Weak Convergence of a Metropolis Algorithm for Bimodal Target Distributions

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- In general, less efficient than iid sampling, but iid sampling is sometimes unrealizable

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- Called *accept/reject step*

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- Must choose σ carefully

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- Let $d \to \infty$

Accelerating the Markov chain

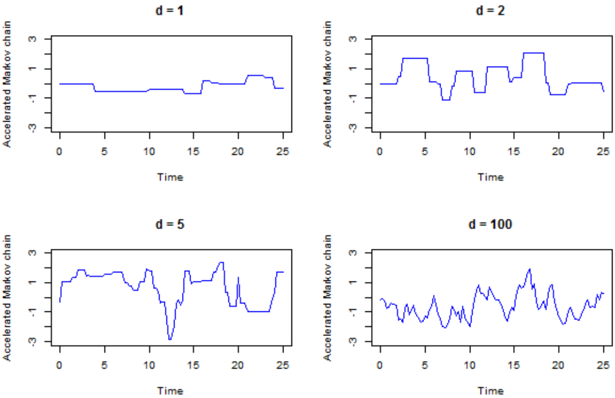

Figure: Trace of the accelerated first component of the Markov chain, $Z_1^{(d)}$, for different values of d. The target distribution is multivariate standard Gaussian.

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- So just simulate a few short runs and tune σ so that acceptance rate is roughly 0.234

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- Chain gets stuck in a mode and never explores the other one

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  where $\mathcal{D}$ is any distribution on $\mathbb{R}^d$ *symmetric around 0*, $p \in (0,1)$
- In practice, choose $\mathcal{D}$ to favor switching modes
- Turns the "slowly mixing" chain into a "rapidly mixing" chain ([Guan and Krone, 2007])

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- The one-dimensional accelerated processes $Z_j^{(d)}$ weakly converge

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$$\alpha(\ell, x, y) = \Phi\left(\frac{\log \frac{f_1(y)}{f_1(x)} - \frac{\ell^2}{2}B}{\ell\sqrt{B}}\right) + \frac{f_1(y)}{f_1(x)}\Phi\left(\frac{-\log \frac{f_1(y)}{f_1(x)} - \frac{\ell^2}{2}B}{\ell\sqrt{B}}\right)$$

The limiting processes

Theorem

If the chain starts at stationnarity ($\mathbf{X}^{(d)}(0) \sim \pi$)*, then as* $d \to \infty$,

$$Z_1^{(d)} \Rightarrow Z_{LM}, \quad \text{and for } j \geq 2, \quad Z_j^{(d)} \Rightarrow Z_L.$$

Here, $\Rightarrow$ represents *weak convergence in the Skorokhod topology*.

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- Number of times we switch modes $\approx np\lambda$

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- Find objective way to choose p through non-asymptotics
- Can replace the large step distribution $\mathcal{D}$ by any *algorithm* ("something-inside-Metropolis")

Guan, Y. and Krone, S. (2007).
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