Rank-based Estimation under Asymptotic Dependence and Independence, with Applications to Spatial Extremes

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SSC Annual Meeting, 08/06/2021

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  - the 1-in-100-years rainfall (or even 1-in-10 000 years)
  - the probability of at least $x$mm rainfall (x larger than anything observed)
- Mathematically, EVT uses largest data points to estimate the whole tail of one (or multiple) random variable(s)

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- Dependence structure in tail regions

Tail dependence

- Copula-based approach: represent the tail dependence structure by

$$R(x, y) := \lim_{t \downarrow 0} t^{-1} \mathbb{P}(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty),$$

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- So estimation of $R \Rightarrow$ estimation of far tail probabilities

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- Example: (X, Y) is jointly Normal with correlation $\rho \in (-1, 1)$
- Modelling/inference based on $R \Rightarrow$ cannot distinguish different asympt. independent distributions

Alternative tail representation: the function c

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  - X and Y are Normal with correlation ρ: $c(x,y) = (xy)^{1/(1+\rho)}$

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- If $\widehat{c}_n \approx c$, hopefully $\widehat{\theta} \approx \theta_0$ and $\widehat{\sigma} \approx q(t_n)$

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- Spatial tail dependence model: probabilities $(*)$ for all collections $\mathcal{S}$

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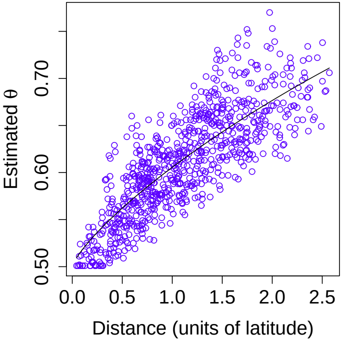
Rainfall over Victoria region (AUS), 1967-2017



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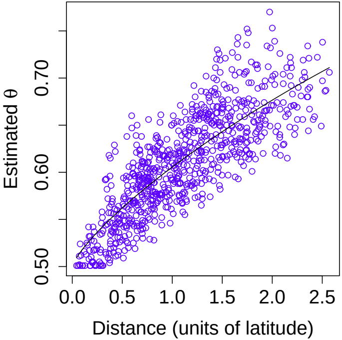

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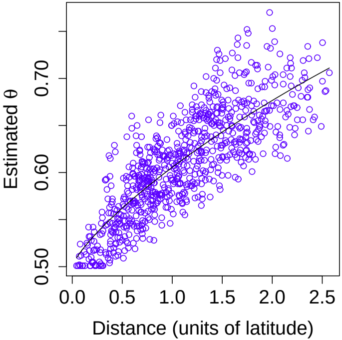

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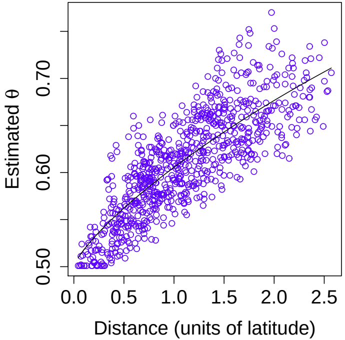

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- Blue dots $= \widehat{\theta}_{\text{pair}}$, black curve $=$ fitted spatial model

Thanks for your attention! Questions?

Lalancette, M., S. Engelke, and S. Volgushev (2021). Rank-based estimation under asymptotic dependence and independence, with applications to spatial extremes. *Ann. Stat.*, to appear.

Other contributions in the paper:

- Novel examples of (bivariate and spatial) tail models
- Asymptotic theory for all (bivariate and spatial) estimators
- Simulation studies

mic-lalancette.github.io