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long double x=2;
while( i < N ){
  x = ((2*x) - 1);
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assert( i == N );
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unsigned int i = 0;
long double x=2;
if( i < N ){
  x = ((2*x) - 1);      k copies
  ++i;
}
...
assert( !(i < N) );
assert( i == N );
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- Show that a property *P* is true for $n >= k$
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- Suppose that
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    - Often *k* = *0*
  - *P(n)* ➠ P*(n+1)* for all *n* >= *k*

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- Suppose that
  - *P(k)* is true for a fixed constant *k*
    - o Often $k = 0$
  - $P(n) \Rightarrow P(n+1)$ for all $n >= k$
- Then *P(n)* is true for all $n >= k$

Induction Example: Gaussian Closed Form

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  - **Inductive step:** show that if $P(k)$ holds, then also $P(k+1)$ holds

    $(0+1+2+\ldots+k)+(k+1) = (k+1)((k+1)+1)/2.$

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    $(0+1+2+\ldots+k)+(k+1) = \mathbf{(k+1)((k+1)+1)/2.}$

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    $k(k+1)/2 + (k+1) = [k(k+1)+2(k+1)]/2 = (k^2+3k+2)/2 = (k+1)(k+2)/2$

    $= \mathbf{(k+1)((k+1)+1)/2}$ **hereby showing that indeed *P(k+1)* holds**

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Recurrence:

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

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Prove that $T(n) \leq cn \log n$ for some $c>0$ using induction and for all $n \geq n_0$