Rank-based M-Estimation for Tail Dependence and Independence

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Joint work with Sebastian Engelke[2] and Stanislav Volgushev[1]

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- Strategies to estimate the extremal dependence structure depend on whether Asymptotic dependence on Asymptotic independence
- Our objective: Propose a unifying way to do so in both situations

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- If F is in a MDA, L, R and the exponent measure are all equivalent

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- X and Y are then said to be *asymptotically independent* (AI). Otherwise, they are deemed *asymptotically dependent* (AD)
- Note that if $(X, Y) \in \mathcal{D}(G)$ for a bivariate EVD G, X and Y are AI iff the two components of G are independent

Another representation of the extremal dependence

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- Note: For c to be unique, we assume $c(1,1) = 1$

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- It appears in [Draisma et al., 2004]

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2. *Under AD, $\sqrt{m}\left(\hat{c}_n - c\right) \rightsquigarrow W^{(2)}$ (in the topo. of hypi-convergence for locally bounded functions ([Bücher et al., 2014])).*

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- "Something" is what one would obtain with known marginal distributions F_1, F_2. It is a fairly standard empirical process
- The other term comes from the error in estimating the marginals
- Under AD, it converges to a non trivial limit
- Under AI, it disappears because convergence of $\hat{x}_n$ and $\hat{y}_n$ is faster than convergence of "Something" to $W^{(1)}$ (based on more data)

Example 1: Inverted max-stable distributions

Suppose that $(1/X, 1/Y)$ has a bivariate max-stable distribution, with L-function L. Then under a mild smoothness assumption on L, (X, Y) satisfies our assumptions, with

$$q(t) = t^{L(1,1)}, \quad c(x, y) = x^{\dot{L}_1(1,1)} y^{\dot{L}_2(1,1)}, \quad q_1(t) = \frac{1}{\log(1/t)}$$

Example 2: A random scale construction

Suppose $R \sim \text{Pareto}(\alpha)$, $W_j \sim \text{Pareto}(1)$, where R, W_1, W_2 are independent. Then $(X, Y) = R(W_1, W_2)$ satisfies our assumptions

Range of α	$q(t)$	$c(x, y)$	$q_1(t)$
$(0, 1)$	$K_\alpha t$	$(1 - r(\alpha))(x \wedge y) + r(\alpha)(x \wedge y)^{1/\alpha}(x \vee y)^{1-1/\alpha}$	$t^{1/\alpha - 1}$
1	$\frac{K_\alpha t}{\log(1/t) + \log\log(1/t)}$	$(x \wedge y)\left(1 + \frac{1}{2}\log\left(\frac{x \vee y}{x \wedge y}\right)\right)$	$\frac{1}{\log(1/t)}$
$(1, 2)$	$K_\alpha t^\alpha$	$(x \wedge y)(x \vee y)^{\alpha - 1}$	$t^{(\alpha-1)\wedge(2-\alpha)}$
2	$K_\alpha t^2 \log(1/t)$	xy	$\frac{1}{\log(1/t)}$
$(2, \infty)$	$K_\alpha t^2$	xy	$t^{\alpha - 2}$

$$r(\alpha) = \frac{\alpha}{2}\left(1 - (2-\alpha)(1-\alpha)^{1/\alpha - 1}\right) \in (0, 1)$$

Only thing to know: $\alpha < 1 \Rightarrow$ AD and $\alpha \geq 1 \Rightarrow$ AI

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- The non-parametric estimator $\hat{c}_n$ is not a proper function c
- More importantly, recall that $\hat{c}_n$ depends on the unknown scaling function q (through $m = nq(k/n)$)
- The following parametric estimation procedure fixes this problem

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- Solution: Since

$$m\hat{c}_n(x,y) = \sum_{i=1}^{n} \mathbb{1}\left\{ \hat{F}_1(X_i) \geq 1 - \frac{k}{n}x, \hat{F}_2(Y_i) \geq 1 - \frac{k}{n}y \right\}$$

can be calculated, simply multiply the second integral by m

The M-estimator we need

- Assume a parametric family of functions $\{c_\theta : \theta \in \Theta \subset \mathbb{R}^p\}$
- Idea: Choose θ as to minimize

$$\left\| \int_{[0,T]^2} g(x,y) c_\theta(x,y)\, dx\, dy - \int_{[0,T]^2} g(x,y) \hat{c}_n(x,y)\, dx\, dy \right\|,$$

  where $g : [0, T]^2 \to \mathbb{R}^q$ is a vector of arbitrary weight functions
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- We obtain the following objective function:

$$\Psi_n(\theta, \sigma) :=$$

$$\left\| \sigma \int_{[0,T]^2} g(x,y) c_\theta(x,y)\, dx\, dy - m \int_{[0,T]^2} g(x,y) \hat{c}_n(x,y)\, dx\, dy \right\|$$

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- By minimizing this objective function, we hope that c_θ will estimate c and σ will estimate m

Asymptotic normality of the M-estimator

- Suppose that the true function generating the data is c_{θ_0}, $\theta_0 \in \Theta$, and that the map

$$(\theta, \xi) \mapsto \xi \int_{[0,T]^2} g(x, y) c_\theta(x, y) \, dx \, dy$$

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Theorem (L, Engelke and Volgushev (2019))

Then if $(\hat{\theta}_n, \hat{\sigma}_n)$ is an estimator such that $\Psi_n(\hat{\theta}_n, \hat{\sigma}_n) = o_P(\sqrt{m})$,

$$\sqrt{m}\left(\left(\hat{\theta}_n, \frac{\hat{\sigma}_n}{m}\right) - (\theta_0, 1)\right) \rightsquigarrow N\left(0, \Sigma(\theta_0)\right).$$

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Thank you!

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