Asymptotics and Concentration of Empirical Variograms

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- $\boldsymbol{X}$ is in the *domain of attraction* (DA) of $\boldsymbol{Y}$

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- In particular, $\{$distribution of $\boldsymbol{Z}\} \Leftrightarrow \{$distribution of $\boldsymbol{Y}\} \Leftrightarrow R$, defined as

$$R(\boldsymbol{x}) := \lim_{q \to 0} q^{-1} \mathbb{P}\big(X_1 \geq (qx_1)^{-1}, \ldots, X_d \geq (qx_d)^{-1}\big), \quad \boldsymbol{x} \in [0, \infty]^d$$

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- So every max-stable dependence model has a unique "associated" MP distribution (if $\boldsymbol{Z}$ is Hüsler–Reiss, $\boldsymbol{Y}$ is *Hüsler–Reiss Pareto* (HRP))

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  1. G is a tree (but $\boldsymbol{Y}$ is "arbitrary" MP)
  2. $\boldsymbol{Y}$ is HRP (but G is "arbitrary" graph)

Extremal variograms

- In both cases, the graph structure is encoded into the *extremal variogram matrix* $\Gamma^{(m)}$ of $\boldsymbol{Y}$ rooted at variable $m \in V = \{1, \ldots, d\}$,

$$\Gamma_{ij}^{(m)} := \mathbb{V}\mathrm{ar}(\log Y_i - \log Y_j \mid Y_m > 1), \quad i, j \in V$$

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- When $\boldsymbol{Y}$ is HRP, all the $\Gamma^{(m)} = \Gamma$, the parameter matrix, i.e. $\Gamma^{(m)}$ fully characterize the HRP distributions

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- Motivates study of the *empirical variogram*

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- (Engelke & Volgushev, 2020) prove that $\widehat{\Gamma}_{ij}^{(m)} \xrightarrow{P} \Gamma_{ij}^{(m)}$

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- "Choice of k" assumption: There exist $0 < \alpha \leq \beta < 2\xi/(2\xi + 1)$ such that

$$n^{\alpha} \lesssim k \lesssim n^{\beta}$$

Density assumption

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- Not the weakest possible

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Conjecture (Engelke, L. & Volgushev, 2021+)

Under the "tail", "choice of k" and "density" assumptions,

$$\sqrt{k}\Big(\widehat{\Gamma}^{(m)} - \Gamma^{(m)}\Big)_{m \in V} \rightsquigarrow \big(W^{(m)}\big)_{m \in V}$$

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- Not informative in growing dimension (e.g. $d > n$)

Concentration

Theorem (Engelke, L. & Volgushev, 2021+)

Let $\delta \geq d^3 e^{-\sqrt{k}}$. *Under the "tail" and "choice of* k*" assumptions, with probability at least* $1-\delta$

$$\max_{i,j,m \in V} \left| \widehat{\Gamma}_{ij}^{(m)} - \Gamma_{ij}^{(m)} \right| \leq C (\log n)^2 \sqrt{\frac{\log d + \log(1/\delta)}{k}}.$$

Further, under the "density" assumption, with probability at least $1-\delta$

$$\max_{i,j,m \in V} \left| \widehat{\Gamma}_{ij}^{(m)} - \Gamma_{ij}^{(m)} \right| \leq \bar{C} \sqrt{\frac{\log d + \log(1/\delta)}{k}}.$$

Corollaries: Extremal graph learning guarantees

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- MTP_2 constrained graph estimation (Frank Röttger's talk): learning guarantees in high dimension?

Bonus: Discussion of the proofs

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- R_{ijm} can be seen as a measure, so e.g. $R_{ijm}([x, \infty), y, 1) := R_{jm}(y, 1) - R_{ijm}(x, y, 1)$

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- $\mathbb{P}(Y_i^{(m)} \geq x, Y_j^{(m)} \geq y) = R_{ijm}(1/x, 1/y, 1)$
- Using that for $X_1, X_2 \geq 0$

$$\mathbb{E}[X_1 X_2] = \int_0^\infty \int_0^\infty \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2) dx_1 dx_2$$

- Obtain expression for $\mathbb{E}[(\log Y_i^{(m)})(\log Y_j^{(m)})]$:

$$\int_0^1 \int_0^1 \frac{R_{ijm}(x, y, 1)}{xy} dxdy - \int_0^1 \int_1^\infty \frac{R_{ijm}([x, \infty), y, 1)}{xy} dxdy \\ - \int_1^\infty \int_0^1 \frac{R_{ijm}(x, [y, \infty), 1)}{xy} dxdy + \int_1^\infty \int_1^\infty \frac{R_{ijm}([x, \infty), [y, \infty), 1)}{xy} dxdy$$

- R_{ijm} can be seen as a measure, so e.g. $R_{ijm}([x, \infty), y, 1) := R_{jm}(y, 1) - R_{ijm}(x, y, 1)$

Bonus: Discussion of the proofs

- Similarly, $\widehat{\mathbb{E}}[(\log \widehat{Y}_i^{(m)})(\log \widehat{Y}_j^{(m)})]$ is equal to

$$\int_0^1 \int_0^1 \frac{\bar{R}_{ijm}(x, y, 1)}{xy} dxdy - \int_0^1 \int_1^\infty \frac{\bar{R}_{ijm}([x, \infty), y, 1)}{xy} dxdy$$
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- The *tail empirical copula*

$$\bar{R}_{ijm}(x, y, 1) := \frac{1}{k} \sum_{t=1}^n \mathbb{1}\left\{ \widehat{F}_i(X_{ti}) \geq 1 - \frac{k}{n} x, \widehat{F}_j(X_{tj}) \geq 1 - \frac{k}{n} y, \widehat{F}_m(X_{tm}) \geq 1 - \frac{k}{n} \right\}$$

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- Bad news: None of those are sufficient, since we consider unbounded sets and an unbounded weighting
- Our fix: choose $\eta > 0$ and index the tail empirical copula process by the functions

$$f_{ijm,x,y,z} := \begin{cases} (xy)^{-\eta} \mathbb{1}_{[0,x]\times[0,y]\times[0,z]}, & 0 < x, y, z \leq 1 \\ x^\eta y^{-\eta} \mathbb{1}_{[x,\infty)\times[0,y]\times[0,z]}, & 1 < x < \infty, 0 < y, z \leq 1 \\ (xy)^\eta \mathbb{1}_{[x,\infty)\times[y,\infty)\times[0,z]}, & 1 < x, y < \infty, 0 < z \leq 1 \end{cases}$$

Thank you for your attention! Questions?

References

Engelke, S. and A. Hitz (2020). Graphical models for extremes (with discussion). *J. R. Stat. Soc. Ser. B 82*, 871–932.

Engelke, S., A. Malinowski, Z. Kabluchko, and M. Schlather (2015). Estimation of hüsler–reiss distributions and brown–resnick processes. *J. R. Stat. Soc. B 77*, 239–265.

Engelke, S. and S. Volgushev (2020). Structure learning for extremal tree models. *arXiv preprint arXiv:2012.06179*.