Propositional satisfiability

- conjunctive normal form (CNF)
- standard transformation to CNF
- clausal normal from
- definitional clausal transformation
- encoding problems as propositional satisifiability problem
- DPLL algorithm for checking satisfiability

Literal, clause

- Literal: either an atom p (positive literal) or its negation $\neg p$ (negative literal).
- The complementary literal to L:

$$\overline{L} \stackrel{\text{def}}{=} \begin{cases} \neg p, & \text{if } L \text{ is of the form } p \text{ (positive);} \\ p, & \text{if } L \text{ has the form } \neg p. \end{cases}$$

  In other words, p and $\neg p$ are complementary.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$, $n \geq 0$ of literals.
  - Empty clause, denoted by $\square$: $n = 0$ (the empty clause is false in every interpretation).
  - Unit clause: $n = 1$.
  - Horn clause: a clause with at most one positive literal.

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CNF

- A formula A is in conjunctive normal form, or simply CNF, if it is either $\top$, or $\bot$, or a conjunction of disjunctions of literals:

  $$A = \bigwedge_i \bigvee_j L_{i,j}.$$

  (In other words, A is a conjunction of clauses.)
- A formula B is called a conjunctive normal form of a formula A if B is equivalent to A and B is in conjunctive normal form.

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Which of these formulas are in CNF

- $(p \lor \neg q \lor r) \land (p \lor r) \land p$
- $(p \land q) \lor (p \leftrightarrow s)$
- $r \lor \neg q \lor s$
- $r \land \neg q \land s$
- $(p \land q) \lor (p \land \neg s)$

Satisfiability of CNF

- An interpretation I satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

if and only if it satisfies every clause

$$\bigvee_j L_{i,j}.$$

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- An interpretation I satisfies a clause

$$L_1 \vee \ldots \vee L_k$$

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The standard CNF transformation

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\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{lr} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \qquad\qquad \ldots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
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A formula to which no rewrite rule is applicable

- contains no $\leftrightarrow$;
- contains no $\rightarrow$;
- may only contain $\neg$ applied to atoms;
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- (hence) is in CNF.

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The standard CNF transformation, example

$$\neg((p \to q) \wedge (p \vee q \to r \vee s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \vee q \to r \vee s)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$$
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$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
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$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$
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\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
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$\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow$
$\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow$
$\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow$
$(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow$
$(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow$
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\begin{array}{rcll}
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A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
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$(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(p \rightarrow r) \Rightarrow$

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$$(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

The standard CNF transformation, example

$$
\begin{aligned}
&\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow \\
&\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow \\
&\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow \\
&(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow \\
&(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r \\
&(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r \\
&(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r
\end{aligned}
$$

$$
\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
& & \ldots \quad \land \\
& & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}
$$

The standard CNF transformation, example

$$
\begin{aligned}
&\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow \\
&\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow \\
&\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow \\
&(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow \\
&(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow \\
&(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r \\
&(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r \\
&(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r
\end{aligned}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
 & & \ldots & \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}
$$

The standard CNF transformation, example

$\neg((p \to q) \wedge (p \vee q \to r \vee s) \to (p \to r)) \Rightarrow$
$\neg(\neg((p \to q) \wedge (p \vee q \to r \vee s)) \vee (p \to r)) \Rightarrow$
$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(p \to r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \to q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \to q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

The standard CNF transformation, example

$\neg((p \to q) \wedge (p \vee q \to r \vee s) \to (p \to r)) \Rightarrow$
$\neg(\neg((p \to q) \wedge (p \vee q \to r \vee s)) \vee (p \to r)) \Rightarrow$
$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(p \to r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$
$(p \to q) \wedge (p \vee q \to r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \to q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \to q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

The standard CNF transformation, example

$$\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
 & & \ldots & \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}$$

The standard CNF transformation, example

$$
\begin{array}{l}
\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \rightarrow (p \rightarrow r)) \Rightarrow \\
\neg(\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \vee (p \rightarrow r)) \Rightarrow \\
\neg\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

The standard CNF transformation, example

$$\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

The standard CNF transformation, example

$$\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

The standard CNF transformation, example

$$\neg((p \to q) \land (p \lor q \to r \lor s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \lor q \to r \lor s)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \lor q \to r \lor s)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \lor q \to r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \lor q) \lor r \lor s) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land ((\neg p \land \neg q) \lor r \lor s) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor r \lor s) \land (\neg q \lor r \lor s) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \quad\quad \ldots \quad\quad\quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

The standard CNF transformation, example

$\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \rightarrow (p \rightarrow r)) \Rightarrow$
$\neg(\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \vee (p \rightarrow r)) \Rightarrow$
$\neg\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \wedge \neg(p \rightarrow r) \Rightarrow$
$(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(p \rightarrow r) \Rightarrow$
$(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$
$(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$
$(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \rightarrow q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$
$(p \rightarrow q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$
$(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

The standard CNF transformation, example

$$\neg((p \to q) \wedge (p \vee q \to r \vee s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \vee q \to r \vee s)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}$$

The standard CNF transformation, example

$$
\begin{array}{l}
\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \rightarrow (p \rightarrow r)) \Rightarrow \\
\neg(\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \vee (p \rightarrow r)) \Rightarrow \\
\neg\neg((p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \vee q \rightarrow r \vee s) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{cc} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \ldots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
\end{array}
$$

The standard CNF transformation, example

$$\neg((p \to q) \wedge (p \vee q \to r \vee s) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \vee q \to r \vee s)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \vee q \to r \vee s)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \vee q \to r \vee s) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \vee q) \vee r \vee s) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge ((\neg p \wedge \neg q) \vee r \vee s) \wedge p \wedge \neg r$$
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$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}$$

CNF and satisfiability

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\dots$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

Therefore, the formula

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has the same models as the set consisting of four clauses

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Definitional clausal form transformation

Problem with standard CNF transformation

Compute CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
(\neg p_1 \lor (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \land \\
(p_1 \lor \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \\
(\neg p_1 \lor \neg p_2 \lor (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \land \\
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(p_1 \lor p_2 \lor (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \land \\
(p_1 \lor \neg p_2 \lor \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \land
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If we continue, the formula will grow exponentially.

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If we continue, the formula will grow exponentially.

CNF is exponential

Is there any way to avoid exponential blowup?

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There are formulas for which the shortest CNF has an exponential size.

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There are formulas for which the shortest CNF has an exponential size.

Approach: relax requirement of equivalence preserving to equisatisfiability preserving.

Idea

Using so-called naming or definition introduction.

- Take a non-trivial subformula A.
- Introduce a new name n for it. A name is a new propositional variable.
- Add a formula stating that n is equivalent to A (definition for n).

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
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$$

- Replace the subformula by its name:

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\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\
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This set is not equivalent to the original formula but equisatisfiable (satisfible if and only if the original formula is).

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The conversion of the original formula to CNF introduces 32 copies of p_6.

The conversion of the new set of formulas to CNF introduces 4 copies of p_6.

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Clausal Form:

- Clausal form of a formula A: a set of clauses which is satisfiable if and only if A is satisfiable.
- Clausal form of a set S of formulas: a set of clauses which is satisfiable if and only if so is S.

Using clausal normal forms instead of conjunctive normal forms we can convert any formula to an equisatisfiable set of clauses with at most linear increase in the size of the formula.

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Definitional clausal form transformation

Theorem

$$F[G] \text{ is satisfiable} \Leftrightarrow F[n] \wedge (n \leftrightarrow G) \text{ is satisfiable.}$$

provided n is a (fresh) propositional variable not occurring in $F[G]$.
n can be seen as a name for G.

Definitional clausal clausal form transformation:

- introduce names for every non-literal subformula in the original formula (this introduces a linear number of new symbols),
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$\Leftarrow$) Assume $F[n] \wedge (n \leftrightarrow G)$ is satisfiable.
Then, there exists an interpretation I such that $I \models F[n] \wedge (n \leftrightarrow G)$.
We need to show that $F[G]$ is satisfiable.
Indeed, take I then by Equivalence Replacement Lemma we have $I \models F[G]$.

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Consider another interpretation:

$$I'(p) = \begin{cases} I(p), & \text{if } p \neq n; \\ I(G) & \text{if } p = n. \end{cases}$$

Then:

- $I' \models F[G]$ (why ?).
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By Equivalence Replacement Lemma we have $I' \models F[n]$.
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Where in the proof we used that n is fresh ?

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Example: Definitional Clausal Form Transformation

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$
			$n_1 \vee n_2$
n_2	$(p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$
			$n_3 \vee n_2$
			$\neg n_7 \vee n_2$
n_3	$(p \to q) \wedge (p \wedge q \to r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$
			$\neg n_3 \vee n_5$
			$\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \vee \neg p \vee q$
			$p \vee n_4$
			$\neg q \vee n_4$
n_5	$p \wedge q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \vee \neg n_6 \vee r$
			$n_6 \vee n_5$
			$\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$
			$\neg n_6 \vee q$
			$\neg p \vee \neg q \vee n_6$
n_7	$p \to \neg r$	$n_7 \leftrightarrow (p \to \neg r)$	$\neg n_7 \vee \neg p \vee \neg r$
			$p \vee n_7$
			$r \vee n_7$

Converting a formula to clausal form.

Example: Definitional Clausal Form Transformation

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \land (p \land q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$
			$n_1 \lor n_2$
n_2	$(p \to q) \land (p \land q \to r) \to (p \to \neg r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$
			$n_3 \lor n_2$
			$\neg n_7 \lor n_2$
n_3	$(p \to q) \land (p \land q \to r)$	$n_3 \leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$
			$\neg n_3 \lor n_5$
			$\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \lor \neg p \lor q$
			$p \lor n_4$
			$\neg q \lor n_4$
n_5	$p \land q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \lor \neg n_6 \lor r$
			$n_6 \lor n_5$
			$\neg r \lor n_5$
n_6	$p \land q$	$n_6 \leftrightarrow (p \land q)$	$\neg n_6 \lor p$
			$\neg n_6 \lor q$
			$\neg p \lor \neg q \lor n_6$
n_7	$p \to \neg r$	$n_7 \leftrightarrow (p \to \neg r)$	$\neg n_7 \lor \neg p \lor \neg r$
			$p \lor n_7$
			$r \lor n_7$

Take all subformulas that are not literals.

Example: Definitional Clausal Form Transformation

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \to q) \wedge (p \wedge q \to r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \to \neg r$	$n_7 \leftrightarrow (p \to \neg r)$	$\neg n_7 \vee \neg p \vee \neg r$ $p \vee n_7$ $r \vee n_7$

Introduce names for these formulas. Note we do not introduce names for literals.

Example: Definitional Clausal Form Transformation

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \land (p \land q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \to q) \land (p \land q \to r) \to (p \to \neg r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \to q) \land (p \land q \to r)$	$n_3 \leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$n_6 \leftrightarrow (p \land q)$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \to \neg r$	$n_7 \leftrightarrow (p \to \neg r)$	$\neg n_7 \lor \neg p \lor \neg r$ $p \lor n_7$ $r \lor n_7$

Introduce definitions.

Example: Definitional Clausal Form Transformation

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$
			$n_1 \vee n_2$
n_2	$(p \to q) \wedge (p \wedge q \to r) \to (p \to \neg r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$
			$n_3 \vee n_2$
			$\neg n_7 \vee n_2$
n_3	$(p \to q) \wedge (p \wedge q \to r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$
			$\neg n_3 \vee n_5$
			$\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \vee \neg p \vee q$
			$p \vee n_4$
			$\neg q \vee n_4$
n_5	$p \wedge q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \vee \neg n_6 \vee r$
			$n_6 \vee n_5$
			$\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$
			$\neg n_6 \vee q$
			$\neg p \vee \neg q \vee n_6$
n_7	$p \to \neg r$	$n_7 \leftrightarrow (p \to \neg r)$	$\neg n_7 \vee \neg p \vee \neg r$
			$p \vee n_7$
			$r \vee n_7$

Convert resulting formulas into CNF using the standard transformation.

Definitional clausal form transformation

Theorem. Any propositional formula can be transformed into an equisatisfiable clausal normal form by applying the definitional clausal form transformation. Moreover, the size of the resulting clause set is linear in the size of the formula and each clause contains at most three literals (3-CNF).

Optimised Definitional Clausal Form Transformation

If we introduce a name for a subformula and the occurence of the subformula is positive or negative, then an implication is used instead of equivalence, if it is neutral then we use equivalence.

Lemma.(Positive Definition) Let $A[B]_\pi$ where $pol(A, \pi)$ is positive. Then $A[B]_\pi$ is satisfiable if and only if $A[n] \wedge (n \rightarrow B)$ is satisfiable, (where n is a new variable that does not occur $A[B]_\pi$).

Lemma. (Negative Definition) Let $A[B]_\pi$ where $pol(A, \pi)$ is negative. Then $A[B]_\pi$ is satisfiable if and only if $A[n] \wedge (B \rightarrow n)$ is satisfiable, (where n is a new variable that does not occur $A[B]_\pi$).

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Example: Optimised Definitional Clausal Form Trans.

	subformula	definition	clauses
			n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow \neg r)$	$(n_3 \rightarrow n_7) \leftrightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$(p \wedge q) \leftrightarrow n_6$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow \neg r$	$(p \rightarrow \neg r) \leftrightarrow n_7$	$\neg n_7 \vee \neg p \vee \neg r$ $p \vee n_7$ $r \vee n_7$

Example: Optimised Definitional Clausal Form Trans.

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \land (p \land q \to r) \to (p \to \neg r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$
			$n_1 \lor n_2$
n_2	$(p \to q) \land (p \land q \to r) \to (p \to \neg r)$	$(n_3 \to n_7) \leftrightarrow n_2$	$\neg n_2 \lor \neg n_3 \lor n_7$
			$n_3 \lor n_2$
			$\neg n_7 \lor n_2$
n_3	$(p \to q) \land (p \land q \to r)$	$n_3 \leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$
			$\neg n_3 \lor n_5$
			$\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \lor \neg p \lor q$
			$p \lor n_4$
			$\neg q \lor n_4$
n_5	$p \land q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \lor \neg n_6 \lor r$
			$n_6 \lor n_5$
			$\neg r \lor n_5$
n_6	$p \land q$	$(p \land q) \leftrightarrow n_6$	$\neg n_6 \lor p$
			$\neg n_6 \lor q$
			$\neg p \lor \neg q \lor n_6$
n_7	$p \to \neg r$	$(p \to \neg r) \leftrightarrow n_7$	$\neg n_7 \lor \neg p \lor \neg r$
			$p \lor n_7$
			$r \lor n_7$

All clauses shown in the red color are not generated by the optimised transformation.

Example: Optimised Definitional Clausal Form Trans.

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \land (p \land q \to r) \to (p \to \neg r))$	$n_1 \to \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \to q) \land (p \land q \to r) \to (p \to \neg r)$	$(n_3 \to n_7) \to n_2$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \to q) \land (p \land q \to r)$	$n_3 \to (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \to q$	$n_4 \to (p \to q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \to r$	$n_5 \to (n_6 \to r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$(p \land q) \to n_6$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \to \neg r$	$(p \to \neg r) \to n_7$	$\neg n_7 \lor \neg p \lor \neg r$ $p \lor n_7$ $r \lor n_7$

The optimised transformation gives fewer clauses.

Summary

- Conjunctive normal form (CNF):
- Standard CNF transformation
  - equivalence preserving
  - exponential time
- Clausal normal from:
- Definitional transformation
  - satisfiability preserving
  - polynomial time
- Optimised definitional transformation (based on polarities)
- Next: formalising using propositional logic

Formalising problems using propositional logic

Expressing Properties "k out of n variables are true"

Suppose we have variables $v_1, \ldots, v_n$ and want to express that exactly k of them are true. These formulas are very useful for encoding various problems in SAT.

We will write this property as a formula $T_{=k}(v_1, \ldots, v_n)$.

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We will write this property as a formula $T_{=k}(v_1, \ldots, v_n)$.

First, let us express some simple special cases:

$$
\begin{array}{lcl}
T_{=0}(v_1, \ldots, v_n) & \stackrel{\text{def}}{=} & \neg v_1 \wedge \ldots \wedge \neg v_n \\
T_{=1}(v_1, \ldots, v_n) & \stackrel{\text{def}}{=} & (v_1 \vee \ldots \vee v_n) \wedge \bigwedge_{i<j} (\neg v_i \vee \neg v_j)
\end{array}
$$

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T_{=1}(v_1, \ldots, v_n) &\stackrel{\text{def}}{=} (v_1 \vee \ldots \vee v_n) \wedge \textstyle\bigwedge_{i<j}(\neg v_i \vee \neg v_j)
\end{aligned}
$$

$$
\begin{aligned}
T_{=n-1}(v_1, \ldots, v_n) &\stackrel{\text{def}}{=} (\neg v_1 \vee \ldots \vee \neg v_n) \wedge \textstyle\bigwedge_{i<j}(v_i \vee v_j) \\
T_{=n}(v_1, \ldots, v_n) &\stackrel{\text{def}}{=} v_1 \wedge \ldots \wedge v_n
\end{aligned}
$$

Expressing Properties "k out of n variables are true"

To define T_k for $0 < k < n$, introduce two formulas:

- $T_{\leq k}(v_1, \ldots, v_n)$: at most k variables among $v_1, \ldots, v_n$ are true, where $k = 0 \ldots n-1$;
- $T_{\geq k}(v_1, \ldots, v_n)$: at least k variables among $v_1, \ldots, v_n$ are true, where $k = 1 \ldots n$;

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At most:

$$T_{\leq k}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \bigwedge_{\substack{x_1, \ldots, x_{k+1} \in \{v_1, \ldots, v_n\} \\ x_1, \ldots, x_{k+1} \text{ are distinct}}} \neg x_1 \vee \ldots \vee \neg x_{k+1}.$$

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At most:

$$T_{\leq k}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \bigwedge_{\substack{x_1, \ldots, x_{k+1} \in \{v_1, \ldots, v_n\} \\ x_1, \ldots, x_{k+1} \text{ are distinct}}} \neg x_1 \vee \ldots \vee \neg x_{k+1}.$$

"At least k variables among $v_1, \ldots, v_n$ are true" is equivalent to "At most $n - k$ variables among $v_1, \ldots, v_n$ are false".

Expressing Properties "k out of n variables are true"

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At most:

$$T_{\leq k}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \bigwedge_{\substack{x_1, \ldots, x_{k+1} \in \{v_1, \ldots, v_n\} \\ x_1, \ldots, x_{k+1} \text{ are distinct}}} \neg x_1 \vee \ldots \vee \neg x_{k+1}.$$

"At least k variables among $v_1, \ldots, v_n$ are true" is equivalent to "At most $n-k$ variables among $v_1, \ldots, v_n$ are false".

At least:

$$T_{\geq k}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \bigwedge_{\substack{x_1, \ldots, x_{n-k+1} \in \{v_1, \ldots, v_n\} \\ x_1, \ldots, x_{n-k+1} \text{ are distinct}}} x_1 \vee \ldots \vee x_{n-k+1}.$$

Sudoku

4		8	7	9			3	
		9	8	2		5		
	2							
9		2			6		1	7
		6	5	8	7	9		
7	8		2			4		6
							4	
		5		4	8	2		
	9			7	2	3		5

Enter digits from 1 to 9 into the blank spaces.

Every row must contain one of each digit.

So must every column,

as must every 3x3 square.

This instance has exactly one solution.

Sudoku

4		8	7	9			3	
		9	8	2		5		
	2							
9		2			6		1	7
		6	5	8	7	9		
7	8		2			4		6
							4	
		5		4	8	2		
	9			7	2	3		5

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		6	5	8	7	9		
7	8		2			4		6
							4	
		5		4	8	2		
	9			7	2	3		5

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		9	8	2		5		
	2							
9		2			6		1	7
		6	5	8	7	9		
7	8		2			4		6
							4	
		5		4	8	2		
	9			7	2	3		5

Enter digits from 1 to 9 into the blank spaces.

Every row must contain one of each digit.

So must every column,

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This instance has exactly one solution.

Sudoku

4	1	8	7	9	5	6	3	2
6	3	9	8	2	1	5	7	4
5	2	7	3	6	4	1	8	9
9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
3	7	5	6	4	8	2	9	1
8	9	4	1	7	2	3	6	5

Enter digits from 1 to 9 into the blank spaces.

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5	2	7	3	6	4	1	8	9
9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
3	7	5	6	4	8	2	9	1
8	9	4	1	7	2	3	6	5

Enter digits from 1 to 9 into the blank spaces.

Every row must contain one of each digit.

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This instance has exactly one solution.

Sudoku

4	1	8	7	9	5	6	3	2
6	3	9	8	2	1	5	7	4
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1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
3	7	5	6	4	8	2	9	1
8	9	4	1	7	2	3	6	5

Enter digits from 1 to 9 into the blank spaces.

Every row must contain one of each digit.

So must every column,

as must every 3x3 square.

This instance has exactly one solution.

Sudoku

4	1	8	7	9	5	6	3	2
6	3	9	8	2	1	5	7	4
5	2	7	3	6	4	1	8	9
9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
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Sudoku as an instance of SAT

9	4		8	7	9			3	
8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
3								4	
2			5		4	8	2		
1		9			7	2	3		5
	1	2	3	4	5	6	7	8	9

Sudoku as an instance of SAT

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8			9	8	2		5		
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5			6	5	8	7	9		
4	7	8		2			4		6
3								4	
2			5		4	8	2		
1		9			7	2	3		5
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Introduce 729 propositional variables v_{rcd}, where $r, c, d \in \{1, \ldots, 9\}$.
The variable v_{rcd} denotes that the cell in the row number r and column number c contains the digit d.

Sudoku as an instance of SAT

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8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
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We should express all rules of sudoku using the variables v_{rcd}.

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We have to write down that each cell contains exactly one digit.

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$$\{\neg v_{r,c,d} \vee \neg v_{r,c',d} \mid r, c, c', d \in \{1, ..., 9\}, c < c'\}.$$

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Every column must contain one of each digit:
similar.

Every 3x3 square must contain one of each digit:
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Finally, we add unit clauses corresponding to the initial configuration, for example v_{129}.

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit.

$v_{rc1} \lor v_{rc2} \lor \ldots \lor v_{rc8} \lor v_{rc9}$
$\neg v_{rc1} \lor \neg v_{rc2}$
$\neg v_{rc1} \lor \neg v_{rc3}$
$\ldots$
$\neg v_{rc8} \lor \neg v_{rc9}$

2,997 clauses,
6,561 literals

Every row must contain one of each digit:

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Lemma Sudoku has a solution if and only if the corresponding set of clauses is satisfiable

Summary

- conjunctive normal form (CNF)
- standard CNF transformation
- clausal normal from
- definitional transformation
- optimised definitional transformation
- encoding problems as propositional satisifiability problem
- Next: DPLL algorithm for checking satisfiability