce_i e_j^T$ is the inverse of E_{ij} . The elimination matrix E_{ij} is invertible. \square

Corollary 1 (Inverse of elimination matrix). Suppose the elimination matrix E_{ij} in lemma 2 is a lower/upper-triangular matrix. The inverse E_{ij}^{-1} is also a lower/upper-triangular matrix.

Proof. By the proof of lemma 2, the matrix E_{ij} and its inverse are written as

$$E_{ij} = I_n + ce_i e_j^T, \quad E_{ij}^{-1} = I_n - ce_i e_j^T.$$

WLOG, we assume E_{ij} is a lower-triangular matrix. Then $ce_i e_j^T$ and $-ce_i e_j^T$ are also lower-triangular matrices. Therefore, E_{ij}^{-1} is a lower-triangular matrix. \square

1.4 Gauss-Jordan Elimination

an

We also use elimination to find the inverse of ~~any~~ invertible matrix.

Suppose $A \in \mathbb{R}^{n \times n}$ is an invertible matrix. The inverse of A , A^{-1} , satisfies

$$AA^{-1} = I_n. \tag{2}$$

Suppose there is an elimination E such that $EA = I_n$. Multiplying E on the both side of the equation (2), we have $EAA^{-1} = A^{-1} = E$. To obtain an such E , we eliminate the *augmented matrix* $[A|I_n]$ until A becomes I_n . Then, the augmented matrix becomes $E[A|I_n] = [I_n|E]$, where E is the inverse of A .

We call this elimination process of finding E as *Gauss-Jordan Elimination*.

1.5 Factorization of matrices

By elimination, for any square matrix A , we have equation (1). By lemma 2, E is invertible. We multiply E^{-1} on both sides of equation (1). We have,

$$PA = E^{-1}U.$$

Note that E is a lower-triangular matrix. By corollary 1, E^{-1} is also a lower-triangular matrix. Let L denote E^{-1} , where the letter L refers to “lower triangular”. Therefore, any square matrix A has a factorization:

$$PA = LU, \tag{3}$$

where U is an upper triangular matrix with pivots on the diagonal, L is lower triangular matrix with ones on the diagonal, and P is a permutation matrix. However, the equation (3) is not the unique factorization of A . For example, cL and $c^{-1}U$ also factorize A , where c is a non-zero scalar.

1.6 Time complexity of elimination

For an n -by- n matrix, a single elimination multiplies a selected row and subtracts the selected row from another row. A single elimination requires $\mathcal{O}(n)$ operations. To eliminate the elements below the first diagonal element, we need repeat single eliminations $(n-1)$ times, and thus require $\mathcal{O}(n^2)$ operations. Similarly, we require $\mathcal{O}((n-1)^2)$ to eliminate the elements below the second diagonal element. Repeat the elimination until we meet the n -th diagonal element. Therefore, we require $\mathcal{O}(n^3)$ operations to obtain an upper-triangular matrix by elimination:

$$1^2 + 2^2 + \cdots + (n)^2 = \sum_i^n i^2 \approx \int_0^n x^2 dx = \frac{1}{3}n^3 = \mathcal{O}(n^3).$$

1.7 Reduced row echelon form of matrices

think about why...

In previous sections, we convert ~~any~~^a matrix \mathbf{A} to an upper triangular matrix \mathbf{U} . Next, we convert \mathbf{U} into *reduced row echelon form* (RREF), which is a simpler form than upper triangle. We use $\mathbf{R} = \text{RREF}(\mathbf{A})$ to denote the reduced row echelon form of \mathbf{A} . In \mathbf{R} , the pivots are equal to 1, and the elements above and below the pivots are eliminated to 0. In the previous example,

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{0 \text{ above and below pivots}} \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is another example,

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{make pivots} = 1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{0 \text{ above and below pivots}} \mathbf{R} = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume there are r pivots in $\mathbf{A} \in \mathbb{R}^{m \times n}$. With proper permutation, the matrix \mathbf{R} is in form $\begin{bmatrix} \mathbf{I}_r & \mathbf{F} \\ 0 & 0 \end{bmatrix}$, where $\mathbf{F} \in \mathbb{R}^{r \times (n-r)}$ is an arbitrary matrix. The columns in \mathbf{A} which correspond to the identity matrix \mathbf{I}_r are called *pivot columns*. The other columns are *free columns*.

1.8 Vector space, Subspace and Column space

- *Vector space* is a collection of vectors that is ~~closed~~^{close} under linear combination (addition and multiplication by any real number); i.e. for any vectors in the collection, all the combinations of these vectors are still in the collection.
- *Subspaces of the vector space* is a vector space that is contained inside of another vector space.

Note that any vector space or subspace must include an origin. For a vector space \mathcal{A} , the subspace of \mathcal{A} can be \mathcal{A} itself or a set that contains only a zero vector.

- Vectors v_1, \dots, v_n *span* a space that consists all the combination of those vectors.
- *Column space* of matrix \mathbf{A} is the space spanned by the columns of \mathbf{A} . Let $C(\mathbf{A})$ denote the column space of \mathbf{A} .

If v_1, \dots, v_n span a space \mathcal{S} , then \mathcal{S} is the smallest space that contain those vectors.

- *Basis* of a vector space is a sequence of vectors v_1, \dots, v_n that satisfy: (1) v_1, \dots, v_n are independent; (2) v_1, \dots, v_n span the space.
- *Dimension* of the space is the number of vectors in a basis of the space. Let $\dim(\mathcal{A})$ denote the dimension of space \mathcal{A} .

1.9 Matrix rank

The *rank* of a matrix \mathbf{A} is defined as the dimension of the column space of \mathbf{A} . Rank of matrix \mathbf{A} is also equal to the number of pivot columns of \mathbf{A} . Let $\text{rank}(\mathbf{A})$ denote the rank of matrix \mathbf{A} . We have

$$\text{rank}(\mathbf{A}) = \# \text{ of pivot columns of } \mathbf{A} = \dim(C(\mathbf{A})). \quad (4)$$

If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\mathbf{A}) = r$, we have $r \leq \min\{m, n\}$. We say the matrix is *full rank* if $r = \min\{m, n\}$.

The rank of a square matrix is closely related to ~~its~~ ^{the} invertibility.

Lemma 3 (Full rankness and invertibility). *A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is full rank, if and only if \mathbf{A} is an invertible matrix.*

Proof. First, we assume \mathbf{A} is full rank and prove the invertibility of \mathbf{A} .

Consider the RREF form of \mathbf{A} , $\mathbf{R} = \text{RREF}(\mathbf{A})$. There ~~are~~ ^{an} elimination matrix \mathbf{E} and permutation matrix \mathbf{P} such that ~~exist an~~

$$\mathbf{EPA} = \mathbf{R}.$$

^{pay attention to the use of article → see my email}

By the full rankness of \mathbf{A} , \mathbf{A} has n pivot columns, and thus $\mathbf{R} = \mathbf{I}_n$. By lemma 2, \mathbf{E} is invertible. The permutation matrix \mathbf{P} is also invertible. Then, **the matrix product \mathbf{EP} is invertible and \mathbf{A} is the inverse of \mathbf{EP} .** Therefore, \mathbf{A} is invertible.

Second, we assume \mathbf{A} is invertible and prove the full rankness of \mathbf{A} by contradiction.

Assume \mathbf{A} has an inverse \mathbf{A}^{-1} and $\text{rank}(\mathbf{A}) < n$. By equation (4), $\dim(C(\mathbf{A})) = \text{rank}(\mathbf{A}) < n$, which implies that the columns of \mathbf{A} are linearly dependent. Then, there exists a non-zero vector v such that ^{comma}

$$\mathbf{A}v = 0. \quad (5)$$

Multiplying \mathbf{A}^{-1} on both sides of equation (5), we have

$$v = \mathbf{A}^{-1}0 = 0.$$

^{clothe the naked "it". → However, equation (*) contradicts ...}

However, **it** contradicts the fact that v is non-zero. Therefore, \mathbf{A} is full rank. □

The rank of \mathbf{A} also **effects** the number of solutions to the system $\mathbf{A}x = b$. We will discuss **it** in next section.

2 Solving $Ax = b$

Here we discuss the solutions of the linear system $Ax = b$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are vectors.

2.1 Solving $Ax = 0$: Nullspace

The *nullspace* of matrix A is the collection of all solutions x to the system $Ax = 0$. Let $N(A)$ denote the nullspace of A .

Lemma 4 (Nullspace). *The nullspace of matrix A is a vector space.* this sentence is grammatically wrong.

Proof. We need to show $N(A)$ is closed to linear combination to prove that $N(A)$ is a vector space.

For $\forall v_1, v_2 \in N(A)$, we have,

close “under” linear combination ...

$$A(c_1v_1 + c_2v_2) = c_1Av_1 + c_2Av_2 = 0, \quad \forall c_1, c_2 \in \mathbb{R}. \quad (6)$$

The equation (6) implies that any linear combination of vectors in $N(A)$ is also a vector in $N(A)$. Therefore, $N(A)$ is closed to linear combination. \square

Lemma 5 (The rank of nullspace). *If $\text{rank}(A) = r$, the rank of nullspace $\text{rank}(N(A)) = n - r$.*

Proof. Let R denote the RREF(A). We write R in form $R = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}$, where $F \in \mathbb{R}^{r \times (n-r)}$ is arbitrary matrix. Let $X = \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$. We have

$$RX = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} = 0.$$

these

Therefore, each column of X is a special solution to the system $Ax = 0$. Next, we show other solutions to $Ax = 0$ are linear combinations of these special solutions.

Suppose there is a solution $x = (x_1, x_2) \in N(A)$, where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$. We have

$$Rx = \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + Fx_2 \\ 0 \end{bmatrix} = 0.$$

That implies $x_1 = -Fx_2$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix} x_2 = Xx_2$. Any vector in $N(A)$ is a linear combination of special solutions, i.e. $C(X) = N(A)$. Therefore, the rank of nullspace is $\text{rank}(N(A)) = \dim(C(X)) = n - r$. \square

columns

Recall the definitions of pivot column and free column. In $Ax = b$, the variables in x that correspond to pivot columns are called *pivot variables* and others are *free variables*. If $\text{rank}(A) = r$, there are $n - r$ free variables, which coincides with $\text{rank}(N(A))$.

In the proof of lemma 5, the columns of $X = \begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}$ are special solutions that compose the basis of $N(A)$. Practically, we find those special solutions by assigning 1 to a free variable and 0 to other free variables, and then solving the system $Ax = 0$.

we solve

2.2 Solving $Ax = b$: complete solutions

Lemma 6 (Solvability of $Ax = b$). *The system $Ax = b$ is solvable only when $b \in C(A)$.*

Proof. If $Ax = b$ is solvable, there exists a x such that $Ax = b$. For any x , $Ax \in C(A)$. Therefore, $b \in C(A)$. \square

Lemma 7 (Complete solution). *The complete solution of $\mathbf{A}x = b$ is given by $x_{comp} = x_p + x_n$, where x_p is a particular solution such that $\mathbf{A}x_p = b$, and $x_n \in N(\mathbf{A})$.*

Proof. Suppose $x = x_p + x_0$ is an arbitrary solution to $\mathbf{A}x = b$. We have

$$\mathbf{A}x - \mathbf{A}x_p = \mathbf{A}(x - x_p) = \mathbf{A}x_0 = 0.$$

Therefore, $x_0 \in N(\mathbf{A})$.

grammaly wrong

□

Usually, we find a particular solution by assigning 0 to free variables, and solving the system $\mathbf{A}x = b$.

The following table discusses the rank of \mathbf{A} , the form of \mathbf{R} , dimension of nullspace $N(\mathbf{A})$, and solutions of $\mathbf{A}x = b$.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
\mathbf{R}	\mathbf{I}	$\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}$	$\begin{bmatrix} \mathbf{I} & \mathbf{F} \end{bmatrix}$	$\begin{bmatrix} \mathbf{I} & \mathbf{F} \\ 0 & 0 \end{bmatrix}$
$\dim(N(\mathbf{A}))$	0	0	$n - r$	$n - r$
# solutions to $\mathbf{A}X = b$	1	0 or 1	infinitely many	0 or infinitely many