MATH 271: Chapter 9 homework

Micah Sherry

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2. For every natural number n, the integer $2n^2 - 4n + 31$ is prime.

   Proof. consider the counter-example $n = 31$, then

$$\begin{aligned} 2n^2 - 4n + 31 &= 2(31)^2 - 4(31) + 31 \\ &= 31(2(31) - 4 + 1) \\ &= 31(59). \end{aligned}$$

   Therefore $2n^2 - 4n + 31$ is not prime for all n (because it is divisible by 31 and 59). Thus the proposition is false. □

6. If A, B, C, and D are sets, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   Proof. Assuming A, B, C, and D are sets. Then,

$$\begin{aligned} (A \times B) \cap (C \times D) &= \{(x, y) | (x, y) \in (A \times B) \cap (C \times D)\} \\ &= \{(x, y) | (x, y) \in (A \times B) \wedge (x, y) \in (C \times D)\} && \text{(by definition of intersection)} \\ &= \{(x, y) | (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D)\} && \text{(by definition of Cartesian product)} \\ &= \{(x, y) | x \in A \wedge y \in B \wedge x \in C \wedge y \in D\} && \text{(by Associativity)} \\ &= \{(x, y) | x \in A \wedge x \in C \wedge y \in B \wedge y \in D\} && \text{(by Commutativity)} \\ &= \{(x, y) | x \in A \cap C \wedge y \in B \cap D\} && \text{(by definition of intersection)} \\ &= \{(x, y) | (x, y) \in (A \cap C) \times (B \cap D)\} && \text{(by definition of Cartesian product)} \\ &= (A \cap C) \times (B \cap D) \end{aligned}$$

   Therefore the Equality holds, So the proposition is true. □

16. If A and B are finite sets, then $|A \cup B| = |A| + |B|$.

   Proof. Assuming A and B are finite sets. Consider the counter-example: $A = B$. So,

$$\begin{aligned} A \cup B &= \{x | x \in A \vee x \in B\} && \text{(by definition of union)} \\ &= \{x | x \in A \vee x \in A\} && \text{(by substitution)} \\ &= \{x | x \in A\} && \text{(since } p \vee p = p\text{)} \\ &= A. \end{aligned}$$

   Therefore $|A \cup B| = |A|$ and $|A| + |B| = 2|A|$. Thus the original equality does not hold, so the proposition is false. □

18. If $a, b, c \in N$ then At least one of $a - b$, $a + c$ and $b - c$ is even.

Proof. Assume for the sake of contradiction that $a-b$, $a+c$ and $b-c$ are all odd. then, $\exists k, j$ such that $a-b=2k+1$ and $b-c=2j+1$. So, $a=2k+1+b$ and $c=-2j-1+b$. Now consider

$$\begin{aligned} a+c &= (2k+1+b)+(-2j-1+b) \\ &= 2k-2j+2b \\ &= 2(k-j+b). \end{aligned}$$

Which is even and contradicts the Assumption that $a+c$ is odd. Thus the proposition holds. $\square$

20. There exist prime numbers p and q for which $p-q=1000$.

Proof. Consider $p=1013$ and $q=13$. So, $p-q=1013-13=1000$. Since 1013 and 13 are prime the proposition is True. $\square$

28. Suppose $a, b \in \mathbb{Z}$ If $a|b$ and $b|a$, then $a=b$

Proof. Consider the counter-example: $a=-b$. This means $a=(-1)b$ which implies $b|a$. Similarly $b=(-1)a$ which implies $a|b$. Since $a|b$ and $b|a$ and $a \neq b$ therefore the proposition is false.

$\square$

34. If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

Proof. Consider the counter-example $A=\{1\}$, $B=\{2\}$, and $X=\{1,2\}$. So, $X \nsubseteq A$ and $X \nsubseteq B$. And notice $A \cup B=\{1,2\}$ which implies $X \subseteq A \cup B$. Since $X \subseteq A \cup B$ and $X \nsubseteq A$ and $X \nsubseteq B$ the proposition is false. $\square$