MATH 405: Assignment 6

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1. Let R be an integral domain. Prove that if $a, b \in R$ and $a^2 = b^2$, then $a = b$ or $a = -b$.

   Proof. Let $a, b \in R$ with $a^2 = b^2$. So,

$$\begin{aligned} a^2 &= b^2 \\ a^2 - b^2 &= 0_R \\ (a-b)(a+b) &= 0_R \end{aligned}$$

   Since R does not have zero divisors $a - b = 0_R$ or $a + b = 0_R$. Therefore $a = b$ or $a = -b$. □

2. Let R be a commutative ring (but not necessarily an integral domain). Let $f(x) \in R[x]$ prove each of the following statements.
   To make the following proofs more concise, we'll prove this statement first: if $f(x) \in R[x]$ is monic and non-constant and $0_R \neq g(x) \in R[x]$, Then $\deg(f(x)g(x)) > 0$.

   Proof. Assume,
   $f(x) = 1_R x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and
   $g(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $n > 0$, $m \geq 0$, and $a_i, b_i \in R$.
   Consider the product of the highest terms of $f(x)g(x)$, $b_m * 1_R x^{m+n}$. Since 1_R is not a zero divisor $\deg(f(x)g(x)) = n + m > 0$. □

   (a) If $f(x)$ is monic and non-constant, then $f(x)$ is not a unit in $R[x]$.

   Proof. Assume $f(x)$ is monic and non-constant. And assume for the sake of contradiction there exist a $g(x)$ such that $f(x)g(x) = 1_R$. From the above result $\deg(f(x)g(x)) > 0$, contradicting the assumption that $f(x)g(x) = 1_R$ therefore $f(x)$ is not a unit □

   (b) If $f(x)$ is monic, then $f(x)$ is not a zero divisor in $R[x]$

   Proof. assume f(x) is monic.

   Case 1: $f(x)$ is monic and constant. $f(x) = 1_R$. Which is not a zero divisor.

   Case 2: f(x) is monic and non-constant. Assume for the sake of contradiction that there exists $g(x) \neq 0_R$ such that $f(x)g(x) = 0_R$. From the above result the $\deg(f(x)g(x)) > 0$ which is contradicting the assumption that $f(x)g(x) = 0_R$. thus $f(x)$ is not a zero divisor.

   Therefore the original statement holds. □

3. units in $\mathbb{Z}_4[x]$

   (a) Find five units (other than 1 and 3) in $\mathbb{Z}_4[x]$

   - $2x + 1$; $(2x + 1)^2 = 4x^2 + 4x + 1 = 1$
   - $2x^2 + 1$; $(2x^2 + 1)^2 = 4x^4 + 4x^2 + 1 = 1$
   - $2x^3 + 1$; $(2x^3 + 1)^2 = 4x^6 + 4x^3 + 1 = 1$
   - $2x^4 + 1$; $(2x^4 + 1)^2 = 4x^8 + 4x^4 + 1 = 1$
   - $2x^5 + 1$; $(2x^5 + 1)^2 = 4x^{10} + 4x^5 + 1 = 1$

(b) Explain why $\mathbb{Z}_4[x]$ has infinitely many units.
Consider all polynomials of the form $(2g(x)+1)$, with $g(x) \in Z_4[x]$. Notice that $(2g(x)+1)^2 = 4g(x)^2 + 4g(x) + 1 = 1$. Therefore, are infinitely many units of the form $(2g(x)+1)$.

4. Consider the function $\theta : \mathbb{Z}_2[x] \to \mathbb{Z}_2[x]$ where $\theta(f(x)) = (f(x))^2$ for any $f(x) \in \mathbb{Z}_2[x]$.

(a) show that θ is a homomorphism.

Proof. Let $f(x),\ g(x) \in \mathbb{Z}_2[x]$ Consider,

$$\begin{aligned}\theta(f(x)g(x)) &= (f(x)g(x))^2 \\ &= f(x)^2 g(x)^2 \\ &= \theta(f(x))\theta(g(x)).\end{aligned}$$

Therefore θ preserves multiplication. Now, consider

$$\begin{aligned}\theta(f(x)+g(x)) &= (f(x)+g(x))^2 \\ &= f(x)^2 + 2f(x)g(x) + g(x)^2 \\ &= f(x)^2 + g(x)^2 \\ &= \theta(f(x)) + \theta(g(x))\end{aligned}$$

Therefore θ preserves addition. Thus θ is a homomorphism. □

(b) find $\ker\theta$
$\ker\theta = \{f(x) \in \mathbb{Z}_2[x] | f(x)^2 = 0\} = \{0\}$

(c) Describe all elements in the image of θ
Notice θ maps each term of the polynomial to its Square. So the image of θ is all polynomials in $\mathbb{Z}_2[x]$ with only even powers of x

5. Let A be the set of all polynomials in $\mathbb{Z}[x]$ with an even constant term.

(a) prove that A is an ideal $\mathbb{Z}[x]$

Proof. let $A = \{xg(x) + 2n | g(x) \in \mathbb{Z}[x] \wedge n \in \mathbb{Z}\}$
let $f_1(x) = xg_1(x) + 2n_1$ and $f_2(x) = xg_2(x) + 2n_2$, for some $g_1(x), g_2(x) \in \mathbb{Z}[x]$, $n_1, n_2 \in \mathbb{Z}$.
Consider,

$$\begin{aligned}f_1(x) + f_2(x) &= xg_1(x) + 2n_1 + xg_2(x) + 2n_2 \\ &= x(g_1(x) + g_2(x)) + 2(n_1 + n_2)\end{aligned}$$

So, A is an closed under addition.
let $t(x) = xg(x) + n$ for some $g(x) \in \mathbb{Z}[x]$ and $n \in \mathbb{Z}$. Now Consider,

$$\begin{aligned}t(x)f_1(x) &= (xg(x) + n)(xg_1(x) + 2n_1) \\ &= x(xg(x)g_1(x) + 2n_1 g(x) + ng_1(x)) + 2n_1 n\end{aligned}$$

This show that A satisfies the absorption property and closure under multiplication. Therefore A is an ideal. □

(b) Anita claims $A = \langle 2 \rangle = \{2 * f(x) | f(x) \in Z[x]\}$. Do you agree or disagree? Explain.
Disagree, notice $x + 2 \in A$ and $x + 2 \notin \langle 2 \rangle$

(c) Elizabeth claims $A = \langle x \rangle = \{x * f(x) | f(x) \in Z[x]\}$. Do you agree or disagree? Explain.
Disagree, notice $2 \in A$ and $2 \notin \langle x \rangle$