

Tight rate for $\hat{\theta}$

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This note discusses the tightest rate of $\hat{\theta}$ in the dTBM.

Two remaining questions:

1. How to extend the 1-d θ_k estimation problem to multiway estimation? The objective function is non-linear for multiple θ_k 's;
2. How to deal with the estimation error from \mathcal{S} ? The simple non-degree estimator $\hat{\mathcal{S}}$ fails because θ_k will render the core tensor estimation error to the mean tensor estimation.

1 Preliminary

We consider the general Gaussian dTBM

$$\mathcal{Y} = \mathcal{S} \times_1 \Theta_1 \mathbf{M}_1 \times_2 \cdots \times \Theta_K \mathbf{M}_K + \mathcal{E},$$

where $\mathcal{S} \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ is the core tensor, $\mathbf{M}_k \in \{0, 1\}^{p_k \times r_k}$ are the membership matrices corresponding to the assignment $z_k \in [p_k] \mapsto [r_k]$, $\theta_k \in \mathbb{R}_+^{p_k}$ are heterogeneity, and $\mathcal{E} \in \mathbb{R}^{p_1 \times \cdots \times p_K}$ is noise tensor with i.i.d. standard Gaussian entries.

We consider the estimation space

$$E = \{(\hat{z}_k, \hat{\theta}_k, \hat{\mathcal{S}}) : \hat{z}_k \text{ is a function } [p_k] \rightarrow [r_k], \hat{\theta}_k(i) > 0, i \in [p_k], k \in [K]\}.$$

We have the MLE of $(z_k, \theta_k, \mathcal{S})$ that minimizes the least square error over E

$$(\hat{z}_{k,\text{MLE}}, \hat{\theta}_{k,\text{MLE}}, \hat{\mathcal{S}}_{\text{MLE}}) = \arg \min_{(z_k, \theta_k, \mathcal{S}) \in E} \|\mathcal{Y} - \mathcal{X}(z_k, \theta_k, \mathcal{S})\|_F^2,$$

where

$$\mathcal{X}(z_k, \theta_k, \mathcal{S}) = \mathcal{S} \times_1 \Theta_1 \mathbf{M}_1 \times_2 \cdots \times \Theta_K \mathbf{M}_K.$$

Let $(z_k^*, \theta_k^*, \mathcal{S}^*)$ denote the true parameters.

2 Tight rate for $\hat{\theta}$ with true assignment

2.1 1-d problem with given core tensor

We first consider a simpler question: Suppose that the true parameters z^*, \mathcal{S}^* and θ_k^* for $k \geq 2$ are given. What's the error rate for θ_1^* ?

Let $\mathcal{X}(\boldsymbol{\theta}_1) := \mathcal{X}(z^*, \boldsymbol{\theta}_1, \{\boldsymbol{\theta}_k^*\}_{k \geq 2}, \mathcal{S}^*)$ denote the mean tensor with degree $\boldsymbol{\theta}_1$ while other parameters are fixed as true parameters.

Lemma 1 (1-d $\boldsymbol{\theta}$ estimation). Consider a general dTBM with true parameter $(z_k^*, \boldsymbol{\theta}^*, \mathcal{S}^*)$, fixed $r \geq 2, K \geq 2$. Suppose that the true parameters z^*, \mathcal{S}^* and $\boldsymbol{\theta}_k^*$ for $k \geq 2$ are given. Consider the estimate $\hat{\boldsymbol{\theta}}_1$ such that $\|\mathcal{Y} - \mathcal{X}(\hat{\boldsymbol{\theta}}_1)\|_F^2 \leq \|\mathcal{Y} - \mathcal{X}(\boldsymbol{\theta}_1^*)\|_F^2$. With probability tends to 1 as $p \rightarrow \infty$, we have

$$\|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \lesssim \frac{1}{p^{K-2}}$$

Proof of Lemma 1. Note that the MSE result in Lemma 12 of the manuscript also applies for the estimate $(z, \boldsymbol{\theta}_k, \mathcal{S})$ such that $\|\mathcal{Y} - \mathcal{X}(z, \boldsymbol{\theta}_k, \mathcal{S})\|_F^2 \leq \|\mathcal{Y} - \mathcal{X}(z^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*)\|_F^2$. Hence, we have

$$\begin{aligned} \mathcal{O}(pr + r^K) &\gtrsim \|\mathcal{X}(\hat{\boldsymbol{\theta}}_1) - \mathcal{X}(\boldsymbol{\theta}_1^*)\|_F^2 \\ &= \|(\hat{\boldsymbol{\Theta}}_1 \mathbf{M}_1^* - \boldsymbol{\Theta}_1^* \mathbf{M}_1^*) \text{Mat}_1(\mathcal{S}^* \times_2 \boldsymbol{\Theta}_2^* \mathbf{M}_2^* \times_3 \cdots \times_K \boldsymbol{\Theta}_K^* \mathbf{M}_K^*)\|_F^2 \\ &\geq \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \min_{a:} \|\mathbf{S}_{a:}^*\|_F^2 p^{K-1} \\ &\geq c_3 p^{K-1} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2, \end{aligned} \tag{1}$$

where in the second inequality, the p^{K-1} comes from the norm of the slice in the mean tensor, and the last inequality follows from the assumption that $\|\mathbf{S}_{a:}^*\|_F \geq c_3$ for some constant $c_3 > 0$. By inequality (1) and the assumption that r is fixed, we obtain that

$$\|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \lesssim \frac{1}{p^{K-2}},$$

with high probability tends to 1 as $p \rightarrow \infty$. \square

Remark 1 (Tightness). The result in Lemma 1 agrees with the heuristic; i.e.,

$$\frac{1}{p} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \leq \frac{\# \text{ parameter}}{\# \text{ number}} = \frac{p}{p^K}.$$

Therefore, we can consider the result in Lemma 1 achieves the sharpest rate.

Lemma 2 (1-d $\boldsymbol{\theta}$ estimation with increasing r). Consider a general dTBM with true parameter $(z_k^*, \boldsymbol{\theta}^*, \mathcal{S}^*)$, fixed $K \geq 2$ and $r = p/2$. Suppose that the true parameters z^*, \mathcal{S}^* and $\boldsymbol{\theta}_k^*$ for $k \geq 2$ are given. Consider the estimate $\hat{\boldsymbol{\theta}}_1$ such that $\|\mathcal{Y} - \mathcal{X}(\hat{\boldsymbol{\theta}}_1)\|_F^2 \leq \|\mathcal{Y} - \mathcal{X}(\boldsymbol{\theta}_1^*)\|_F^2$. With probability tends to 1 as $p \rightarrow \infty$, we have

$$\|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \lesssim \mathcal{O}(p) \tag{2}$$

Proof of Lemma 2. Following the same idea of the proof of Lemma 1, we replace the r by $p/2$. The left hand side in inequality (1) becomes $\mathcal{O}(p^K)$, and we obtained the desired result. \square

Remark 2 (Alternative error rate via the closed form of $\hat{\boldsymbol{\theta}}$). Given the true parameters z^*, \mathcal{S}^* and $\boldsymbol{\theta}_k^*$ for $k \geq 2$ and fixed $r \geq 2$, the $\hat{\boldsymbol{\theta}}_1$ minimizes the least square error has the closed form

$$\hat{\boldsymbol{\theta}}_1(i) = \frac{\langle \mathbf{Y}_{i:}, \mathbf{C}_{z^*(i):} \rangle}{\|\mathbf{C}_{z^*(i):}\|^2} = \boldsymbol{\theta}_1^*(i) + \frac{\langle \mathbf{E}_{i:}, \mathbf{C}_{z^*(i):} \rangle}{\|\mathbf{C}_{z^*(i):}\|^2}$$

where $\mathbf{C} = \text{Mat}_1(\mathcal{S}^* \times_2 \boldsymbol{\Theta}_2^* \mathbf{M}_2^* \times_3 \cdots \times_K \boldsymbol{\Theta}_K^* \mathbf{M}_K^*)$, $\mathbf{Y} = \text{Mat}(\mathcal{Y})$, and $\mathbf{E} = \text{Mat}_1(\mathcal{E})$. Note that $\mathbf{C}_{z^*(i):}$ is the vectorization of a $K-1$ tensor. By Lemma E5 in Han et al. (2022), we have

$$\frac{\langle \mathbf{E}_{i:}, \mathbf{C}_{z^*(i):} \rangle}{\|\mathbf{C}_{z^*(i):}\|} \leq \sup_{\mathcal{T} \in \mathcal{Q}(r, \dots, r) \cap \|\mathcal{T}\|=1} \langle \mathcal{E}_{z^*(i):}, \mathcal{T} \rangle \lesssim \sqrt{p}. \quad (3)$$

Since $\|\mathbf{C}_{z^*(i):}\| \geq \min_{a \in [r]} \|\mathbf{S}_{a:}^*\|_{Fp}^{(K-1)/2}$, we have

$$|\boldsymbol{\theta}_1^*(i) - \hat{\boldsymbol{\theta}}_1(i)|^2 \lesssim \frac{1}{p^{K-2}}, \quad (4)$$

with high probability tends to 1. Compared with Lemma 1, the rate (4) obtained via closed form is sup-optimal with an extra p factor; since the average point-wise estimation error in Lemma 1 $\frac{1}{p} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2$ is of order $1/p^{K-1}$. The closed form method ignores the decomposition structure on the first mode, which may be the reason causes the sub-optimality.

When $r = p/2$, the right hand side of the inequality (3) becomes to $\sqrt{p^{K-1}}$, and thus we have

$$|\boldsymbol{\theta}_1^*(i) - \hat{\boldsymbol{\theta}}_1(i)|^2 \leq \mathcal{O}(1). \quad (5)$$

This result agrees with the conclusion in (2) since there is no significant dimension reduction in the decomposition structure with $r = p/2$. In fact, the closed form result (5) is stronger than the result in Lemma 2. Because point-wise result (5) indicates the vector-wise result in (2) while vector-wise result (2) allows $|\boldsymbol{\theta}_1^*(i) - \hat{\boldsymbol{\theta}}_1(i)|^2 = \mathcal{O}(p)$ for some $i \in [p]$.

2.2 Multiway problem with given core tensor

We now extend the simple question to a multiway problem: Suppose that the true parameters z^*, \mathcal{S}^* are given. What's the error rate for $\boldsymbol{\theta}_k^*$ for all $k \in [K]$?

In 1-d problem, the key step to establish the error rate for $\boldsymbol{\theta}$ is to find following inequality

$$\|\mathcal{X}(z^*, \boldsymbol{\theta}_k, \mathcal{S}^*) - \mathcal{X}(z^*, \boldsymbol{\theta}_k^*, \mathcal{S}^*)\|_F^2 \geq C \min_{k \in [K]} \|\boldsymbol{\theta}_k - \boldsymbol{\theta}_k^*\|^2 p^{K-1}, \quad (6)$$

where C is some positive constant.

I did not figure out how to prove the inequality (6). I think the difficulty lies in the non-linearity of \mathcal{X} with $\boldsymbol{\theta}_k$ s. Also, since we are finding lower bound, triangle inequality may not be helpful to decompose the left hand side by considering one $\boldsymbol{\theta}_k$ at a time.

2.3 Error rate only with given assignment

The next extension will be: Suppose that only the true assignment z^* is given. What's the error rate for $\boldsymbol{\theta}_k^*$ for all $k \in [K]$?

In this extension, we need to consider the estimation error of \mathcal{S}^* . Given the true parameter, consider the estimator $\hat{\mathcal{S}}$ such that

$$\hat{\mathcal{S}}_{a_1, \dots, a_K} = \frac{1}{\prod_{k \in [K]} |(z^*)^{-1}(a_k)|} \sum_{i_k \in (z^*)^{-1}(a_k)} \mathcal{Y}_{i_1, \dots, i_K}.$$

Hence, we have

$$\mathbb{P}(|\hat{\mathcal{S}}_{a_1, \dots, a_K} - \mathcal{S}_{a_1, \dots, a_K}^*| \geq t) \asymp \mathbb{P}\left(\frac{1}{p^K} \left| \sum_{i \in [p^K]} \epsilon_i \right| \geq t\right) \leq 2 \exp(-p^K t^2),$$

where ϵ_i 's are i.i.d. standard normal variables. Thus, we have

$$|\hat{\mathcal{S}}_{a_1, \dots, a_K} - \mathcal{S}_{a_1, \dots, a_K}^*| \leq \mathcal{O}(p^{-K/2}), \quad (7)$$

with probability tends to 1 as $p \rightarrow \infty$.

In the 1-d problem, we now have

$$\begin{aligned} \|\mathcal{X}(\hat{\boldsymbol{\theta}}_1, \hat{\mathcal{S}}) - \mathcal{X}(\boldsymbol{\theta}_1^*, \mathcal{S}^*)\|_F^2 &\geq \|(\hat{\boldsymbol{\Theta}}_1 \mathbf{M}_1^* - \boldsymbol{\Theta}_1^* \mathbf{M}_1^*) \text{Mat}_1(\mathcal{S}^* \times_2 \boldsymbol{\Theta}_2^* \mathbf{M}_2^* \times_3 \cdots \times_K \boldsymbol{\Theta}_K^* \mathbf{M}_K^*)\|_F^2 \\ &\quad - \|\hat{\boldsymbol{\Theta}}_1 \mathbf{M}_1^* \text{Mat}_1(\hat{\mathcal{S}} - \mathcal{S}^* \times_2 \boldsymbol{\Theta}_2^* \mathbf{M}_2^* \times_3 \cdots \times_K \boldsymbol{\Theta}_K^* \mathbf{M}_K^*)\|_F^2 \\ &\geq c_3 p^{K-1} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 - \|\hat{\mathcal{S}} - \mathcal{S}\|_F^2 p^{2K}, \\ &\geq c_3 p^{K-1} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 - \mathcal{O}(p^K). \end{aligned} \quad (8)$$

where the first inequality follows from the equation that $\hat{\mathcal{S}} = \mathcal{S}^* + (\hat{\mathcal{S}} - \mathcal{S}^*)$, the second inequality follows from (1) and the fact that $\lambda_{\max}(\boldsymbol{\Theta}^* \mathbf{M}^*) \lesssim p/r$, and the last inequality follows from the estimation rate of \mathcal{S} in (7).

Plugging in above inequality (8) into the proof of Lemma 1, we obtain the non-desirable result $\|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^*\|^2 \lesssim \mathcal{O}(p)$. Notice the estimation rate of \mathcal{S} in (7) satisfies the heuristic, however, the degree $\boldsymbol{\theta}_k^*$ s render the core tensor estimation error to the mean tensor and result in the term p^K in inequality (8).

Moreover, notice that the $\hat{\mathcal{S}}$ is not the least square estimator that minimizes $\|\mathcal{Y} - \mathcal{X}(z^*, \boldsymbol{\theta}_k, \mathcal{S})\|_F$. The real minimizer $\hat{\mathcal{S}}_{sq}$ involves $\hat{\boldsymbol{\theta}}_{k, sq}$ and thus is hard to directly analyze the closed form of $\hat{\mathcal{S}}_{sq}$. I will try to find a better way to deal with the joint estimation of \mathcal{S} and $\boldsymbol{\theta}$.

References

Han, R., Willett, R., and Zhang, A. R. (2022). An optimal statistical and computational framework for generalized tensor estimation. *The Annals of Statistics*, 50(1):1–29.