COMP26120: Introducing Complexity Analysis (2020/21)

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- In this course, we care most about ***asymptotic performance***
  - We focus on the **infinite set of large *n*** ignoring small values of *n*
    - The best choice for all, but minimal inputs
- How does the algorithm behave as the problem size gets vast?
  - Running time
  - Memory/storage requirements
  - Bandwidth/power requirements/logic gates/etc.

Asymptotic Notation

- You should have an **intuitive feel** for asymptotic (big-O) notation:
  - *What does $O(n)$ running time mean? $O(n^2)$? $O(\log_2 n)$?*
  - *How does asymptotic running time relate to asymptotic memory usage?*

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- Our first task is to **define this notation more formally**

Search Problem (Arbitrary Sequence)

Input

- *sequence of numbers* $(a_1, \ldots a_n)$
- *search for a specific number* (q)

$a_1, a_2, a_3, \ldots.., a_n; \; q$

Output

- *index or NIL*

j

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2 5 4 10 7; 5 ⟶ *?*

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2 5 4 10 7; 9 $\longrightarrow$ *?*

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Output

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Linear Search

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j=1
while j<=length(A) and A[j]!=q
    do j++
if j<=length(A) then return j
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- Can we do better using this approach?

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- Worst case: ***f(n)=n***, average case: ***n/2***
- Can we do better using this approach?
  - this is a **lower bound** for the search problem in an **arbitrary sequence**

A Search Problem (Sorted Sequence)

Input

- *sequence of numbers* $(a_1 <= a_2, \ldots, a_{n-1} <= a_n)$
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$2 \quad 4 \quad 5 \quad 7 \quad 10; \quad 10 \longrightarrow 5$

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2	*4*	*5*	*7*	*10;*	*8*	*?*

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Did the sorted sequence help in the search?

Binary Search

- Assume that the array is **sorted** and then perform **successive divisions**

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left=1
right=length(A)
do
  j=(left+right)/2
  if A[j]==q then return j
  else if A[j]>q then right=j-1
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- How many times is the loop executed?

3	*4*	*6*	*8*	*9*	*10*	*12*	*15*

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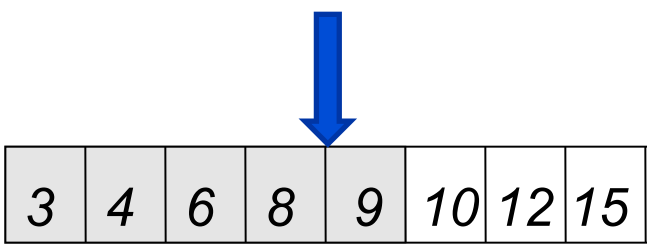

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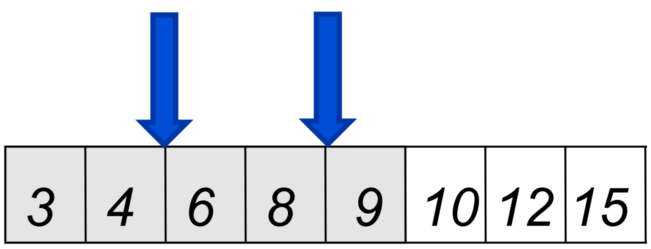

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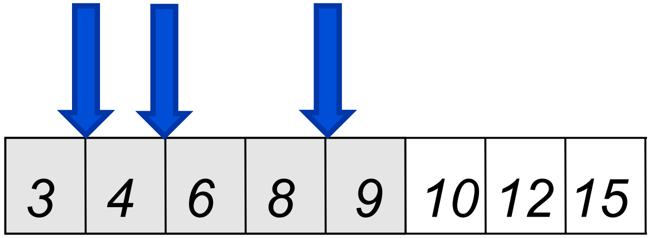
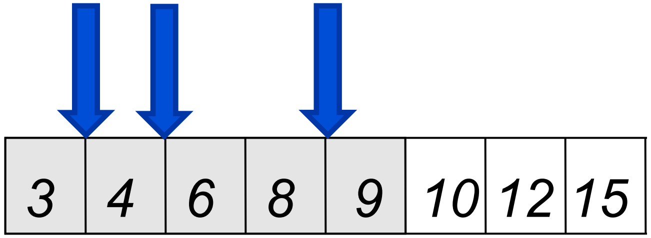


Binary Search Analysis

- How many times is the loop executed?
  - At each interaction, the number of positions **n** is cut in half
  - How many times do we cut in half **n** to reach 1?



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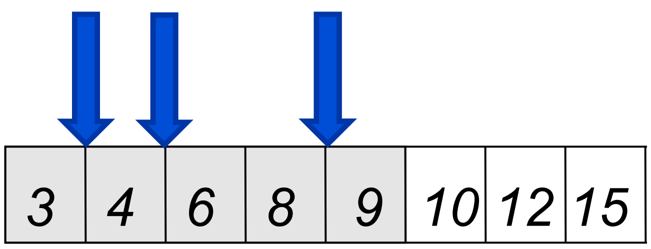

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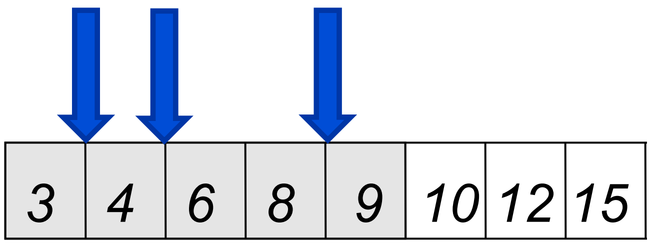

$$\lg_2 n = x \Leftrightarrow n = 2^x$$

$$\lg_2 8 = 3$$

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  - All reasonable instructions take **unit time**
    - Except, of course, function calls
  - Constant word size
    - Unless we are explicitly manipulating bits

Input Size

- **Time** and **space** complexity
  - This is generally a **function of the input size**
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  - This is generally a **function of the input size**
    - E.g., sorting, multiplication
  - How we characterize input size depends:
    - **Sorting:** number of input items
    - **Multiplication:** total number of bits
    - **Graph algorithms:** number of nodes and edges
    - Etc.

Running Time

- Number of **primitive steps** that are executed
  - Except for time of executing a function call most statements roughly require the same amount of time
    - o f = (g + h) – (i + j)
    - o y = m * x + b
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    - o c = 5 / 9 * (t - 32)
    - o z = f(x) + g(y)
- We can be more exact if needed

```
add t0, g, h # temp t0 = g + h
add t1, i, j # temp t1 = i + j
sub f, t0, t1 # f = t0 - t1
```

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- **Worst case**
  - Provides an **upper bound** on running time
  - An (absolute) guarantee
- **Average case**
  - Provides the expected running time
  - Very useful, but treat with care: what is “average”?
    - Random (equally likely) inputs
    - Real-life inputs