MATH 271: Proof Portfolio

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December 4, 2024

Sample Direct Proof (problem 7.34)

If $\gcd(a, c) = \gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.

Proof. Assume $\gcd(a, c) = \gcd(b, c) = 1$, So, $\exists w, x, y, z \in \mathbb{Z}$ such that $1 = aw + cx$ and $1 = by + cz$ (by proposition 7.1 page 152). let d denote the $\gcd(ab, c)$, therefore $d|c$ and $d|ab$. So, $\exists m, n$ such that $c = dm$ and $ab = dn$. Consider,

$$\begin{aligned} 1 &= aw + cx \\ b &= abw + cby && \text{(multiplying both sides by } b.) \\ &= dnw + dmby && \text{(substituting } c \text{ for } dm \text{ and } ab \text{ for } dn.) \\ &= d(nw + mby) && \text{(let } k \text{ denote } nw + mby.) \end{aligned}$$

Therefore $b = dk$. Now consider,

$$\begin{aligned} 1 &= by + cz \\ &= dky + dmz && \text{(substituting } c \text{ for } dm\text{, and } b \text{ for } dk) \\ &= d(ky + mz) \\ &= dj && \text{(letting } j \text{ denote } ky + mz) \end{aligned}$$

This implies that $d|1$, therefore, d=1 (because 1 only has divisors -1, and 1; and $1 > -1$). Therefore the $\gcd(ab, c) = 1$, so the proposition holds. □

Sample Indirect Proof From (problem 5.10)

Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$, If $x \nmid yz$ then $x \nmid y$ and $x \nmid z$

Proof. Consider the contra-positive: if $x \mid y$ or $x \mid z$ then $x \mid yz$. Without loss of generality let x divide y. So, $\exists k \in \mathbb{Z}$ such that $y = xk$

$$\begin{aligned} yz &= (xk)z \\ yz &= x(kz) \end{aligned}$$

Let n denote kz, yz can be expressed as $yz = xn$ and n is an integer. So, x divides yz (by definition of divisibility). Thus the contra-positive holds and so does the original statement.

□

Sample Proof by Cases From (problem 7.22)

If $n \in \mathbb{Z}$ then $4|n^2$ or $4|(n^2 - 1)$

Proof. Notice there are 2 cases for n: odd and even.

Case 1: n is odd. So, $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Consider,

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4(k^2 + k). \end{aligned}$$

So 4 divides $n^2 - 1$ when n is odd.

Case 2: n is even. So, $\exists k \in \mathbb{Z}$ such that $n = 2k$. Consider,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4(k^2). \end{aligned}$$

So, 4 divides n^2 when n is even. Since 4 divides n^2 or $n^2 - 1$ the proposition holds.

□

Sample Proof by Contradiction (problem 6.8)

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$ then a or b is even

Proof. Assume for the sake of contradiction that $a^2 + b^2 = c^2$, and a and b are odd. Then $\exists k, n \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2n + 1$. So,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2k+1)^2 \\ &= 4k^2 + 4k + 1 + 4n^2 + 4n + 1 \\ &= 4(k^2 + n^2 + k + n) + 2. \end{aligned}$$

let $m = k^2 + n^2 + k + n$ Therefore c^2 = 4m+2.

Case 1: c is even. Then $c = 2j$ for some $j \in \mathbb{Z}$. Then $c^2 = 4j^2$. This is a contradiction because we have established that c^2 is of the form $4m + 2$

Case 2: c is odd. Then $c = 2j + 1$ for some $j \in \mathbb{Z}$. Then $c^2 = 4j^2 + 4j + 1 = 4(j^2 + j) + 1$. This is a contradiction because we have established that $c^2$is of the form $4m + 2$.

Since both cases for c leads to a contradiction the original statement holds. □

Sample Disproof From (problem 9.2)

For every natural number n, the integer $2n^2 - 4n + 31$ is prime.

Proof. consider the counter-example $n = 31$, then

$$\begin{aligned} 2n^2 - 4n + 31 &= 2(31)^2 - 4(31) + 31 \\ &= 31(2(31) - 4 + 1) \\ &= 31(59). \end{aligned}$$

Therefore $2n^2 - 4n + 31$ is not prime for all n (because it is divisible by 31 and 59). Thus the proposition is false. □

Sample Induction Proof (problem 1.1)

Proof. □