

MATH 271: chapter 7 homework

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4. Given an integer a , then $a^2 + 4a + 5$ is odd if and only if a is even.

Proof. First we will show that: if a is even, then $a^2 + 4a + 5$ is odd and we will show this directly. Assuming a is even, then $\exists n \in \mathbb{Z}$ such that $a = 2n$. So,

$$\begin{aligned}a^2 + 4a + 5 &= (2n)^2 + 4(2n) + 5 \\&= 4n^2 + 8n + 5 \\&= 2(2n^2 + 4n + 2) + 1\end{aligned}$$

Since $2(2n^2 + 4n + 2) + 1$ is odd (by definition of odd) the implication holds.

Next we will show that: if $a^2 + 4a + 5$ is odd, then a is even. Consider the contra-positive of this statement: if a is odd, then $a^2 + 4a + 5$ is even. Assuming a is odd, then, $\exists n \in \mathbb{Z}$ such that $a = 2n + 1$. So,

$$\begin{aligned}a^2 + 4a + 5 &= (2n + 1)^2 + 4(2n + 1) + 5 \\&= 4n^2 + 4n + 1 + 8n + 4 + 5 \\&= 4n^2 + 12n + 10 \\&= 2(2n^2 + 6n + 5)\end{aligned}$$

Since $2(2n^2 + 6n + 5)$ is even the contra-positive holds and so does the original implication.

Since if a is even, implies $a^2 + 4a + 5$ is odd and if $a^2 + 4a + 5$ is odd, implies a is even. The original statement holds.

□

6. Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$

Proof. First we will show that: if $y = x^2$ or $y = -x$, then $x^3 + x^2y = y^2 + xy$.

case 1: $y = -x$

$$\begin{aligned}x^3 + x^2y &= x^3 + x^2(-x) && \text{(by substitution)} \\&= x^3 - x^3 \\&= 0\end{aligned}$$

and

$$\begin{aligned}y^2 + xy &= (-x)^2 + x(-x) && \text{(by substitution)} \\&= x^2 - x^2 \\&= 0\end{aligned}$$

Therefore $x^3 + x^2y = y^2 + xy$ when $y = -x$.

case 2: $y = x^2$

Consider $x^4 + x^3$,

$$\begin{aligned} x^4 + x^3 &= x^4 + x^3 \\ &= (x^2)^2 + (x^2)x && \text{(by factoring)} \\ &= y^2 + yx && \text{(by substitution)} \end{aligned}$$

$$\begin{aligned} x^4 + x^3 &= x^4 + x^3 \\ &= x^2(x^2) + x^3 && \text{(by factoring)} \\ &= x^2y + x^3 && \text{(by substitution).} \end{aligned}$$

Thus, $x^3 + x^2y = y^3 + xy$ when $y = x^2$. since both cases hold so does the original implication.

next consider the implication: if $x^3 + x^2y = y^2 + xy$, then $y = x^2$ or $y = -x$. Assume for the sake of contradiction that: $x^3 + x^2y = y^2 + xy$ and $y \neq x^2$ and $y \neq -x$. Since,

$$\begin{aligned} x^3 + x^2y &= y^2 + xy \\ x^2(x + y) &= y(x + y) && \text{(by factoring)} \\ \frac{x^2(x + y)}{(x + y)} &= \frac{y(x + y)}{(x + y)} && \text{(since } y \neq -x) \\ x^2 &= y. \end{aligned}$$

Which is a contradiction therefore the original implication holds.

Since, $y = x^2$ or $y = -x$, implies $x^3 + x^2y = y^2 + xy$ and $x^3 + x^2y = y^2 + xy$, implies $y = x^2$ or $y = -x$ the original statement holds

□

8. testing