MATH 405: Assignment 2

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1. Let m and n be positive integers. Prove that if $m \mid n$ and $n \mid m$ then $m = n$.

   Proof. Assume $m \mid n$ and $n \mid m$. Since $m \mid n$, $n = m(x_1)$ for some $x_1 \in \mathbb{Z}$ and since $n \mid m$, $m = n(x_2)$ for some $x_2 \in \mathbb{Z}$. Therefore,

   $$\begin{aligned} n &= m(x_1) \\ &= n(x_2 x_1) && \text{(by substitution)} \end{aligned}$$

   So, $(x_2 x_1) = 1$, and because n and m are both positive $x_1 = x_2 = 1$. This implies n=m, which is what we wanted to show. $\square$

2. let $m, n \in \mathbb{Z}$ if $n \mid m$ then $m\mathbb{Z} \subseteq n\mathbb{Z}$.

   (a) Example: $n = 2$ and $m = 4$
   Note $2 \mid 4$. Let $x \in 4\mathbb{Z}$ then $x = 4a$ for some $a \in \mathbb{Z}$. $x = 2(2a)$ therefore $x \in 2\mathbb{Z}$

   (b) prove the statement.

   Proof. Let $n, m \in \mathbb{Z}$ and assume $n \mid m$. Since $n \mid m$, $m = n(a)$ for some $a \in \mathbb{Z}$. Let $x \in m\mathbb{Z}$ so, $x = m(b)$ for some $b \in \mathbb{Z}$. So,

   $$\begin{aligned} x &= m(b) \\ &= n(a)(b) && \text{(by substitution)} \\ &= n(ab) \end{aligned}$$

   Since ab is an integer (because $\mathbb{Z}$ is closed under multiplication) x is an element of $n\mathbb{Z}$. Therefore the subset relationship holds, which is what we wanted to show. $\square$

3. Prove the converse of the statement above. That is Prove: Let $m, n \in \mathbb{Z}$. If $m\mathbb{Z} \subseteq n\mathbb{Z}$, then $n \mid m$.

   Proof. The proof will be by contradiction.
   Assume that $m\mathbb{Z} \subseteq n\mathbb{Z}$ and $n \nmid m$. Notice $m \in m\mathbb{Z}$ (because $m = m(1)$). This means that m has to be an element of $n\mathbb{Z}$ (otherwise it would not be a subset). Therefore $m = n(a)$ for some $a \in \mathbb{Z}$ contradicting the assumption. Therefore the original statement is true. $\square$

4. For each pair of integers a and b determine whether or not $ax + by = 1$ has an integer solution. If it does then find at least 3 different integer solutions (x, y) if it doesn't then explain why not

   (a) $a = 7, b = 10$
   Integer solutions $(x, y) : (-7, 5),\ (3, -2),\ (13, -9)$

   (b) $a = 8, b = 10$

   $$\begin{aligned} ax + by &= 8x + 10y \\ &= 2(4x + 5y) \end{aligned}$$

   Which implies that all (integer) linear combos of a and b must be even. Therefore there is no integer solutions.

(c) $a = 15, b = 21$

$$\begin{aligned} ax + by &= 15x + 21y \\ &= 3(5x + 7y) \end{aligned}$$

Which implies that all (integer) linear combos of a and b must be a multiple of 3. Therefore there is no integer solutions.

(d) $a = 15, b = 16$
Integer solutions $(x, y) : (-1, 1),\ (15, -14),\ (-17, 16)$

5. Let a and b be consecutive odd integers. show that the $gcd(a, b) = 1$.

Proof. Since a and b are odd integers $a = 2k + 1$ and $b = 2k + 3$ for some $k \in \mathbb{Z}$. Let $d = gcd(a, b)$. Therefore $a = dn$ for some $n \in \mathbb{Z}$ and similarly $b = dm$ for some $m \in \mathbb{Z}$. So,

$$\begin{aligned} b - a &= dm - dn \\ b - a &= d(m - n) \\ (2k + 3) - (2k + 1) &= d(m - n) \\ 2 &= d(m - n) \end{aligned}$$

Therefor $d \mid 2$ and since a and b are odd $d \neq 2$. So, $d = 1$. $\square$

Extra Credit: prove that $3, 5, 7$ is the only prime triple.

Proof. Let $n \in \mathbb{Z}$ and $n > 3$. Notice that if $n \equiv_3 0$ it is a multiple of 3, therefore it is not prime.

Case 1: n is even. This is trivial if n is even then it is not prime and therefore not in a prime triple

Case 2: n is odd.
Consider the three consecutive odd integers $n, n+2n+4$. Notice there are three sub-cases for n: $n \equiv_3 0$, $n \equiv_3 1$, $n \equiv_3 2$

Subcase 1: $n \equiv_3 0$. Therefore n is not prime so it is not part of a prime triple.

Subcase 2: $n \equiv_3 1$. which implies $n + 2 \equiv_3 0$ Therefore $n + 2$ is not prime so it is not part of a prime triple.

Subcase 3: $n \equiv_3 2$. which implies $n + 4 \equiv_3 0$ Therefore $n + 4$ is not prime so it is not part of a prime triple.

Therefore since in all cases for n, $n + 2$ and $n + 4$ is not a prime triple 3, 5 ,7 is the only prime triple.

$\square$