MATH 271: Chapter 7 homework

Micah Sherry

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2. Prove that $\{6n|n \in \mathbb{Z}\} = \{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\}$

Proof. ($\subseteq$) first we will show that $\{6n|n \in \mathbb{Z}\} \subseteq \{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\}$.
Assume that $x \in \{6n|n \in \mathbb{Z}\}$. Therefore $x = 6n$ for some $n \in \mathbb{Z}$. So, $x = 2(3n)$ and because $3n \in \mathbb{Z}$, $x \in \{2n|n \in \mathbb{Z}\}$. Similarly $x = 3(2n)$ and since $2n \in \mathbb{Z}$, $x \in \{3n|n \in \mathbb{Z}\}$. Therefore $\{6n|n \in \mathbb{Z}\} \subseteq \{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\}$.

($\supseteq$) Next we will show that $\{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\} \subseteq \{6n|n \in \mathbb{Z}\}$.
Assume that $x \in \{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\}$. This implies that $2|x$ and $3|x$. Which implies that $x = 6n$ for some $n \in \mathbb{Z}$ (because 2 and 3 share no factors other than 1). Thus, $\{2n|n \in \mathbb{Z}\} \cap \{3n|n \in \mathbb{Z}\} \subseteq \{6n|n \in \mathbb{Z}\}$. Therefore the original equality holds. □

6. Suppose A,B and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$

Proof. Assume that $A \subseteq B$ and let $x \in A - C$. Therefore $x \in A$ and $x \notin \mathbb{C}$. Since $A \subseteq B$ if $x \in A$ then $x \in B$

□

8. If A,B and C are sets, then $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$.

Proof. Assuming A,B, and C are sets.

$$\begin{aligned}
A \cup (B \cap C) &= \{x|x \in A \vee x \in (B \cap C)\} && \text{(definition of union)}\\
&= \{x|x \in A \vee x \in (x \in B \wedge x \in C)\} && \text{(definition of intersection)}\\
&= \{x|x \in (A \vee x \in x \in B) \wedge (x \in A \vee x \in x \in C)\} && \text{(distributive property)}\\
&= \{x|x \in (A \cup B) \wedge x \in (A \cup C)\} && \text{(definition of union)}\\
&= \{x|x \in (A \cup B) \cap (A \cup C)\} && \text{(definition of union)}\\
&= (A \cap B) \cup (A \cap C)
\end{aligned}$$

Therefore the equality holds . □

10. If A and B are sets in a universal set U, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof. Assuming A and B are subsets of U. Then

$$\begin{aligned}
\overline{A \cap B} &= \{x \in U|x \notin A \cap B\} && \text{(by definition of complement)}\\
&= \{x \in U| \sim (x \in A \cap B)\}\\
&= \{x \in U| \sim (x \in A \wedge x \in B)\} && \text{(by definition of intersection)}\\
&= \{x \in U| \sim (x \in A) \vee \sim (x \in B)\} && \text{(by De Morgan's law } \sim (p \wedge q) = \sim p \vee \sim q\text{)}\\
&= \{x \in U|x \notin A \vee x \notin B\}\\
&= \{x \in U|x \in \overline{A} \vee x \in \overline{B}\} && \text{(by definition of complement)}\\
&= \{x \in U|x \in \overline{A} \cup \overline{B}\} && \text{(by definition of union)}\\
&= \overline{A} \cup \overline{B}.
\end{aligned}$$

Therefore the implication holds. $\square$

16. If A,B and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. Assuming A,B and C are sets. Then,

$$\begin{aligned} A \times (B \cup C) &= \{(x,y) | x \in A \wedge y \in (B \cup C)\} && \text{(definition of Cartesian product)} \\ &= \{(x,y) | x \in A \wedge (y \in B \vee y \in C)\} && \text{(definition of union)} \\ &= \{(x,y) | (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)\} && \text{(the distributive property)} \\ &= \{(x,y) | (x,y) \in A \times B \vee (x,y) \in A \times C\} && \text{(definition of Cartesian product)} \\ &= \{(x,y) | (x,y) \in (A \times B) \cup (A \times C)\} && \text{(definition of union product)} \\ &= (A \times B) \cup (A \times C). \end{aligned}$$

Therefore the the implication holds. $\square$

20. Prove that $\{9^n | n \in \mathbb{Q}\} = \{3^n | n \in \mathbb{Q}\}$.

Proof. ($\subseteq$) First we will show that $\{9^n | n \in \mathbb{Q}\} \subseteq \{3^n | n \in \mathbb{Q}\}$.
Assume that $x \in \{9^n | n \in \mathbb{Q}\}$, then $x = 9^n = 3^{2n}$ for some $n \in \mathbb{Q}$. Since $x = 3^{2n}$ and $2n \in \mathbb{Q}$, $x \in 9\{3^n | n \in \mathbb{Q}\}$. Therefore $\{9^n | n \in \mathbb{Q}\} \subseteq \{3^n | n \in \mathbb{Q}\}$.

($\supseteq$) Next, we will show that $\{3^n | n \in \mathbb{Q}\} \subseteq \{9^n | n \in \mathbb{Q}\}$.
Assume that $x \in \{3^n | n \in \mathbb{Q}\}$, then $x = 3^n = 9^{\frac{n}{2}}$ for some $n \in \mathbb{Q}$. Since $x = 9^{\frac{n}{2}}$ and $\frac{n}{2} \in \mathbb{Q}$, $x \in \{9^n | n \in \mathbb{Q}\}$. Therefore $\{3^n | n \in \mathbb{Q}\} \subseteq \{9^n | n \in \mathbb{Q}\}$. Therefore since both subset relations hold so does the original equality. $\square$

26. Prove that $\{4k + 5 | k \in \mathbb{Z}\} = \{4k + 1 | k \in \mathbb{Z}\}$

Proof. ($\subseteq$) First, we will show $\{4k + 5 | k \in \mathbb{Z}\} \subseteq \{4k + 1 | k \in \mathbb{Z}\}$.
Assume that $x \in \{4k + 5 | k \in \mathbb{Z}\}$ then $x = 4k + 5 = 4(k + 1) + 1$ for some $k \in \mathbb{Z}$. Since $x = 4(k + 1) + 1$ and $k + 1 \in \mathbb{Z}$. So, $x \in \{4k + 1 | k \in \mathbb{Z}\}$. Therefore, $\{4k + 5 | k \in \mathbb{Z}\} \subseteq \{4k + 1 | k \in \mathbb{Z}\}$.

($\supseteq$) Next, we will show $\{4k + 1 | k \in \mathbb{Z}\} \subseteq \{4k + 5 | k \in \mathbb{Z}\}$.
Assume that $x \in \{4k + 1 | k \in \mathbb{Z}\}$ then $x = 4k + 1 = 4(k - 1) + 5$ for some $k \in \mathbb{Z}$. Since $x = 4(k - 1) + 5$ and $k - 1 \in \mathbb{Z}$. So, $x \in \{4k + 1 | k \in \mathbb{Z}\}$. Therefore, $\{4k + 5 | k \in \mathbb{Z}\} \subseteq \{4k + 1 | k \in \mathbb{Z}\}$. Therefore since both subset relations hold so does the original equality. $\square$