MATH 405: Exam 2

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1. Consider $R = \mathbb{Z}_4[x]$ And the Ideal $J = \langle x^2 \rangle$ Now consider the quotient ring $\mathbb{Z}_4[x]/\langle x2 \rangle$.

   (a) Explain how many elements are in $\mathbb{Z}_4[x]/\langle x2 \rangle$.
   $\mathbb{Z}_4[x]/\langle x2 \rangle = \{a + bx + \langle x^2 \rangle | a, b \in \mathbb{Z}_4\}$ Since $\mathbb{Z}_4$ has 4 elements, $\mathbb{Z}_4[x]/\langle x2 \rangle$ has 16 (4 choices for each a and b)

   (b) give the multiplication table for $\mathbb{Z}_4[x]/\langle x2 \rangle$
   See last page.

   (c) indicate for each element of $\mathbb{Z}_4[x]/\langle x2 \rangle$ if it is a unit, zero divisor or neither

0+ 0x	None
1+ 0x	unit
2+ 0x	zero divisor
3+ 0x	unit
0+ 1x	zero divisor
1+ 1x	unit
2+ 1x	zero divisor
3+ 1x	unit
0+ 2x	zero divisor
1+ 2x	unit
2+ 2x	zero divisor
3+ 2x	unit
0+ 3x	zero divisor
1+ 3x	unit
2+ 3x	zero divisor
3+ 3x	unit

2. Recall we have shown that for $R = \mathbb{R}[x]$ every ideal is principal. That is if J is an ideal of $\mathbb{R}[x]$, then $J = \langle g(x) \rangle = \{f(x)g(x) | f(x) \in \mathbb{R}[x]\}$ for some polynomial g(x).

   Let $J = \langle 2x^2 + 3x + 1, 10x^2 + x - 2 \rangle = \{(2x^2 + 3x + 1)f(x) + (10x^2 + x - 2)g(x) | f(x), g(x) \in \mathbb{R}[x]\}$.

   Now J is a principal ideal so every element of J is expressible as a factor of one polynomial, find a $h(x)$ such that $J = \langle h(x) \rangle$

   $h(x)$ can be found by using the Euclidean algorithm algorithm for polynomial over $\mathbb{R}[x]$

$$10x^2 + x - 2 = 5(2x^2 + 3x + 1) - 14x - 7$$
$$2x^2 + 3x + 1 = (-\frac{1}{7}x - \frac{1}{7})(-14x - 7) + 0$$

   Therefore the gcd of $2x^2 + 3x + 1$ and $10x^2 + x - 2$ is $-14x - 7$. Thus, $J = \langle -14x - 7 \rangle$

3. Let R be a commutative ring and J be an ideal of R. Define the relationship congruence $\mod J$ on R as follows: For $r, s \in R$, $r \equiv_J s$ if and only if $r - s \in J$. Show that the relation congruence $\mod J$ is an equivalence relation

   Proof. Let $x \in R$, Notice $x - x = O_R$ which is an element of J. Therefore equivalence $\mod J$ is reflexive.

   Let $x, y \in R$ with $x \equiv_J y$. So, $x - y = j$ for some $j \in J$. Notice $y - x = -1(j) \in J$ (by absorption property of J). Therefore equivalence $\mod J$ is symmetric.

   let $x, y, z \in R$ with $x \equiv_J y$ and $y \equiv_J z$. $x - y \in J$ and $y - z \in J$ now consider $(x-y)+(y-z) = x - z \in J$ (by closure of J under addition). Therefore equivalence $\mod J$ is transitive.
   Since equivalence $\mod J$ is reflexive, symmetric, and transitive it is an equivalence relation

   □

4. Recall we have shown that for $R = \mathbb{Z}$ every ideal is principal. That is if J is an ideal of $\mathbb{Z}$, then $J = \langle k \rangle = \{kn | n \in \mathbb{Z}\}$ for some positive integer k. Define an ideal M of a commutative ring R to be maximal if $M \neq R$ and if J is an ideal with $M \subseteq J \subseteq R$, then either $J = M$ or $J = R$. Prove an ideal M of $\mathbb{Z}$ is maximal if and only if $M = \langle p \rangle$ for some prime number p.

   Proof. ($\Rightarrow$) Assume for the sake of contradiction that M $= \langle p \rangle$ Is Maximal and p is not prime. So, $p = p_1 p_2$ for some $p_1, p_2 > 1 \in \mathbb{Z}$. Let $x \in \langle p \rangle$, So $x = pn = p_1(p_2 n)$ for some $n \in \mathbb{Z}$. Notice $p_2 n \in \mathbb{Z}$, So $x \in \langle p_1 \rangle$, which implies $\langle p \rangle \subseteq \langle p_1 \rangle \subseteq \mathbb{Z}$. Since $\langle p_1 \rangle \neq \langle p \rangle$ and $\langle p_1 \rangle \neq \mathbb{Z}$, $\langle p \rangle$ is Not a maximal ideal contradicting the assumption. So if ideal $M = \langle p \rangle$ of $\mathbb{Z}$ is maximal then p is a prime number.

   ($\Leftarrow$) let p be a prime number and let $j \in \mathbb{Z}$ such that $\langle p \rangle \subseteq \langle j \rangle \subseteq \mathbb{Z}$. Notice $p \in \langle p \rangle$ and $p \in \langle j \rangle$. So, $p = jk$ for some $k \in \mathbb{Z}$, which implies that $j | p$ and since p is prime $j = 1$ or $j = p$.
   if $j = 1$ then $\langle j \rangle = \mathbb{Z}$ and if $j = p$ then $\langle j \rangle = \langle p \rangle$.
   So, if p is prime then $\langle p \rangle$ is maximal. This completes the proof. □

5. Let G be a group (not necessarily commutative). Recall the center of the group is $Z(G) = \{g \in G | g \circ h = h \circ g$ for all $h \in G\}$. Prove that a group G is commutative if and only if $Z(G) = G$.

   Proof. ($\Rightarrow$) Let G be a commutative group, let $g, h \in G$, Since G is commutative $g \circ h = h \circ g$ so, $g \in Z(G)$. Thus $Z(G) = G$.

   ($\Leftarrow$) Let $Z(G) = G$ and let $g \in G$. Since $g \in Z(G)$, $g \circ h = h \circ g$ for all $h \in G$, Therefore G is commutative. This completes the proof. □

Multiplication table for $\mathbb{Z}_4[x]/\langle x^2 \rangle$

$\cdot$	$\overline{0+0x}$	$\overline{1+0x}$	$\overline{2+0x}$	$\overline{3+0x}$	$\overline{0+1x}$	$\overline{1+1x}$	$\overline{2+1x}$	$\overline{3+1x}$	$\overline{0+2x}$	$\overline{1+2x}$	$\overline{2+2x}$	$\overline{3+2x}$	$\overline{0+3x}$	$\overline{1+3x}$	$\overline{2+3x}$	$\overline{3+3x}$
$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$	$\overline{0+0x}$
$\overline{1+0x}$	$\overline{0+0x}$	$\overline{1+0x}$	$\overline{2+0x}$	$\overline{3+0x}$	$\overline{0+1x}$	$\overline{1+1x}$	$\overline{2+1x}$	$\overline{3+1x}$	$\overline{0+2x}$	$\overline{1+2x}$	$\overline{2+2x}$	$\overline{3+2x}$	$\overline{0+3x}$	$\overline{1+3x}$	$\overline{2+3x}$	$\overline{3+3x}$
$\overline{2+0x}$	$\overline{0+0x}$	$\overline{2+0x}$	$\overline{0+0x}$	$\overline{2+0x}$	$\overline{0+2x}$	$\overline{2+2x}$	$\overline{0+2x}$	$\overline{2+2x}$	$\overline{0+0x}$	$\overline{2+0x}$	$\overline{0+0x}$	$\overline{2+0x}$	$\overline{0+2x}$	$\overline{2+2x}$	$\overline{0+2x}$	$\overline{2+2x}$
$\overline{3+0x}$	$\overline{0+0x}$	$\overline{3+0x}$	$\overline{2+0x}$	$\overline{1+0x}$	$\overline{0+3x}$	$\overline{3+3x}$	$\overline{2+3x}$	$\overline{1+3x}$	$\overline{0+2x}$	$\overline{3+2x}$	$\overline{2+2x}$	$\overline{1+2x}$	$\overline{0+1x}$	$\overline{3+1x}$	$\overline{2+1x}$	$\overline{1+1x}$
$\overline{0+1x}$	$\overline{0+0x}$	$\overline{0+1x}$	$\overline{0+2x}$	$\overline{0+3x}$	$\overline{0+0x}$	$\overline{0+1x}$	$\overline{0+2x}$	$\overline{0+3x}$	$\overline{0+0x}$	$\overline{0+1x}$	$\overline{0+2x}$	$\overline{0+3x}$	$\overline{0+0x}$	$\overline{0+1x}$	$\overline{0+2x}$	$\overline{0+3x}$
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$\overline{2+1x}$	$\overline{0+0x}$	$\overline{2+1x}$	$\overline{0+2x}$	$\overline{2+3x}$	$\overline{0+2x}$	$\overline{2+3x}$	$\overline{0+0x}$	$\overline{2+1x}$	$\overline{0+0x}$	$\overline{2+1x}$	$\overline{0+2x}$	$\overline{2+3x}$	$\overline{0+2x}$	$\overline{2+3x}$	$\overline{0+0x}$	$\overline{2+1x}$
$\overline{3+1x}$	$\overline{0+0x}$	$\overline{3+1x}$	$\overline{2+2x}$	$\overline{1+3x}$	$\overline{0+3x}$	$\overline{3+0x}$	$\overline{2+1x}$	$\overline{1+2x}$	$\overline{0+2x}$	$\overline{3+3x}$	$\overline{2+0x}$	$\overline{1+1x}$	$\overline{0+1x}$	$\overline{3+2x}$	$\overline{2+3x}$	$\overline{1+0x}$
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$\overline{2+2x}$	$\overline{0+0x}$	$\overline{2+2x}$	$\overline{0+0x}$	$\overline{2+2x}$	$\overline{0+2x}$	$\overline{2+0x}$	$\overline{0+2x}$	$\overline{2+0x}$	$\overline{0+0x}$	$\overline{2+2x}$	$\overline{0+0x}$	$\overline{2+2x}$	$\overline{0+2x}$	$\overline{2+0x}$	$\overline{0+2x}$	$\overline{2+0x}$
$\overline{3+2x}$	$\overline{0+0x}$	$\overline{3+2x}$	$\overline{2+0x}$	$\overline{1+2x}$	$\overline{0+3x}$	$\overline{3+1x}$	$\overline{2+3x}$	$\overline{1+1x}$	$\overline{0+2x}$	$\overline{3+0x}$	$\overline{2+2x}$	$\overline{1+0x}$	$\overline{0+1x}$	$\overline{3+3x}$	$\overline{2+1x}$	$\overline{1+3x}$
$\overline{0+3x}$	$\overline{0+0x}$	$\overline{0+3x}$	$\overline{0+2x}$	$\overline{0+1x}$	$\overline{0+0x}$	$\overline{0+3x}$	$\overline{0+2x}$	$\overline{0+1x}$	$\overline{0+0x}$	$\overline{0+3x}$	$\overline{0+2x}$	$\overline{0+1x}$	$\overline{0+0x}$	$\overline{0+3x}$	$\overline{0+2x}$	$\overline{0+1x}$
$\overline{1+3x}$	$\overline{0+0x}$	$\overline{1+3x}$	$\overline{2+2x}$	$\overline{3+1x}$	$\overline{0+1x}$	$\overline{1+0x}$	$\overline{2+3x}$	$\overline{3+2x}$	$\overline{0+2x}$	$\overline{1+1x}$	$\overline{2+0x}$	$\overline{3+3x}$	$\overline{0+3x}$	$\overline{1+2x}$	$\overline{2+1x}$	$\overline{3+0x}$
$\overline{2+3x}$	$\overline{0+0x}$	$\overline{2+3x}$	$\overline{0+2x}$	$\overline{2+1x}$	$\overline{0+2x}$	$\overline{2+1x}$	$\overline{0+0x}$	$\overline{2+3x}$	$\overline{0+0x}$	$\overline{2+3x}$	$\overline{0+2x}$	$\overline{2+1x}$	$\overline{0+2x}$	$\overline{2+1x}$	$\overline{0+0x}$	$\overline{2+3x}$
$\overline{3+3x}$	$\overline{0+0x}$	$\overline{3+3x}$	$\overline{2+2x}$	$\overline{1+1x}$	$\overline{0+3x}$	$\overline{3+2x}$	$\overline{2+1x}$	$\overline{1+0x}$	$\overline{0+2x}$	$\overline{3+1x}$	$\overline{2+0x}$	$\overline{1+3x}$	$\overline{0+1x}$	$\overline{3+0x}$	$\overline{2+3x}$	$\overline{1+2x}$