MATH 271: chapter 6 homework

Micah Sherry

November 4, 2024

2. Suppose $n \in \mathbb{Z}$, If n^2 is odd then n is odd

Suppose for the sake of contradiction that n^2 is odd and n is even. Since n is even $\exists k \in \mathbb{Z}$ such that $n = 2k$. Therefore,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= (4k^2) \\ &= 2(2k^2). \end{aligned}$$

This implies n^2 is even but, we assumed n^2 is not even. Thus we have a contradiction, therefore the original statement holds. $\square$

4. Prove that $\sqrt{6}$ is irrational

Suppose for the sake of contradiction that $\sqrt{6}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that a and b have no common factors and $\sqrt{6} = \frac{a}{b}$. So,

$$\begin{aligned} 6 &= \frac{a^2}{b^2} \\ 6b^2 &= a^2. \end{aligned}$$

Which implies 6 divides a^2.

Now we will show that if $a \in \mathbb{Z}$ and $6 \mid a^2$, then $6 \mid a$. Consider the contra-positive of this statement: If $6 \nmid a$, Then $6 \nmid a^2$ Since 6 does not divide then $\exists k \in \mathbb{Z}$and $r \in \{1, 2, 3, 4, 5\}$ such that $a = 6k + r$. So,

$$\begin{aligned} a^2 &= (6k + r)^2 \\ &= 36k^2 + 12kr + r^2 \\ &= 6(6k^2 + 2kr) + r^2 \end{aligned}$$

let $n = 6k^2 + 2kr$, So $a^2 = 6n + r^2$ Now we need to check all Possible cases for r.

r	r^2	a^2
1	1	$6n + 1$
2	4	$6n + 4$
3	9	$6(n + 1) + 3$
4	16	$6(n + 2) + 4$
5	25	$6(n + 4) + 1$

Since in all cases 6 does not divide a^2. Therefore the Contra-positive holds and thus 6 divides a.
using the result of the proof above $a = 6k$ for some $k \in \mathbb{Z}$

$$\begin{aligned} 6b^2 &= a^2. \\ &= 6(6k^2)^2. \\ b^2 &= 6k^2. \end{aligned}$$

Therefore 6 divides b^2 and therefore divides b (by the above result). So, a and b share 6 as a factor which is a contradiction. Thus the original statement holds. $\square$

6. If $a, b \in \mathbb{Z}$ then $a^2 - 4b - 2 \neq 0$

Assume for the sake of contradiction that a and b are integers and $a^2 - 4b - 2 = 0$. Therefore, $a^2 = 4b + 2$. Now consider the cases where a is odd and even

Case 1: a is even. If a is even then $a = 2k$, for some k in the integers. So, $a^2 = 4(k^2)$, which is a contradiction be cause we established a^2 is of the form $4b + 2$.

Case 2: a is odd. If a is odd then $a = 2k + 1$, for some k in the integers. So, $a^2 = 4(k^2 + k) + 1$, which is a contradiction be cause we established a^2 is of the form $4b + 2$.

(Note im fairly confident that a singular case would have been sufficient to prove a contradiction. But I included the both even and odd cases to be sure) Thus the original statement holds. $\square$

8. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$ then a or b is even

Assume for the sake of contradiction that $a^2 + b^2 = c^2$, and a and b are odd. Then $\exists k, n \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2n + 1$. So,

$$\begin{aligned}
c^2 &= a^2 + b^2 \\
&= (2k+1)^2 + (2k+1)^2 \\
&= 4k^2 + 4k + 1 + 4n^2 + 4n + 1 \\
&= 4(k^2 + n^2 + k + n) + 2.
\end{aligned}$$

let $m = k^2 + n^2 + k + n$ Therefore c^2 = 4m+2.

Case 1: c is even. Then $c = 2j$ for some $j \in \mathbb{Z}$. Then $c^2 = 4j^2$. This is a contradiction because we have established that c^2 is of the form $4m + 2$

Case 2: c is odd. Then $c = 2j + 1$ for some $j \in \mathbb{Z}$. Then $c^2 = 4j^2 + 4j + 1 = 4(j^2 + j) + 1$. This is a contradiction because we have established that c^2 is of the form $4m + 2$.

Since both cases for c leads to a contradiction the original statement holds. $\square$

10. There exist no integers a and b such that $21a + 30b = 1$

Suppose for the sake of contradiction that $\exists a, b \in \mathbb{Z}$ such that $21a + 30b = 1$. Lets consider

$$\begin{aligned}
1 &= 21a + 30b \\
&= 3(7a + 10b).
\end{aligned}$$

This implies that 3 divides 1 (because $7a + 10b$ is an integer). Which is a contradiction. Thus the original statement holds

14. If A and B are sets then $A \cap (B - A) = \varnothing$

Assume for the sake of contradiction that A and B are sets and $A \cap (B - A) \neq \varnothing$. Then $\exists x$ such that $x \in A \cap (B - A)$

$$\begin{aligned}
A \cap (B - A) &= \{x \mid x \in A \text{ and } x \in (B - A)\} && \text{(by definition of Union)} \\
B - A &= \{x \mid x \in B \text{ and } x \notin A\} && \text{(by definition of Set Subtraction)} \\
A \cap (B - A) &= \{x \mid x \in A \text{ and } x \in B \text{ and } x \notin A\}
\end{aligned}$$

Therefore x must be an element of A and not an element of A which Is a contradiction and thus the original statement is true. $\square$

16. If a and b are positive real numbers then $a + b \geq 2\sqrt{ab}$

Assume for the sake of contradiction that a and b are positive real numbers and $a + b < 2\sqrt{ab}$. Then,

$$\begin{aligned} a - 2\sqrt{ab} + b &< 0 \\ \sqrt{a}^2 - 2\sqrt{ab} + \sqrt{b}^2 &< 0 \\ (\sqrt{a} - \sqrt{b})^2 &< 0. && \text{(by Factoring)} \end{aligned}$$

This implies that a real number squared is negative which is false. Therefore the original statement holds. $\square$