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Definitions

Let $f_n(x) = \sum_{i=0}^{n} x^i$.
Let G_n be the set of all $g(x)$ such that $g(x) = \sum_{i=1}^{n} a_i x^n$ and $\forall a_i$ if $a_i = 0$ then $a_{i+1} \neq 0$. Let $F(n) =$ the nth Fibonacci number.

Proofs

Theorem 1. *let $g(x) = \sum_{i=0}^{n} a_i x^i$ with $a_n = a_0 = 1$ if $(x+1) * g(x) = 2^{n+1} - 1$ then $g(x)$ does not contain two consecutive terms with coefficient zero*

Proof. Assume for the sake of contradiction that there exists i such that $0 < j < n$, $a_{j-1} = a_j = 0$ and $(x+1) * g(x) = f_{n+1}(x)$. Consider the following:

$$\begin{aligned}
(x+1) * g(x) &= x * g(x) + 1 * g(x) \\
&= x * \sum_{i=0}^{n} a_i x^i + 1 * \sum_{i=0}^{n} a_i x^i && \text{(substituting in g(x))} \\
&= \sum_{i=0}^{n} a_i x^{i+1} + \sum_{i=0}^{n} a_i x^i && \text{(multiplying by x in summation)} \\
&= a_n x^{n+1} + \sum_{i=0}^{n-1} a_i x^{i+1} + \sum_{i=1}^{n} a_i x^i + a_0 && \text{(pulling 0th and (n+1)th terms out of the summation)} \\
&= a_n x^{n+1} + \sum_{i=1}^{n} a_{i-1} x^i + \sum_{i=1}^{n} a_i x^i + a_0 && \text{(reindexing the first sum } i \leftarrow i+1\text{)} \\
&= a_n x^{n+1} + \sum_{i=1}^{n} (a_{i-1} + a_i) x^i + a_0 && \text{(combining summations)}
\end{aligned}$$

by the assumption there exist j such that $a_{j-1} + a_j = 0$ contradicting the assumption that $(x+1) * g(x) = f_{n+1}(x)$ because the jth term of $(x+1) * g(x)$'s coefficient is 0 $\square$

Theorem 2. $|G_n| = F(n+2)$

Proof. The proof will be by strong induction.

n = 1

1. $g(x) = 1x$
2. $g(x) = 0x$

n = 2

1. $g(x) = 1x^2 + x$
2. $g(x) = 0x^2 + x$
3. $g(x) = 1x^2 + 0x$

The base case holds.

Induction hypothesis:

Assume that $|G_j| = F(j+2)$ for all $j \in \{1, 2, \cdots, k\}$ for some k.

Inductive step:

consider a $g(x) \in G_{k+1}$.

Case 1. $a_{k+1} = 1$. In which case the rest of the polynomial can be any polynomial in G_k

Case 2. $a_{k+1} = 0$. which implies the $a_k = 1$. In which case the rest of the polynomial can be any polynomial in G_{k-1}.

Since every polynomial is in G_{k+1} one of these cases $|G_{k+1}| = |G_k| + |G_{k-1}| = F(k+2) + F(k+1)$. And by applying the Induction hypothesis $|G_k| = F(k+2) + F(k+1) = F(k+3)$. Therefore by the Principle of Mathematical Induction $|G_n| = F(n+2)$

□

Theorem 3. *let* $g(x) \in G_{n-1}$. $(x+1)(x^n + g(x) + 1) = f_{n+1}(x)$

Proof. let $g(x) = \sum_{i=0}^{n-1} a_i x^i$. such that if $a_i = 0$ then $a_{i+1} \neq 0$ consider the following

$$\begin{aligned}
(x+1)(x^n + g(x) + 1) &= (x+1)x^n + (x+1)g(x) + (x+1) \\
&= x^{n+1} + x^n + x(g(x)) + g(x) + x + 1 \\
&= x^{n+1} + x^n + \sum_{i=1}^{n-1} a_i x^{i+1} + \sum_{i=1}^{n-1} a_i x^i + x + 1 \\
&= x^{n+1} + x^n + \sum_{i=2}^{n} a_{i-1} x^i + \sum_{i=1}^{n-1} a_i x^i + x + 1 && \text{(reindexing 1st sum } i \to i+1\text{)} \\
&= x^{n+1} + x^n + a_{n-1}x^n + \sum_{i=2}^{n-1} a_{i-1} x^i + \sum_{i=2}^{n-1} a_i x^i + a_i x + x + 1 && \text{(matching sum indices)} \\
&= x^{n+1} + x^n + \sum_{i=2}^{n-1} (a_{i-1} + a_i) x^i + x + 1 && \text{(combining sums and simplifying)} \\
&= x^{n+1} + x^n + \sum_{i=2}^{n-1} x^i + x + 1 && \text{(Since } (a_{i-1} + a_i) = 1\text{)} \\
&= f_{n+1}(x) && \text{(by definition of } f_{n+1}(x)\text{)}
\end{aligned}$$

Therefore $(x+1)(x^n + g(x) + 1) = f_{n+1}(x)$. □