Modeling and Estimation of Bivariate Tails

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1. Extreme Value Theory
2. Extremal Dependence
3. Our Work: Estimation of c
4. Bonus: Application to Spatial Extremes (Some Ideas)

Extreme Value Theory

- Very broadly, EVT is the field of statistics that studies how much we can extrapolate to understand the data-generating mechanism *outside of the range of the data*

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  - What is a 1-in-100-years rainfall (or even sometimes 1-in-10 000 years)?
  - What is the probability that on a given day at least $x$mm or rain happen (where x is possibly larger than anything observed we have oberved)?

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  2. Parametric estimator is based on assumption that tails of the parametric model are correct (uncheckable)

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Theorem (Balkema, de Haan (1974), Pickands (1975))

For a very large class of distributions, as $u \to \infty$,

$$S(y \mid u) \longrightarrow \left(1 + \frac{\gamma y}{\sigma}\right)^{-1/\gamma}, \quad y > 0,$$

for some $\sigma > 0$ and $\gamma \in \mathbb{R}$. That is, $X - u \mid X > u$ approximately has a $GP(\sigma, \gamma)$ distribution.

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- For $\gamma = 0$, $(1 + \gamma y/\sigma)^{-1/\gamma}$ understood as $e^{-y/\sigma}$

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- Parameters σ, γ are estimated by assuming that for every observation X_i above u, $X_i - u$ is approximately $\mathrm{GP}(\sigma, \gamma)$ distributed

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- Idea: model and estimate ℓ
- Then, for t arbitrarily small, use approximations

$$\mathbb{P}\left(F_1(X) \geq 1 - tx \quad \text{or} \quad F_2(Y) \geq 1 - ty\right) \approx t\ell(x, y)$$
$$\mathbb{P}\left(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty\right) \approx t(x + y - \ell(x, y))$$

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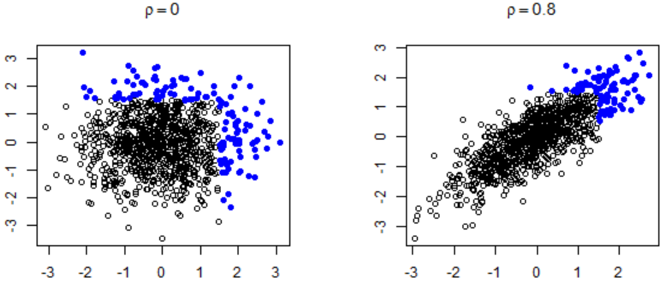

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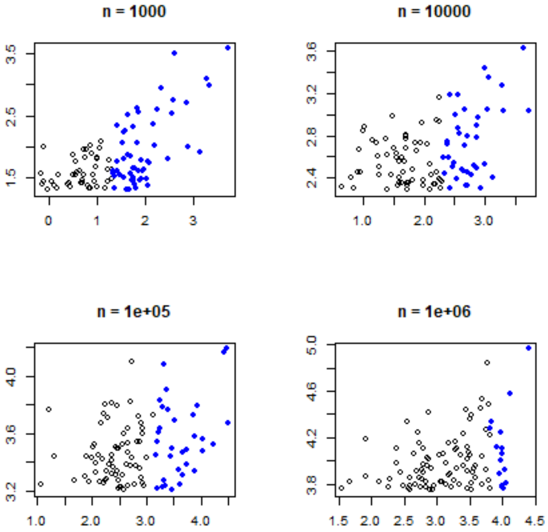
- That is, extremes do not occur simultaneously


n = 1000
n = 10000
n = 1e+05
n = 1e+06
1.5
2.5
3.5
0
1
2
3
2.4
2.8
3.2
3.6
1.0
2.0
3.0
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- Instead, what if we directly model the probability of joint threshold exceedance?

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- Under AI however, c contains much information on dependence structure

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  - $\alpha > 2 \Rightarrow$ AI, negative association between extremes (rare)

An Example that Connects AI and AD

- Let $R \sim \text{Pareto}\,(\lambda)$, $\lambda \in (0,2]$, $W_j \sim \text{Pareto}\,(1)$, $i = 1,2$, and R, W_1, W_2 are independent. Then $(X, Y) = R(W_1, W_2)$ satisfies our expansion

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  - $\lambda \geq 1 \Rightarrow$ AI
- Motivates inference for tail dependence that is based on c

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  where $\hat{F}_j$ are the empirical CDF's
- This is a rank-based estimator (can be rewritten as a function of ranks)
- It appears in [Draisma et al., 2004], but just as a tool in their proofs. Never used directly for inference before

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- Assume that as $t \to 0$,

$$\frac{1}{q(t)} \mathbb{P} \left(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty \right) = c(x, y) + O\left(\frac{1}{\log(1/t)} \right)$$

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Theorem (L, Engelke and Volgushev (2020))

There exist Gaussian processes $W^{(1)}$ and $W^{(2)}$ on $[0,\infty)^2$ such that

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Theorem (L, Engelke and Volgushev (2020))

There exist Gaussian processes $W^{(1)}$ and $W^{(2)}$ on $[0, \infty)^2$ such that

1. *Under AI, $\sqrt{m}\,(\hat{c}_n - c) \rightsquigarrow W^{(1)}$ (in $\ell^\infty\left([0, T]^2\right)$).*

The Most Difficult Theorem I Ever Proved

- Assume that as $t \to 0$,

$$\frac{1}{q(t)} \mathbb{P}\left(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty\right) = c(x, y) + O\left(\frac{1}{\log(1/t)}\right)$$

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1. *Under AI, $\sqrt{m}\left(\hat{c}_n - c\right) \rightsquigarrow W^{(1)}$ (in $\ell^\infty\left([0, T]^2\right)$).*
2. *Under AD, $\sqrt{m}\left(\hat{c}_n - c\right) \rightsquigarrow W^{(2)}$ (in the topo. of hypi-convergence for locally bounded functions ([Bücher et al., 2014])).*

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- "Something" is what one would obtain with known marginal distributions F_1, F_2. It is a fairly standard empirical process
- The other term comes from the error in estimating the marginals
- Under AD, it converges to a non trivial limit
- Under AI, it disappears because convergence of $\hat{x}_n$ and $\hat{y}_n$ is faster than convergence of "Something" to $W^{(1)}$ (based on more data)

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The M-Estimator We Deserve

- We obtain the following objective function:

$$\Psi_n(\theta, \sigma) := \left\| \sigma \int_{[0,T]^2} g(x,y) c_\theta(x,y) \, dx \, dy - m \int_{[0,T]^2} g(x,y) \hat{c}_n(x,y) \, dx \, dy \right\|$$

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- By minimizing this objective function, we hope that c_θ will estimate c and σ will estimate m

A Much Easier Theorem

- Suppose that the true function generating the data is c_{θ_0}, $\theta_0 \in \Theta$, and that the map

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Theorem (L, Engelke and Volgushev (2020))

If $(\hat{\theta}_n, \hat{\sigma}_n) = argmin_{\theta,\sigma} \Psi_n(\theta, \sigma)$,

$$\sqrt{m}\left(\left(\hat{\theta}_n, \frac{\hat{\sigma}_n}{m}\right) - (\theta_0, 1)\right) \rightsquigarrow N\left(0, \Sigma(\theta_0)\right).$$

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Thank you! Questions?

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  $d \in \mathbb{N}$, $u_j \in \mathcal{T}$, $x \in [0,\infty)^d$
- Same problem as before: If for two locations u_1, u_2, $Y(u_1)$ and $Y(u_2)$ are AI, then $\ell^{(u_1,u_2)}$ is trivial

A Partial Solution

- Can characterize the extremal dependence of Y by functions

$$c^{(u_1,u_2)}(x,y) := \lim_{t \to 0} \frac{1}{q^{(u_1,u_2)}(t)} \mathbb{P}\left(F^{(u_j)}(Y(u_j)) > 1 - tx_j, \quad j = 1, 2\right),$$

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- Advantage: contains more information on the pairwise dependencies under AI
- Disadvantage: only contains information on pairs. Luckily, currently used tail models are completely characterize by pairwise structure

Estimation of Functions $c^{(u_1,u_2)}$

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- Combine all nonparametric estimators $\hat{c}_n^{(u_i,u_j)}$ to estimate θ by minimizing some global distance, e.g.

$$\hat{\theta} = \arg\min_\theta \sum_{1 \leq i,j \leq d} \left\| f\left(\hat{c}_n^{(u_i,u_j)}\right) - f\left(c_\theta^{(u_i,u_j)}\right)\right\|^2,$$

for some vector-valued functional

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