MATH 271: chapter 5 homework

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2. Suppose $n \in \mathbb{Z}$. If n^2 is odd then n is odd.

Consider the contra-positive: If n is even (not odd) then n^2 is even (not odd). Assuming n is even then $\exists k$ such that $n = 2k$. Then,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

Then letting $m = 2k^2$, n^2 can be expressed as $n^2 = 2(m)$, since m is an integer (by closure) n^2 is even. Thus the contra-positive holds and so does the original statement. □

4. Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

Consider the contra-positive: If a divides b then a divides bc. Assuming a divides b then $\exists k \in \mathbb{Z}$ such that $b = ak$. Then,

$$\begin{aligned} bc &= (ak)c \\ bc &= a(kc) \end{aligned}$$

Letting $n = kc$, bc can be expressed as $bc = a(n)$ since n is an integer (by closure), a divides bc. Thus the contra-positive holds and so does the original statement. □

6. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

Consider the contra-positive: if $x \leq -1$, then $x^3 - x \leq 0$. Now consider the product:

$$x(x^2 - 1)$$

We know that x is a negative number because $x \leq -1$. So for the product to also be negative (or zero) $x^2 - 1$ must be greater than (or equal to) zero. which implies that $x^2 \geq 1$. Since $x < -1$,

$$\begin{aligned} x &\leq -1 \\ x^2 &\geq 1 (\text{by squaring both sides}) \\ x^2 - 1 &\geq 0 \end{aligned}$$

Note: The inequality flips because we multiplied both sides by a negative number.
Now we have established that $x < -1$ and $x^2 - 1 > 0$. Since a negative times a positive (or zero) is negative (or zero),

$$\begin{aligned} x(x^2 - 1) &\leq 0 \\ x^3 - x &\leq 0 \end{aligned}$$

So the contra-positive holds, and so does the original statement. □

8. Suppose $x \in \mathbb{R}$ if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$

Consider the contra-positive: if $x < 0$ then $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$. since x is negative so are $x^5, 3x^3, 3x$ (i.e. odd powers of x are negative when x is negative), similarly $-4x^4, -x^2, -4$ are all negative because even powers of x are all positive when x is negative and each of the terms has a negative coefficient. Since the sum of negative numbers is negative. $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$ Thus the contra-positive holds and so does the original statement. $\square$

10. Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$, If $x \nmid yz$ then $x \nmid y$ and $x \nmid z$

Consider the contra-positive: if $x \mid y$ or $x \mid$ then $x \mid z$. Without loss of generality let x divide y. So, $\exists k \in \mathbb{Z}$ such that $y = xk$

$$\begin{aligned} yz &= (xk)z \\ yz &= x(kz) \end{aligned}$$

Let n denote kz, yz can be expressed as $yz = xn$ and n is an integer. So, x divides yz (by definition of divisibility). Thus the contra-positive holds and so does the original statement. $\square$

12. Suppose $a \in \mathbb{Z}$. if a^2 is not divisible by 4 then a is odd

Consider the contra-positive: if a is even then a^2 is divisible by 4. Assuming a is even then $\exists k \in \mathbb{Z}$ such that $a = 2k$. Now, Consider, a^2

$$\begin{aligned} a^2 &= (2k)^2 \\ &= 4k^2 \end{aligned}$$

Since k^2 is an integer and $a^2 = 4k^2$, 4 divides a. Therefore the contra-positive holds and so does the original statement. $\square$

18. If $a, b \in \mathbb{Z}$, then $(a + b)^3 \equiv a^3 + b^3 \pmod{3}$

Assuming a and b are integers. If $(a + b)^3 \equiv a^3 + b^3 \pmod{3}$ then $3 \mid (a + b)^3 - (a^3 + b^3) \pmod{3}$
Note:

$$\begin{aligned} (a + b)^3 &= (a + b)(a + b)^2 \\ &= (a + b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

So,

$$\begin{aligned} (a + b)^3 - (a^3 + b^3) &= a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3 \\ &= 3a^2b + 3ab^2 \\ &= 3(a^2b + ab^2) \end{aligned}$$

Letting $n = a^2b + ab^2$, since $(a + b)^3 - (a^3 + b^3) = 3n$ and n is an integer. $(a + b)^3 - (a^3 + b^3)$ is divisible by 3. So the original statement holds. $\square$

20. If $a \in \mathbb{Z}$ and $a \equiv 1$ (mod 5), then $a^2 \equiv 1$ (mod 5)

Assuming $a \equiv 1(\text{mod } 5)$ then $5 \mid (a-1)$. So, $\exists k$ such that $a - 1 = 5k$ or equivalently $a = 5k + 1$ So,

$$\begin{aligned} a^2 &= (5k+1)^2 \\ &= 25k^2 + 10k + 1 \\ &= 5(5k^2 + 2k) + 1 \end{aligned}$$

Letting $n = 5k^2 + 2k$, since $a^2 = 5n + 1$ and n is an integer. $a^2 \equiv 1$ (mod 5) Thus the original proposition holds. □

24. If $a \equiv b$ (mod n) and $d \equiv c$ (mod n) then $ac \equiv bd$ (mod n)

Assuming $a \equiv b$ (mod n), then $\exists k_1, k_2, r_1 \in \mathbb{Z}$ such that $a = nk_1 + r_1$ and $b = nk_2 + r_1$. Similarly, Assuming $c \equiv d$ (mod n), then $\exists k_3, k_4, r_2 \in \mathbb{Z}$ such that $c = nk_3 + r_2$ and $b = nk_4 + r_2$. So,

$$\begin{aligned} ac &= (nk_1 + r_1)(nk_3 + r_2) \\ &= n^2k_1k_3 + nr_1k_3 + nr_2k_1 + r_1r_2 \\ &= n(nk_1k_3 + r_1k_3 + r_2k_1) + r_1r_2 \\ \text{and} \\ bd &= (nk_2 + r_1)(nk_4 + r_2) \\ &= n^2k_2k_4 + nr_2k_4 + nr_2k_2 + r_1r_2 \\ &= n(nk_2k_4 + r_1k_4 + r_2k_2) + r_1r_2 \end{aligned}$$

Since multiples of n are congruent to zero modulo n, $ac \equiv bc \equiv r_1r_2$ (mod n). Thus the original proposition holds. □