MATH 271: chapter 7 homework

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4. Given an integer a, then $a^2 + 4a + 5$ is odd if and only if a is even.

   Proof. First we will show that: if a is even, then $a^2 + 4a + 5$ is odd and we will show this directly. Assuming a is even, then $\exists n \in \mathbb{Z}$ such that $a = 2n$. So,

$$\begin{aligned} a^2 + 4a + 5 &= (2n)^2 + 4(2n) + 5 \\ &= 4n^2 + 8n + 5 \\ &= 2(2n^2 + 4n + 2) + 1 \end{aligned}$$

   Since $2(2n^2 + 4n + 2) + 1$ is odd (by definition of odd) the implication holds.

   Next we will show that: if $a^2 + 4a + 5$ is odd, then a is even. Consider the contra-positive of this statement: if a is odd, then $a^2 + 4a + 5$ is even. Assuming a is odd, then, $\exists n \in \mathbb{Z}$ such that $a = 2n + 1$. So,

$$\begin{aligned} a^2 + 4a + 5 &= (2n + 1)^2 + 4(2n + 1) + 5 \\ &= 4n^2 + 4n + 1 + 8n + 4 + 5 \\ &= 4n^2 + 12n + 10 \\ &= 2(2n^2 + 6n + 5) \end{aligned}$$

   Since $2(2n^2 + 6n + 5)$ is even the contra-positive holds and so does the original implication.

   Since if a is even, implies $a^2 + 4a + 5$ is odd and if $a^2 + 4a + 5$ is odd, implies a is even. The original statement holds.

   $\square$

6. Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$

   Proof. First we will show that: if $y = x^2$ or $y = -x$, then $x^3 + x^2y = y^2 + xy$.

   case 1: $y = -x$

$$\begin{aligned} x^3 + x^2y &= x^3 + x^2(-x) && \text{(by substitution)} \\ &= x^3 - x^3 \\ &= 0 \end{aligned}$$

   and

$$\begin{aligned} y^2 + xy &= (-x)^2 + x(-x) && \text{(by substitution)} \\ &= x^2 - x^2 \\ &= 0 \end{aligned}$$

Therefore $x^3 + x^2y = y^2 + xy$ when $y = -x$.

case 2: $y = x^2$
Consider $x^4 + x^3$,

$$\begin{aligned}
x^4 + x^3 &= x^4 + x^3 \\
&= (x^2)^2 + (x^2)x && \text{(by factoring)} \\
&= y^2 + yx && \text{(by substitution)}
\end{aligned}$$

$$\begin{aligned}
x^4 + x^3 &= x^4 + x^3 \\
&= x^2(x^2) + x^3 && \text{(by factoring)} \\
&= x^2y + x^3 && \text{(by substitution).}
\end{aligned}$$

Thus, $x^3 + x^2y = y^3 + xy$ when $y = x^2$. since both cases hold so does the original implication.

next consider the implication: if $x^3 + x^2y = y^2 + xy$, then $y = x^2$ or $y = -x$. Assume for the sake of contradiction that: $x^3 + x^2y = y^2 + xy$ and $y \neq x^2$ and $y \neq -x$. Since,

$$\begin{aligned}
x^3 + x^2y &= y^2 + xy \\
x^2(x+y) &= y(x+y) && \text{(by factoring)} \\
\frac{x^2(x+y)}{(x+y)} &= \frac{y(x+y)}{(x+y)} && \text{(since } y \neq -x\text{)} \\
x^2 &= y.
\end{aligned}$$

Which is a contradiction therefore the original implication holds.

Since, $y = x^2$ or $y = -x$, implies $x^3 + x^2y = y^2 + xy$ and $x^3 + x^2y = y^2 + xy$, implies $y = x^2$ or $y = -x$ the original statement holds

□

8. Suppose $a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{10}$ if and only if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.

Proof. First we will show that: if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{10}$. Assuming $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. So, $2|(a-b)$ (by definition of mod) and $5|(a-b)$ (by definition of mod). Therefore $10|(a-b)$ because 2 and 5 share no factors. Thus $a \equiv b \pmod{10}$ So, the original implication holds

Next we will show that: if $a \equiv b \pmod{10}$, then $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.
Assuming $a \equiv b \pmod{10}$, then $10|(a-b)$. So, $\exists n \in \mathbb{Z}$ such that $10n = (a-b)$. So,

$$\begin{aligned}
(a-b) &= 10n \\
&= 2(5n)
\end{aligned}$$

and

$$\begin{aligned}
(a-b) &= 10n \\
&= 5(2n)
\end{aligned}$$

So $2|(a-b)$ and $5|(a-b)$. Therefore $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$ thus the original implication holds □