MATH 271: Chapter 10 homework

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4. If $n \in \mathbb{N}$ Then $1*2+2*3+3*4+4*5+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

Proof. We will prove this with Induction.
Base Case: Consider the base case $n=1$,

$$\sum_{i=1}^{1}(i)(i+1)=1*2=2$$

and

$$\frac{n(n+1)(n+2)}{3}=\frac{1(2)(3)}{3}=2$$

Therefore the base case holds.

Induction Hypothesis: Now Assume that $1*2+2*3+3*4+4*5+\ldots+k(k+1)=\frac{k(k+1)(k+2)}{3}$ is true for some $k \in \mathbb{N}$.

Inductive step: Now consider the case when $n=k+1$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(i)(i+1) &= \sum_{i=1}^{k}(i)(i+1)+(k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2) && \text{(By the Induction Hypothesis)} \\
&= (\frac{k}{3}+1)(k+1)(k+2) && \text{(By factoring)} \\
&= \frac{(k+1)(k+2)(k+3)}{3} && \text{(By factoring and rearranging)}
\end{aligned}$$

Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction , the statement is true $\forall n \in \mathbb{N}$

□

8. If $n \in \mathbb{N}$ then $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$

Proof. We will prove this with Induction.
Base Case: Consider the base case $n=1$,

$$\sum_{i=1}^{1}\frac{i}{(i+1)!}=\frac{1}{2!}=\frac{1}{2}$$

and

$$1-\frac{1}{(n+1)!}=1-\frac{1}{2!}=\frac{1}{2}$$

So the base case holds.

Induction Hypothesis: Now assume that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ is true for some $k \in \mathbb{N}$.

Inductive step: Now consider the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \sum_{i=1}^{k} \frac{i}{(i+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} && \text{(by the induction hypothesis)} \\
&= 1 + \frac{-k-2+k+1}{(k+2)!} && \text{(by factoring)} \\
&= 1 - \frac{1}{(k+2)!}
\end{aligned}$$

Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction ,the statement is true $\forall n \in \mathbb{N}$ ☐

10. prove that $3|5^{2n} - 1$ for ever integer $n \geq 0$

Proof. **Base Case:** Consider the base case $n = 0$. Then $5^{2n} - 1 = 0$. Since 0 is divisible by all integers the base case holds.

Induction Hypothesis: Assume there is a k such that $3|5^{2k} - 1$. (Note $5^{2k} = 3m + 1$ for some $m \in \mathbb{Z}$)
Inductive step: Now consider the case when $n = k + 1$. So,

$$\begin{aligned}
5^{2k+2} - 1 &= 5^{2k}5^2 - 1 \\
&= (3m+1)5^2 - 1 && \text{(by the Induction Hypothesis)} \\
&= 3(25m) + 24 && = 3(25m + 8)
\end{aligned}$$

which implies $3|5^{2k+2} - 1$. Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction, the statement is true $\forall n, n \geq 0$ ☐

18. Suppose $A_1, A_2, \ldots A_n$ are sets in a universal set U and $n \geq 2$. Prove that

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$$

Proof. **Base Case:** Consider the base case $n = 2$.

$$\begin{aligned}
\overline{A_1 \cup A_2} &= \{x \in U | \sim (x \in A_1 \cup A_2)\} && \text{(by definition of complement)} \\
&= \{x \in U | \sim (x \in A_1 \vee x \in A_2)\} && \text{(by definition of union)} \\
&= \{x \in U | \sim (x \in A_1) \wedge \sim (x \in A_2)\} && \text{(by DeMorgans law)} \\
&= \{x \in U | x \in \overline{A_1} \wedge x \in \overline{A_2}\} && \text{(by definition of complement)} \\
&= \{x \in U | x \in \overline{A_1} \cap \overline{A_2}\} && \text{(by definition of intersection)} = \overline{A_1} \cap \overline{A_2}
\end{aligned}$$

Therefore the base case holds.

Induction Hypothesis: $\overline{A_1 \cup A_2 \cup \ldots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_k}$ is true for some $k \geq 2$

Inductive step: Now consider the case when $n = k + 1$. Let B denote $A_1 \cup A_2 \cup \ldots \cup A_k$. by the inductive hypothesis $\overline{B} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_k}$.

$$\begin{aligned}
\overline{B \cup A_{k+1}} &= \{x \in U | \sim (x \in B \cup A_{k+1})\} && \text{(by definition of complement)} \\
&= \{x \in U | \sim (x \in B \vee x \in A_{k+1})\} && \text{(by definition of union)} \\
&= \{x \in U | \sim (x \in B) \wedge \sim (x \in A_{k+1})\} && \text{(by DeMorgans law)} \\
&= \{x \in U | x \in \overline{B} \wedge x \in \overline{A_{k+1}}\} && \text{(by definition of complement)} \\
&= \{x \in U | x \in \overline{B} \cap \overline{A_{k+1}}\} && \text{(by definition of intersection)} \\
&= \overline{B} \cap \overline{A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_{k+1}}
\end{aligned}$$

Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction, the statement is true $\forall n, n \geq 2$ $\square$

26. Concerning the Fibonacci sequence, prove that

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

Proof. **Base Case:** Consider the base case $n = 1$.

$$\sum_{i=1}^{1} F_i^2 = F_1^2 = 1 \text{ and } F_1 F_2 = 1 * 1 = 1$$

therefore the base case holds

Induction Hypothesis: Assume

$$\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$$

is True for some k.

Inductive step: Now consider the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 && \text{(by the induction hypothesis)} \\
&= F_{k+1}(F_k + F_{k+1}) && \text{(by factoring)} \\
&= F_{k+1}(F_{k+2}) && \text{(by definition of the Fibonacci sequence)}
\end{aligned}$$

Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction, the statement is true $\forall n \in \mathbb{N}$ $\square$

34. prove that $3^1 + 3^2 + 3^3 + 3^4 \ldots + 3^n = \frac{3^{n+1}-3}{2}$ for every $n \in \mathbb{N}$.

Proof. **Base Case:** Consider the base case $n = 1$. $3^1 = 3$ and $\frac{3^2-3}{2}$ So the base Case holds.

Induction Hypothesis: Assume $3^1 + 3^2 + 3^3 + 3^4 \ldots + 3^k = \frac{3^{k+1}-3}{2}$ for some $k \in \mathbb{N}$

Inductive step: Now consider the case when $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} 3^i &= \sum_{i=1}^{k} 3^i + 3^{k+1} \\
&= \frac{3^{k+1} - 3}{2} + 3^{k+1} && \text{(by the induction hypothesis)} \\
&= \frac{3^{k+1} - 3 + 2(3^{k+1})}{2} \\
&= \frac{3^{k+1}(1 + 2) - 3}{2} \\
&= \frac{3^{k+2} - 3}{2}
\end{aligned}$$

Since the base case holds and $S_k \implies S_{k+1}$ by the principle of mathematical induction, the statement is true $\forall n \in \mathbb{N}$ $\square$