MATH 271: chapter 4 homework

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2. if x is an odd integer then x^3 is odd

Assuming x be an odd integer, then $\exists k$ such that $x = 2k + 1$.

$$\begin{aligned} x^3 &= (2k+1)^3 \\ &= (4k^2 + 4k + 1)(2k + 1) \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

Then letting n denote $4k^3 + 6k^2 + 3k$. Since n is an integer (by closure properties) and x^3 can be written as 2n+1 which odd by definition. So the proposition holds.

QED

4. suppose $x, y \in \mathbb{Z}$. if x and y are odd, then xy is odd

Assuming x and y are odd. Then $\exists j, k \in \mathbb{Z}$, such that $x = 2j + 1$ and $y = 2k + 1$.

So,

$$\begin{aligned} xy &= (2j+1)(2k+1) \\ &= 4jk + 2j + 2k + 1 \\ &= 2(2jk + j + k) + 1 \end{aligned}$$

Then letting n denote $(2jk+j+k)$. Since n is an integer (by closure properties) and xy can be written as 2n+1 which odd by definition. So the proposition holds.

QED

6. suppose $a, b, c \in \mathbb{Z}$ if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$

Assuming a divides b and a divides c then $\exists k, j \in \mathbb{Z}$ such that $b = aj$ and $c = ak$ So,

$$\begin{aligned} b + c &= aj + ak \\ &= a(j + k) \end{aligned}$$

Then letting n denote $(j + k)$. Since n is an integer (by closure properties) and $(b + c) = an$. Thus a divides $(b + c)$. So the original proposition holds

QED

8. Suppose a is an integer. If $5 \mid 2a$, then $5 \mid a$

Assuming 5 divide $2a$, and since $2a$ is even (by definition), then $\exists k \in \mathbb{Z}$ such that $2a = 5(2k)$. so,

$$\begin{aligned} 2a &= 2(5k) \\ \frac{2a}{2} &= \frac{2(5k)}{2} \\ a &= 5k \end{aligned}$$

Thus 5 divides a by definition. So the proposition holds QED

10. Suppose a and b are integers. If $a \mid b$, then $a \mid (3b^3 - b^2 + 5b)$

Assuming a divides b then $\exists k \in \mathbb{Z}$ such that $b = ak$ so,

$$\begin{aligned} 3b^3 - b^2 + 5b &= 3(ak)^3 - (ak)^2 + 5(ak) \\ &= 3a^3k^3 - a^2k^2 + 5ak \\ &= a(3a^2k^3 - ak^2 + 5k) \end{aligned}$$

Letting n denote $(3a^2k^3 - ak^2 + 5k)$. Since n is an integer (by closure properties) and $3b^3 - b^2 + 5b = an$; a divides $3b^3 - b^2 + 5b$. thus the proposition holds

12. If $x \in \mathbb{R}$ and $0 < x < 4$ then $\frac{4}{x(4-x)} \geq 1$

Assuming $x \in (0, 4)$.

$$\begin{aligned} \frac{4}{x(4-x)} &\geq 1 \\ (x(4-x))\frac{4}{x(4-x)} &\geq x(4-x)(\text{because } x(4-x) > 0) \\ 4 &\geq 4x - x^2 \\ x^2 - 4x + 4 &\geq 0 \\ (x-2)^2 &\geq 0 \end{aligned}$$

Since any real number squared is positive and $(x - 2)$ is a real number the inequality holds. And thus the original inequality holds QED

18. Suppose x and y are positive real numbers, if $x < y$ then $x^2 < y^2$

Assuming $x < y$ is true, then $x - y < 0$ is true as well. To show $x^2 < y^2$ we can also show that the difference is positive (i.e. $y^2 - x^2 > 0$). note: $y^2 - x^2 = (y - x)(y + x)$
so,

$$\begin{aligned} (y+x)(y-x) &> 0 \\ \frac{(y+x)(y-x)}{(y+x)} &> \frac{0}{(y+x)} \text{ (because } y+x > 0) \\ (y-x) &> 0 \end{aligned}$$

since we know that $y - x > 0$ the original proposition holds QED

20. If a is an integer and $a^2 \mid a$, then $a \in \{-1, 0, 1\}$

Assuming a^2 divides a then, $\exists k \in \mathbb{Z}$ such that $a = ka^2$.

case 1: $a = 0$

0 is divides all integers because $0 = k0$

case 2: $a \neq 0$

$$\begin{aligned} a &= ka^2 \\ \frac{a}{a^2} &= \frac{ka^2}{a^2} \\ \frac{1}{a} &= k \end{aligned}$$

since $\frac{1}{a}$ is only an integer when $a = 1$ or $a = -1$ this implies a^2 does not divide unless $a = -1$ or $a = 1$. this with the result of the first case proves that a must be an element of $\{-1, 0, 1\}$. So the proposition holds. QED

26. Every odd integer is the difference of 2 squares

let S denote the set of all odd integers $S = \{2k + 1 \mid k \in \mathbb{Z}\}$. To show that all $x \in S$ can be expressed as $x = a^2 - b^2$ where $a, b \in \mathbb{Z}$. We can ignore the cases where a and b have the same parity because $a^2 - b^2$ will be even. That leaves us with 2 case that we need to consider a is odd, b is even and a is even and b is odd

case 1: $a = 2k + 1$ and $b = 2n$ where $k, n \in \mathbb{Z}$

$$\begin{aligned} a^2 - b^2 &= (2k + 1)^2 - (2n)^2 \\ &= 4k^2 + 4k + 1 - 4n^2 \\ &= 4(k^2 - n^2) + 4k + 1 \end{aligned}$$

choosing n=k allows us to reduce the equality down to:

$$a^2 - b^2 = 4k + 1$$

let $S_1 = \{4k + 1 \mid k \in \mathbb{Z}\}$

case 2: $a = 2m$ and $b = 2j + 1$ where $j, m \in \mathbb{Z}$

$$\begin{aligned} a^2 - b^2 &= (2m)^2 - (2j + 1)^2 \\ &= 4m^2 - (4j^2 + 4j + 1) \\ &= 4(m^2 - j^2) - 4j - 1 \end{aligned}$$

choosing m=j allows us to reduce the equality down to:

$$a^2 - b^2 = -4j - 1$$

let $S_2 = \{-4j - 1 \mid j \in \mathbb{Z}\}$

putting it together

Since S_1 is the set of all integers one more than a multiple of 4. And S_2 is the set of all integers one less than a multiple of 4. So, $S = S_1 \cup S_2$. So the original proposition holds. QED