U-Soar 2025 Summary

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Abstract

The Boolean semifield, denoted as $\mathbb{B}$, is an algebraic structure with two operations: logical OR, analogous to addition, and logical AND, analogous to multiplication. Our research explores the factorization of polynomials over the Boolean semifield, denoted as $\mathbb{B}[x]$.

A key challenge in $\mathbb{B}[x]$ is that factorization is not unique. Specifically, we investigate the factorization of a particular class of polynomials in $\mathbb{B}[x]$ namely:

$$f_n(x) = \sum_{i=0}^{n} x^i = x^n + x^{n-1} + \cdots + x + 1$$

Understanding the non-unique factorization of these polynomials is a central focus of our study. This research contributes to a deeper understanding of algebraic structures beyond traditional fields and rings.

1 Definitions

Let $\mathbb{B}$ be the Boolean semifield, $\mathbb{B} = \{0, 1\}$ equipped with two binary operations logical OR $(+)$ which is analogous to addition and logical AND $(\cdot)$ which is analogous to multiplication. And let $\mathbb{B}[x]$ be the set of all polynomials over this semifield. We define a specific polynomial $f_n(x)$ as the sum of powers of x from 0 to n, which can be written as $f_n(x) = \sum_{i=0}^{n} x^i = x^n + x^{n-1} + \cdots + x + 1 \in \mathbb{B}[x]$. Furthermore, we introduce an isomorphism $\theta(f(x))$ that maps polynomials from $\mathbb{B}[x]$ to Numbral arithmetic (described in the paper On the Sequence A079500 and Its Combinatorial Interpretations'), with $\theta^{-1}(n)$ being its inverse. Notably, the isomorphism $\theta(f(x))$ is equivalent to evaluating the polynomial at $x = 2$, expressed as $\theta(f(x)) = [f(2)]$, where $[n]$ represents the base 2 representation of an integer n.

2 Factorizations of the form $(x + 1)h(x)$

Let G_n be the set of $g(x) \in \mathbb{B}[x]$ such that $g(x) = \sum_{i=1}^{n} a_i x^i$ and for all a_i if $a_i = 0$ then $a_{i+1} \neq 0$. Let $F(n)$ = the nth Fibonacci number.

Theorem 2.1. $|G_n| = F(n + 2)$

Proof (by strong induction): **Base Cases:**

- $n = 1$; $G_1 = \{1x, 0x\}$; $|G_1| = 2 = F(3)$
- $n = 2$; $G_2 = \{1x^2 + 1x, 1x^2 + 0x, 0x^2 + 1x\}$; $|G_2| = 3 = F(4)$

Therefore the base cases hold.

Induction Hypothesis

Assume that $|G_j| = F(j + 2)$ for all $j \in \{1, 2, \ldots, k\}$ for some k.

Inductive Step

Let $g(x) \in G_{k+1}$. Consider the cases for a_{k+1}.

1. Case 1. $a_{k+1} = 1$. In which case the rest of the polynomial can be any polynomial in G_k.
2. Case 2. $a_{k+1} = 0$ which implies $a_k = 1$. In which case the rest of the polynomial can be any polynomial in G_{k-1}.

So, $G_{k+1} = G_k + G_{k-1} = F_{k+2} + F_{k+1} = F_{k+3}$. Therefore, by the Principle of Mathematical Induction $|G_n| = F_{n+2}$. □

Theorem 2.2. *$(x+1)h(x) = f_n(x)$ if and only if $h(x) = x^{n-1} + g'(x) + 1$ where $g'(x) \in G_{n-2}$.*

Proof. ($\rightarrow$): Assume that $(x+1)h(x) = f_n(x)$ and let $h(x) = \sum_{i=0}^{n-1} a_i x^i$. (Notice $a_{n-1} = a_0 = 1$ otherwise it would contradict the assumption). Now assume for the sake of contradiction that there exists $k < n-1$ such that $a_k = a_{k+1} = 0$. Consider the following:

$$(x+1)h(x) = x\sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} a_i x^i.$$

Notice the $(k+1)^{\text{th}}$ term will have coefficient $a_k + a_{k+1} = 0$, contradicting the original assumption.

($\leftarrow$): Assume $h(x) = \sum_{i=0}^{n-1} a_i x^i$ with $a_{n-1} = a_0 = 1$ and for all a_i, if $a_i = 0$ then $a_{i+1} \neq 0$. Consider the following:

$$(x+1)h(x) = \sum_{i=0}^{n-1} a_i x^{i+1} + \sum_{i=0}^{n-1} a_i x^i = a_{n-1}x^n + \sum_{i=1}^{n-1}(a_{i-1} + a_i)x^i + a_0 = f_n(x).$$

Therefore, the theorem holds. □

Theorem 2.3. *There are $F(n)$ polynomials $h(x)$ such that $(x+1)h(x) = f_n(x)$.*

Proof. Consider the following cases:

- $n = 1$. $h(x) = 1$ is the only valid $h(x)$ such that $(x+1)h(x) = f_1(x)$. So, the theorem holds in this case.
- $n = 2$. $h(x) = x + 1$ is the only valid $h(x)$ such that $(x+1)h(x) = f_2(x)$. So, the theorem holds in this case.
- $n \geq 3$. In this case, the theorem follows directly from the two previous theorems.

□

3 Divisibility Conditions for $f_n(x)$

Lemma 3.1. *Lemma 3.1 from the paper 'On the Sequence A079500 and Its Combinatorial Interpretations' states conditions for divisibility in the numbral arithmetic: For each $n, d > 0$, it holds that $[d]$ is a divisor of $[2^n - 1]$ if and only if it ends with the digit 1, and it does not contain any subsequence of 0s having length greater than $n - k$, with k being the length of $[d]$.*

Theorem 3.1. *If $2k - 1 \leq n$ then $x^k + 1$ divides $f_n(x)$.*

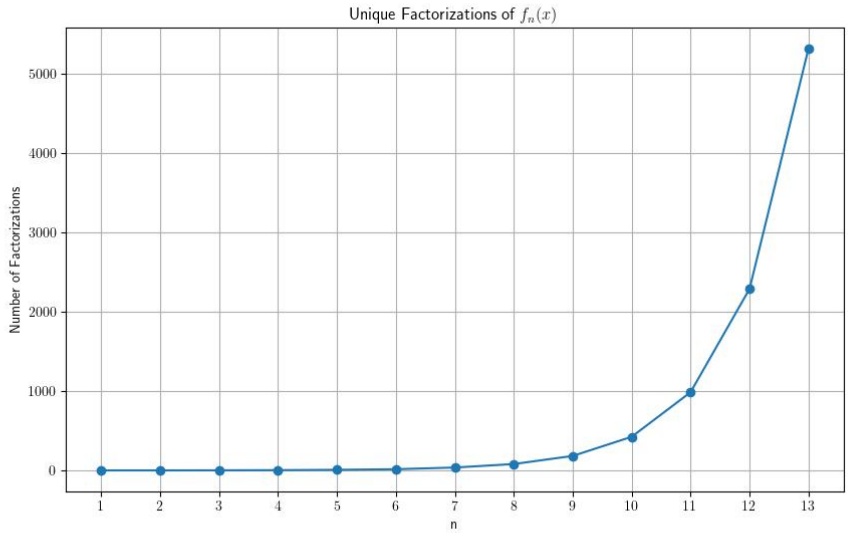
Proof. Assume $2k - 1 \leq n$ and let $d = \theta(x^k + 1) = 10\cdots01$ (notice $[d]$ is length $k+1$ and has $k-1$ zeros). Notice $\theta(f_n(x)) = 2^{n+1} - 1$. By the above lemma, d divides $[2^{n+1} - 1]$ because $k - 1 < n + 1 - (k+1)$, which is equivalent to $2k - 1 \leq n$. And by applying θ^{-1} we see $x^k + 1$ divides $f_n(x)$. □

Theorem 3.2. *Let $g(x) \in \mathbb{B}[x]$ such that the constant term of $g(x)$ is 1. If $\deg(g(x)) = k$ and $2k - 1 \leq n$, then $g(x)$ divides $f_n(x)$.*

Proof. Assume $2k - 1 \leq n$ and let $d = \theta(g(x))$ (notice $[d]$ is length $k+1$). Let m be the maximum length of zeros in $[d]$. Notice $m \leq k - 1$. Notice $\theta(f_n(x)) = 2^{n+1} - 1$. By the above lemma, d divides $2^{n+1} - 1$ because $k - 1 \leq n + 1 - (k+1)$, which is equivalent to $2k - 1 \leq n$. And by applying θ^{-1} we see $g(x)$ divides $f_n(x)$. □

4 Number of unique Factorizations of f_n

Since Factorization in $\mathbb{B}[x]$ is not unique an interesting question is how many unique factorizations are there for a given polynomial? we looked at factorizations for the class of polynomials $f_n(x)$. One of the challenges of this is that factoring in $\mathbb{B}[x]$ is an NP-complete problem. This caused us only to be able to run the factorization algorithm for low values of n.



We also looked at the ratio at which the number of factors increases. The ratio seemed to approach 2.32. More work is needed to see if this trend continues as n increases.

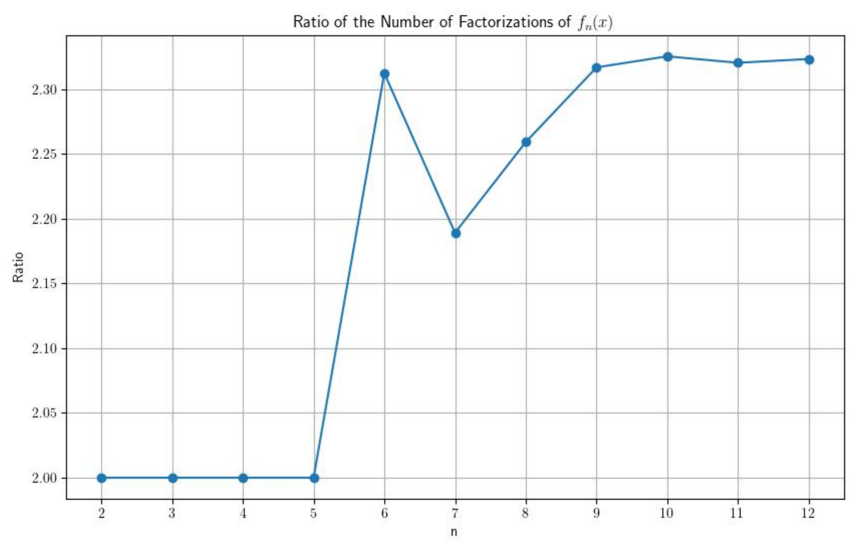


Table 1: Number of unique factorizations

n	$2^{n+1}-1$	# factorizations
1	3	1
2	7	1
3	15	2
4	31	4
5	63	8
6	127	16
7	255	37
8	511	81
9	1023	183
10	2047	424
11	4095	986
12	8191	2288
13	16383	5316

Table 2: Unique Factorizations by number of irreducible factors

n	$2^{n+1}-1$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	1												
2	7		1											
3	15		1	1										
4	31		2	1	1									
5	63		2	4	1	1								
6	127		3	7	4	1	1							
7	255		6	16	9	4	1	1						
8	511		11	33	22	9	4	1	1					
9	1023		18	75	51	24	9	4	1	1				
10	2047		42	162	125	56	24	9	4	1	1			
11	4095		84	373	290	142	58	24	9	4	1	1		
12	8191		179	833	696	336	147	58	24	9	4	1	1	
13	16383		385	1888	1613	833	351	149	58	24	9	4	1	1

5 Future work

Related future topics could include:

- Establishing better upper and lower bounds for the number of factorizations of $f_n(x)$.
- Finding better criteria for determining when a polynomial is irreducible.
- Determining the number of factors of forms other than $(x+1)h(x) = f_n(x)$.
  - We suspect $(x^k+1)h(x) = f_n(x)$ is a good class of candidates for making similar arguments.
- General improvements such as faster algorithms.