MATH 271: Chapter 7 homework

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12. There exists a positive real number x for which $x^2 < \sqrt{x}$.

Proof. Consider the case when $x = \frac{1}{4}$. Then $x^2 = (\frac{1}{4})^2 = \frac{1}{16}$ and $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2}$. Since $\frac{1}{16} < \frac{1}{2}$. The proposition holds. $\square$

16. Suppose $a, b \in \mathbb{Z}$. If ab is odd, then $a^2 + b^2$ is even.

Proof. assume that ab is odd. Then a and b are both odd. So, $\exists n, m \in \mathbb{Z}$ such that $a = 2m + 1$ and $b = 2n + 1$.

$$\begin{aligned} a^2 + b^2 &= (2m+1)^2 + (2n+1)^2 && \text{(by substitution)} \\ &= 4m^2 + 4m + 1 + 4n^2 + 4n + 1 \\ &= 2(2m^2 + 2n^2 + 2m + 2n + 1). \end{aligned}$$

Since, $2(2m^2 + 2n^2 + 2m + 2n + 1)$ is even by definition, The proposition holds. $\square$

18. There is a set X for which $\mathbb{Z} \in X$ and $\mathbb{Z} \subseteq X$.

Proof. Consider the set $X = \mathbb{Z} \cup \{\mathbb{Z}\}$. Therefore, $X = \{x | x \in \mathbb{Z} \text{ or } x \in \{\mathbb{Z}\}\}$ (by definition of union). So $\mathbb{Z} \in X$ because $\mathbb{Z} \in \{Z\}$. And $\mathbb{Z}$ is a subset of X because $\mathbb{Z}$ is a subset of itself. Since $\mathbb{Z}$ is a subset and an element of X, the proposition holds $\square$

20. There exists an $n \in N$ for which $11 | (2n - 1)$.

Proof. Consider the case where $n = 10$. Then $2^n - 1 = 1023 = 11(93)$. So, 11 divides $2^{10} - 1$. Therefore the proposition holds. (Note: I found this solution with a Python script but I think an argument can be made that $(n + 1)|(2^n - 1)$). $\square$

22. if $n \in \mathbb{Z}$ then $4|n^2$ or $4|(n^2 - 1)$

Proof. Notice there are 2 cases for n: odd and even).

Case 1: n is odd. So, $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Consider,

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4(k^2 + k). \end{aligned}$$

So 4 divides $n^2 - 1$ when n is odd.

Case 2: n is even. So, $\exists k \in \mathbb{Z}$ such that $n = 2k$. Consider,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4(k^2). \end{aligned}$$

So 4 divides n^2 when n is even. Since 4 divides n^2 or $n^2 - 1$ the proposition holds.

□

30. Suppose $a, b, p \in \mathbb{Z}$ and p is prime. Prove that if $p|ab$ then $p|a$ or $p|b$.

Proof. Assume for the sake of contradiction that $p|ab$ and $p \nmid a$ and $p \nmid b$. So, $\exists k$ such that $ab = pk$. And the $\gcd(p, b) = 1$, because $p \nmid b$ and p only has factors 1 and p. so by proposition 7.1 (page 152) $\exists m, n$ such that $1 = bm + pn$.

$$\begin{aligned} 1 &= bm + pn \\ a &= abm + pna && \text{(multiplying both side by } a) \\ &= pkm + pna && \text{(by substituting ab for pk)} \\ &= p(km + na) \end{aligned}$$

Therefore $p|a$ which is a contradiction therefore the original statement holds. □

31. if $\gcd(a, c) = \gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.

Proof. Assume $\gcd(a, c) = \gcd(b, c) = 1$, So, $\exists w, x, y, z \in \mathbb{Z}$ such that $1 = aw + cx$ and $1 = by + cz$ (by proposition 7.1 page 152). let d denote the $\gcd(ab, c)$, therefore $d|c$ and $d|ab$. So, $\exists m, n$ such that $c = dm$ and $ab = dn$. Consider,

$$\begin{aligned} 1 &= aw + cx \\ b &= abw + cby && \text{(multiplying both sides by } b.) \\ &= dnw + dmby && \text{(substituting } c \text{ for } dm \text{ and } ab \text{ for } dn.) \\ &= d(nw + mby) && \text{(let } k \text{ denote } nw + mby.) \end{aligned}$$

Therefore $b = dk$. Now consider,

$$\begin{aligned} 1 &= by + cz \\ &= dky + dmz && \text{(substituting } c \text{ for } dm\text{, and } b \text{ for } dk) \\ &= d(ky + mz) \\ &= dj && \text{(letting } j \text{ denote } ky + mz) \end{aligned}$$

This implies that $d|1$, therefore, d=1 (because 1 only has divisors -1, and 1; and 1 ¿-1). Therefore the $\gcd(ab, c) = 1$, so the proposition holds. □