Quantified Boolean Formulas (QBF)

- syntax
- semantics
- evaluating QBF formula on an interpretations: and-or trees
- positions/polarity
- bound/free occurrences of variables
- rectified formulas
- prenex normal form

Two-Player Games

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.

There are two players: P and Q.

At step k each player makes a move:

1. the player P can choose a Boolean value for the variable p_k;
2. the player Q can choose a Boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \to q_1$
3. $p_1 \land \neg q_1$
4. $q_1 \to q_1$
5. $p_1 \land \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \rightarrow q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \rightarrow q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \to q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \to q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \rightarrow q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \rightarrow q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \rightarrow q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \rightarrow q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \rightarrow q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \rightarrow q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Suppose Both Players Make no Errors: Who Wins?

Consider several special cases

1. p_1
2. $p_1 \rightarrow q_1$
3. $p_1 \wedge \neg q_1$
4. $q_1 \rightarrow q_1$
5. $p_1 \wedge \neg p_1$
6. $p_1 \leftrightarrow \neg q_1$

Winning Strategy

Problem: does player P have a winning strategy?

Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \dots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Winning Strategy

Problem: does player P have a winning strategy?
Player P has a winning strategy if

- there exists a move for P (a Boolean value for for p_1) such that
- for all moves of Q (Boolean values for for q_1)
- there exists a move for P (a Boolean value for for p_2) such that
- for all moves of Q (Boolean values for for q_2)
- ...
- for all moves of Q (Boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified Boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \ldots \exists p_n \forall q_n G$.

Quantified Boolean Formulas (Syntax)

Propositional formula:

- Every Boolean variable is a formula.
- $\top$ and $\bot$ are formulas.
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$, then $(F_1 \wedge \ldots \wedge F_n)$ and $(F_1 \vee \ldots \vee F_n)$ are formulas.
- If F is a formula, then $\neg F$ is a formula.
- If F and G are formulas, then $(F \to G)$ and $(F \leftrightarrow G)$ are formulas.

Quantified Boolean formulas:

- If p is a Boolean variable and F is a formula, then $\forall pF$ and $\exists pF$ are formulas.

Variable p is called existentially quantified in the case of $\exists p$ and universally quantified in the case of $\forall p$.

Quantified Boolean Formulas (Syntax)

Propositional formula:

- Every Boolean variable is a formula.
- $\top$ and $\bot$ are formulas.
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$, then $(F_1 \land \ldots \land F_n)$ and $(F_1 \lor \ldots \lor F_n)$ are formulas.
- If F is a formula, then $\neg F$ is a formula.
- If F and G are formulas, then $(F \rightarrow G)$ and $(F \leftrightarrow G)$ are formulas.

Quantified Boolean formulas:

- If p is a Boolean variable and F is a formula, then $\forall p F$ and $\exists p F$ are formulas.

Variable p is called existentially quantified in the case of $\exists p$ and universally quantified in the case of $\forall p$.

Quantifiers

- $\forall$ is called the universal quantifier.
- $\exists$ is called the existential quantifier.
- Read $\forall pF$ as "for all p, F".
- Read $\exists pF$ as "there exists p such that F" or "for some p, F".

Connective	Name	Precedence
$\forall$	for all	5
$\exists$	exists	5
$\neg$	negation	5
$\wedge$	conjunction	4
$\vee$	disjunction	3
$\rightarrow$	implication	2
$\leftrightarrow$	equivalence	1

New Notation

Define

$$I_p^b(q) \stackrel{\text{def}}{=} \begin{cases} I(q), & \text{if } p \neq q; \\ b, & \text{if } p = q. \end{cases}$$

Example: let $I = \{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$. Then

$$\begin{array}{lcl} I_q^1 & = & \{p \mapsto 1, q \mapsto 1, r \mapsto 1\} \\ I_q^0 & = & \{p \mapsto 1, q \mapsto 0, r \mapsto 1\} \quad = \quad I \\ I_p^0 & = & \{p \mapsto 0, q \mapsto 0, r \mapsto 1\} \end{array}$$

QBF Semantics

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(F_1 \land \ldots \land F_n) = 1$ if and only if $I(F_i) = 1$ for all i.
3. $I(F_1 \lor \ldots \lor F_n) = 1$ if and only if $I(F_i) = 1$ for some i.
4. $I(\neg F) = 1$ if and only if $I(F) = 0$.
5. $I(F \rightarrow G) = 1$ if and only if $I(F) = 0$ or $I(G) = 1$.
6. $I(F \leftrightarrow G) = 1$ if and only if $I(F) = I(G)$.
7. $I(\forall pF) = 1$ if and only if $I_p^0(F) = 1$ and $I_p^1(F) = 1$.
8. $I(\exists pF) = 1$ if and only if $I_p^0(F) = 1$ or $I_p^1(F) = 1$.

QBF Semantics

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(F_1 \wedge \ldots \wedge F_n) = 1$ if and only if $I(F_i) = 1$ for all i.
3. $I(F_1 \vee \ldots \vee F_n) = 1$ if and only if $I(F_i) = 1$ for some i.
4. $I(\neg F) = 1$ if and only if $I(F) = 0$.
5. $I(F \rightarrow G) = 1$ if and only if $I(F) = 0$ or $I(G) = 1$.
6. $I(F \leftrightarrow G) = 1$ if and only if $I(F) = I(G)$.
7. $I(\forall pF) = 1$ if and only if $I_p^0(F) = 1$ and $I_p^1(F) = 1$.
8. $I(\exists pF) = 1$ if and only if $I_p^0(F) = 1$ or $I_p^1(F) = 1$.

Evaluating a Formula: and-or trees

Let us evaluate $\forall p \exists q (p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.
Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{l} I_{00} \models \exists q (p \leftrightarrow q) \\ I_{10} \models \exists q (p \leftrightarrow q) \end{array} \quad \text{and}}$$

$$\Leftrightarrow \quad \boxed{\boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \quad \text{or}} \quad \text{and}}$$

Evaluating a Formula: and-or trees

Let us evaluate $\forall p \exists q (p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.
Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q)$$

$\Leftrightarrow$

$I_{00} \models \exists q (p \leftrightarrow q)$
$I_{10} \models \exists q (p \leftrightarrow q)$ and

$\Leftrightarrow$

$I_{00} \models p \leftrightarrow q$
$I_{01} \models p \leftrightarrow q$ or

and

$I_{10} \models p \leftrightarrow q$
$I_{11} \models p \leftrightarrow q$ or

Evaluating a Formula: and-or trees

Let us evaluate $\forall p \exists q (p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.
Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{ll} I_{00} \models \exists q (p \leftrightarrow q) & \\ I_{10} \models \exists q (p \leftrightarrow q) & \end{array} \text{ and}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} \boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \text{ or}} \\ \\ \boxed{\begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} \text{ or}} \end{array} \text{ and}}$$

Evaluating a Formula: and-or trees

Let us evaluate $\forall p \exists q(p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.
Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$I_{10} \models \forall p \exists q(p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{l} I_{00} \models \exists q(p \leftrightarrow q) \\ I_{10} \models \exists q(p \leftrightarrow q) \end{array} \text{ and}}$

$\Leftrightarrow \quad \boxed{\begin{array}{l} \boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \text{ or}} \\ \\ \boxed{\begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} \text{ or}} \end{array} \text{ and}}$

Evaluating a Formula: and-or trees

Let us evaluate $\forall p \exists q (p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.
Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{ll} I_{00} \models \exists q (p \leftrightarrow q) & \\ I_{10} \models \exists q (p \leftrightarrow q) & \text{and} \end{array}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{ll} \boxed{\begin{array}{ll} I_{00} \models p \leftrightarrow q & \\ I_{01} \models p \leftrightarrow q & \text{or} \end{array}} & \\ & \text{and} \\ \boxed{\begin{array}{ll} I_{10} \models p \leftrightarrow q & \\ I_{11} \models p \leftrightarrow q & \text{or} \end{array}} & \end{array}}$$

Evaluating a formula

$$\begin{tikzcd}[column sep=small]
 & & & I_{10} \models \forall p \exists q (p \leftrightarrow q) \arrow[dll, dash] \arrow[drr, dash] \arrow[d, phantom, "\wedge"] & & & \\
 & I_{00} \models \exists q (p \leftrightarrow q) \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\vee"] & & {} & & I_{10} \models \exists q (p \leftrightarrow q) \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\vee"] & \\
I_{00} \models p \leftrightarrow q & {} & I_{01} \models p \leftrightarrow q & & I_{10} \models p \leftrightarrow q & {} & I_{11} \models p \leftrightarrow q
\end{tikzcd}$$

Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $*$ to denote “any” Boolean value.

$$I_{**} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{l} I_{0*} \models \exists q (p \leftrightarrow q) \\ I_{1*} \models \exists q (p \leftrightarrow q) \end{array} \quad \text{and}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} \boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \quad \text{or}} \\ \\ \boxed{\begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} \quad \text{or}} \end{array} \quad \text{and}}$$

The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $*$ to denote "any" Boolean value.

$$I_{**} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{l} I_{0*} \models \exists q (p \leftrightarrow q) \\ I_{1*} \models \exists q (p \leftrightarrow q) \end{array} \text{ and}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} \boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \text{ or}} \\ \\ \boxed{\begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} \text{ or}} \end{array} \text{ and}}$$

The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $*$ to denote "any" Boolean value.

$$I_{**} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{ll} \begin{array}{l} I_{0*} \models \exists q (p \leftrightarrow q) \\ I_{1*} \models \exists q (p \leftrightarrow q) \end{array} & \text{and} \end{array}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{ll} \begin{array}{l} \boxed{\begin{array}{ll} \begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} & \text{or} \end{array}} \\ \\ \boxed{\begin{array}{ll} \begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} & \text{or} \end{array}} \end{array} & \text{and} \end{array}}$$

The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $*$ to denote "any" Boolean value.

$$I_{**} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \boxed{\begin{array}{l} I_{0*} \models \exists q(p \leftrightarrow q) \\ I_{1*} \models \exists q(p \leftrightarrow q) \end{array} \quad \text{and}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{ll} \boxed{\begin{array}{l} I_{00} \models p \leftrightarrow q \\ I_{01} \models p \leftrightarrow q \end{array} \quad \text{or}} & \\ & \text{and} \\ \boxed{\begin{array}{l} I_{10} \models p \leftrightarrow q \\ I_{11} \models p \leftrightarrow q \end{array} \quad \text{or}} & \end{array}}$$

The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

QBF: positions, polarity, free and bound variables

Subformula

Propositional formulas:

- The formulas $F_1, \ldots, F_n$ are the immediate subformulas of the formulas $F_1 \wedge \ldots \wedge F_n$ and $F_1 \vee \ldots \vee F_n$.
- The formulas F is the immediate subformula of the formula $\neg F$.
- The formulas F_1, F_2 are the immediate subformulas of the formulas $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$.
- ...

Quantified Boolean formulas:

- The formula F_1 is the immediate subformula of the formulas $\forall p F_1$ and $\exists p F_1$.

Subformula

Propositional formulas:

- The formulas $F_1, \ldots, F_n$ are the immediate subformulas of the formulas $F_1 \wedge \ldots \wedge F_n$ and $F_1 \vee \ldots \vee F_n$.
- The formulas F is the immediate subformula of the formula $\neg F$.
- The formulas F_1, F_2 are the immediate subformulas of the formulas $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$.
- ...

Quantified Boolean formulas:

- The formula F_1 is the immediate subformula of the formulas $\forall p F_1$ and $\exists p F_1$.

Positions and Polarity

Let $F|_\pi = G$.
Propositional formulas:

- If G has the form $G_1 \wedge \ldots \wedge G_n$ or $G_1 \vee \ldots \vee G_n$, then for all $i \in \{1, \ldots, n\}$ the position $\pi.i$ is a position in F and $pol(F, \pi.i) \stackrel{\text{def}}{=} pol(F, \pi)$.
- If G has the form $\neg G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} -pol(F, \pi)$.
- ...

Quantified Boolean formulas:

- If G has the form $\forall p G_1$ or $\exists p G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} pol(F, \pi)$.

Positions and Polarity

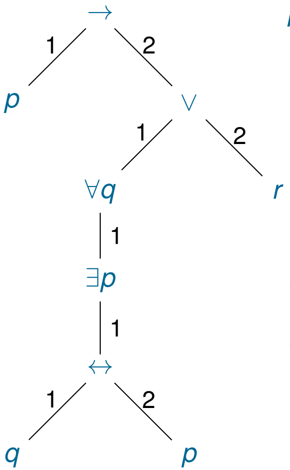
Let $F|_\pi = G$.
Propositional formulas:

- If G has the form $G_1 \wedge \ldots \wedge G_n$ or $G_1 \vee \ldots \vee G_n$, then for all $i \in \{1, \ldots, n\}$ the position $\pi.i$ is a position in F and $pol(F, \pi.i) \stackrel{\text{def}}{=} pol(F, \pi)$.
- If G has the form $\neg G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} -pol(F, \pi)$.
- …

Quantified Boolean formulas:

- If G has the form $\forall p G_1$ or $\exists p G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} pol(F, \pi)$.

Example

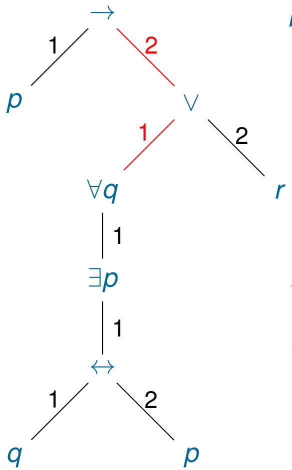


$p \rightarrow \forall q \exists p (q \leftrightarrow p) \vee r$

$F|_{2.1} = \forall q \exists p (q \leftrightarrow p)$

$F|_{2.1.1.1.1} = q$

Example

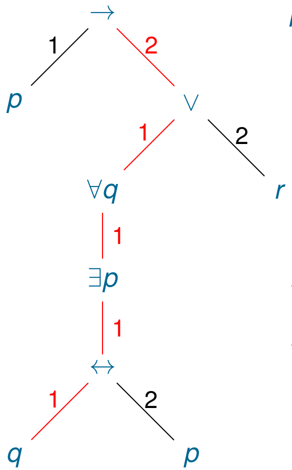


$p \to \forall q \exists p (q \leftrightarrow p) \lor r$

$F|_{2.1} = \forall q \exists p (q \leftrightarrow p)$

$F|_{2.1.1.1.1} = q$

Example



$$p \to \forall q \exists p (q \leftrightarrow p) \lor r$$

$$F|_{2.1} = \forall q \exists p (q \leftrightarrow p)$$

$$F|_{2.1.1.1.1} = q$$

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall p G$ or $\exists p G$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

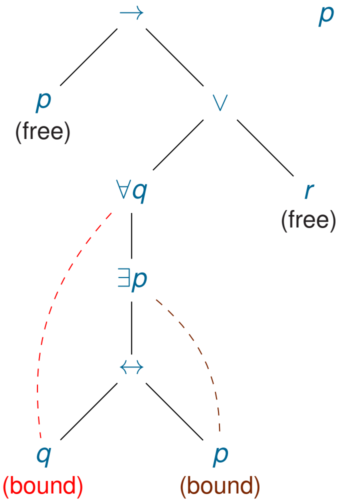
- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Free and bound occurrences of variables

Let p be a Boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Warning The same variable can be both free and bound in a formula.
- Closed formula: formula with no free variables.

Example: Free and Bound Variables



$p \to \forall q \exists p (q \leftrightarrow p) \vee r$

Only Free Variables Matter

The truth value of a formula depends only on the truth values of free variables of the formula:

Lemma

Let for all free variables p of a formula F we have $I_1(p) = I_2(p)$. Then $I_1 \models F$ if and only if $I_2 \models F$.

Theorem

Let F be a closed formula and I_1, I_2 be interpretations. Then $I_1 \models F$ if and only if $I_2 \models F$.

Only Free Variables Matter

The truth value of a formula depends only on the truth values of free variables of the formula:

Lemma

Let for all free variables p of a formula F we have $I_1(p) = I_2(p)$. Then $I_1 \models F$ if and only if $I_2 \models F$.

Theorem

Let F be a closed formula and I_1, I_2 be interpretations. Then $I_1 \models F$ if and only if $I_2 \models F$.

Truth, Validity and Satisfiability

Validity and satisfiability are defined as for propositional formulas.

There is no difference between these notions for closed formulas:

Lemma

For every interpretation I and closed QBF formula F the following propositions are equivalent: (i) $I \models F$; (ii) F is satisfiable; and (iii) F is valid.

If a closed QBF formula is valid/satisfiable we say that the formula is true and otherwise it is false.

Validity and satisfiability can be expressed through truth:

Lemma

Let F be a QBF formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable if and only if the closed formula $\exists p_1 \ldots \exists p_n F$ is true (satisfiable, valid).*
- *F is valid if and only if the closed formula $\forall p_1 \ldots \forall p_n F$ is true (satisfiable, valid).*

Truth, Validity and Satisfiability

Validity and satisfiability are defined as for propositional formulas.

There is no difference between these notions for closed formulas:

Lemma

For every interpretation I and closed QBF formula F the following propositions are equivalent: (i) $I \models F$; (ii) F is satisfiable; and (iii) F is valid.

If a closed QBF formula is valid/satisfiable we say that the formula is true and otherwise it is false.

Validity and satisfiability can be expressed through truth:

Lemma

Let F be a QBF formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable if and only if the closed formula $\exists p_1 \ldots \exists p_n F$ is true (satisfiable, valid).*
- *F is valid if and only if the closed formula $\forall p_1 \ldots \forall p_n F$ is true (satisfiable, valid).*

Truth, Validity and Satisfiability

Validity and satisfiability are defined as for propositional formulas.

There is no difference between these notions for closed formulas:

Lemma

For every interpretation I and closed QBF formula F the following propositions are equivalent: (i) $I \models F$; (ii) F is satisfiable; and (iii) F is valid.

If a closed QBF formula is valid/satisfiable we say that the formula is true and otherwise it is false.

Validity and satisfiability can be expressed through truth:

Lemma

Let F be a QBF formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable if and only if the closed formula $\exists p_1 \ldots \exists p_n F$ is true (satisfiable, valid).*
- *F is valid if and only if the closed formula $\forall p_1 \ldots \forall p_n F$ is true (satisfiable, valid).*

QBF: substitutions, rectification

Substitutions for propositional formulas

Substitution: $(F)_p^G$: denotes the formula obtained from F by replacing all occurrences of the variable p by G.

Example:

$$((p \lor s) \land (q \to p))_p^{(l \land s)} =$$
$$(((l \land s) \lor s) \land (q \to (l \land s)))$$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

Substitutions for propositional formulas

Substitution: $(F)^G_p$: denotes the formula obtained from F by replacing all occurrences of the variable p by G.

Example:

$$((p \lor s) \land (q \rightarrow p))^{(l \land s)}_p =$$
$$(((l \land s) \lor s) \land (q \rightarrow (l \land s)))$$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

Substitutions for propositional formulas

Substitution: $(F)^G_p$: denotes the formula obtained from F by replacing all occurrences of the variable p by G.

Example:

$$((p \lor s) \land (q \to p))^{(l \land s)}_p =$$
$$(((l \land s) \lor s) \land (q \to (l \land s)))$$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:
$\exists(r \to r)(\neg p \leftrightarrow r \to r)$???

Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \to q) \land (q \lor (q \to p))$

Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:
$\exists(r \to r)(\neg p \leftrightarrow r \to r)$???

Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \to q) \wedge (q \vee (q \to p))$

Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:
$\exists(r \to r)(\neg p \leftrightarrow r \to r)$???

Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \to q) \wedge (q \vee (q \to p))$

Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:
$\exists(r \to r)(\neg p \leftrightarrow r \to r)$???

Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \to q) \wedge (q \vee (q \to p))$

Renaming bound variables

Notation: $\exists\!\!\!\forall$: any of $\exists$, $\forall$ and $\wedge\!\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\!\!\!\forall qG'\ ]$ as the result.

Lemma

$F[\ \exists\!\!\!\forall pG\ ] \equiv F[\ \exists\!\!\!\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \to q) \wedge p)\ ) \vee p.$

Then we can rename p into r obtaining
$\exists q(\ \forall r((r \to q) \wedge r)\ ) \vee p.$

Renaming bound variables

Notation: $\exists\forall$: any of $\exists$, $\forall$ and $\wedge\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\forall qG'\ ]$ as the result.

Lemma
$F[\ \exists\forall pG\ ] \equiv F[\ \exists\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \to q) \wedge p)\ ) \vee p.$

Then we can rename p into r obtaining
$\exists q(\ \forall r((r \to q) \wedge r)\ ) \vee p.$

Renaming bound variables

Notation: $\exists\!\!\!\forall$: any of $\exists$, $\forall$ and $\wedge\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\!\!\!\forall qG'\ ]$ as the result.

Lemma

$F[\ \exists\!\!\!\forall pG\ ] \equiv F[\ \exists\!\!\!\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \rightarrow q) \wedge p)\ ) \vee p.$

Then we can rename p into r obtaining
$\exists q(\ \forall r((r \rightarrow q) \wedge r)\ ) \vee p.$

Renaming bound variables

Notation: $\exists\!\!\!\forall$: any of $\exists$, $\forall$ and $\wedge\!\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\!\!\!\forall qG'\ ]$ as the result.

Lemma

$F[\ \exists\!\!\!\forall pG\ ] \equiv F[\ \exists\!\!\!\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \to q) \wedge p)\ ) \vee p$.

Then we can rename p into r obtaining
$\exists q(\ \forall r((r \to q) \wedge r)\ ) \vee p$.

Rectified formulas

Rectified formula F:

- no variable appears both free and quantified in F;
- for every variable p, the formula F contains at most one occurrence of quantifiers $\exists\forall p$.

Any formula can be transformed into a rectified formula by renaming bound variables.

We can use the usual notation $(F)_p^G$ for rectified formulas assuming that p occurs only free.

Rectified formulas

Rectified formula F:

- no variable appears both free and quantified in F;
- for every variable p, the formula F contains at most one occurrence of quantifiers $\exists\forall p$.

Any formula can be transformed into a rectified formula by renaming bound variables.

We can use the usual notation $(F)_p^G$ for rectified formulas assuming that p occurs only free.

Rectified formulas

Rectified formula F:

- no variable appears both free and quantified in F;
- for every variable p, the formula F contains at most one occurrence of quantifiers $\exists\!\!\!\forall p$.

Any formula can be transformed into a rectified formula by renaming bound variables.

We can use the usual notation $(F)_p^G$ for rectified formulas assuming that p occurs only free.

Rectification: Example

$$p \to \exists p(p \land \boxed{\forall p(p \lor r \to \neg p)}) \Rightarrow$$

$$p \to \boxed{\exists p(p \land \forall p_1(p_1 \lor r \to \neg p_1))} \Rightarrow$$

$$p \to \exists p_2(p_2 \land \forall p_1(p_1 \lor r \to \neg p_1))$$

Rectification: Example

$$p \to \exists p(p \land \boxed{\forall p(p \lor r \to \neg p)}) \Rightarrow$$

$$p \to \boxed{\exists p(p \land \forall p_1(p_1 \lor r \to \neg p_1))} \Rightarrow$$

$$p \to \exists p_2(p_2 \land \forall p_1(p_1 \lor r \to \neg p_1))$$

Rectification: Example

$$p \to \exists p(p \land \boxed{\forall p(p \lor r \to \neg p)}\,) \Rightarrow$$

$$p \to \boxed{\exists p(p \land \forall p_1(p_1 \lor r \to \neg p_1))} \Rightarrow$$

$$p \to \exists p_2(p_2 \land \forall p_1(p_1 \lor r \to \neg p_1))$$

Variable capture

$\exists q(\neg p \leftrightarrow q)$: there exists a truth value equivalent to $\neg p$. This formula is valid.

Substitute p by q.

$\exists q(\neg q \leftrightarrow q)$: there exists a truth value equivalent to its own negation. This formula is unsatisfiable.

Variable capture

$\exists q(\neg p \leftrightarrow q)$: there exists a truth value equivalent to $\neg p$. This formula is valid.

Substitute p by q.

$\exists q(\neg q \leftrightarrow q)$: there exists a truth value equivalent to its own negation. This formula is unsatisfiable.

Jointly rectified formulas

Suppose we want to substitute $(F)^G_p$.
Then we require: no free variable in G become bound in $(F)^G_p$.

In previous example $\exists q(\neg p \leftrightarrow q)$:
Substitute p by q. ($\exists q(\neg q \leftrightarrow q)$ does not satisfy above)

Uniform solution – renaming of bound variables
$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$
Now we can substitute p by q obtaining $\exists r(\neg q \leftrightarrow r)$

Formulas $F_1, \ldots, F_n$ are jointly rectified if for every variable p,

- if p occurs free in one of F_i then p does not occur bound in $F_1, \ldots, F_n$;
- there is a most one occurrence of quantifiers $\exists\forall p$ in $F_1, \ldots, F_n$.

Jointly rectified formulas

Suppose we want to substitute $(F)_p^G$.
Then we require: no free variable in G become bound in $(F)_p^G$.

In previous example $\exists q(\neg p \leftrightarrow q)$:
Substitute p by q. ($\exists q(\neg q \leftrightarrow q)$ does not satisfy above)

Uniform solution – renaming of bound variables
$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$
Now we can substitute p by q obtaining $\exists r(\neg q \leftrightarrow r)$

Formulas $F_1, \ldots, F_n$ are jointly rectified if for every variable p,

- if p occurs free in one of F_i then p does not occur bound in $F_1, \ldots, F_n$;
- there is a most one occurrence of quantifiers $\exists\forall p$ in $F_1, \ldots, F_n$.

Jointly rectified formulas

Suppose we want to substitute $(F)_p^G$.
Then we require: no free variable in G become bound in $(F)_p^G$.

In previous example $\exists q(\neg p \leftrightarrow q)$:
Substitute p by q. ($\exists q(\neg q \leftrightarrow q)$ does not satisfy above)

Uniform solution – renaming of bound variables
$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$
Now we can substitute p by q obtaining $\exists r(\neg q \leftrightarrow r)$

Formulas $F_1, \ldots, F_n$ are jointly rectified if for every variable p,

- if p occurs free in one of F_i then p does not occur bound in $F_1, \ldots, F_n$;
- there is a most one occurrence of quantifiers $\exists\!\!\forall p$ in $F_1, \ldots, F_n$.

Rectification assumption

From now on we always assume that:

- formulas are rectified and collections of formulas are jointly rectified;
- in particular when considering substitutions $(F)_p^G$, then F and G are jointly rectified.

QBF: Equivalent Replacement; Prenex Form

Equivalent replacement

Lemma

Let I be an interpretation and $I \models F_1 \leftrightarrow F_2$. Then $I \models G[F_1] \leftrightarrow G[F_2]$.

Theorem (Equivalent Replacement)

Let $F_1 \equiv F_2$. Then $G[F_1] \equiv G[F_2]$.

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\forall_1 p_1 \ldots \exists\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\forall_1 p_1 \ldots \exists\forall_n p_n G$: the longest subsequence $\exists\forall_1 p_1 \ldots \exists\forall_k p_k$ of $\exists\forall_1 p_1 \ldots \exists\forall_n p_n$ such that $\exists\forall_1 = \ldots = \exists\forall_k$.
- A formula F is a prenex form of a formula G if F is prenex and $F \equiv G$.

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$: the longest subsequence $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_k p_k$ of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n$ such that $\exists\!\!\!\forall_1 = \ldots = \exists\!\!\!\forall_k$.
- A formula F is a prenex form of a formula G if F is prenex and $F \equiv G$.

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$: the longest subsequence $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_k p_k$ of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n$ such that $\exists\!\!\!\forall_1 = \ldots = \exists\!\!\!\forall_k$.
- A formula F is a prenex form of a formula G if F is prenex and $F \equiv G$.

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\!\!\forall_1 p_1 \ldots \exists\!\!\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\!\!\forall_1 p_1 \ldots \exists\!\!\forall_n p_n G$: the longest subsequence $\exists\!\!\forall_1 p_1 \ldots \exists\!\!\forall_k p_k$ of $\exists\!\!\forall_1 p_1 \ldots \exists\!\!\forall_n p_n$ such that $\exists\!\!\forall_1 = \ldots = \exists\!\!\forall_k$.
- A formula F is a prenex form of a formula G if F is prenex and $F \equiv G$.

Prenexing rules

We assume that the formula is rectified before application of

Prenexing rules:

$$\exists\!\!\!\forall p F_1 \mathbin{\wedge\!\!\!\!\vee} \ldots \mathbin{\wedge\!\!\!\!\vee} F_n \;\Rightarrow\; \exists\!\!\!\forall p (F_1 \mathbin{\wedge\!\!\!\!\vee} \ldots \mathbin{\wedge\!\!\!\!\vee} F_n)$$

$$F_1 \leftrightarrow F_2 \;\Rightarrow\; (F_1 \rightarrow F_2) \land (F_2 \rightarrow F_1)$$

$$\forall p F_1 \rightarrow F_2 \;\Rightarrow\; \exists p (F_1 \rightarrow F_2) \qquad \exists p F_1 \rightarrow F_2 \;\Rightarrow\; \forall p (F_1 \rightarrow F_2)$$

$$F_1 \rightarrow \forall p F_2 \;\Rightarrow\; \forall p (F_1 \rightarrow F_2) \qquad F_1 \rightarrow \exists p F_2 \;\Rightarrow\; \exists p (F_1 \rightarrow F_2)$$

$$\neg \forall p F \;\Rightarrow\; \exists p \neg F \qquad \neg \exists p F \;\Rightarrow\; \forall p \neg F$$

Warning: Sound only when the formulas are rectified!

Some useful equivalences: $\neg \forall p\, F \equiv \exists p\, \neg F$ and $\neg \exists p\, F \equiv \forall p\, \neg F$

Prenexing. Example I

$$
\begin{array}{l}
\exists q(q \to p) \to \neg\forall r(r \to p) \vee p \Rightarrow \\
\forall q((q \to p) \to \neg\forall r(r \to p) \vee p) \Rightarrow \\
\forall q((q \to p) \to \exists r\neg(r \to p) \vee p) \Rightarrow \\
\forall q((q \to p) \to \exists r(\neg(r \to p) \vee p)) \Rightarrow \\
\forall q\exists r((q \to p) \to \neg(r \to p) \vee p).
\end{array}
$$

Prenexing. Example I

$$
\begin{aligned}
&\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow \\
&\forall q((q \to p) \to \neg\forall r(r \to p) \lor p) \Rightarrow \\
&\forall q((q \to p) \to \exists r\neg(r \to p) \lor p) \Rightarrow \\
&\forall q((q \to p) \to \exists r(\neg(r \to p) \lor p)) \Rightarrow \\
&\forall q\exists r((q \to p) \to \neg(r \to p) \lor p).
\end{aligned}
$$

Prenexing. Example I

$$\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow$$
$$\forall q((q \to p) \to \neg\forall r(r \to p) \lor p) \Rightarrow$$
$$\forall q((q \to p) \to \exists r\neg(r \to p) \lor p) \Rightarrow$$
$$\forall q((q \to p) \to \exists r(\neg(r \to p) \lor p)) \Rightarrow$$
$$\forall q\exists r((q \to p) \to \neg(r \to p) \lor p).$$

Prenexing. Example I

$$\exists q(q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p \Rightarrow$$
$$\forall q((q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p) \Rightarrow$$
$$\forall q((q \rightarrow p) \rightarrow \exists r \neg (r \rightarrow p) \vee p) \Rightarrow$$
$$\forall q((q \rightarrow p) \rightarrow \exists r(\neg (r \rightarrow p) \vee p)) \Rightarrow$$
$$\forall q \exists r((q \rightarrow p) \rightarrow \neg (r \rightarrow p) \vee p).$$

Prenexing. Example I

$$
\begin{aligned}
&\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow\\
&\forall q((q \to p) \to \neg\forall r(r \to p) \lor p) \Rightarrow\\
&\forall q((q \to p) \to \exists r\neg(r \to p) \lor p) \Rightarrow\\
&\forall q((q \to p) \to \exists r(\neg(r \to p) \lor p)) \Rightarrow\\
&\forall q\exists r((q \to p) \to \neg(r \to p) \lor p).
\end{aligned}
$$

Prenexing. Example II

$$
\begin{aligned}
&\exists q(q \to p) \to \neg\forall r(r \to p) \vee p \Rightarrow\\
&\exists q(q \to p) \to \exists r\neg(r \to p) \vee p \Rightarrow\\
&\exists q(q \to p) \to \exists r(\neg(r \to p) \vee p) \Rightarrow\\
&\exists r(\exists q(q \to p) \to \neg(r \to p) \vee p) \Rightarrow\\
&\exists r\forall q((q \to p) \to \neg(r \to p) \vee p).
\end{aligned}
$$

Prenexing. Example II

$$
\begin{array}{l}
\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow \\
\exists q(q \to p) \to \exists r\neg(r \to p) \lor p \Rightarrow \\
\exists q(q \to p) \to \exists r(\neg(r \to p) \lor p) \Rightarrow \\
\exists r(\exists q(q \to p) \to \neg(r \to p) \lor p) \Rightarrow \\
\exists r\forall q((q \to p) \to \neg(r \to p) \lor p).
\end{array}
$$

Prenexing. Example II

$$
\begin{aligned}
&\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow \\
&\exists q(q \to p) \to \exists r\neg(r \to p) \lor p \Rightarrow \\
&\exists q(q \to p) \to \exists r(\neg(r \to p) \lor p) \Rightarrow \\
&\exists r(\exists q(q \to p) \to \neg(r \to p) \lor p) \Rightarrow \\
&\exists r\forall q((q \to p) \to \neg(r \to p) \lor p).
\end{aligned}
$$

Prenexing. Example II

$$\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow$$
$$\exists q(q \to p) \to \exists r\neg(r \to p) \lor p \Rightarrow$$
$$\exists q(q \to p) \to \exists r(\neg(r \to p) \lor p) \Rightarrow$$
$$\exists r(\exists q(q \to p) \to \neg(r \to p) \lor p) \Rightarrow$$
$$\exists r\forall q((q \to p) \to \neg(r \to p) \lor p).$$

Prenexing. Example II

$$\exists q(q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p \Rightarrow$$
$$\exists q(q \rightarrow p) \rightarrow \exists r \neg (r \rightarrow p) \vee p \Rightarrow$$
$$\exists q(q \rightarrow p) \rightarrow \exists r(\neg (r \rightarrow p) \vee p) \Rightarrow$$
$$\exists r(\exists q(q \rightarrow p) \rightarrow \neg (r \rightarrow p) \vee p) \Rightarrow$$
$$\exists r \forall q((q \rightarrow p) \rightarrow \neg (r \rightarrow p) \vee p).$$

Summary

- quantified Boolean formulas (QBF): $\exists x\, \forall y \exists z\, F$
- syntax, semantics
- evaluating QBF formula on an interpretations: and-or trees
- positions/polarity
- bound/free occurrences of variables
- for closed formulas: validity and satisfiability coincide;
  for open formulas we can express satisfiability/validity using $\exists/\forall$ quantifiers respectively.
- rectified formulas: i) no variable occurs free and bound, ii) every variable is quantified at most once.
- rectification: rename bound variables
- prenex normal form: all quantifiers are on the left-hand-side
- prenexing: rectify + apply prenexing rules