What's next

Algorithms for satisfiability, validity of QBF:

- Splitting
- DPLL

Reminder:
(i) $F(p_1, \ldots, p_n)$ is satisfiable iff $\exists p_1 \ldots \exists p_n F(p_1, \ldots, p_n)$ is true.
(ii) $F(p_1, \ldots, p_n)$ is valid iff $\forall p_1 \ldots \forall p_n F(p_1, \ldots, p_n)$ is true.

Algorithms will check whether a closed formula is true or false.

Splitting: foundations

Lemma

- *A closed formula $\forall p F$ is true if and only if the formulas $F_p^{\perp}$ and $F_p^{\top}$ are true.*
- *A closed formula $\exists p F$ is true if and only if at least one of the formulas $F_p^{\perp}$ or $F_p^{\top}$ is true.*

Splitting

Simplification rules for $\top$:

$$\neg\top \;\Rightarrow\; \bot$$
$$\top \wedge F_1 \wedge \ldots \wedge F_n \;\Rightarrow\; F_1 \wedge \ldots \wedge F_n$$
$$\top \vee F_1 \vee \ldots \vee F_n \;\Rightarrow\; \top$$
$$F \to \top \;\Rightarrow\; \top \qquad \top \to F \;\Rightarrow\; F$$
$$F \leftrightarrow \top \;\Rightarrow\; F \qquad \top \leftrightarrow F \;\Rightarrow\; F$$

Simplification rules for $\bot$:

$$\neg\bot \;\Rightarrow\; \top$$
$$\bot \wedge F_1 \wedge \ldots \wedge F_n \;\Rightarrow\; \bot$$
$$\bot \vee F_1 \vee \ldots \vee F_n \;\Rightarrow\; F_1 \vee \ldots \vee F_n$$
$$F \to \bot \;\Rightarrow\; \neg F \qquad \bot \to F \;\Rightarrow\; \top$$
$$F \leftrightarrow \bot \;\Rightarrow\; \neg F \qquad \bot \leftrightarrow F \;\Rightarrow\; \neg F$$

Splitting

Simplification rules for $\top$:

$$
\begin{array}{rcl}
\neg\top & \Rightarrow & \bot \\
\top \wedge F_1 \wedge \ldots \wedge F_n & \Rightarrow & F_1 \wedge \ldots \wedge F_n \\
\top \vee F_1 \vee \ldots \vee F_n & \Rightarrow & \top \\
F \rightarrow \top \Rightarrow \top & & \top \rightarrow F \Rightarrow F \\
F \leftrightarrow \top \Rightarrow F & & \top \leftrightarrow F \Rightarrow F \\
\forall p \top & \Rightarrow & \top \\
\exists p \top & \Rightarrow & \top
\end{array}
$$

Simplification rules for $\bot$:

$$
\begin{array}{rcl}
\neg\bot & \Rightarrow & \top \\
\bot \wedge F_1 \wedge \ldots \wedge F_n & \Rightarrow & \bot \\
\bot \vee F_1 \vee \ldots \vee F_n & \Rightarrow & F_1 \vee \ldots \vee F_n \\
F \rightarrow \bot \Rightarrow \neg F & & \bot \rightarrow F \Rightarrow \top \\
F \leftrightarrow \bot \Rightarrow \neg F & & \bot \leftrightarrow F \Rightarrow \neg F \\
\forall p \bot & \Rightarrow & \bot \\
\exists p \bot & \Rightarrow & \bot
\end{array}
$$

Splitting algorithm

procedure *splitting*(F)
input: closed rectified prenex formula F
output: 0 or 1
parameters: function *select_variable_value* (selects a variable
from the outermost prefix of F and a boolean value for it)
begin
$F := simplify(F)$
if $F = \bot$ **then return** 0
if $F = \top$ **then return** 1
Let F have the form $\exists\!\!\!\forall p_1 \ldots \exists\!\!\!\forall p_k F_1$
$(p, b) := select\_variable\_value(F)$
Let F' be obtained from F by deleting $\exists\!\!\!\forall p$ from its outermost prefix
if $b = 0$ **then** //$p \mapsto \bot$ branch first
case $(splitting((F')^{\bot}_{p}),\ \exists\!\!\!\forall)$ **of**
$(0, \forall) \Rightarrow$ **return** 0 $(1, \exists) \Rightarrow$ **return** 1
$(1, \forall) \Rightarrow$ **return** $splitting((F')^{\top}_{p})$ $(0, \exists) \Rightarrow$ **return** $splitting((F')^{\top}_{p})$
end

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  $(p, b) \ := \ select\_variable\_value(F)$
  Let F' be obtained from F by deleting $\exists\!\!\forall p$ from its outermost prefix
  if $b = 0$ **then**   //$p \mapsto \bot$ branch first
    case $(splitting((F')^{\bot}_{p}), \ \exists\!\!\forall)$ **of**
      $(0, \forall) \Rightarrow$ **return** 0 $(1, \exists) \Rightarrow$ **return** 1
      $(1, \forall) \Rightarrow$ **return** $splitting((F')^{\top}_{p})$ $(0, \exists) \Rightarrow$ **return** $splitting((F')^{\top}_{p})$
    end
  else   //$b = 1$
    case $(splitting((F')^{\top}_{p}), \ \exists\!\!\forall)$ **of**
      $(0, \forall) \Rightarrow$ **return** 0 $(1, \exists) \Rightarrow$ **return** 1
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    end

Splitting: examples

$$\forall p \exists q (p \leftrightarrow q)$$

Splitting: examples

$$\begin{tikzcd} & \forall p \exists q (p \leftrightarrow q) \arrow[dl, "p = 0"'] \\ \exists q (\neg q) & \wedge \end{tikzcd}$$

Splitting: examples

$$\begin{tikzcd}[column sep=small]
 & & \forall p \exists q (p \leftrightarrow q) \quad \wedge \arrow[dl, "p = 0"'] \\
 & \exists q (\neg q) \quad \vee \arrow[dl, "q = 0"'] & \\
\boxed{1}\ \top & &
\end{tikzcd}$$

Splitting: examples

$$\forall p \exists q (p \leftrightarrow q) \quad \wedge$$

$$\xrightarrow{p = 0}$$

$$\boxed{1}\ \exists q (\neg q) \quad \vee$$

$$\xrightarrow{q = 0}$$

$$\boxed{1}\ \top$$

Splitting: examples

$$\begin{tikzcd}[column sep=small]
& & \forall p \exists q (p \leftrightarrow q) \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] & \\
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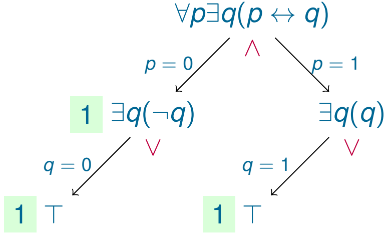
\end{tikzcd}$$

Splitting: examples



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 \boxed{1}\ \top & & \boxed{1}\ \top & &
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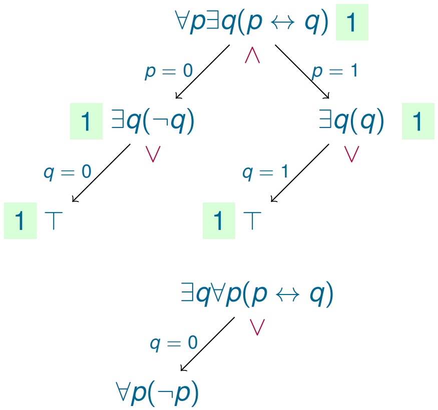
$$\exists q \forall p (p \leftrightarrow q)$$

Splitting: examples



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\boxed{0}\ \bot & {} &
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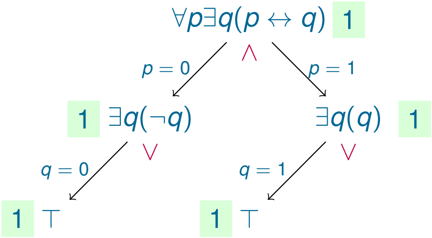

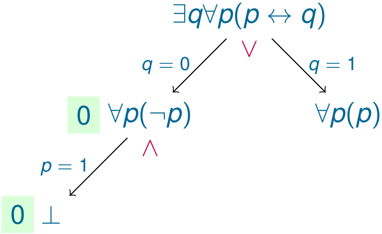

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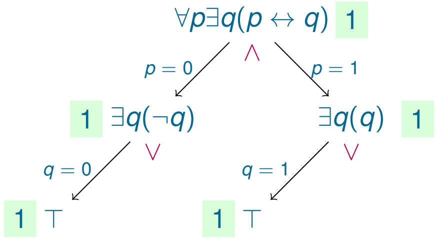

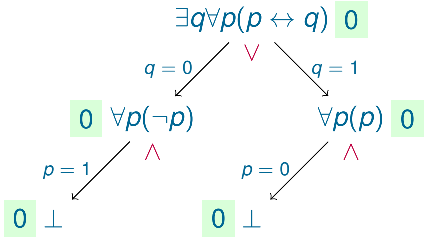

Note: the order of variables is important!

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\boxed{0}\ \bot & & \boxed{0}\ \bot & &
\end{tikzcd}
$$

Note: the order of variables is important!
Two-player game: $\exists$-player tries to make formula true and $\forall$-player tries to make formula false.

QBF: CNF, DPLL

QBF: CNF

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A quantified boolean formula F is in CNF, if it is either $\perp$, or $\top$, or rectified prenex form which has the form $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n (C_1 \land \ldots \land C_m)$, where $C_1, \ldots, C_m$ are propositional clauses.

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Example:

$$\forall p \exists q \exists s \, ((\neg p \vee s \vee q) \wedge (s \vee \neg q) \wedge \neg s))$$

QBF: CNF transformation

Rectification + Prenexing + propositional CNF rules:

$$
\begin{array}{rcl}
F \leftrightarrow G & \Rightarrow & (\neg F \vee G) \wedge (\neg G \vee F), \\
F \rightarrow G & \Rightarrow & \neg F \vee G, \\
\neg(F \wedge G) & \Rightarrow & \neg F \vee \neg G, \\
\neg(F \vee G) & \Rightarrow & \neg F \wedge \neg G, \\
\neg\neg F & \Rightarrow & F, \\
(F_1 \wedge \ldots \wedge F_m) \vee G_1 \vee \ldots \vee G_n & \Rightarrow & \begin{array}{cc} (F_1 \vee G_1 \vee \ldots \vee G_n) & \wedge \\ \ldots & \wedge \\ (F_m \vee G_1 \vee \ldots \vee G_n). & \end{array}
\end{array}
$$

DPLL

Input of DPLL:

- Q: quantifier sequence $\exists\!\!\forall_1 p_1 \ldots\ldots \exists\!\!\forall_n p_n$
- S: a set of clauses

DPLL:

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- if Q contains $\exists p$ or p does not occur in Q
  1. remove from S every clause of the form $L \vee C'$;
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Why different for universal quantifiers? Use intuition from games!

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The player playing $\forall$ wants to make the formula false.

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Why different for universal quantifiers? Use intuition from games!

The player playing $\forall$ wants to make the formula false.
When it is his turn to make a move $\forall p$, he has a winning move: select the value for p which makes the unit clause false (and hence the conjunction of clauses false too).

DPLL algorithm

procedure $DPLL(Q, S)$
input: quantifier sequence $Q = \exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n$, set of clauses S
output: 0 or 1
parameters: function *select_variable_value* (selects a variable
from the outermost prefix of F and a boolean value for it)
begin
$S := unit\_propagate(Q, S)$
if S is empty **then** **return** 1
if S contains $\square$ **then** **return** 0
$(p, b) := select\_variable\_value(Q, S)$
Let Q' be obtained from Q by deleting $\exists\!\!\!\forall p$ from its outermost prefix
if $b = 0$ **then** $L := \neg p$
else $L := p$
case $(DPLL(Q', S \cup \{L\}), \exists\!\!\!\forall)$ **of**
$(0, \forall) \Rightarrow$ **return** 0
$(1, \forall) \Rightarrow$ **return** $DPLL(Q', S \cup \{\overline{L}\})$
$(1, \exists) \Rightarrow$ **return** 1
$(0, \exists) \Rightarrow$ **return** $DPLL(Q', S \cup \{\overline{L}\})$
end

Example

$$\exists p \forall q \exists r$$
$$p \vee q \vee \neg r$$
$$p \vee \neg q \vee r$$
$$\neg p \vee q \vee r$$
$$\neg p \vee q \vee \neg r$$

$$\vee$$

Example

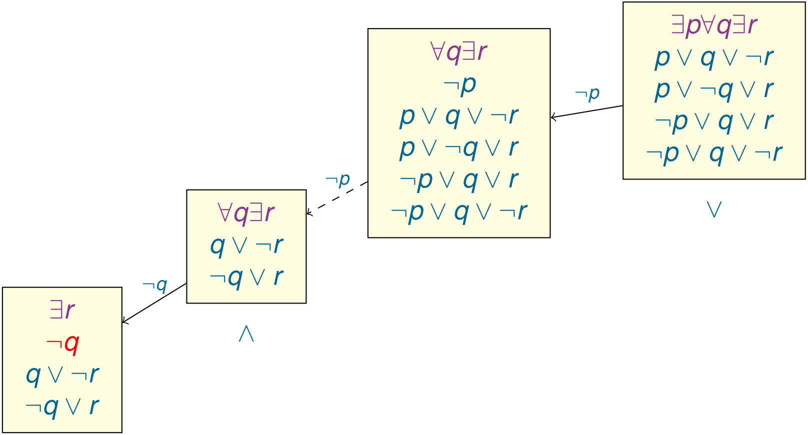
$$\boxed{\begin{array}{c}\forall q \exists r\\ \neg p\\ p \vee q \vee \neg r\\ p \vee \neg q \vee r\\ \neg p \vee q \vee r\\ \neg p \vee q \vee \neg r\end{array}} \xleftarrow{\neg p} \boxed{\begin{array}{c}\exists p \forall q \exists r\\ p \vee q \vee \neg r\\ p \vee \neg q \vee r\\ \neg p \vee q \vee r\\ \neg p \vee q \vee \neg r\end{array}}$$

$$\vee$$

Example

$$\begin{tikzcd}
\boxed{\begin{array}{c}\forall q \exists r \\ q \lor \neg r \\ \neg q \lor r\end{array}} & \boxed{\begin{array}{c}\forall q \exists r \\ \neg p \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r\end{array}} \arrow[l, dashed, "\neg p"'] & \boxed{\begin{array}{c}\exists p \forall q \exists r \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r\end{array}} \arrow[l, "\neg p"'] \\
\land & & \lor
\end{tikzcd}$$

Example



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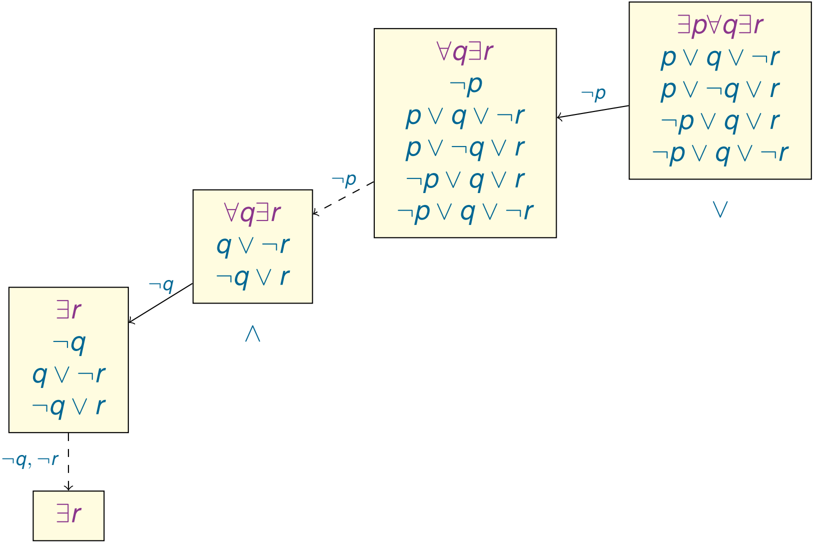

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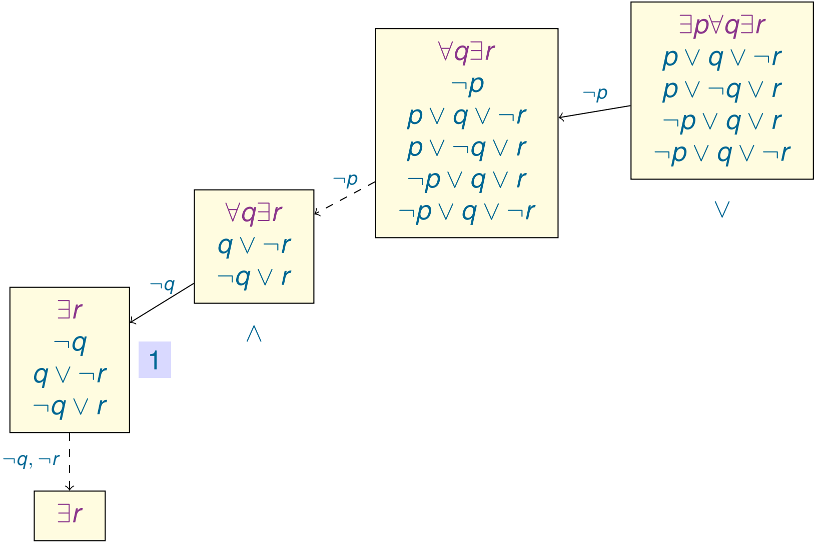
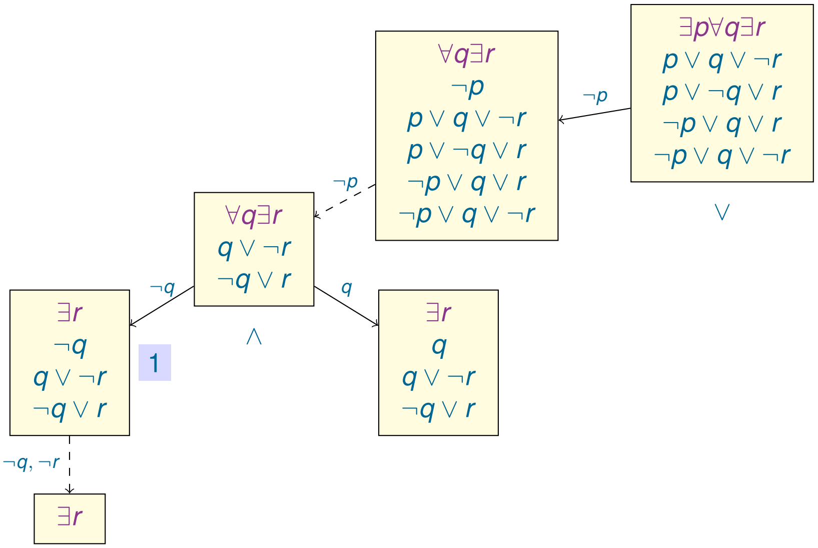


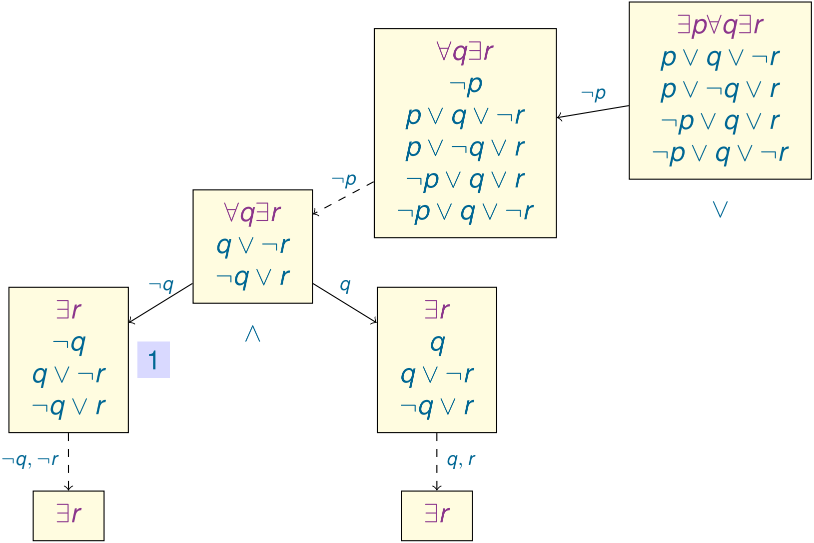
$\exists p \forall q \exists r$
$p \lor q \lor \neg r$
$p \lor \neg q \lor r$
$\neg p \lor q \lor r$
$\neg p \lor q \lor \neg r$
$\lor$
$\neg p$
$\forall q \exists r$
$\neg p$
$p \lor q \lor \neg r$
$p \lor \neg q \lor r$
$\neg p \lor q \lor r$
$\neg p \lor q \lor \neg r$
$\neg p$
$\forall q \exists r$
$q \lor \neg r$
$\neg q \lor r$
$\land$
$\neg q$
$\exists r$
$\neg q$
$q \lor \neg r$
$\neg q \lor r$
1
$\neg q, \neg r$
$\exists r$

Example

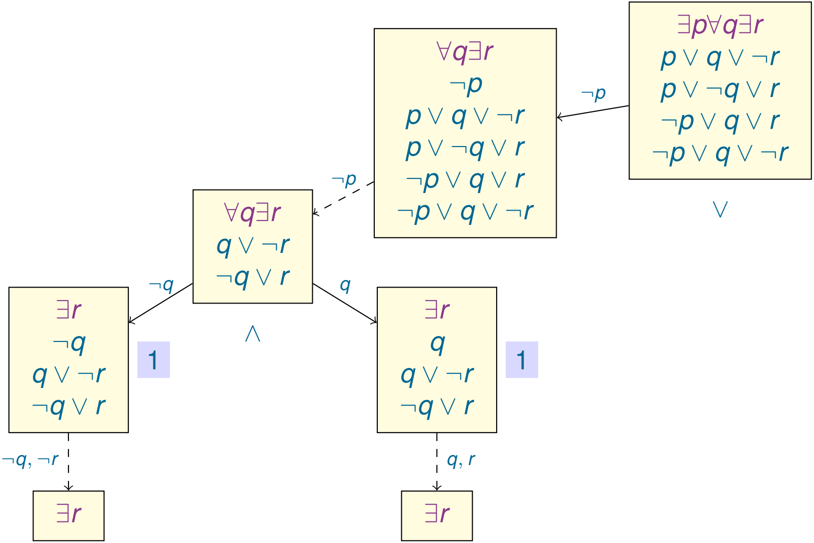


∃p∀q∃r
p ∨ q ∨ ¬r
p ∨ ¬q ∨ r
¬p ∨ q ∨ r
¬p ∨ q ∨ ¬r
∨
¬p
∀q∃r
¬p
p ∨ q ∨ ¬r
p ∨ ¬q ∨ r
¬p ∨ q ∨ r
¬p ∨ q ∨ ¬r
¬p
∀q∃r
q ∨ ¬r
¬q ∨ r
∧
¬q
q
∃r
¬q
q ∨ ¬r
¬q ∨ r
1
∃r
q
q ∨ ¬r
¬q ∨ r
¬q, ¬r
∃r

Example


$\exists p \forall q \exists r$
$p \vee q \vee \neg r$
$p \vee \neg q \vee r$
$\neg p \vee q \vee r$
$\neg p \vee q \vee \neg r$
$\vee$
$\neg p$
$\forall q \exists r$
$\neg p$
$p \vee q \vee \neg r$
$p \vee \neg q \vee r$
$\neg p \vee q \vee r$
$\neg p \vee q \vee \neg r$
$\neg p$
$\forall q \exists r$
$q \vee \neg r$
$\neg q \vee r$
$\wedge$
$\neg q$
q
$\exists r$
$\neg q$
$q \vee \neg r$
$\neg q \vee r$
1
$\exists r$
q
$q \vee \neg r$
$\neg q \vee r$
$\neg q, \neg r$
$\exists r$
q, r
$\exists r$

Example



$\exists p \forall q \exists r$
$p \vee q \vee \neg r$
$p \vee \neg q \vee r$
$\neg p \vee q \vee r$
$\neg p \vee q \vee \neg r$
$\vee$
$\neg p$
$\forall q \exists r$
$\neg p$
$p \vee q \vee \neg r$
$p \vee \neg q \vee r$
$\neg p \vee q \vee r$
$\neg p \vee q \vee \neg r$
$\neg p$
$\forall q \exists r$
$q \vee \neg r$
$\neg q \vee r$
$\wedge$
$\neg q$
q
$\exists r$
$\neg q$
$q \vee \neg r$
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$\exists r$
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$q \vee \neg r$
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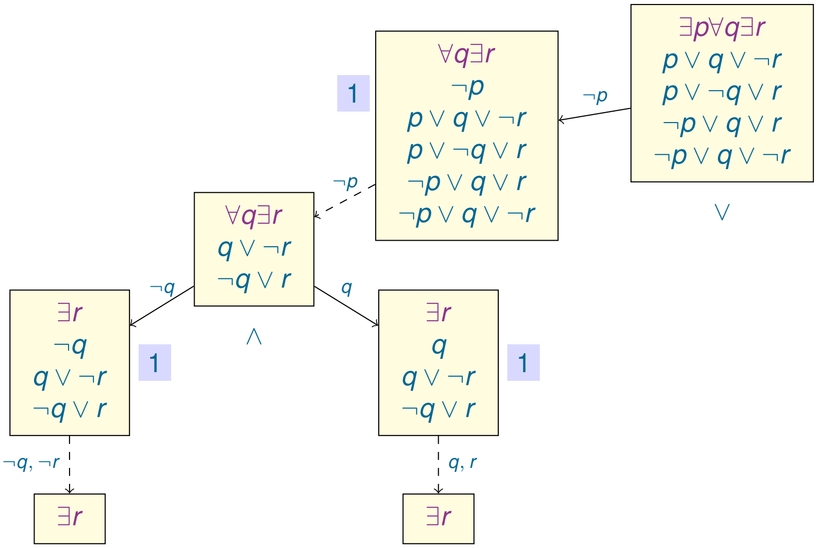
1
$\neg q, \neg r$
$\exists r$
q, r
$\exists r$

Example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∀q∃r
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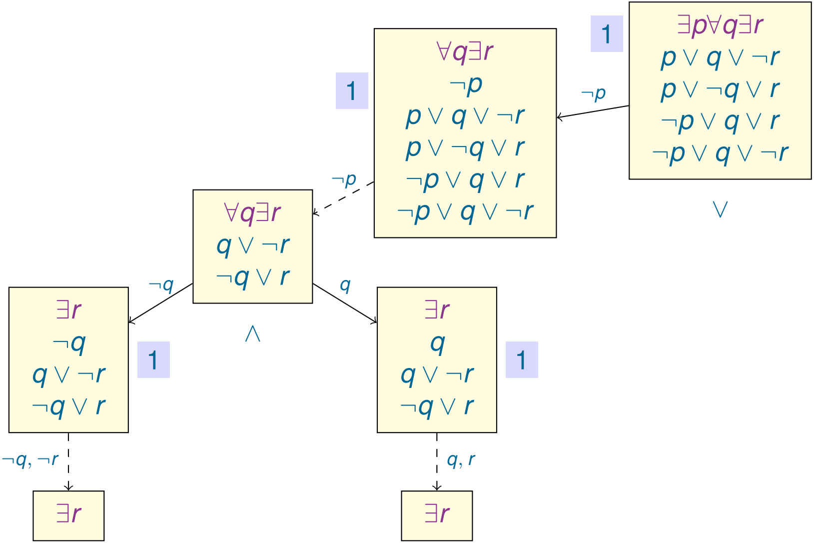
Example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QBF: DPLL optimisations

Pure literal rule

Let Q be quantifier prefix and S set of clauses.
Let literal L be pure in S then $\overline{L}$ does not occur in S.
Pure literal rule:

- If the variable of L is existentially quantified in Q then we can remove all clauses in which L occurs.
- If the variable of L is universally quantified then we can remove L from all clauses where L occurs.

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- The $\exists$-player will make the literal true (so all clauses containing this literal will be satisfied).

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Why?

- The $\exists$-player will make the literal true (so all clauses containing this literal will be satisfied).
- The $\forall$-player will make the literal false (so it can be removed from all clauses containing this literal).

Universal literal deletion

Consider a quantifier prefix Q and a conjunction of clauses S.

- a variable p is existential in Q, if Q contains $\exists p$.
- a variable q is universal in Q, if Q contains $\forall q$.
- A variable p is quantified before a variable q if p occurs before q in Q.

Example: If Q is $\forall q \exists p \forall r$ then q is quantified before both p and r; and p is quantified before r (in Q).

Theorem

Let Q be a quantifier prefix and S a conjunction of clauses. Suppose that

1. *C is a non-tautological clause in S;*
2. *a variable q in C is universal in Q;*
3. *all existential variables in C are quantified before q.*

Then the deletion of the literal containing q from C does not change the truth value of QS.

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Universal literal deletion

Let $q_1, \ldots, q_m$ be all universal variables of C such that all existential variables are quantified before them. Then C has the form:

$$L_1 \vee \ldots \vee L_n \vee (\neg)q_1 \vee \ldots \vee (\neg)q_m$$

Consider the position before the $q_1, \ldots, q_m$-moves of the $\forall$-player.

- If at least one of the literals $L_1, \ldots, L_n$ is true, deletion of $(\neg)q_1, \ldots, (\neg)q_m$ will not change the outcome of the game, since after any assignment to $q_1, \ldots, q_m$ the clause will be true.
- If all of the literals $L_1, \ldots, L_n$ are false, the $\forall$-player will make all $(\neg)q_1, \ldots, (\neg)q_m$ false and win the game, so deletion of these literals will not change the outcome of the game either.

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Example

$$\exists p \forall r \exists q \exists s ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s))$$

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- Apply universal literal deletion to $p \vee \neg r$

Example

$$\exists p \forall r \exists q \exists s ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s)) \Rightarrow$$
$$\exists p \forall r \exists q \exists s (p \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s))$$

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- Apply universal literal deletion to $p \vee \neg r$
- Unit propagation p

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$$\forall r \exists q \exists s((\neg q \vee r) \wedge (q \vee s) \wedge (q \vee r \vee \neg s))$$

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- Apply universal literal deletion to $p \vee \neg r$
- Unit propagation p
- Apply the pure literal rule to r

Example

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- Apply universal literal deletion to $p \vee \neg r$
- Unit propagation p
- Apply the pure literal rule to r

Example

$$
\begin{aligned}
&\exists p \forall r \exists q \exists s ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s)) \Rightarrow \\
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&\exists s (s \wedge \neg s)
\end{aligned}
$$

- Apply universal literal deletion to $p \vee \neg r$
- Unit propagation p
- Apply the pure literal rule to r
- Unit propagation $\neg q$

Example

$$\exists p \forall r \exists q \exists s ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s)) \Rightarrow$$
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$$\exists s (s \wedge \neg s) \Rightarrow$$
$$\square$$

- Apply universal literal deletion to $p \vee \neg r$
- Unit propagation p
- Apply the pure literal rule to r
- Unit propagation $\neg q$, s

QBF: using OBDDs for representing QBFs

QBF and OBDD

Any QBF $F(p_1, \ldots, p_n)$ represents a boolean function.

OBDDs can be used to canonically represent boolean functions.

We know how to apply boolean operations to OBDDs.
Can we also apply quantification to OBDDs in an algorithmic way?

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OBDDs can be used to canonically represent boolean functions.

We know how to apply boolean operations to OBDDs.
Can we also apply quantification to OBDDs in an algorithmic way?

Quantification: given an OBDD representing a formula F, find an OBDD representing $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n F$

Quantification for OBDDs

We can use the following properties of QBFs:

- $\exists p\,(\mathit{if}\ p\ \mathit{then}\ F\ \mathit{else}\ G) \equiv F \vee G$;
- $\forall p\,(\mathit{if}\ p\ \mathit{then}\ F\ \mathit{else}\ G) \equiv F \wedge G$;

Quantification for OBDDs

We can use the following properties of QBFs:

- $\exists p\,(\textit{ if } p \textit{ then } F \textit{ else } G) \equiv F \lor G$;
- $\forall p\,(\textit{ if } p \textit{ then } F \textit{ else } G) \equiv F \land G$;
- If $p \neq q$, then
  $\exists\!\!\!\forall p\,(\textit{ if } q \textit{ then } F \textit{ else } G) \equiv \textit{ if } q \textit{ then } \exists\!\!\!\forall pF \textit{ else } \exists\!\!\!\forall pG$

$\exists$-quantification algorithm for OBDDs

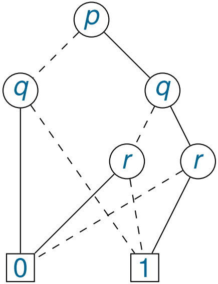
procedure $\exists quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_m\})$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $\exists p_1 \ldots \exists p_k (F_1 \vee \ldots \vee F_m)$ in (modified) D
begin

- **if** $m = 0$ **then** **return** $\boxed{0}$
- **if** some n_i is $\boxed{1}$ **then** **return** $\boxed{1}$
- **if** some n_i is $\boxed{0}$ **then**
  - **return** $\exists quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m\})$
- $p \ := \ max\_atom(n_1, \ldots, n_m)$
- **forall** $i = 1 \ldots m$
  - **if** n_i is labelled by p
    - **then** $(l_i, r_i) \ := \ (neg(n_i), pos(n_i))$
    - **else** $(l_i, r_i) \ := \ (n_i, n_i)$
- **if** $p \in \{p_1, \ldots, p_k\}$
  - **then** **return** $\exists quant(\{p_1, \ldots, p_k\} - \{p\}, \{l_1, \ldots, l_m, r_1, \ldots, r_m\})$
  - **else**
    - $k_1 \ := \ \exists quant(\{p_1, \ldots, p_k\}, \{l_1, \ldots, l_m\})$
    - $k_2 \ := \ \exists quant(\{p_1, \ldots, p_k\}, \{r_1, \ldots, r_m\})$
    - **return** $integrate(k_1, p, k_2, D)$

end

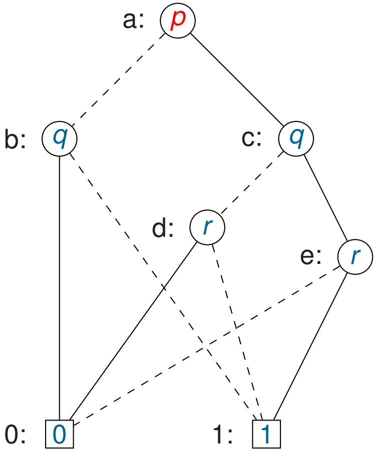
Example

Take the order $p > q > r$ and the formula $\exists r \exists p(p \leftrightarrow ((p \rightarrow r) \leftrightarrow q))$.



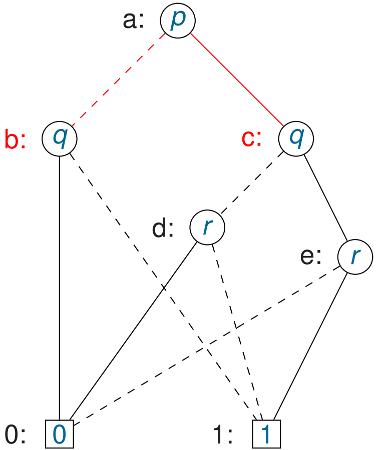
Example $\exists r \exists p (p \leftrightarrow ((p \to r) \leftrightarrow q))$

$\exists quant(\{r, p\}, \{a\})$



Example $\exists r \exists p(p \leftrightarrow ((p \rightarrow r) \leftrightarrow q))$

$\exists quant(\{r, p\}, \{a\})$
$\exists quant(\{r\}, \{b, c\})$



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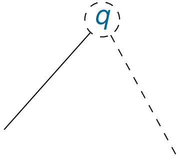

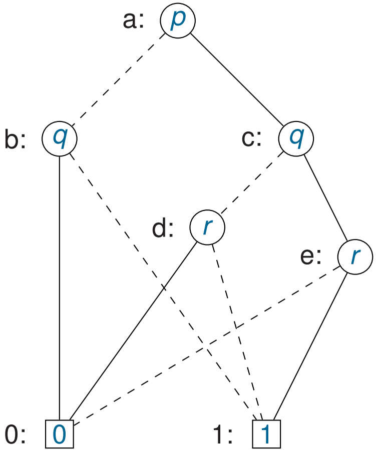

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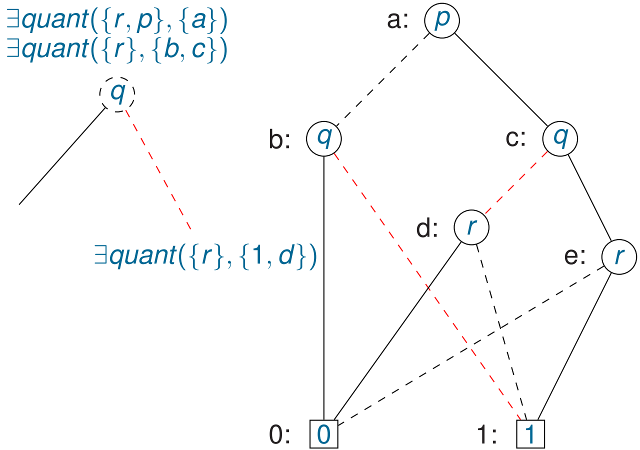

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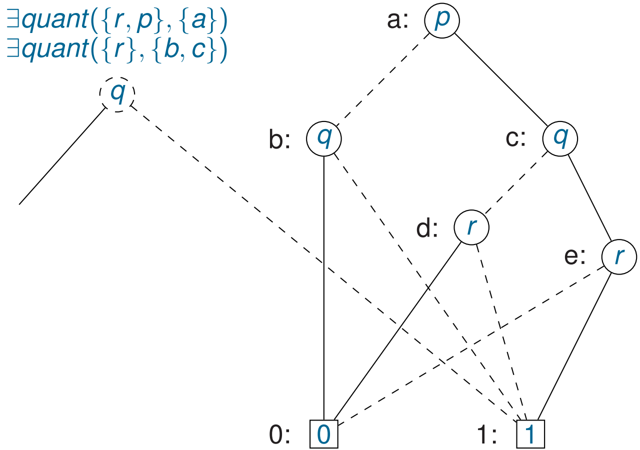

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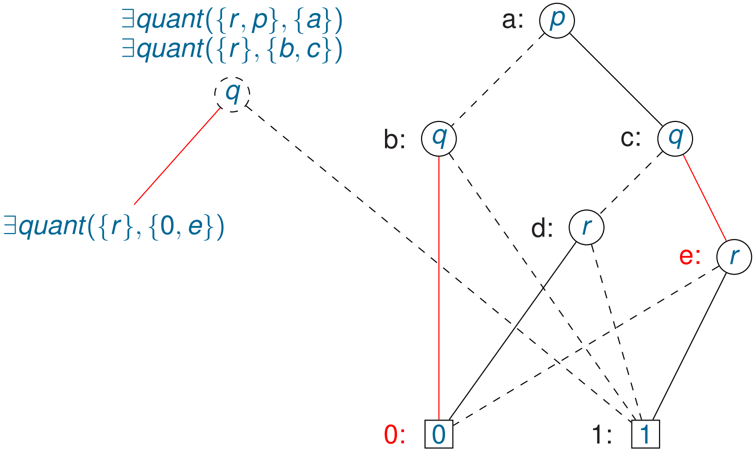
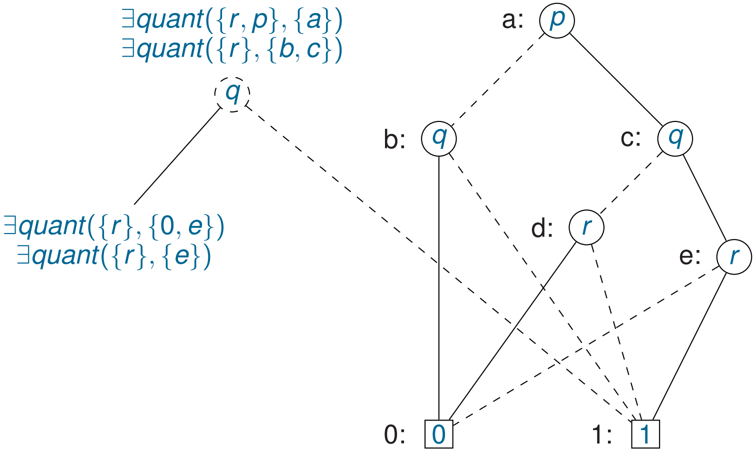
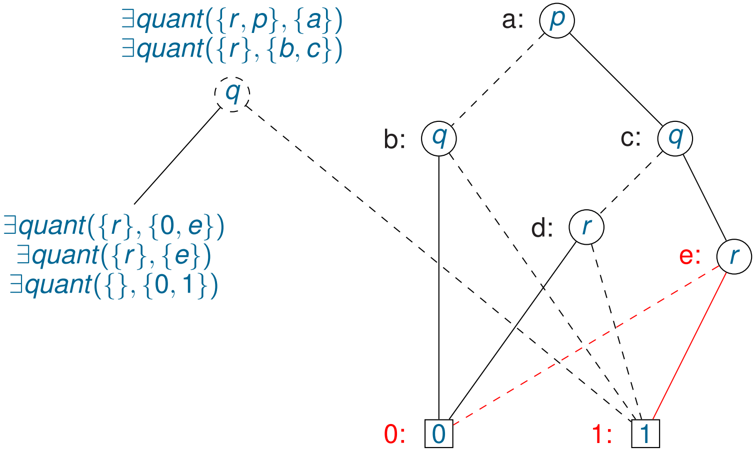


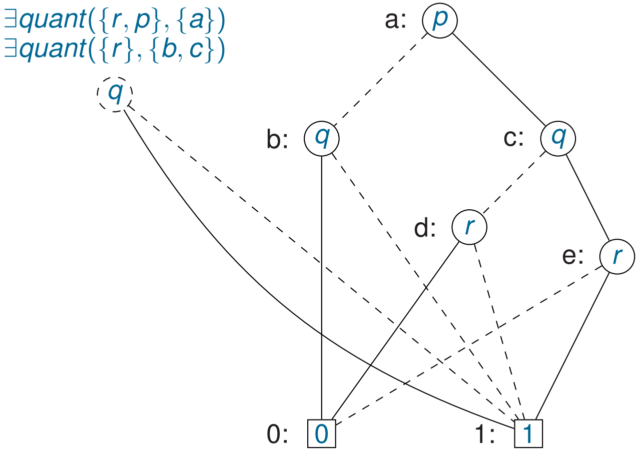
Example $\exists r \exists p(p \leftrightarrow ((p \rightarrow r) \leftrightarrow q))$



Example $\exists r \exists p (p \leftrightarrow ((p \to r) \leftrightarrow q))$

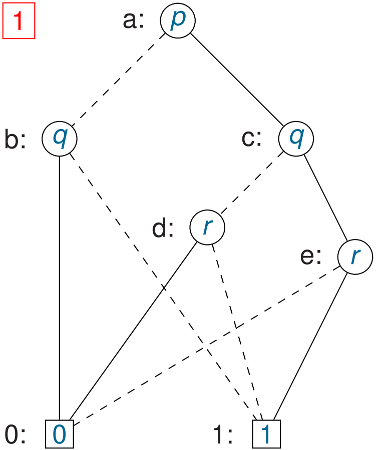


Example $\exists r \exists p (p \leftrightarrow ((p \to r) \leftrightarrow q))$



Example $\exists r \exists p(p \leftrightarrow ((p \rightarrow r) \leftrightarrow q))$

$\exists quant(\{r,p\},\{a\}) = \boxed{1}$



QBF $\exists r \exists p(p \leftrightarrow ((p \rightarrow r) \leftrightarrow q))$ is represented by the node $\boxed{1}$.
This formula is true.

$\exists$-quantification algorithm for OBDDs

procedure $\exists quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_m\})$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $\exists p_1 \ldots \exists p_k(F_1 \vee \ldots \vee F_m)$ in (modified) D
begin
  if $m = 0$ **then** **return** $\boxed{0}$
  if some n_i is $\boxed{1}$ **then** **return** $\boxed{1}$
  if some n_i is $\boxed{0}$ **then**
    return $\exists quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m\})$
  $p := max\_atom(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then $(l_i, r_i) := (neg(n_i), pos(n_i))$
      else $(l_i, r_i) := (n_i, n_i)$
  if $p \in \{p_1, \ldots, p_k\}$
    then **return** $\exists quant(\{p_1, \ldots, p_k\} - \{p\}, \{l_1, \ldots, l_m, r_1, \ldots, r_m\})$
    else
      $k_1 := \exists quant(\{p_1, \ldots, p_k\}, \{l_1, \ldots, l_m\})$
      $k_2 := \exists quant(\{p_1, \ldots, p_k\}, \{r_1, \ldots, r_m\})$
      return $integrate(k_1, p, k_2, D)$
end

$\forall$-quantification algorithm for OBDDs

procedure $\forall quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_m\})$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $\forall p_1 \ldots \forall p_k (F_1 \wedge \ldots \wedge F_m)$ in (modified) D
begin
  if $m = 0$ **then** **return** $\boxed{1}$
  if some n_i is $\boxed{0}$ **then** **return** $\boxed{0}$
  if some n_i is $\boxed{1}$ **then**
    return $\forall quant(\{p_1, \ldots, p_k\}, \{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m\})$
  $p := max\_atom(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then $(l_i, r_i) := (neg(n_i), pos(n_i))$
      else $(l_i, r_i) := (n_i, n_i)$
  if $p \in \{p_1, \ldots, p_k\}$
    then **return** $\forall quant(\{p_1, \ldots, p_k\} - \{p\}, \{l_1, \ldots, l_m, r_1, \ldots, r_m\})$
    else
      $k_1 := \forall quant(\{p_1, \ldots, p_k\}, \{l_1, \ldots, l_m\})$
      $k_2 := \forall quant(\{p_1, \ldots, p_k\}, \{r_1, \ldots, r_m\})$
      return $integrate(k_1, p, k_2, D)$
end

Quantifier elimination

Lemma (quantifier elimination): For any quantified Boolean formula F there is an equivalent quantifier-free formula Q, ($F \equiv Q$).

Remarks:

- We can eliminate quantifiers from formulas one by one from innermost to outermost using OBDDs.
- In particular, we can evaluate/check satisfiability/validity of QBFs using quantification algorithms on OBDDs.
- Evaluation/satisfiability/validity of QBF is PSPACE-complete.

Summary

Quantified Boolean Formulas: boolean formulas + quantifiers $\exists, \forall$.

Any closed formula is either true or false (in all interpretations).

Satifiability/validity of formulas with free variables can be reduced to checking truth/falsity of closed formulas.

Prenex normal form: rectification + prenexing rules + CNF rules.

Alg. for checking truth/falsity of closed formulas in prenex form:

- Splitting: $\wedge, \vee$ nodes.
- DLL: splitting + unit propagation; $\wedge, \vee$ nodes.
  Pure literal rule, Universal literal deletion.

QBF with free variables represent boolean functions.
Quantification algorithms for building OBDDs from QBFs.

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Next: Modelling using Propositional Logic of Finite Domains.