Compact representation of Boolean functions and formulas:

BDTs/BDDs/OBDDs

Data Structures for Large Propositional Formulas

Suppose that large propositional formulas are reused over and over again. For example, we may

- Build a conjunction of several formulas;
- Negate a formula;
- Check if two formulas are equivalent . . .

One needs data structures which

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- facilitate boolean operations on this formulas, for example, taking conjunction of several formulas;
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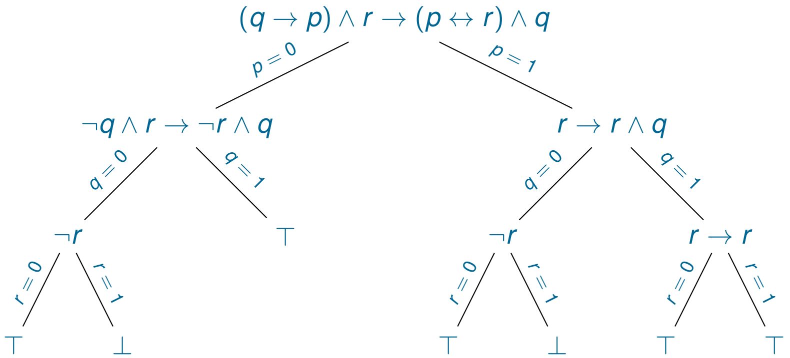
Splitting Tree



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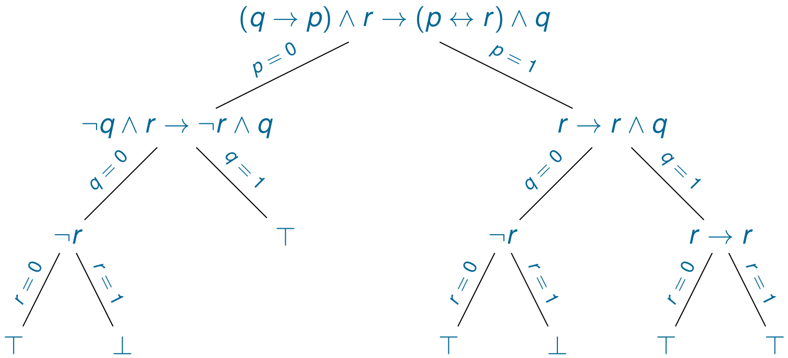
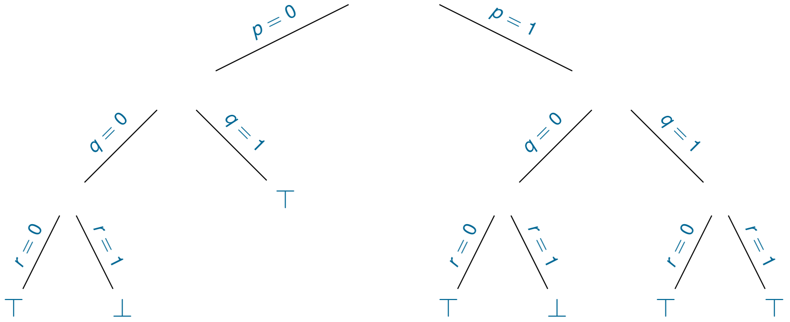


Let us “forget” about formulas in the tree

Splitting Tree



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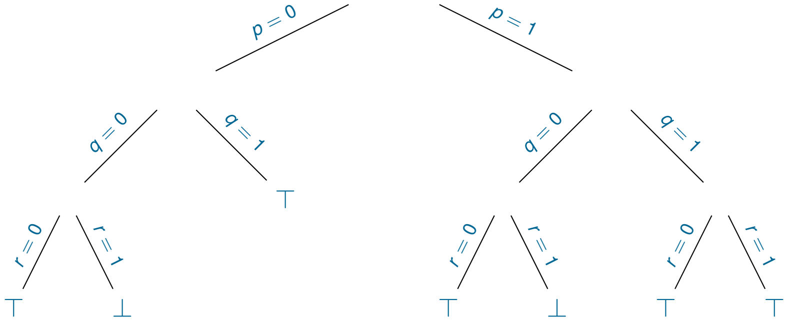
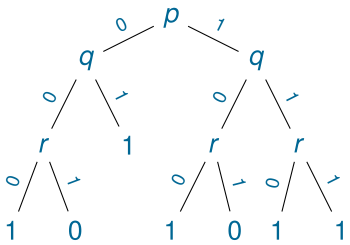


We do not see the syntax of the formula but the semantics is preserved: the tree encodes all models of the formula. Any formula having the same tree with “forgotten” formulas has the same models.

Binary Decision Tree



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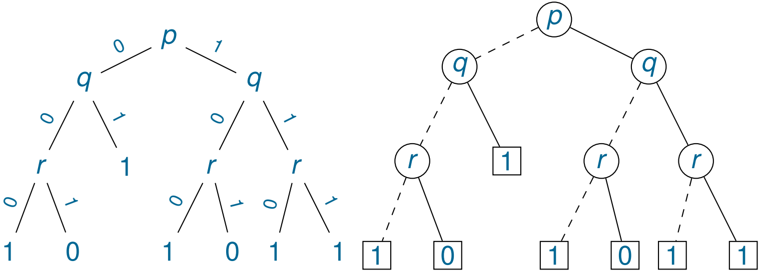

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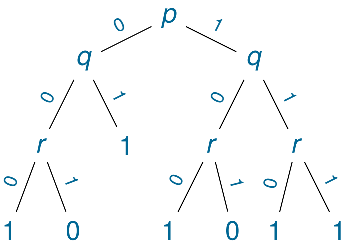

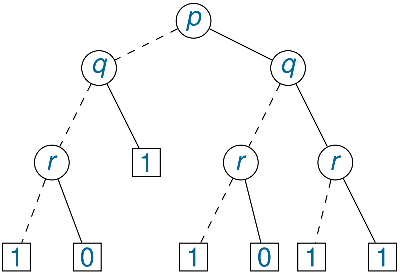

A circled node, such as (p), denotes the decision on the variable of this node.

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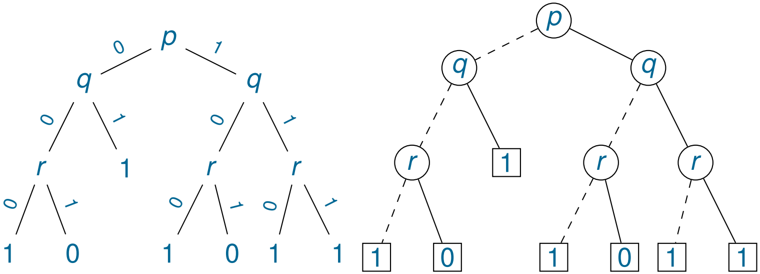

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Terminal nodes are squared, for example $\boxed{1}$.

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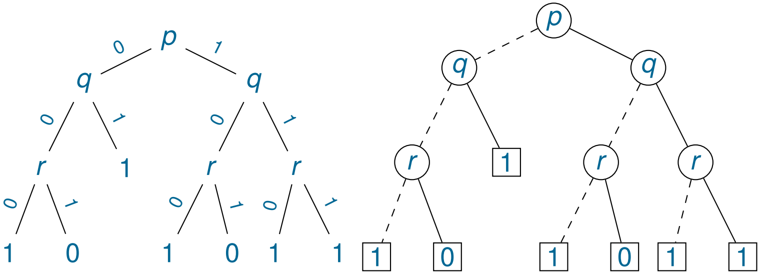

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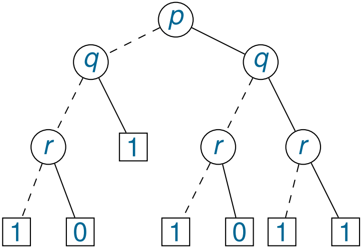
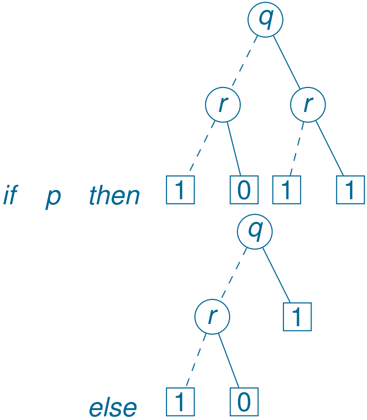
Solid lines correspond to choices when the variable is true, dashed lines to the choices when the variable is false.

Tests correspond to “if-then-else”





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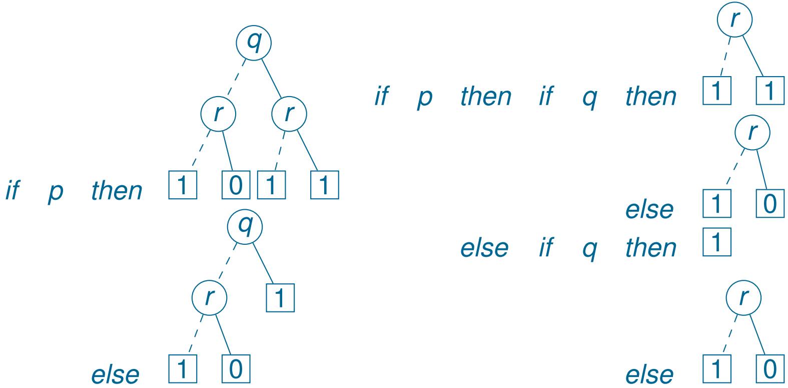

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$$
\begin{array}{llllllllll}
\textit{if} & p & \textit{then} & \textit{if} & q & \textit{then} & \textit{if} & r & \textit{then} & \top \\
 & & & & & & & & \textit{else} & \top \\
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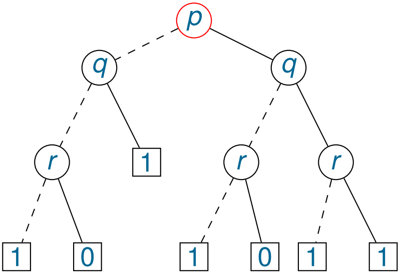
Consider, for example the interpretation $\{p \mapsto 0, q \mapsto 0, r \mapsto 1\}$ and the following tree.



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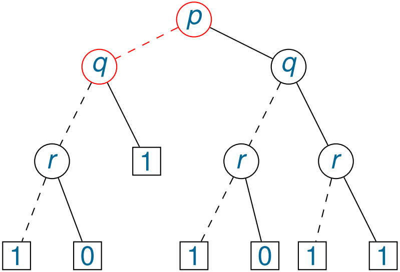

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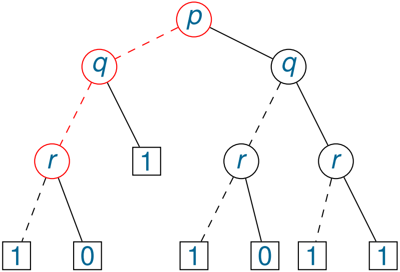

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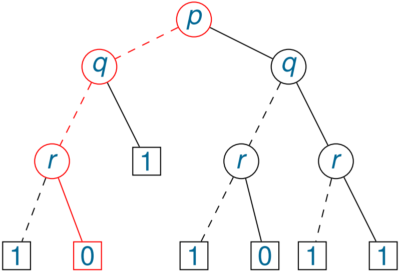

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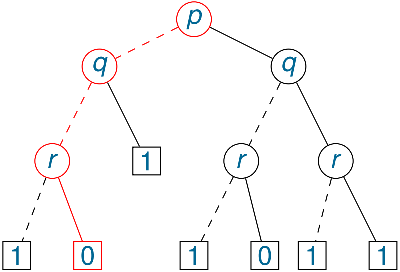

So any formula with this decision tree is false in this interpretation.

Properties

Properties of binary decision trees:

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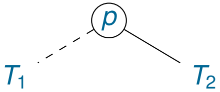
- Some boolean operations, e.g., conjunction, are hard to implement.
- The size of the tree is in the worst case exponential in the number of variables.

Algorithm for Building Binary Decision Trees

procedure *bdt*(A)
input: propositional formula A
output: a binary decision tree
parameters: function *select_variable*
begin
  $A := simplify(A)$
  if $A = \bot$ **then return** $\boxed{0}$
  if $A = \top$ **then return** $\boxed{1}$
  $p := select\_variable(A)$
  return $tree(bdt(A_p^{\bot}), p, bdt(A_p^{\top}))$
end

Here $tree(T_1, p, T_2)$ builds the tree



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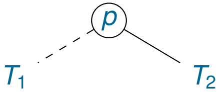
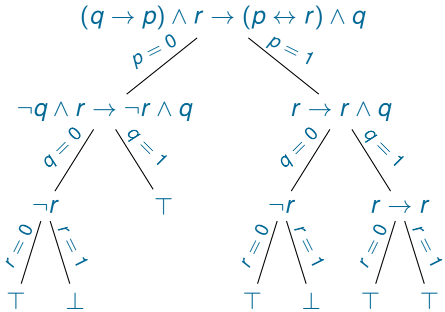


Note resemblance to the splitting algorithm!

Example



$(q \to p) \land r \to (p \leftrightarrow r) \land q$

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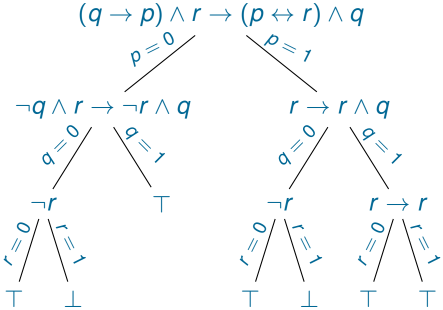
$(q \to p) \wedge r \to (p \leftrightarrow r) \wedge q$
$p = 0$
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$\neg q \wedge r \to \neg r \wedge q$
$r \to r \wedge q$
$q = 0$
$q = 1$
$q = 0$
$q = 1$
$\neg r$
$\top$
$\neg r$
$r \to r$
$r = 0$
$r = 1$
$r = 0$
$r = 1$
$r = 0$
$r = 1$
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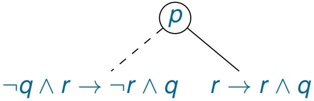
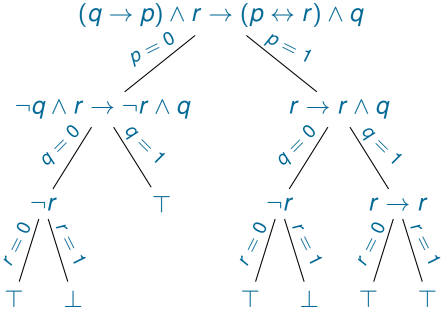
$\top$


p
$\neg q \wedge r \to \neg r \wedge q$
$r \to r \wedge q$

Example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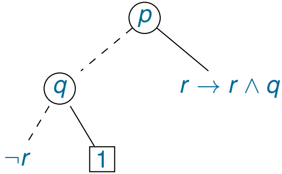
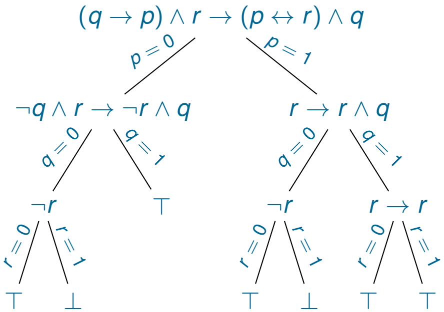

p
q
$r \to r \wedge q$
$\neg r$
1

Example



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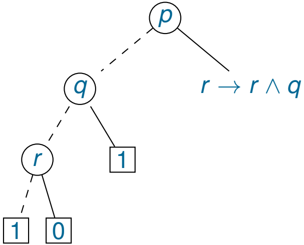
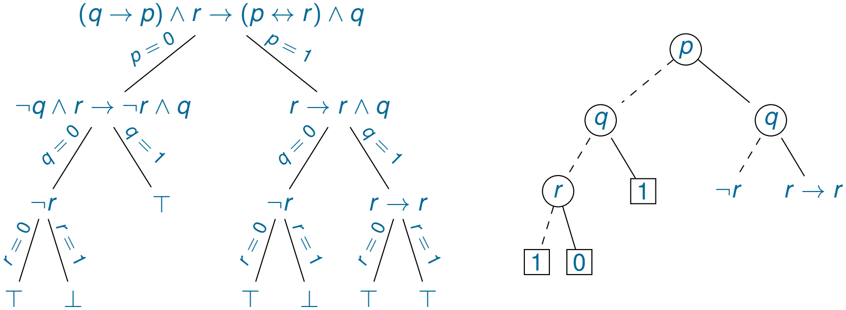


p
q
$r \to r \land q$
r
1
1
0

Example



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q
q
r
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1
0

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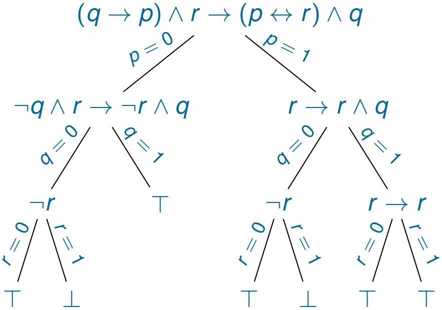
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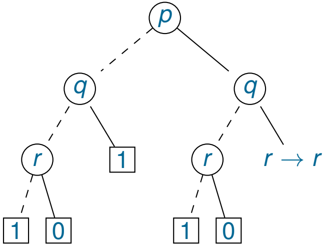
p
q
q
r
1
r
$r \to r$
1
0
1
0

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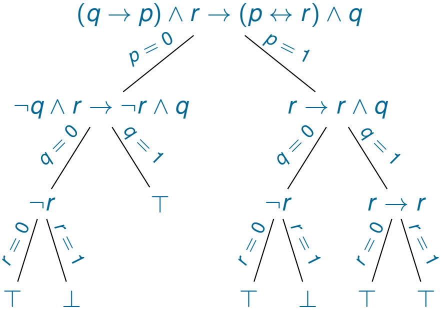
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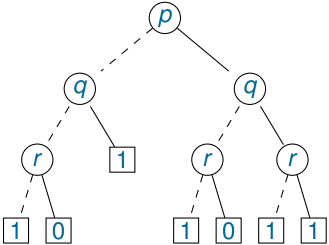
p
q
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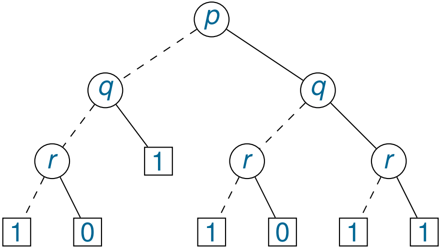
r
1
0
1
0
1
1

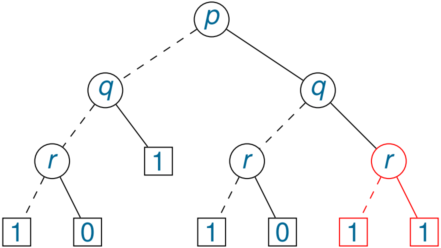
Are BDTs compact?

Are binary decision trees compact?



Are BDTs compact?

Are binary decision trees compact?
No: they may contain redundant tests:



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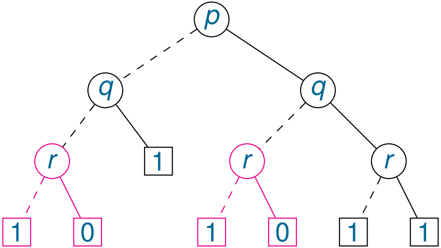
Are binary decision trees compact?
No: they may contain redundant tests:
No: they may contain isomorphic subtrees:



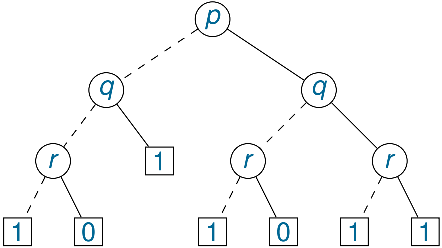
Binary Decision Diagrams (BDDs)

Binary Decision Diagrams

A binary decision diagrams or simply a BDD is a dag (directed acyclic graph) built like a binary decision tree but containing

- no redundant tests; and
- no isomorphic subtrees.

Transforming Binary Decision Tree into a BDD



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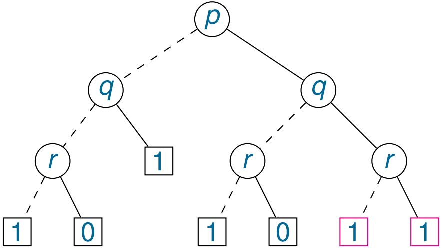

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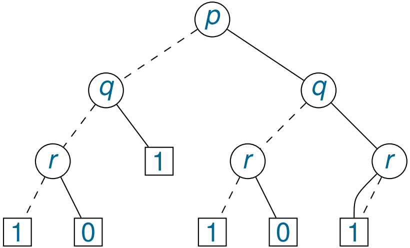
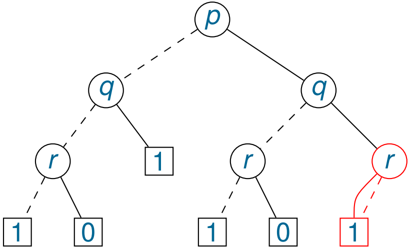


Transforming Binary Decision Tree into a BDD



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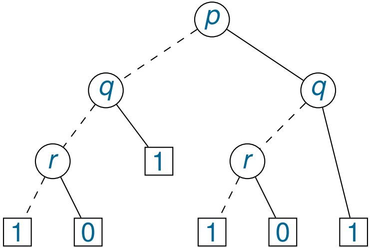

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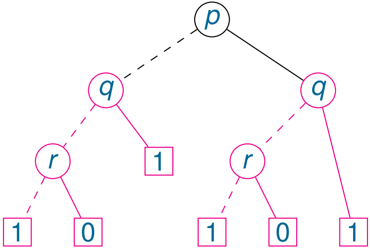

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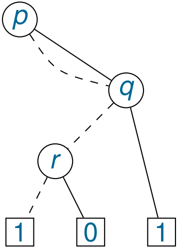

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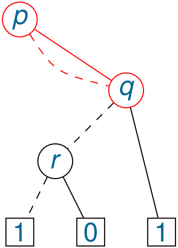

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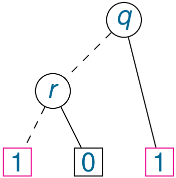

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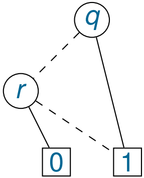

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What is the complexity of satisfiability, validity and equivalence checking for BDDs?

- Satisfiability checking can be done in constant time;
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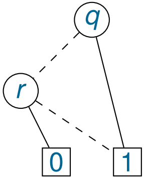

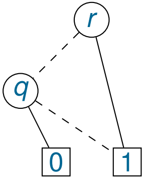

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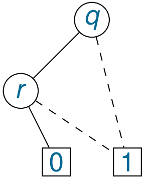

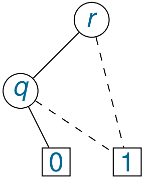

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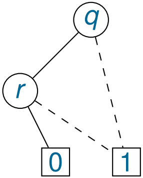

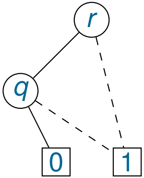

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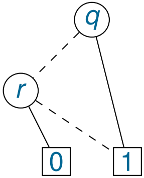
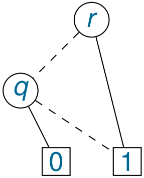


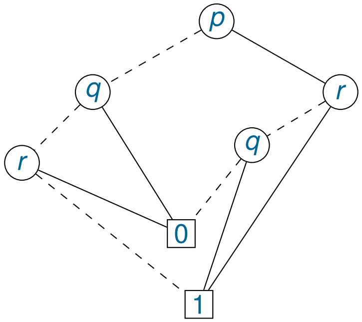


OBDDs

Ordered Binary Decision Diagrams (OBDDs)

Donald Knuth regarding OBDDs:
"one of the only really fundamental data structures that came out in the last twenty-five years"

BDDs



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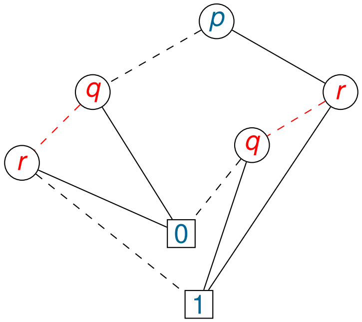
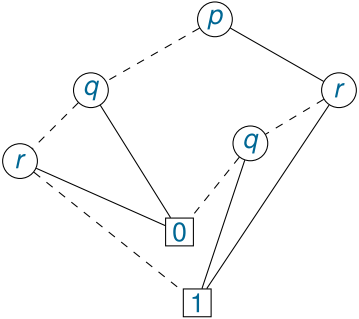


Problem: on different branches of the dag tests are made in a different order.

BDDs



Problem: on different branches of the dag tests are made in a different order.

OBDDs:

- introduce an order $>$ on variables;
- make tests in this order.

OBDDs Properties

- Every Bolean function has a unique representation as OBDD!
  In other words OBDDs are canonical for Boolean functions.

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- Boolean operations e.g., conjunction, are easy to implement.

Integrating a node in a dag

All OBDD algorithms will use the same procedure for integrating a node in a dag.

procedure *integrate*(n_1, p, n_2, D)
parameters: global dag D
input: nodes n_1, n_2 in D representing formulas F_1, F_2, variable p
output: node n in (modified) D representing *if* p *then* F_2 *else* F_1

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  if $n_1 = n_2$ **then** **return** n_1;

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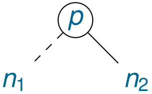

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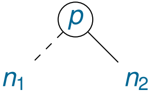
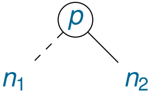


   then **return** n;
  else add to D a new node n of the form



  return n
end

Building OBDDs

procedure $obdd(F)$
input: propositional formula F
parameters: order on variables $p_1 > p_2 \ldots > p_k$
global dag D respecting using this order
output: a node n in (modified) D which represents F
begin
$F := simplify(F)$
if $F = \bot$ **then** **return** $\boxed{0}$
if $F = \top$ **then** **return** $\boxed{1}$
$p := max\_variable(F)$
$n_1 := obdd(F_p^\bot)$
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return $integrate(n_1, p, n_2, D)$
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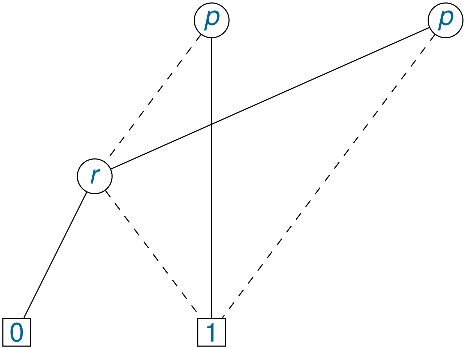
- This procedure puts together the algorithms for building decision trees and the algorithm for eliminating redundancies.
- Redundancy elimination is done by the procedure *integrate*.

Example: Building OBDD: $p > q > r$

$obdd((q \rightarrow p) \wedge r \rightarrow (p \leftrightarrow r) \wedge q)$



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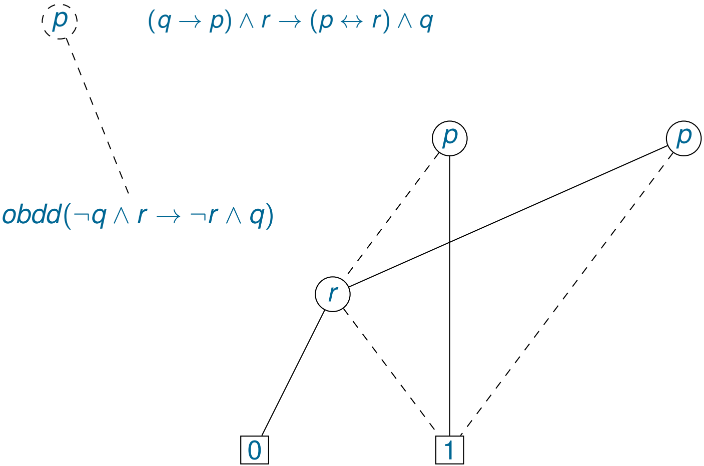

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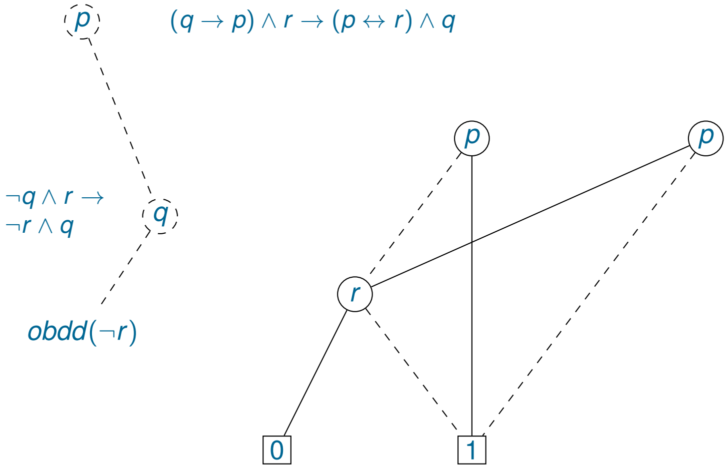

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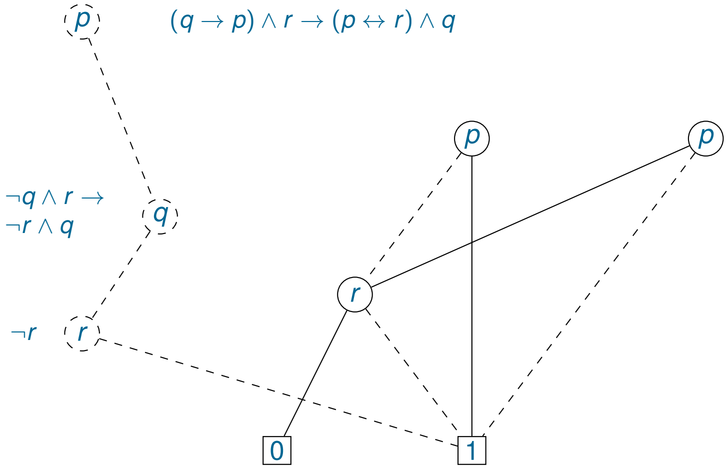

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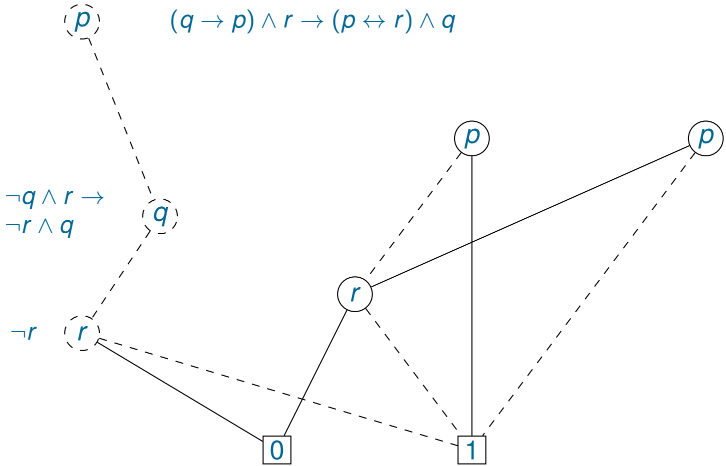

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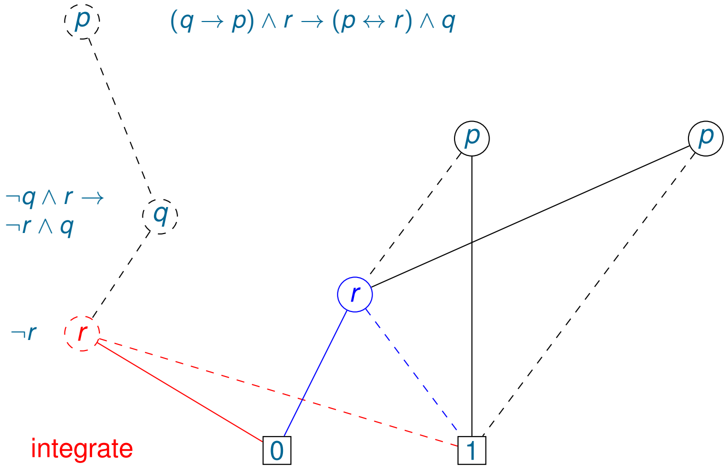

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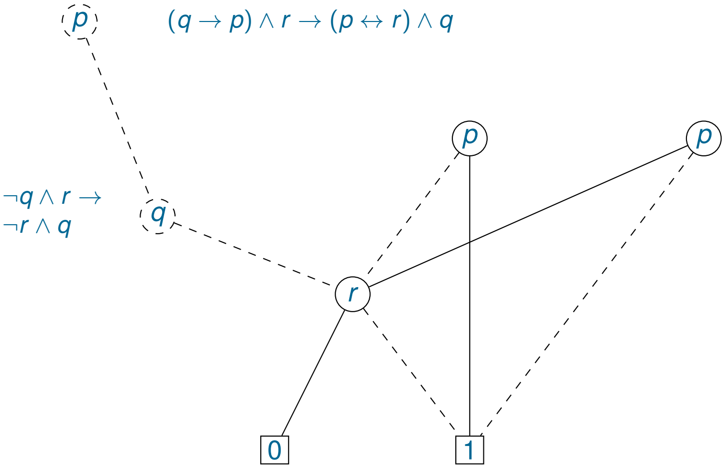

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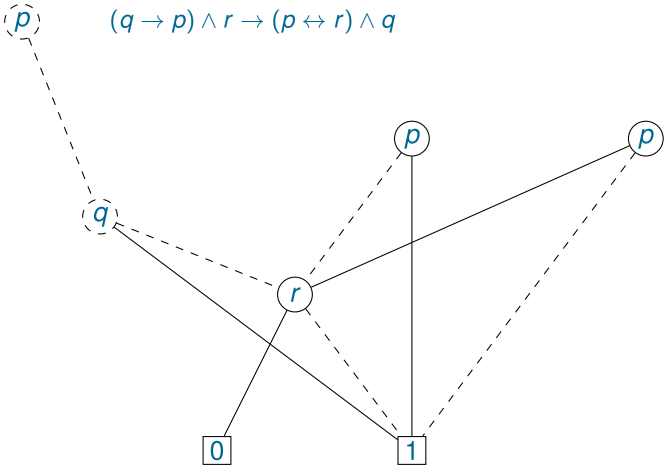

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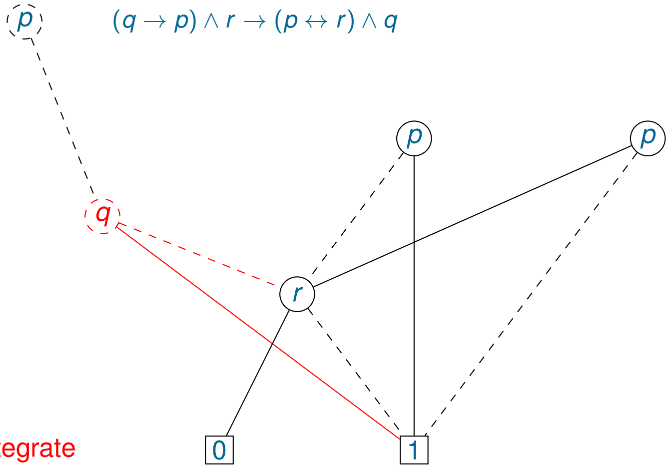

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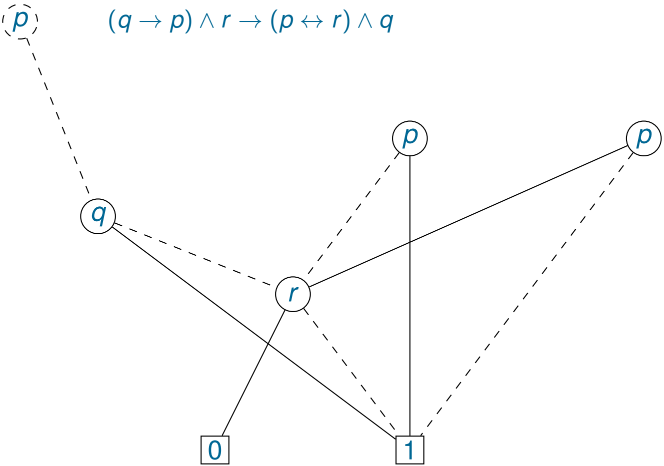

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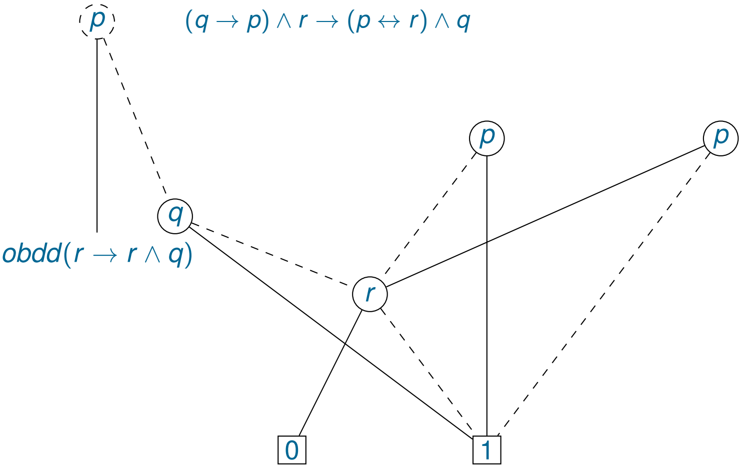

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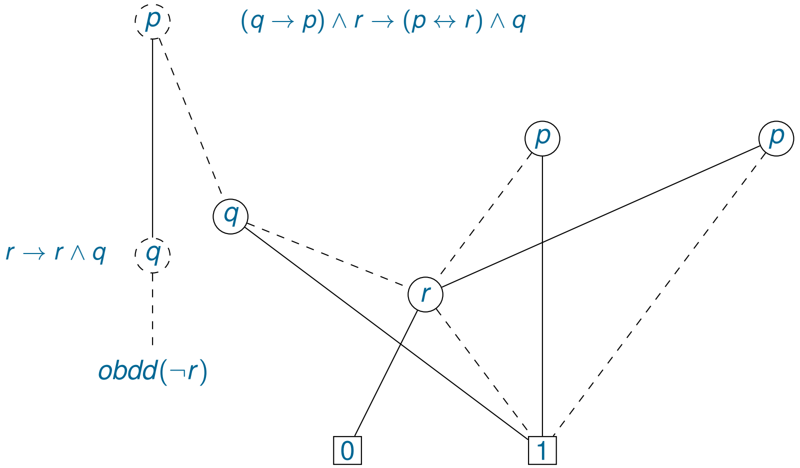

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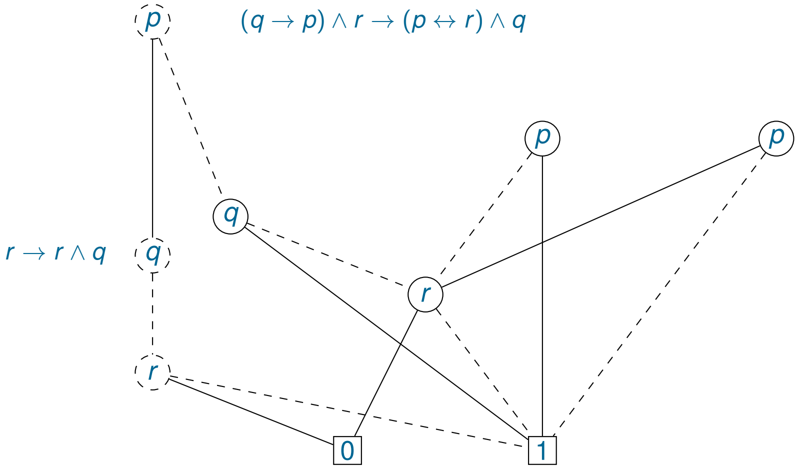

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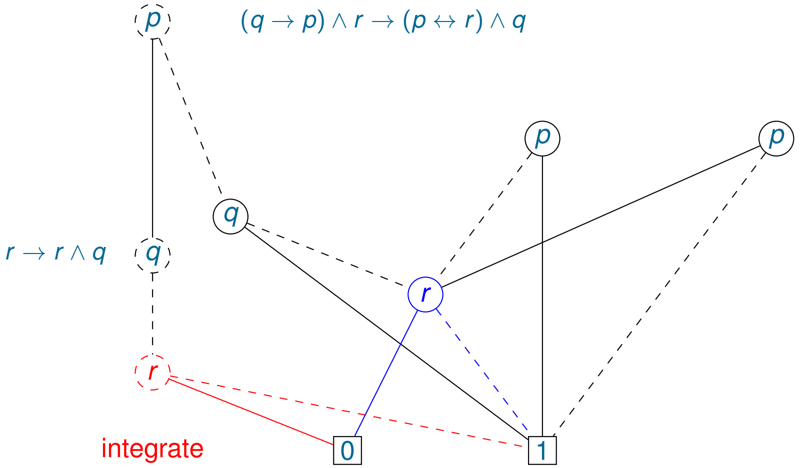

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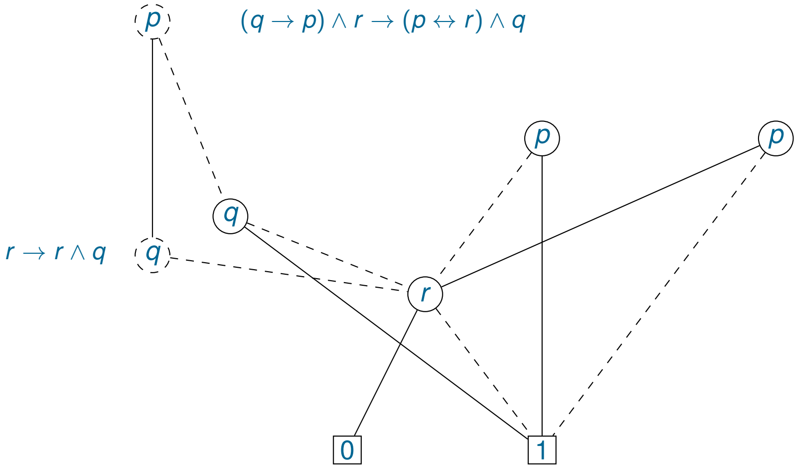

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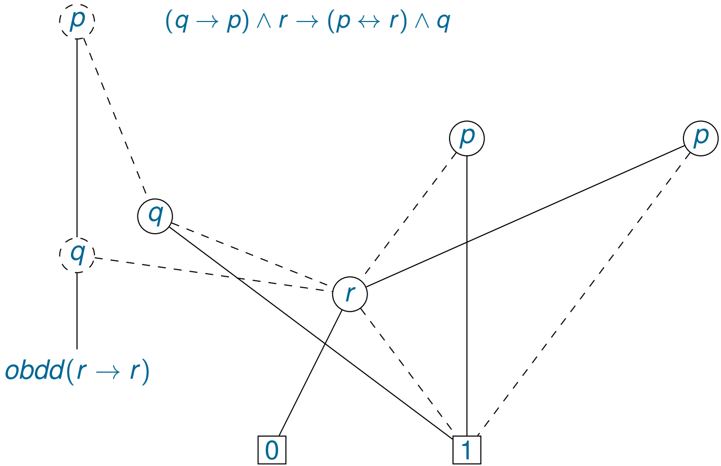

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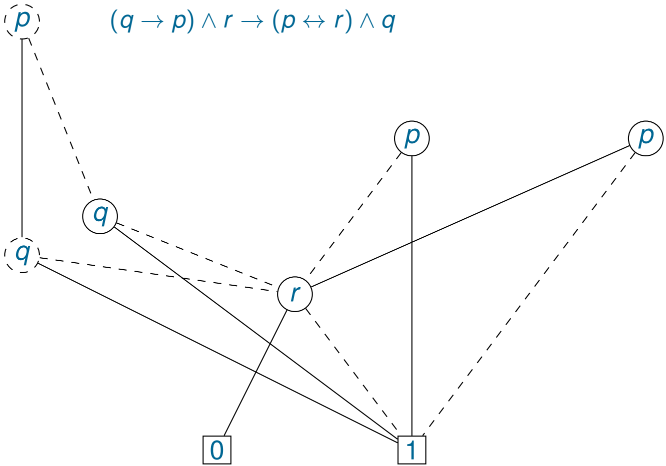

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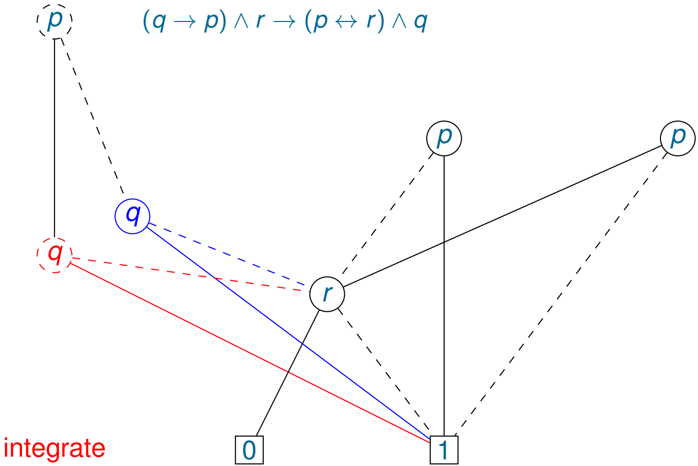

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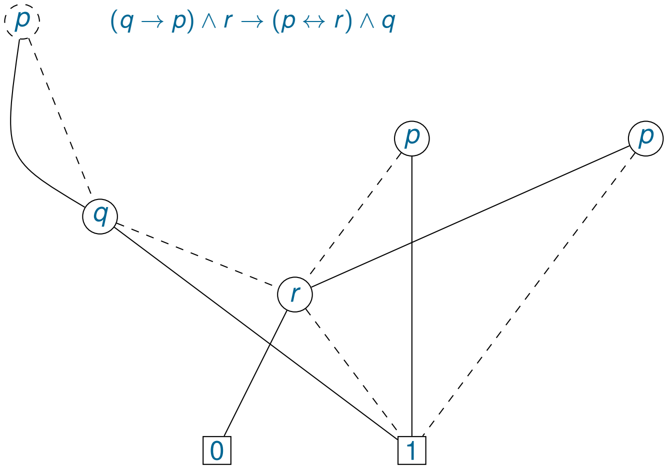

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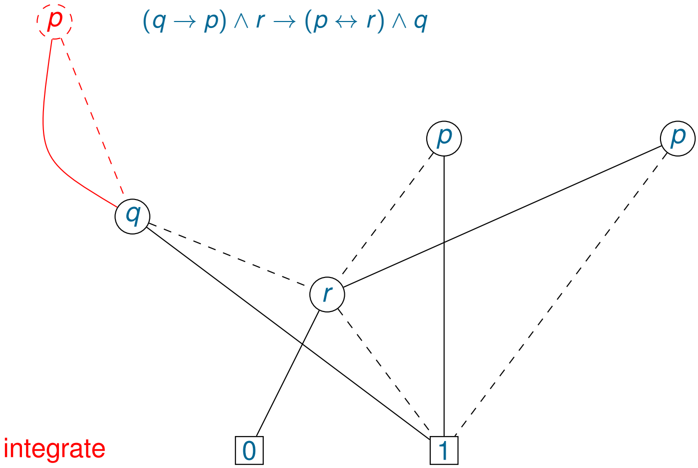

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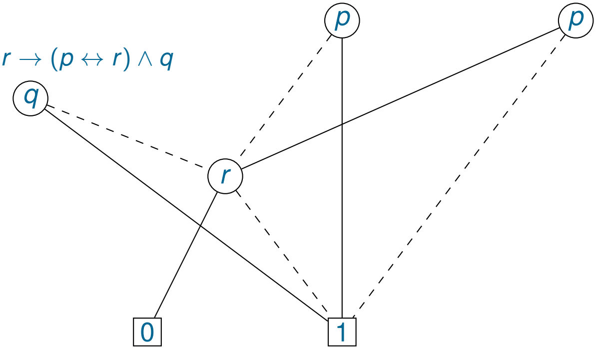

We return the new node rooted at q.

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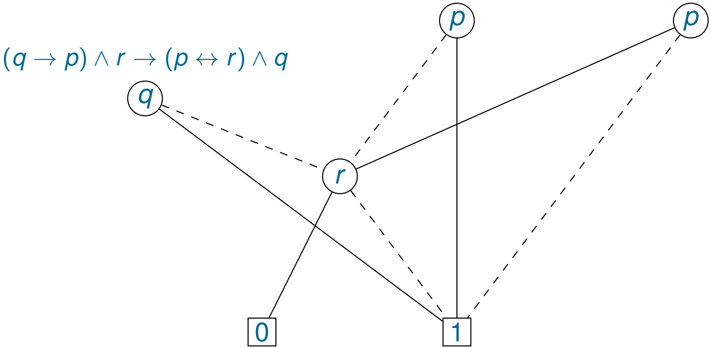

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Note that the application of this procedure modified the global dag.

Applying Boolean functions to OBDDs

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- Use the following fundamental property of if-then-else: if-then-else commutes with functions.

Fundamental property of if-then-else

If-then-else commutes with any function.

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$$
\begin{array}{l}
f(\ \ \textit{if } p \textit{ then } l_1 \textit{ else } r_1, \\
\qquad \ldots, \\
\qquad \textit{if } p \textit{ then } l_n \textit{ else } r_n\ ) = \\
\textit{if } p \textit{ then } f(l_1, \ldots, l_n) \textit{ else } f(r_1, \ldots, r_n)
\end{array}
$$

Fundamental property of if-then-else

If-then-else commutes with any function.

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This is what this formula teaches us: to apply f to n OBDDs rooted at p,

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- apply f to the subtrees corresponding to $p = 1$ (the subdags under the solid edge), obtaining a dag D_1;

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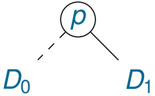
This is what this formula teaches us: to apply f to n OBDDs rooted at p,

- apply f to the subdags corresponding to $p = 0$ (the subdags under the dashed edge), obtaining a dag D_0;
- apply f to the subtrees corresponding to $p = 1$ (the subdags under the solid edge), obtaining a dag D_1;
- Build and return the dag of the form

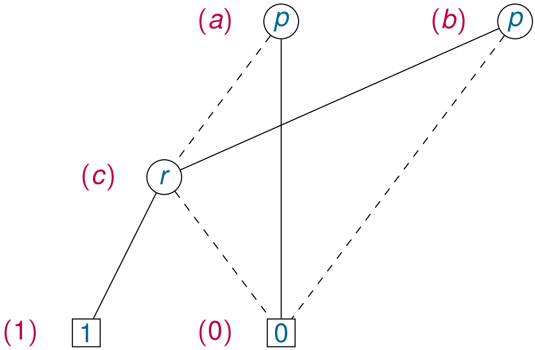


Disjunction

```
procedure disjunction(n_1, ..., n_m)
parameters: global dag D
input: nodes n_1, ..., n_m representing F_1, ..., F_m in D
output: a node n representing F_1 ∨ ... ∨ F_m in (modified) D
begin
  if some n_i is [1] then return [1]
  if m = 0 then return [0]
  if m = 1 then return n_1
  if some n_i is [0] then
    return disjunction(n_1, ..., n_{i-1}, n_{i+1}, ..., n_m)
  p := max_variable(n_1, ..., n_m)
  forall i = 1 ... m
    if n_i is labelled by p
      then (l_i, r_i) := (neg(n_i), pos(n_i))
      else (l_i, r_i) := (n_i, n_i)
  k_1 := disjunction(l_1, ..., l_m)
  k_2 := disjunction(r_1, ..., r_m)
  return integrate(k_1, p, k_2, D)
end
```

Example: Disjunction $p > r$

(a) represents $(\neg p \wedge r)$; (b) represents $(p \wedge r)$



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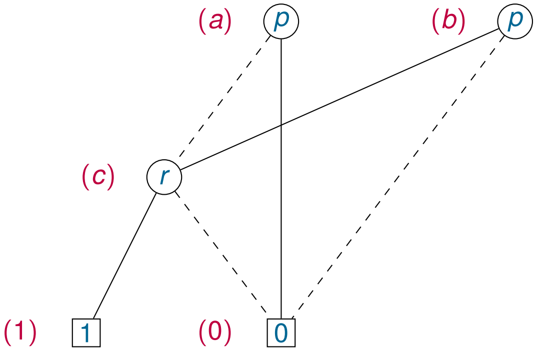
dis(a,b)



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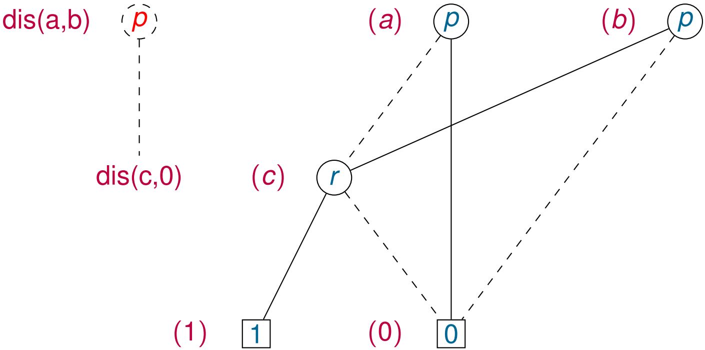

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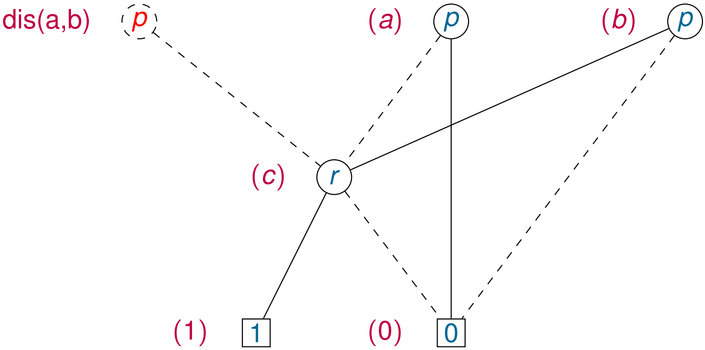

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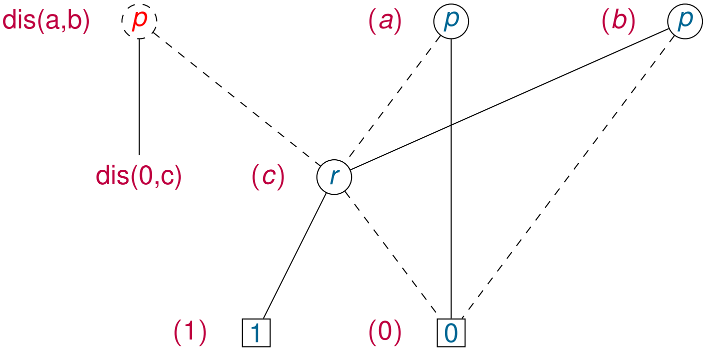

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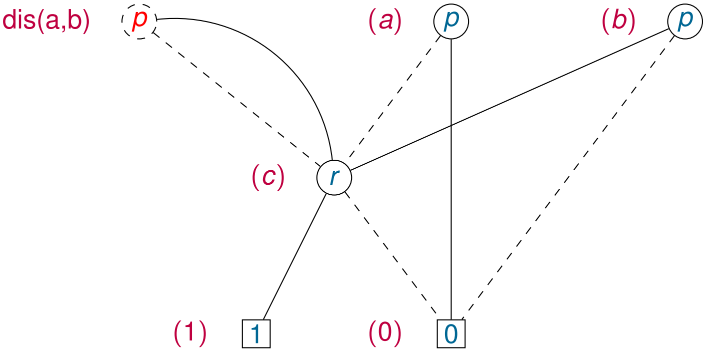

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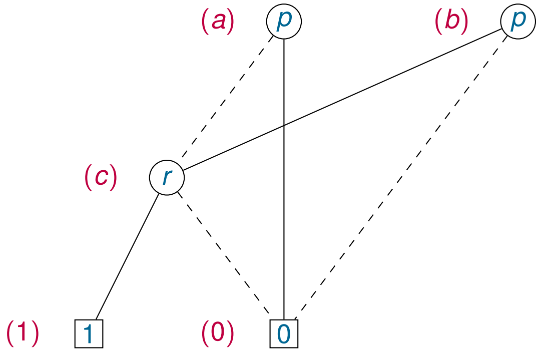
dis(a,b) = c



Disjunction

procedure $disjunction(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \vee \ldots \vee F_m$ in (modified) D
begin
  if some n_i is $\boxed{1}$ **then** **return** $\boxed{1}$
  if $m = 0$ **then** **return** $\boxed{0}$
  if $m = 1$ **then** **return** n_1
  if some n_i is $\boxed{0}$ **then**
    return $disjunction(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  $p \;:=\; max\_variable(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then $(l_i, r_i) \;:=\; (neg(n_i), pos(n_i))$
      else $(l_i, r_i) \;:=\; (n_i, n_i)$
  $k_1 \;:=\; disjunction(l_1, \ldots, l_m)$
  $k_2 \;:=\; disjunction(r_1, \ldots, r_m)$
  return $integrate(k_1, p, k_2, D)$
end

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      then $(l_i, r_i) \ := \ (neg(n_i), pos(n_i))$
      else $(l_i, r_i) \ := \ (n_i, n_i)$
  $k_1 \ := \ disjunction(l_1, \ldots, l_m)$
  $k_2 \ := \ disjunction(r_1, \ldots, r_m)$
  return $integrate(k_1, p, k_2, D)$
end

$F \vee \top \equiv \top$
$F \vee \bot \equiv F$

Conjunction

procedure $conjunction(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \wedge \ldots \wedge F_m$ in (modified) D
begin
  if some n_i is $\boxed{0}$ **then** **return** $\boxed{0}$
  if $m = 0$ **then** **return** $\boxed{1}$
  if $m = 1$ **then** **return** n_1
  if some n_i is $\boxed{1}$ **then**
    return $conjunction(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  $p \ := \ max\_variable(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then $(l_i, r_i) \ := \ (neg(n_i), pos(n_i))$
      else $(l_i, r_i) \ := \ (n_i, n_i)$
  $k_1 \ := \ conjunction(l_1, \ldots, l_m)$
  $k_2 \ := \ conjunction(r_1, \ldots, r_m)$
  return $integrate(k_1, p, k_2, D)$
end

$F \wedge \bot \equiv \bot$
$F \wedge \top \equiv F$

Summary

Binary decision trees, BDDs, OBDDs.

OBDDs:

- canonical representation of Boolean functions;
- can transform any formula to equivalent OBDD representation. But in some cases the size of the OBDD can grow exponentially.
- constant time checks for main logical problems: satisfiability, validity, equivalence;
- Boolean operations e.g. conjunction, disjunction, etc. are easy to implement.