State-Changing Systems

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Our main interest from now on is modelling state-changing systems.

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The system state is changing in time. There are actions (controlled or not) that change the state.	Actions change values of state variables.

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Reasoning about state-changing systems

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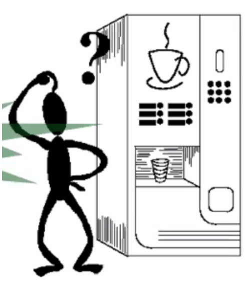
Running example: Vending Machine



Vending machine example

Consider an example state-changing system: a vending machine which dispenses drinks in a university department.

- The machine has several components, including at least the following: a storage space for storing and preparing drinks, a box for dispensing drinks and a coin slot for putting coins in.
- When the machine is operating, it goes through several states depending on the behavior of the current customer.
- Each action undertaken by the customer or by the machine itself may change the state of the machine. For example, when the customer inserts a coin in the coin slot, the amount of money stored in the slot changes.
- Actions which may change the state of the system are called transitions.

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Modeling state-changing systems

To build a formal model of a particular state-changing system, we should define

1. What are the state variables.
2. What are the possible values of the state variables.
3. What are the transitions and how they change the values of the state variables.

Transition systems

A transition system is a tuple $\mathbb{S} = (S, In, T, \mathcal{X}, dom)$, where

1. S is a finite non-empty set, called the set of states of $\mathbb{S}$.
2. $In \subseteq S$ is a non-empty set of states, called the set of initial states of M.
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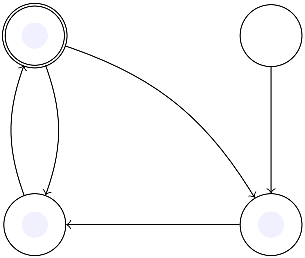

We denote the initial state(s) using double lines.

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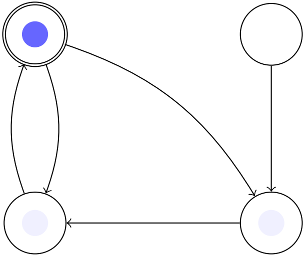

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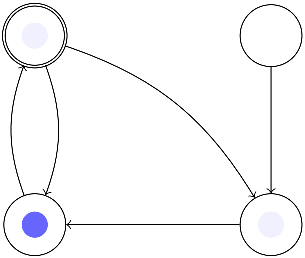

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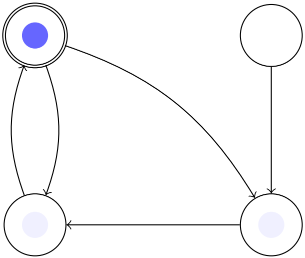

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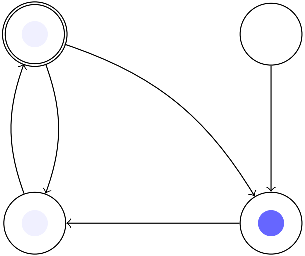

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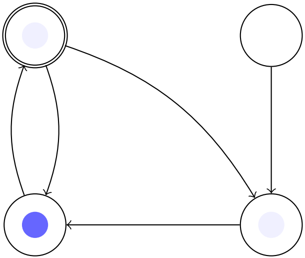

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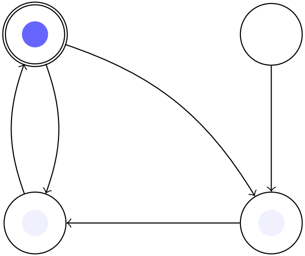

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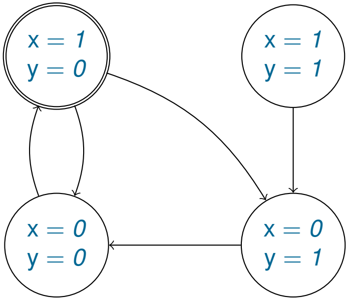

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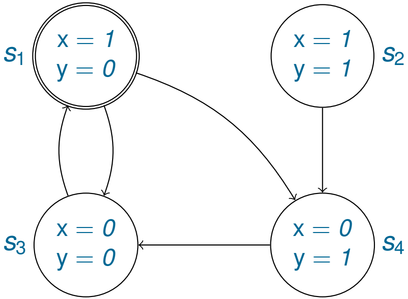
States as Interpretations



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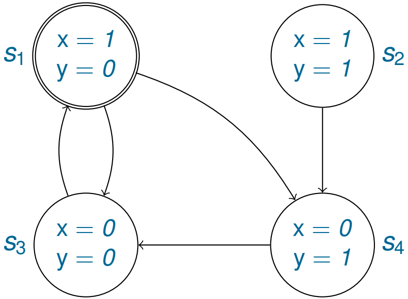

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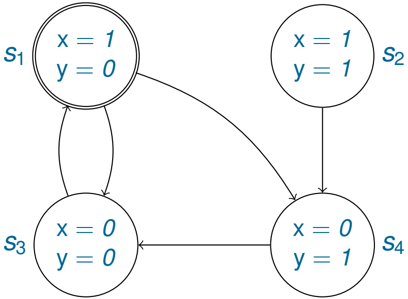

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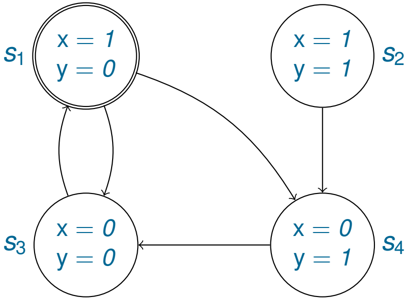

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Symbolic representation of state-changing systems

Symbolic Representation of Sets of States

Main issue with graph representation of state-changing systems:
The number of nodes in the graph = number of states = 2^N

Symbolic Representation of Sets of States:

Let $\mathbb{S} = (S, In, T, \mathcal{X}, dom)$ be a finite-state transition system.
Then every formula F defines a set states:

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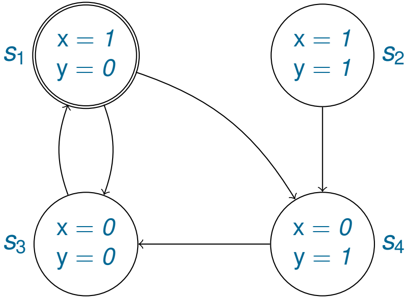

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How many states the formula $\neg x_1$ symbolically represents ?

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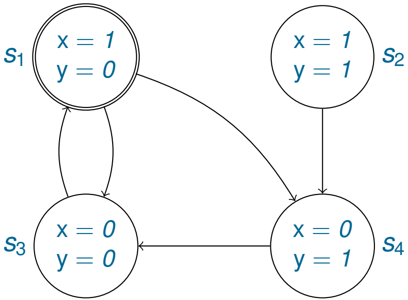

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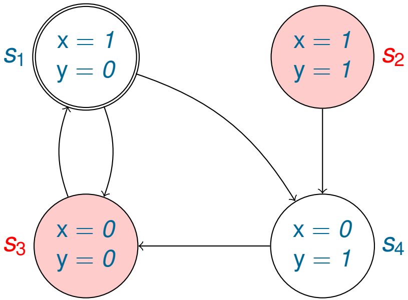

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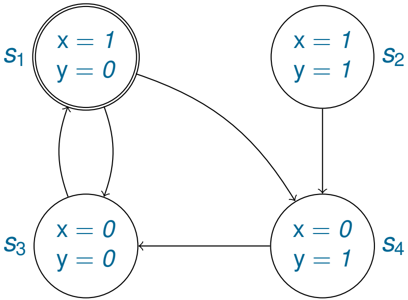

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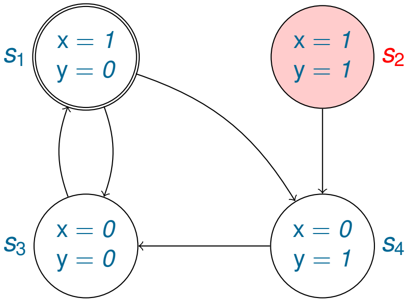

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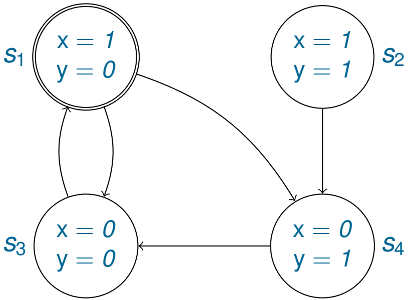

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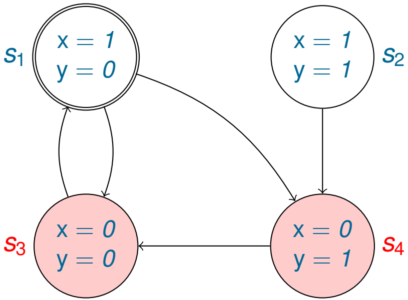

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Vending machine

1. The vending machine contains a drink storage, a coin slot, and a drink dispenser. The drink storage stores drinks of two kinds: beer and coffee. We are only interested in whether a particular kind of drink is currently being stored or not, but not interested in the amount of it.
2. The coin slot can accommodate up to three coins.
3. The drink dispenser can store at most one drink. If it contains a drink, this drink should be removed before the next one can be dispensed.
4. A can of beer costs two coins. A cup of coffee costs one coin.
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6. From time to time the drink storage can be recharged.

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Vending Machine: Variables and Domains

variable	domain	explanation
st_coffee	$\{0, 1\}$	drink storage contains coffee
st_beer	$\{0, 1\}$	drink storage contains beer
disp	$\{\mathit{none}, \mathit{beer}, \mathit{coffee}\}$	content of drink dispenser
coins	$\{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}\}$	number of coins in the slot
customer	$\{\mathit{none}, \mathit{student}, \mathit{prof}\}$	customer

Example

Let us represent the set of states in which the machine is ready to dispense a drink. In every such state, a drink should be available, the drink dispenser empty, and the coin slot contain enough coins.

This can be expressed by:

$$(\mathrm{st\_coffee} \lor \mathrm{st\_beer}) \land$$
$$\mathrm{disp} = \mathit{none} \land$$
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Symbolic representation of transition relations

Transitions

When we model systems, we will usually represent the transition relation as a union of so-called transitions.

- A transition t is any set of pairs of states.
- A transition t is applicable to a state s if there exists a state s' such that $(s, s') \in t$.
- A transition t is deterministic if for every state s there exists at most one state s' such that $(s, s') \in t$.

Symbolic Representation of Transitions

A transition is a relation on pairs of states. It brings the system to the current state and the next state. Formulas of PLFD can only express properties of a single state. How can we represent transitions using formulas?

- In addition to the set of current state variables $\mathcal{X} = \{x_1, \ldots, x_n\}$, introduce a set of next state variables $\mathcal{X}' = \{x'_1, \ldots, x'_n\}$.
- Pairs of states as interpretations. For every variable $x \in \mathcal{X}$ define

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\begin{aligned}
(s, s')(x) &\stackrel{\text{def}}{=} s(x); \\
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- Symbolic representation. Formula F of variables $\mathcal{X} \cup \mathcal{X}'$ represents a transition t if $t = \{(s, s') \mid (s, s') \models F\}$.

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Vending machine

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2. The coin slot can accommodate up to three coins.
3. The drink dispenser can store at most one drink. If it contains a drink, this drink should be removed before the next one can be dispensed.
4. A can of beer costs two coins. A cup of coffee costs one coin.
5. There are two kinds of customers: students and professors. Students drink only beer, professors drink only coffee.
6. From time to time the drink storage can be recharged.

Transitions for the Vending Machine

1. *Recharge* which results in the drink storage having both beer and coffee.
2. *Customer_arrives*, after which a customer appears at the machine.
3. *Customer_leaves*, after which the customer leaves.
4. *Coin_insert*, when the customer inserts a coin in the machine.
5. *Dispense_beer*, when the customer presses the button to get a can of beer.
6. *Dispense_coffee*, when the customer presses the button to get a cup of coffee.
7. *Take_drink*, when the customer removes a drink from the dispenser.

Example

The transition *Recharge*:

$$\text{customer} = \textit{none} \land \text{st\_coffee}' \land \text{st\_beer}'.$$

But this formula includes describes a very strange transition after which, for example

- coins may appear in and disappear from the slot;
- dfrinks may appear in and disappear from the dispenser.
- …

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- coins may appear in and disappear from the slot;
- dfrinks may appear in and disappear from the dispenser.
- ...

Frame problem

One has to express explicitly, maybe for a large number of state variables, that the values of these variables do not change after a transition. For example,

$$
\begin{array}{l}
(\mathrm{coins} = \mathit{0} \leftrightarrow \mathrm{coins}' = \mathit{0}) \wedge \\
(\mathrm{coins} = \mathit{1} \leftrightarrow \mathrm{coins}' = \mathit{1}) \wedge \\
(\mathrm{coins} = \mathit{2} \leftrightarrow \mathrm{coins}' = \mathit{2}) \wedge \\
(\mathrm{coins} = \mathit{3} \leftrightarrow \mathrm{coins}' = \mathit{3}).
\end{array}
$$

This frame problem arises in artificial intelligence, knowledge representation, and reasoning about actions.

Notation for the frame formula

Abbreviations (we assume $dom(x) = dom(y)$):

$$
\begin{aligned}
x \neq v &\stackrel{\text{def}}{=} \neg(x = v) \\
x = y &\stackrel{\text{def}}{=} \bigwedge_{v \in dom(x)} (x = v \leftrightarrow y = v).
\end{aligned}
$$

Let $\mathbb{S}$ be a transition system and $\{x_1, \ldots, x_n\} \subseteq \mathcal{X}$ be a set of state variables of $\mathcal{L}(\mathbb{S})$. Define

$$
only(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \bigwedge_{y \in \mathcal{X} \setminus \{x_1, \ldots, x_n\}} y = y'.
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This formula expresses that $x_1, \ldots, x_n$ are the only variables whose values can be changed by the transition.

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This formula expresses that $x_1, \ldots, x_n$ are the only variables whose values can be changed by the transition.

Preconditions and postconditions

When we represent a transition symbolically using a formula F of variables $\mathcal{X} \cup \mathcal{X}'$, the formula F is usually represented as the conjunction $F_1 \wedge F_2$ of two formulas:

1. F_1 expresses some conditions on the variables $\mathcal{X}$ which are necessary to execute the transition (precondition);
2. F_2 expresses some conditions relating variables in $\mathcal{X}$ to those in $\mathcal{X}'$, i.e., conditions which show how the values of the variables after the transition relate to their values before the transition (postcondition).

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Transitions: Symbolic Representation

Recharge
Customer_arrives
Customer_leaves
Coin_insert

$$\textit{Recharge} \stackrel{\text{def}}{=} \text{customer} = \textit{none} \land \text{st\_coffee}' \land \text{st\_beer}' \land \textit{only}(\text{st\_coffee}, \text{st\_beer}).$$

$$\textit{Customer\_arrives} \stackrel{\text{def}}{=} \text{customer} = \textit{none} \land \text{customer}' \neq \textit{none} \land \textit{only}(\text{customer})$$

$$\textit{Customer\_leaves} \stackrel{\text{def}}{=} \text{customer} \neq \textit{none} \land \text{customer}' = \textit{none} \land \textit{only}(\text{customer}).$$

$$\textit{Coin\_insert} \stackrel{\text{def}}{=} \text{customer} \neq \textit{none} \land \text{coins} \neq 3 \land (\text{coins} = 0 \rightarrow \text{coins}' = 1) \land (\text{coins} = 1 \rightarrow \text{coins}' = 2) \land (\text{coins} = 2 \rightarrow \text{coins}' = 3) \land \textit{only}(\text{coins}).$$

Transitions: Symbolic Representation

Recharge
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Customer_leaves
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\begin{aligned}
\mathit{Recharge} &\stackrel{\text{def}}{=} \mathsf{customer} = \mathit{none} \land \\
&\quad \mathsf{st\_coffee}' \land \mathsf{st\_beer}' \land \\
&\quad \mathit{only}(\mathsf{st\_coffee}, \mathsf{st\_beer}). \\
\mathit{Customer\_arrives} &\stackrel{\text{def}}{=} \mathsf{customer} = \mathit{none} \land \mathsf{customer}' \neq \mathit{none} \land \\
&\quad \mathit{only}(\mathsf{customer}) \\
\mathit{Customer\_leaves} &\stackrel{\text{def}}{=} \mathsf{customer} \neq \mathit{none} \land \mathsf{customer}' = \mathit{none} \land \\
&\quad \mathit{only}(\mathsf{customer}). \\
\mathit{Coin\_insert} &\stackrel{\text{def}}{=} \mathsf{customer} \neq \mathit{none} \land \mathsf{coins} \neq \mathit{3} \land \\
&\quad (\mathsf{coins} = \mathit{0} \rightarrow \mathsf{coins}' = \mathit{1}) \land \\
&\quad (\mathsf{coins} = \mathit{1} \rightarrow \mathsf{coins}' = \mathit{2}) \land \\
&\quad (\mathsf{coins} = \mathit{2} \rightarrow \mathsf{coins}' = \mathit{3}) \land \\
&\quad \mathit{only}(\mathsf{coins}).
\end{aligned}
$$

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\mathit{Customer_arrives} &\stackrel{\text{def}}{=} \text{customer} = \mathit{none} \land \text{customer}' \neq \mathit{none} \land \\
&\quad \mathit{only}(\text{customer}) \\
\mathit{Customer_leaves} &\stackrel{\text{def}}{=} \text{customer} \neq \mathit{none} \land \text{customer}' = \mathit{none} \land \\
&\quad \mathit{only}(\text{customer}). \\
\mathit{Coin_insert} &\stackrel{\text{def}}{=} \text{customer} \neq \mathit{none} \land \text{coins} \neq \mathit{3} \land \\
&\quad (\text{coins} = \mathit{0} \to \text{coins}' = \mathit{1}) \land \\
&\quad (\text{coins} = \mathit{1} \to \text{coins}' = \mathit{2}) \land \\
&\quad (\text{coins} = \mathit{2} \to \text{coins}' = \mathit{3}) \land \\
&\quad \mathit{only}(\text{coins}).
\end{aligned}$$

Transitions: Symbolic Representation

Recharge
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$$\mathit{Recharge} \stackrel{\text{def}}{=} \text{customer} = \mathit{none} \land \text{st\_coffee}' \land \text{st\_beer}' \land \mathit{only}(\text{st\_coffee}, \text{st\_beer}).$$

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Transitions: Symbolic Representation

Recharge
Customer_arrives
Customer_leaves
Coin_insert

$$\begin{array}{rcl}
\mathit{Recharge} & \stackrel{\mathrm{def}}{=} & \mathrm{customer} = \mathit{none} \wedge \\
& & \mathrm{st_coffee}' \wedge \mathrm{st_beer}' \wedge \\
& & \mathit{only}(\mathrm{st_coffee}, \mathrm{st_beer}). \\
\mathit{Customer_arrives} & \stackrel{\mathrm{def}}{=} & \mathrm{customer} = \mathit{none} \wedge \mathrm{customer}' \neq \mathit{none} \wedge \\
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\mathit{Coin_insert} & \stackrel{\mathrm{def}}{=} & \mathrm{customer} \neq \mathit{none} \wedge \mathrm{coins} \neq \mathit{3} \wedge \\
& & (\mathrm{coins} = \mathit{0} \rightarrow \mathrm{coins}' = \mathit{1}) \wedge \\
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\mathit{Customer\_arrives} & \stackrel{\text{def}}{=} & \text{customer} = \mathit{none} \land \text{customer}' \neq \mathit{none} \land \\
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 & & \mathit{only}(\text{coins}).
\end{array}
$$

Transitions

Dispense_beer
Dispense_coffee
Take_drink

$$
\begin{array}{rcl}
\mathit{Dispense\_beer} & \stackrel{\text{def}}{=} & \text{customer} = \mathit{student} \wedge \text{st\_beer} \wedge \\
& & \text{disp} = \mathit{none} \wedge (\text{coins} = \mathit{2} \vee \text{coins} = \mathit{3}) \wedge \\
& & \text{disp}' = \mathit{beer} \wedge \\
& & (\text{coins} = \mathit{2} \rightarrow \text{coins}' = \mathit{0}) \wedge \\
& & (\text{coins} = \mathit{3} \rightarrow \text{coins}' = \mathit{1}) \wedge \\
& & \mathit{only}(\text{st\_beer}, \text{disp}, \text{coins}). \\
\mathit{Dispense\_coffee} & \stackrel{\text{def}}{=} & \text{customer} = \mathit{prof} \wedge \text{st\_coffee} \wedge \\
& & \text{disp} = \mathit{none} \wedge \text{coins} \neq \mathit{0} \wedge \\
& & \text{disp}' = \mathit{coffee} \wedge \\
& & (\text{coins} = \mathit{1} \rightarrow \text{coins}' = \mathit{0}) \wedge \\
& & (\text{coins} = \mathit{2} \rightarrow \text{coins}' = \mathit{1}) \wedge \\
& & (\text{coins} = \mathit{3} \rightarrow \text{coins}' = \mathit{2}) \wedge \\
& & \mathit{only}(\text{st\_coffee}, \text{disp}, \text{coins}). \\
\mathit{Take\_drink} & \stackrel{\text{def}}{=} & \text{customer} \neq \mathit{none} \wedge \text{disp} \neq \mathit{none} \wedge \\
& & \text{disp}' = \mathit{none} \wedge \\
& & \mathit{only}(\text{disp}).
\end{array}
$$

Transitions

Dispense_beer
Dispense_coffee
Take_drink

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$$\mathit{Take\_drink} \stackrel{\text{def}}{=} \begin{array}{l} \text{customer} \neq \mathit{none} \wedge \text{disp} \neq \mathit{none} \wedge \\ \text{disp}' = \mathit{none} \wedge \\ \mathit{only}(\text{disp}). \end{array}$$

Transitions

Dispense_beer
Dispense_coffee
Take_drink

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\textit{Dispense_beer} \stackrel{\text{def}}{=}\ & \text{customer} = \textit{student} \land \text{st_beer} \land \\
& \text{disp} = \textit{none} \land (\text{coins} = \textit{2} \lor \text{coins} = \textit{3}) \land \\
& \text{disp}' = \textit{beer} \land \\
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\textit{Take_drink} \stackrel{\text{def}}{=}\ & \text{customer} \neq \textit{none} \land \text{disp} \neq \textit{none} \land \\
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Summary

State-changing systems:

- $\mathbb{S} = (S, In, T, \mathcal{X}, dom)$
- states are identified with interpretations in PLFD
- Explicit representation: using state transition graphs
- Symbolic representation: using formulas:
  - states: PLFD over $\mathcal{X}$
  - transitions: PLFD over current state variables $\mathcal{X}$ and next state variables $\mathcal{X}'$.
  - frame problem

Next: Temporal properties of transition systems: LTL

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