INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

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#1: UNDERGRADUTE MECHANICS

`PROBLEM:` Two particles of mass m_1 and m_2, move about each other in circular orbits under the influence of gravitational forces with a period τ. The motion is instantly stopped and the particles are then released and allowed to fall into each other. Find the time after which they collide. Express your answer in terms of τ.

#2: UNDERGRADUTE MECHANICS

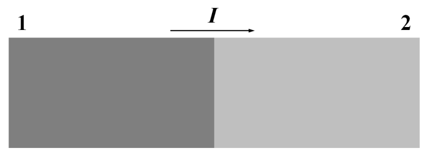
`PROBLEM:` A bead of mass m_1 can slide on a horizontal frictionless rod; call its position X. A (massless) string connects the bead to another mass, m_2, which hangs down because of gravity; call its angle relative to vertical ϕ.

(a) Write the Lagrangian for the above system, in the coordinates given above. Please simplify the expression as much as you can (e.g. use basic trig identities).

(b) Compute the generalized momenta p_X and p_ϕ, and associated generalized forces.

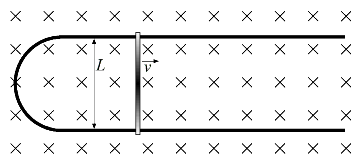
(c) Write the Euler-Lagrange equations of motion.

(d) Find expressions for all conserved quantities.



#3: UNDERGRADUTE E&M

`PROBLEM:` A current I flows through a wire made of a piece of material 1 and a piece of material 2 of identical cross-sections A welded end-to-end as shown in the figure. The resistivity of material 1 is ρ_1, the resistivity of material 2 is ρ_2. How much electric charge accumulates at the boundary between the two materials?



#4: UNDERGRADUTE E&M

`PROBLEM:` A cylindrical rod made of a material with a density ρ, and electrical conductivity σ, has a small diameter and a length L and is moving with a speed v along an infinite U-shaped wire in a uniform magnetic field B, which is applied perpendicularly to the wire and the rod. The distance between the wires is the same as the rod length. The electrical resistance of the wire and the friction between the wide and the rod are negligible, and there is no gravity. Find the distance the wire will move before it stops.

#5: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` Assume particles A,B, and C are bosonic states with angular momentum $J = 0$ and parity $P = -1$. Particle D is a bosonic state with $J = 1$ and $P = -1$, and particle E is boson with $J = 0$ and $P = +1$. Assume that J and P are conserved. Which of the following transitions are allowed? Please state your answer with reasons for each case.

1. $A \to B + C$
2. $D \to B + C$
3. $E \to B + C$
4. $D \to A + A$

#6: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in a time dependent electric field $\overrightarrow{E} = E(t)\hat{k}$. Consider transitions between the n=1 ground state, and the n=2 excited states given the perturbation $H' = eEz$.

(a) Use symmetry to show that for all n=1 and n=2 states the matrix element $H'_{ii} = 0$.

(b) Use symmetry arguments to show that only one of the n=2 states can be reached from the n=1 ground state.

(c) Calculate the one remaining transition matrix element H'_{ij} between n=1 and n=2 states that is not zero.

#7: UNDERGRADUTE STAT MECH/THERMO

PROBLEM:

1.5 mol of Helium considered as an ideal gas is initially at a temperature of 300 K and has a volume of 15 liters. It undergoes an isothermal expansion to a volume of 30 liters and then an adiabatic contraction to the initial volume. By what factor does the number of gas molecules with the x-component of velocity between 200 m/s and 200.1 m/s change? Various constants are (in SI units) Avogadro number: 6.03×10^{23}; Boltzmann constant: 1.38×10^{-23}; mass of proton: 1.67×10^{-27}; molar mass of Helium 0.004; universal gas constant: 8.31; charge of electron: 1.6×10^{-19}. You might find the following integral useful: $\int_{-\infty}^{+\infty} e^{(-\lambda x^2)} = \sqrt{\pi/\lambda}$.

#8: UNDERGRADUTE STAT MECH/THERMO

`PROBLEM:` The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice, at the melting point (absolute temperature T_0). We want to use an ideal, maximally efficient, cyclic (reversible) refrigerator (powered by some external mechanical work) to freeze a mass m amount more of the liquid water in the bucket into ice. The refrigerator rejects all heat to a finite, external reservoir, of constant heat capacity C, that is initially also at temperature T_0.

(a) What is the change in the entropies of (i) the contents of the bucket where the water is changed to ice, (ii) the refrigerator, and (iii) the external reservoir? Write your answers as inequalities if appropriate.

(b) What is the change in the Gibbs function of the ice-water mixture in this process?

(c) What is the minimum mechanical work required to run the refrigerator for this process?

#9: UNDERGRADUTE MATH

`PROBLEM:` Evaluate the integral

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$$

around a circle C with equation $|z| = 3$.

#10: UNDERGRADUTE GENERAL

`PROBLEM:` a) Estimate the life time in years of a star that has a total mass 2×10^{30} kg, assuming that it life ends when the nuclear reactions consume 10% of all of the hydrogen, and that the star radiates at a constant rate that gives a flux, $f = 1360$ W/m^2, at $a = 1.5 \times 10^{11}$ m from the star.

b) The center of a star is dense and fully ionized. Estimate the time in years for a photon to escape from the center of the star out to a radius of $r = 10^8$ m, using a mean (plasma) density of $\rho = 10^5$ kg/m^3. Assume only elastic scattering off electrons and that the photon wavelength is much smaller than the mean free path.

The mass of a proton is $m_H = 1.673 \times 10^{-27}$ kg, the mass of an electron is $m_e = 9.11 \times 10^{-31}$ kg and the mass of a helium nucleus is $m_{He} = 6.646 \times 10^{-27}$ kg. The Thomson cross-section is $\sigma_T = 6.65 \times 10^{-29}$ m^2.

#11: GRADUATE MECHANICS

`PROBLEM:` A non-uniform spherical mass M, of radius a and moment of inertia I rolls on the outside of a fixed hemi-spherical surface of radius R. The sphere starts at rest at the top of the hemi-sphere, $\theta_1 = 0$, and then rolls down, without slipping, until it loses contact with the hemisphere. Here θ_1 is the coordinate for the position of the CM of the sphere.

(a) Write the Lagrangian, including the constraints via Lagrange multipliers.

(b) Find the value of θ_1 where the sphere first loses contact with the hemi-sphere, for general moment of inertia I.

(c) Which would lose contact first: a uniform solid ball, or a thin spherical shell?

#12: GRADUATE MECHANICS

`PROBLEM:`

The Hamiltonian for a charged particle moving in a uniform magnetic field $B\hat{z}$ is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2.$$

(a) Find Hamilton's equations of motion for this Hamiltonian.

(b) Find at least three constants of the motion.

(c) Solve Hamilton's equations of motion of part (a) for $x(t)$, $p_x(t)$, $y(t)$ and $p_y(t)$ in terms of initial value data $x(0)$, $p_x(0)$, $y(0)$ and $p_y(0)$.

(d) Find the Lagrangian corresponding to this Hamiltonian. Find the Euler-Lagrange equations of this Lagrangian. Check that your solutions for $x(t)$ and $y(t)$ from part (a) satisfy the Euler-Lagrange equations.

#13: GRADUATE E&M

`PROBLEM:`

The electric potential that results when a metallic sphere of radius R is placed in an external uniform electric field E_0 pointing in the z direction is of the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos\theta) \tag{1}$$

where P_l are Legendre polynomials.

(a) Give the expression $\phi_0(r,\theta)$, for the electric potential due to the external electric field only in spherical coordinates.

(b) By considering the potential $\phi(r,\theta)$ for large values of r, deduce which coefficients A_l, B_l in the expression for ϕ must vanish.

(c) By considering $\phi(r,\theta)$ on the surface of the sphere, deduce the values of all the coefficients A_l, B_l in the formula for ϕ, and hence an expression for $\phi(r,\theta)$.

(d) Find the magnitude of the electric field at $\theta = 0$ right outside the surface of the sphere.

Show all steps in your derivations and justify all steps.

#14: GRADUATE E&M

`PROBLEM:`

The radiation fields $\vec{E}, \vec{B}$ of an oscillating electric dipole $\vec{P}(t)$ are expressed in the SI unit as

$$\begin{aligned} \vec{B}(\vec{r}, t) &= -\tfrac{\mu_0}{4\pi r c} \hat{e}_r \times \tfrac{\partial^2}{\partial_t^2} \vec{P}(t - \tfrac{r}{c}) \\ \vec{E}(\vec{r}, t) &= -c \hat{e}_r \times \vec{B}(\vec{r}, t), \end{aligned} \tag{2}$$

where the dipole is put at the origin, μ_0 is the vacuum permeability; c is the light velocity; $\hat{e}_r$ is the unit vector along the radial direction. Using these formulae, solve the following problems.

a) Put a point charge q at the origin. It is driven by a planar polarized electromagnetic plane wave $\vec{E} = \hat{x} E_0 e^{i(kz - \omega t)}$ where $k = \frac{\omega}{c}$. What is the radiation field $\vec{E}$ and $\vec{B}$ of this point charge.

b) Use the spherical coordinate, derive the angular distribution of the radiation power intensity (Poynting vector) $\vec{S}(\theta, \phi)$.

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:`

A particle with charge q and mass M is confined in the 2D xy-plane and moves in the presence of an external magnetic field $\vec{B} = B\hat{z}$. Its Hamiltonian is written as

$$H = \frac{(-i\hbar\vec{\nabla} - q\vec{A})^2}{2M}, \tag{3}$$

Use the symmetric gauge $\vec{A} = \frac{B}{2}\hat{z} \times \vec{r}$.

(a) Solve for the semiclassical motion. According to the electron's classic equation of motion and Bohr-Sommerfeld quantization condition, what is the radius l_B of the smallest cyclotron orbit? What is the angular frequency ω associated with this smallest cyclotron orbit?

(b) The "mechanical momentum" is defined as $\vec{P}_m = -i\hbar\vec{\nabla} - q\vec{A}$. Define operators a and $a^\dagger$ as $a = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} + iP_{m,y})$ and $a^\dagger = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} - iP_{m,y})$. Work out their commutation relations $[a, a^+]$.

(c) Show that the Hamiltonian can be written in terms of a and a^+ and find all its eigenvalues.

(d) Define the "guiding center coordinates" $\vec{R}_g = \vec{r} - \hat{z} \times \frac{\vec{P}_m}{M\omega}$ and the operators $b = \frac{1}{\sqrt{2}l_B}(R_{g,x} - iR_{g,y})$ and $b^\dagger = \frac{1}{\sqrt{2}l_B}(R_{g,x} + iR_{g,y})$. Work out the commutation relation $[b, b^\dagger]$. Prove that $[a, b] = [a^\dagger, b^\dagger] = 0$, and $[a, b^\dagger] = [a^\dagger, b] = 0$.

(e) Express the canonical angular momentum $L_z = (\vec{r} \times -i\hbar\vec{\nabla}) \cdot \hat{z}$ in terms of a, $a^\dagger$, b and $b^\dagger$ and find all its eigenvalues.

(f) For each energy level of Eq. 3, figure out its allowed eigenvalues of L_z.

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in an external perturbing potential

$$V = f(r)\left(x^2 + y^2\right)$$

You are given the matrix elements

$$\begin{aligned}
\langle 2s, m=0|f(r)x^2|2s, m=0\rangle &= v_x \\
\langle 2s, m=0|f(r)y^2|2s, m=0\rangle &= v_y \\
\langle 2s, m=0|f(r)z^2|2s, m=0\rangle &= v_z \\
\langle 2p, m=0|f(r)x^2|2p, m=0\rangle &= w_x \\
\langle 2p, m=0|f(r)y^2|2p, m=0\rangle &= w_y \\
\langle 2p, m=0|f(r)z^2|2p, m=0\rangle &= w_z
\end{aligned}$$

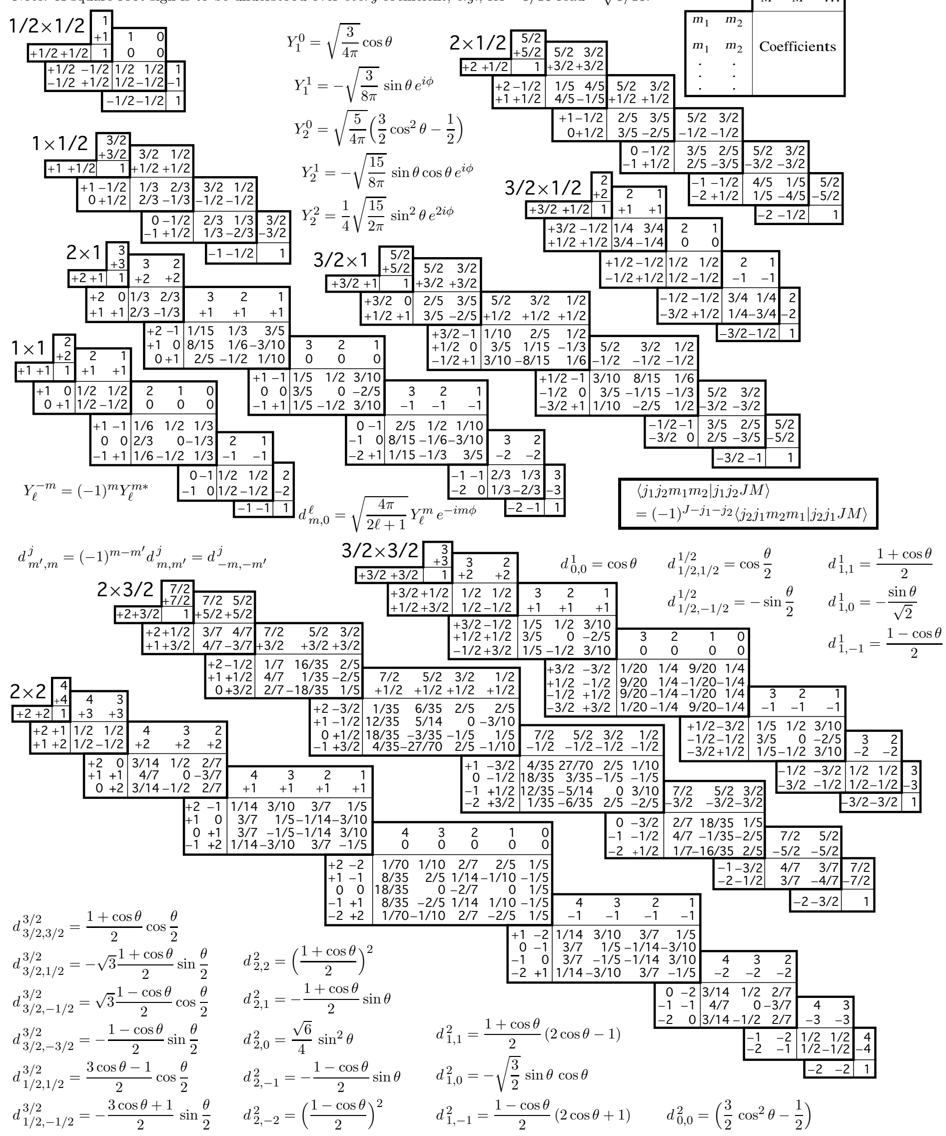
(a) There are two relations between v_x, v_y and v_z. Find these to write v_x and v_y in terms of v_z.

(b) There is one relation between w_x, w_y and w_z. Find this to eliminate w_y.

(c) Find the first order shift in the energy levels of the $n = 2$ states in terms of v_z and $w_{x,z}$.

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*} \qquad d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

#17: GRADUATE STAT MECH

`PROBLEM:`

In the Ising model of ferromagnetism, the spin at each lattice site σ_i can take the values ± 1. The energy for each configuration of spins is

$$E = -J \sum_{<ij>} \sigma_i \sigma_j \tag{4}$$

where $J > 0$ and the sum $< ij >$ is over nearest neighbor sites. Assume each site has z nearest neighbors.

Call $m =< \sigma_j >$ the thermal average of the spin σ_j at temperature T. In the mean field approximation, the interaction of the spin σ_i with a neighbor σ_j is approximated by replacing σ_j by $< \sigma_j >= m$.

(a) Set up a self-consistency condition for the value of m using the fact that all lattice sites are equivalent, that will determine the value of m as function of J, z, and the temperature T. Hint: Use the canonical ensemble.

(b) Show that the self-consistency condition has a solution with $m \neq 0$ only for T lower than a critical value T_c, and find an expression for T_c in terms of z and J.

(c) By expanding the self-consistency condition for small m, show that m for T close to T_c can be written as

$m = C(T_c - T)^\beta$

with β and C constants, and find the values of β and C.

Show all steps in your derivations and justify all steps.

#18: GRADUATE STAT MECH

`PROBLEM:` Consider a gas of non-interacting, non-relativistic spin-1 bosons in an external magnetic field $B\hat{z}$. The one-particle Hamiltonian is

$$\mathcal{H}(\vec{p}, s_z) = \frac{\vec{p}^{\,2}}{2m} - \mu_0 s_z B,$$

where $\mu_0 \equiv \frac{e\hbar}{mc}$ and s_z, the spin quantum number in the $\hat{z}$ direction, can take three possible values $-1, 0, 1$.

(a) In the grand canonical ensemble, what are the average occupation numbers $\langle n_+(\vec{k})\rangle$, $\langle n_0(\vec{k})\rangle$ and $\langle n_-(\vec{k})\rangle$ of one-particle states with wavenumber $\vec{k} = \frac{\vec{p}}{\hbar}$ and with $s_z = -1,\ 0,\ +1$?

Use these average occupation numbers to find the average total numbers N_+, N_0, N_- of bosons with $s_z = -1,\ 0,\ +1$ in terms of the functions

$$f_m^+(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty dx\ \frac{x^{m-1}}{z^{-1}e^x - 1},$$

where $z \equiv e^{\beta\mu}$.

(b) The total number density is equal to

$$n = \frac{N_+ + N_0 + N_-}{V}.$$

Find the temperature $T_c(n, B = 0)$ of Bose-Einstein condensation at zero field in terms of the number density n.

#19: GRADUATE MATH

`PROBLEM:` Find the first term in the asymptotic expansion of the following integral (i.e. the behavior of the integral in the limit $x \to \infty$).

$$I = \frac{1}{\pi} \int_0^{\pi} (t^4 + 2t^6)^{1/2} e^{x \cos t} \cos{(nt)} dt,$$

where n is a constant. You may want definition of the Gamma function: $\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du$, with $\Gamma(1/2) = \sqrt{\pi}$.

#20: GRADUATE GENERAL

`PROBLEM:`

a) We examine the distribution of bubbles that form along the path of a charged particle in a bubble chamber and we find that bubbles are apparently randomly distributed with a uniform probability of occurance per unit length. This is equivalent to the following statements:

(i) There is at most one bubble in an infinitesimal interval of length $[l, l+\Delta l]$.

(ii) The probability $P_1(\Delta l)$ of finding one bubble in this interval is proportional to Δl (as long as $P_1(\Delta l)$ remains small).

(iii) The occurrence of a bubble in the interval $[l, l+\Delta l]$ is independent of the occurrence of bubbles in any other non-overlapping interval.

Derive an equation that gives the probability $P_o(l)$ of zero bubbles in the finite interval of length l, assuming that the average density of bubbles per unit length is g and that bubbles have negligible size.

b) Derive the probability density (per unit length) $f(l)$ that the first bubble on a track is at a distance l from some arbitrary origin.

c) If we count one bubble in a length of $l = 1$ mm, what is (1) the uncertainty in the observation and (2) the 68.3% confidence interval (one-sigma uncertainty) associated with our best estimate for g?

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#1: UNDERGRAD MECHANICS

`PROBLEM:` A man standing on earth bends his knees, lowering his center of mass 50 cm. Then he jumps up, raising his center of mass 60 cm above its initial position (i.e. 110 cm above its lowered position). What would be the height of the jump on the moon. Assume that the man exerts the same constant force until he leaves the ground when jumping on the earth and on the moon. The moon radius is 0.275 of the earth radius and the moon density is 0.604 of the earth density.

`SOLUTION:`

The acceleration of gravity on earth is

$$g_e = Gm_e/r_e^2 = G(\tfrac{4}{3}\pi r_e^3\rho_e)/r_e^2 = \tfrac{4}{3}G\pi r_e\rho_e,$$

where G is the gravitational constant, m_e is the earth mass, r_e is the earth radius, ρ_e is the earth density. The acceleration of gravity on the moon is

$$g_m = \tfrac{4}{3}G\pi r_m\rho_m = g_e r_m\rho_m/(r_e\rho_e) \approx g_e/6,$$

where r_m is the moon radius, ρ_m is the moon density.

Since the force is the same on earth as on the moon, the work done is the same, and $mg_m(h_m + 50) = mg_e(h_e + 50)$, where m is the man mass and h_m and h_e are the heights of the jump on the moon and earth, respectively. For $h_e = 60$ cm, we find $h_m = 6.1$ m.

#2: UNDERGRAD MECHANICS

PROBLEM:

Consider two famous experiments that determined the size and mass of the Earth:

(A) Around 235 BC Eratosthenes conducted the following experiment. On June 22 in Syene, Egypt, the sun was directly overhead. A stick planted vertically in the ground in Syene cast no shadow, but a stick planted vertically in Alexandria, a city 800 kilometers north of Syene over flat ground, cast a shadow 1/8 the height of the stick. Determine the radius of the earth from this measurement.

(B) In 1798 Cavendish reported a determination of the mass of the Earth. At the time, Newton's law of gravity was known but the Gravitational constant G was not accurately known. The acceleration of the gravity near the surface of the Earth was known to be 9.8 m/s/s. The experiment consisted of a 1.8 m wood rod, of negligible weight, suspended from a wire with a 0.73 kg lead ball attached to each end. Two 158 kg lead balls could be placed near the smaller balls on alternate sides of the rod. The faint gravitational attraction between the small and large balls caused the arm to rotate, twisting the wire. When the rod was disturbed it was observed to oscillate with a period of about 15 minutes. When large balls were placed 0.23 m away from the smaller balls (center-to-center distance), the rod rotated until the smaller balls were pulled 4.1 mm closer to the larger balls. Determine the mass of the earth from this measurement.

SOLUTIONS:

(A) The shadow cast by the vertical stick in Alexandria indicates the angle of the rays of sunlight relative to the Earth's normal is equal to arctan(1/8)=0.124 radians. This is ∼1/50th of a full circle (2*pi radians). Since the vertical sticks are parallel to the Earth's radius the distance between Syene and Alexandria must be ∼1/50th of the circumferance of the Earth, or 800 x 50 = 40,000 km. The Earth's radius is therefore about 40,000/2/pi = 6,400 km.

(B) The force between two lead balls is $F_{balls} = G\frac{m_1 m_2}{d^2}$, and is determined by the experiment.

The force between m_1 and the Earth is $F_{Earth} = G\frac{m_1 m_{Earth}}{r_{Earth}^2} = m_1 g =$ (0.73 kg)(9.8 m/s/s) = 7.2N.

By finding the ratio $\frac{F_{balls}}{F_{Earth}} = \frac{m_2}{d^2}\frac{r_{Earth}^2}{m_{Earth}}$ one can determine $m_{Earth} = \frac{F_{Earth}}{F_{balls}}\frac{m_2}{d^2}r_{Earth}^2$ without knowing G.

The experiment determines F_{balls} as follows. The suspended rod acts as a torsional pendulum with equation of motion $\tau = -k\Delta\theta = I\alpha$. k is the torsional spring constant and $I = 2m_1 r^2$ is the moment of inertia where $r =$ half length of the rod. The period of oscillation is $T = 2\pi\sqrt{I/k}$, which determines k:

$k = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2 m_1 r^2}{T^2}$ = 8*pi^2*0.73*0.9^2/(15*60)^2=5.73e-5 (MKS units)

When the balls are pulled a distance $\Delta s = 4.1e-3$m closer due to gravitational force this torque is balanced by the torsional spring force of the wire: $\tau = 2F_{balls}r = -k\Delta\theta = k(\Delta s/r)$

Thus $F_{balls} = k(\Delta s/2r^2)$ = (5.76e-5)(4.1e-3)/2/0.9^2 = 1.46e-7 N.

Using the radius of the Earth of 6,400 km found in part (A) one can thus calculate:

$m_{Earth} = \frac{F_{Earth}}{F_{balls}}\frac{m_2}{d^2}r_{Earth}^2$ = 7.2*158*(6400e3)^2/1.46e-7/0.41^2=6e24 kg

#3: UNDERGRAD E&M

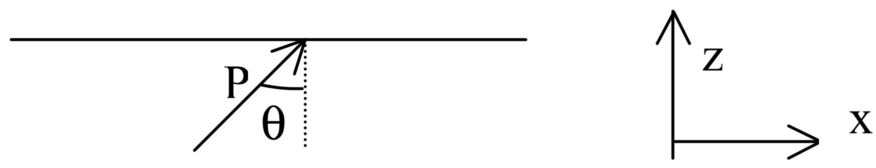
`PROBLEM:` A spherical capacitor consists of two concentric spheres or radii a and b, $a < b$. The charge on the inner and outer spheres, $Q(t)$ and $-Q(t)$ respectively, decreases with time as radial current runs through the material between the spheres (which has dielectric constant ε and permittivity μ.) Find $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ due to Q.

`SOLUTION:`

Gauss' law implies $D = Q/r^2$ between the spheres so $E = Q/\varepsilon r^2$ (in the radial direction). The electric field is zero everywhere else by Gauss's law. To find $\mathbf{B}$, note that the radial current density is $j = -\dot{Q}/4\pi r^2$. However the displacement current is also radial, and given by $j_D = \dot{Q}/4\pi r^2$ so $\nabla \times \mathbf{H} = \nabla \times \mathbf{B}/\mu = \mathbf{0}$, which implies $\mathbf{B} = \nabla\psi$ for some scalar potential ψ. Since $\nabla \cdot \mathbf{B} = \mathbf{0}$ as well, $\nabla^2\psi = 0$. Symmetry implies ψ is a function only of r and t so the solution of Laplace's equation is $\psi = A + B/r$. However $B = 0$ because this term can only be due to a magnetic monopole. (Alternate argument: The integral constraint $\int \mathbf{B} \cdot \hat{n}\, d^2r = 0$ implies $B = 0$). Therefore $\mathbf{B} = 0$ everywhere.

#4: UNDERGRAD E&M

`PROBLEM:` A thin electrically insulating sheet of material has thickness d and lateral extent $L \times L$, where $d \ll L$. It is in a vacuum and isolated from external electric fields. It has frozen-in polarization per unit volume $\vec{P}$ oriented in the (x,z) plane at an angle θ to the normal to the surfaces, which is in the $\hat{z}$ direction.



Find the magnitudes and directions of the electric displacement $\vec{D}$ and the electric field $\vec{E}$ both inside and outside the material, stating clearly your reasoning.

`SOLUTION:`

Bound charge on upper face:

$\sigma = \mathbf{P} \cdot \hat{n} = P\ \cos\theta$

Bound charge on lower face is opposite, implying an electric field within the material from Gauss's law:

$\mathbf{E} = -\frac{\sigma}{\epsilon_0}\hat{z} = -\frac{P}{\varepsilon_0}\cos\theta\hat{z}$

Electric displacement within the material:

$\mathbf{D} = \varepsilon_0\,\mathbf{E} + \mathbf{P} = -P\ \cos\theta\hat{z} + P(\sin\theta\hat{x} + \cos\theta\hat{z}) = P\ \sin\theta\hat{x}$

Outside, $\mathbf{E} = \mathbf{P} = 0$ so $\mathbf{D} = 0$.

#5: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider two entangled spin 1/2 particles separated by a large distance. The four Bell states are given as,

$$\begin{aligned} |\Phi_\pm^{(AB)}\rangle &= \frac{1}{\sqrt{2}}\left(|z+,z+\rangle \pm |z-,z-\rangle\right), \\ |\Psi_\pm^{(AB)}\rangle &= \frac{1}{\sqrt{2}}\left(|z+,z-\rangle \pm |z-,z+\rangle\right), \end{aligned}$$

where $|z\pm\rangle$ are the two spin states along or against the z axis, the first $z\pm$ refers to particle A which is nearby, and the second $z\pm$ refers to particle B which is far away. Consider applying a magnetic field only at location A, either in the $\hat{x}$ or $\hat{z}$ directions for a time t.

(a) Write down the Hamiltonians, H_z and H_x for the two operations above, and also write down the time evolution operators, $U_z(t,0)$ and $U_x(t,0)$.

(b) Find a time duration, t_A, which results in $U_z|z+\rangle = -i|z+\rangle$.

(c) Using time, t_A, find $U_z|z-\rangle$, and $U_x|z\pm\rangle$ (Hint: you might need to change from the $|z\pm\rangle$ basis to the $|x\pm\rangle$ basis.

(d) Using the two operations above show that any one of the Bell states can be transformed into any other, operating only at location A.

`SOLUTION:`

(a) The Hamiltonian $H = -\vec{\mu}\cdot\vec{B}$ only acts on particle A; particle B is unaffected. $H_z = \omega S_z$, with $\omega = |e|B_z/m_e c$, where $S_z|z\pm\rangle = \pm\frac{\hbar}{2}|z\pm\rangle$. The time evolotion operator is $U(t,0) = e^{-iHt/\hbar}$, so $U_z = e^{-i\omega t S_z/\hbar}$ and $U_x = e^{-i\omega t S_x/\hbar}$.

(b) $U_z|z+\rangle = e^{-i\omega t/2}|z+\rangle$, so choose $\omega t_A = \pi$, so $U_z(t_A) = e^{-i\pi/2} = -i$.

(c) $U_z|z-\rangle = e^{i\pi/2}|z-\rangle = i|z-\rangle$. Write $|z\pm\rangle = \frac{1}{\sqrt{2}}(|x+\rangle \pm |x-\rangle)$, then $U_x|z\pm\rangle = -i|z\mp\rangle$.

Hence, using the the above on the Bell states,

$$\begin{aligned}
U_z|\Phi_+^{(AB)}\rangle &= -i|\Phi_-^{(AB)}\rangle, \\
U_z|\Psi_+^{(AB)}\rangle &= -i|\Psi_-^{(AB)}\rangle, \\
U_x|\Phi_+^{(AB)}\rangle &= -i|\Psi_+^{(AB)}\rangle, \\
U_x|\Phi_-^{(AB)}\rangle &= i|\Psi_-^{(AB)}\rangle.
\end{aligned}$$

These together with the inverse transforms will link any Bell state to another.

#6: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` (a) The ground state wave function of the one dimensional harmonic oscillator (Hamiltonian: $H_{\rm harm} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$) has the Gaussian form $\Psi(x) = C\, e^{-k^2x^2}$. Using Schrodinger's Equation, find k, and also find the ground state energy E_0. Show all steps.

(b)Use first order perturbation theory to find the first order correction, ΔE, to the ground state energy given by a perturbation to the potential, $H_{pert} = \lambda|x|$, with λ a constant, and $H_{\rm tot} = H_{\rm harm} + H_{\rm pert}$.

Hint: $\int\limits_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$.

`SOLUTION:`

(a) The 1-D harmonic oscillator has ground state wave function that satisfies $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi = E_0\Psi$. Substituting the trial form $\Psi = C\, e^{-k^2x^2}$ yields

$-\frac{\hbar^2}{2m}(4k^4x^2 - 2k^2) + \frac{1}{2}m\omega^2x^2 = E_0$.

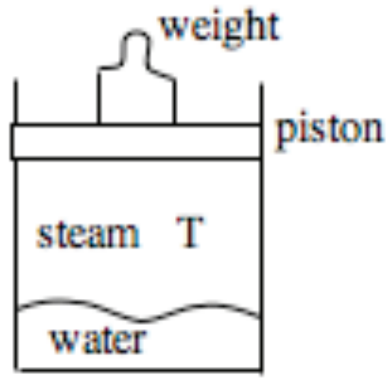
Balancing terms yields $k^4 = \frac{m^2\omega^2}{4\hbar^2}$.

Also, $E_0 = \frac{\hbar^2k^2}{m} = \frac{\hbar\omega}{2}$. Thus, $\Psi(x) = \frac{1}{\pi^{1/4}\sqrt{x_0}}e^{-\frac{x^2}{2x_0^2}}$, with $x_0 = \sqrt{\hbar/m\omega}$. This form is normalized such that $\int |\Psi(x)|^2 dx = 1$.

(b) $\Delta E = \int_{-\infty}^{\infty} \Psi^*(x) H_{\rm pert}\Psi(x) dx$. That is, $\Delta E = \int_{-\infty}^{\infty} \frac{\lambda}{\pi^{1/2}x_0}|x|e^{-x^2/x_0^2}dx$. The integral can be performed using elementary methods, yielding $\Delta E = \lambda x_0/\sqrt{\pi}$.

#7: UNDERGRAD STAT MECH/THERMO

PROBLEM:



The water-steam system shown above is in equilibrium. The temperature is $T = 200^0C$. The piston can move freely and the external pressure (provided by the weight on top of the piston plus the atmospheric pressure) balances the steam pressure. There is 1 mol of water and 1 mol of steam initially. The latent heat of vaporization of water at this temperature is 38,090 J/mol. The gas constant is R=8.31J/mol/K. Assume throughout this problem that the temperature dependence of the latent heat can be neglected.

(a) Assume the temperature is lowered to $T = 180^0C$. When the system reaches equilibrium, how much water and how much steam is there? Note: energy can flow in or out of the system (as heat and/or work) in this process, the external pressure doesn't change, and the piston can move freely.

(b) Same as (a) if instead the temperature is raised to $T = 220^0C$.

(c) Instead of changing the temperature, 20,000 J of heat are added to the system, with the temperature fixed at $T = 200^0C$ and the piston free to move. How much water and how much steam is there when the system reaches equilibrium?

(d) Assume now the piston is clamped into position and cannot move, so the volume occupied by the system doesn't change. The temperature is raised from $T = 200^0C$ to $T = 220^0C$ by adding heat to the system. Calculate how much water and steam there is when the system reaches equilibrium, in moles. Assume the steam behaves like an ideal gas, and that the volume occupied by the water can be neglected since it is much smaller than that of the steam. Use the Clausius-Clapeyron equation

$$\frac{dP}{dT} = \frac{L}{T\Delta V} \tag{1}$$

`SOLUTION:`

(a) For each value of the temperature there is one value of the pressure where the two phases coexist, determined by the condition

$$g_{liquid}(T,P) = g_{vapor}(T,P) \tag{2}$$

with g the Gibbs free energy per mol. Since the external pressure doesn't change, there cannot be coexistence at the lower temperature. Therefore all the steam condenses into liquid water and the piston moves all the way down to the surface of the water.

(b) Similarly, at the higher temperature all the water evaporates.

(c) The heat goes both into latent heat of vaporization, converting some liquid to gas, and into work for lifting the piston. The work done is

$$W = P_0(V - V_0) = RT_0(n - n_0) \tag{3}$$

with P_0 the external pressure. So, with L the latent heat of vaporization per mol:

$$\begin{aligned} Q &= 20,000J = L(n-n_0) + RT_0(n-n_0) = (L + RT_0)(n - n_0) \\ &= (38,090 + 8.31 \times 473)(n - n_0)J = 41,522(n - n_0)J \end{aligned} \tag{4}$$

with n_0 and n the initial and final number of moles of vapor. Hence,

$$n - n_0 = 0.48 \tag{5}$$

so that 0.48 mols of water evaporate, and there are 0.52 moles of water and 1.48 moles of vapor at the end of the process.

(d) The Clausius-Clapeyron eq. yields for the coexistence condition, using the eq. of state for the ideal gas and neglecting the volume occupied by the water:

$$P = Ce^{-L/RT} \tag{6}$$

where C is a constant. So we have initially

$$P_0 = Ce^{-L/RT_0} \tag{7}$$

with $T_0 = 473^0 K$and at the end

$$P_1 = Ce^{-L/RT_1} \tag{8}$$

with $T_1 = 493^0 K$. So

$$P_1 = P_0 e^{\frac{L}{R}(\frac{1}{T_0} - \frac{1}{T_1})} \tag{9}$$

$$\frac{L}{R}(\frac{1}{T_0} - \frac{1}{T_1}) = \frac{38{,}090}{8.31}(\frac{1}{473} - \frac{1}{493}) = 0.393 \tag{10}$$

Hence

$$P_1 = e^{0.393} P_0 = 1.48 P_0 \tag{11}$$

From $PV = nRT$ and with initially $n_0 = 1mol$,

$$n_1 = \frac{P_1 V}{RT_1} = \frac{P_1}{P_0}\frac{T_0}{T_1} n_0 = 1.48 \times \frac{473}{493} n_0 = 1.42 n_0 \tag{12}$$

So there are 0.58 moles of water and 1.42 moles of vapor when the system reaches equilibrium.

#8: UNDERGRADUATE STAT MECH

`PROBLEM:`

Herpes viruses of mass m and density ρ are suspended in water (density ρ_w), placed in a tube, and spun at ω radians per second in a centrifuge. After centrifuging for a long time the suspension reaches thermal equilibrium. The acceleration due to the Earth's gravity is negligible compared with the centrifugal acceleration. If the end of the centrifuge tube is at radius r_0 from the axis of rotation (the largest radius the viruses can reach) what is the equilibrium distribution of concentration of the viruses at radius r relative to that at r_0?

`SOLUTION:`

The net force on an object of mass m floating in water at radius r is the centrifugal force on the object minus the buoyant force, which equals the centrifugal force on the mass of water m_w displaced by the object (Archimedes principle):

$F = m\omega^2 r - m_w\omega^2 r = (m - m_w)\omega^2$

In terms of densities $m - m_w = m - (\rho_w/\rho)m = m(1 - \rho_w/\rho)$; call this m' (the effective mass)

The potential energy of an object at r relative to r_0 is equal to the work required to move the objecg from r_0 to r against the net force:

$E(r) - E(r_0) = -\int_{r_0}^{r} m'\omega^2 r\,dr = m'\omega^2(r_0^2 - r^2)/2$

At thermal equilibrium the probability of finding a particle in state i of Energy E_i relative to state j of energy E_j is given by the Boltzmann equation:

$\frac{P_i}{P_j} = \exp(-(E_i - E_j)/kT)$

The concentration at radius r relative to that at r_0 is thus

$\frac{C(r)}{C(r_0)} = \exp(-m'\omega^2(r_0^2 - r^2)/2kT)$

#9: UNDERGRADUATE MATH

`PROBLEM:`

The unit vector $\vec{n}$ points with equal probability in any direction in three dimensions. Calculate the average values of $(\vec{a}\cdot\vec{n})^2$, $(\vec{a}\cdot\vec{n})(\vec{b}\cdot\vec{n})$, and $(\vec{a}\cdot\vec{n})\vec{n}$ where $\vec{a}$ and $\vec{b}$ are fixed vectors during the averaging.

hint: The tensor $\overline{n_i n_j}$ is the same (invariant) in every coordinate system due to the averaging.

`SOLUTION:`

All averages can be reduced to the average of $\overline{n_i n_j}$ which has to be of the form

$$\overline{n_i n_j} = \lambda \cdot \delta_{ij}.$$

Let us choose $i = j$ and sum over i:

$$\sum_{i=1}^{3} \overline{n_i n_i} = \overline{\vec{n}^2} = \overline{1} = 1 = 3\lambda,$$

so that

$$\overline{n_i n_j} = \tfrac{1}{3} \cdot \delta_{ij}.$$

With this we have

$$\overline{(\vec{a}\cdot\vec{n})^2} = \sum_{i,j} \overline{a_i n_i a_j n_j} = \sum_{i,j} a_i a_j \overline{n_i n_j} = \sum_{i,j} = \tfrac{1}{3}\delta_{ij} a_i a_j = a^2/3;$$

$$\overline{(\vec{a}\cdot\vec{n})(\vec{b}\cdot\vec{n})} = \vec{a}\cdot\vec{b}/3;$$

$$\overline{(\vec{a}\cdot\vec{n})\vec{n}} = \vec{a}/3\ .$$

#10: UNDERGRADUATE GENERAL PHYSICS

PROBLEM:

On a clear night in San Diego, surfaces exposed to the sky can become significantly colder than the ambient air. This can lead to frost even when the temperature remains well above freezing. Consider a thin sheet of horizontal material suspended in the air, perhaps acting as a patio shade. The material is opaque and has a high infrared emissivity. Both parts of the problem seek numerical answers. Estimate reasonable values where needed, and see the hints below.

(i) Estimate the equilibrium temperature decrement of the material exposed to the clear night sky, relative to the ambient air. Assume the terrestrial surroundings are all at the ambient air temperature.

(ii) Once the surface cools to the dewpoint, water will begin to condense, depositing heat as it does so and effectively holding the surface at the dewpoint. If the dewpoint is 5 K below the ambient air temperature, how much water per hour (in millimeters of height) will collect on the surface, and how many liters per night might one collect per square meter of surface?

Hints and Additional Information: Convection in relatively still air exchanges 5 W m^{-2}K^{-1} at each air-material interface; Greenhouse gases prevent the sky from looking like the few °K cosmos beyond, effectively cutting the net outbound radiation in half; It helps to linearize the radiative piece for parts of the problem when ΔT is small; $\sigma = 5.67 \times 10^{-8}$ W m^{-2}K^{-4}; Properties of water at the temperatures of interest: specific heat capacity 4184 J kg^{-1}K^{-1}; heat of fusion 334 J g^{-1}; heat of vaporization 2500 J g^{-1}; density 1000 kg m^{-3}.

SOLUTION:

(i) For each square meter of surface, convection acts on both the top and bottom, adding to 10 W times ΔT. Linearizing the radiated power leads to $P/A = 4\sigma T^3 \Delta T$. At a typical nighttime temperature of 280 K, this yields $4 \times 5 \times 10^{-8} \times 280^3 \Delta T \approx 5.0 \Delta T$ W m^{-2} (numerical factor is 6.1 for $T = 300$ K). The material will be cooler than the surroundings, so the net power *into* each square meter of the material from the local environment is $10\Delta T$ W for convection (both sides) plus $5\Delta T$ W via radiative coupling to the ground below (bottom only).

For radiation to the sky, we cannot use the linearized relation, and are told that the net outbound radiation is half what it would be to a 0 K background. For 280 K, $\sigma T^4 \sim 350$ W m^{-2} (460 W m^{-2} for $T = 300$ K), so the net loss per square meter is 175 W (230 W). Really, the effective blackbody temperature for a dry sky is about 235 K, so the 175 W turns out to be about right for $T = 280$ K.

Setting 175 W $= 10\Delta T + 5\Delta T$, we find that $\Delta T \sim 12$ K.

(ii) Using the results above, if condensation parks the material at $\Delta T =$ 5 K, for every square meter, we have the convection part contributing 50 W and radiation to the ground contributing an additional 25 W. As before, the net skyward radiation is 175 W, leaving 100 W contributed by condensation for each square meter. Since each condensed gram deposits 2500 J, one gram may be deposited every 25 seconds, and 144 grams every hour. At a density of 1 g cm^{-3}, this amounts to a layer 0.14 mm thick each hour. Over the course of a 12 hour night, one might expect a maximum of 1.7 liters of condensation per square meter of roof.

#11: GRADUATE MECHANICS

`PROBLEM:` A particle of mass m is constrained to move on the surface of a cylinder of radius R. The Lagrangian of the particle is

$$L = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2).$$

(a) Find the Hamiltonian corresponding to this Lagrangian.

(b) Derive Hamilton's equations of motion for the particle from the Hamiltonian in part (a). Find the solutions $z(t)$, $p_z(t)$, $\phi(t)$ and $p_\phi(t)$ to the equations of motion in terms of initial value data $z(0)$, $p_z(0)$, $\phi(0)$, $p_\phi(0)$.

(c) Find at least two constants of the motion.

(d) Find a canonical transformation to new canonical variables such that the new canonical momenta are constants.

Hint:

$$\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\frac{y}{a}$$

`SOLUTION:`

(a)

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$H = p_z\dot{z} + p_\phi\dot{\phi} - L = \frac{p_z^2}{2m} + \frac{p_\phi^2}{2mR^2} + \frac{1}{2}k(R^2 + z^2)$$

(b)

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2}, \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$p_\phi(t) = \text{constant} = p_\phi(0)$$

$$\phi(t) = \frac{p_\phi(0)}{mR^2} t + \phi(0)$$

$$z(t) = z(0)\cos\omega t + \frac{p_z(0)}{m\omega}\sin\omega t$$

$$p_z(t) = -m\omega z(0)\sin\omega t + p_z(0)\cos\omega t$$

(c) Two constants of the motion are p_ϕ and H.

$$\dot{p}_\phi = [p_\phi, H] = 0$$

$$\dot{H} = [H, H] = 0$$

$H_z \equiv \frac{p_z^2}{2m} + \frac{1}{2}kz^2 = E_z$ and $H_\phi \equiv \frac{p_\phi^2}{2m} = E_\phi$ also are constants of the motion:

$$[H_z, H] = 0$$

$$[H_\phi, H] = 0$$

(d) Canonical momentum p_ϕ is already a constant of the motion, so only need to perform nontrivial canonical transformation to new P_z which is a constant of the motion. Choose $P_z = E_z$. Thus, canonical transformation from (z, p_z, ϕ, p_ϕ) to new canonical variables $(Q_z, P_z, Q_\phi, P_\phi) = (\beta_z, E_z, \phi, p_\phi)$.

$$S(z, \phi, E_z, p_\phi) = W_z(z, E_z) + p_\phi \phi - E_z t$$

$$K(p_\phi, E_z) = H(z, p_z, \phi, p_\phi) + \frac{\partial S}{\partial t} = \frac{p_\phi^2}{2mR^2} + \frac{1}{2}kR^2$$

$$E_z = \frac{1}{2m}\left(\frac{\partial W_z}{\partial z}\right)^2 + \frac{1}{2}m\omega^2 z^2$$

$$p_z = \frac{\partial W_z}{\partial z} = \sqrt{2mE_z - m^2\omega^2 z^2}$$

$$W_z(z, E_z) = \int dz\ \sqrt{2mE_z - m^2\omega^2 z^2}$$

$$Q_z = \beta_z = \frac{\partial S}{\partial E_z} = \frac{\partial W_z}{\partial E_z} - t = \int dz\ \frac{m}{\sqrt{2mE_z - m^2\omega^2 z^2}} = \frac{1}{\omega}\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E_z}}z\right) - t$$

$$z(t) = \sqrt{\frac{2E_z}{m\omega^2}}\ \sin\left(\omega(t + \beta_z)\right)$$

$$p_z(t) = \sqrt{2mE_z}\ \cos\left(\omega(t+\beta_z)\right)$$

where

$$E_z = \frac{p_z^2(t)}{2m} + \frac{1}{2}kz^2(t) = \frac{p_z^2(0)}{2m} + \frac{1}{2}kz^2(0)$$

$$\frac{p_z(0)}{z(0)} = m\omega\cot(\omega\beta_z)$$

relate E_z and β_z to initial value data.

#12: GRADUATE MECHANICS

`PROBLEM:` Starting from rest at $(x, y) = (0, 0)$, a particle slides down a frictionless hill whose shape is given by the equation $y = -ax^n$, $a > 0$ and $n > 0$. Determine the range of allowed n for which the particle leaves the surface, and the x location at which this occurs. Assume gravity is constant, in the $-y$ direction.

`SOLUTION:`

$E = 0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$

$y = -ax^n \;\Rightarrow\; \dot{y} = -nax^{n-1}\dot{x}$

$\Rightarrow\; \dot{x}^2 = \frac{2gax^n}{1+n^2a^2x^{2(n-1)}}$

Force of constraint in the x direction is $Q_x = m\ddot{x} = 0$ if particle leaves the surface.

$2\cancel{\dot{x}}\,\ddot{x} = \frac{\partial}{\partial t}\left(\frac{2gax^n}{1+n^2a^2x^{2(n-1)}}\right) = \cancel{\dot{x}}\,\frac{\partial}{\partial x}\left(\frac{2gax^n}{1+n^2a^2x^{(2n-2)}}\right) = 0$

$\Rightarrow\; \frac{nx^{n-1}}{1+n^2a^2x^{2(n-1)}} - \frac{2(n-1)n^2a^2x^{3n-3}}{(1+n^2a^2x^{2n-2})^2} = 0$

$nx^{n-1} + n^3a^2x^{3n-3} - 2(n-1)n^2a^2x^{3n-3} = 0$

$\Rightarrow\; a^2x^{2n-2} = \frac{1}{n(n-2)}$. Real finite solution only for $n > 2$.

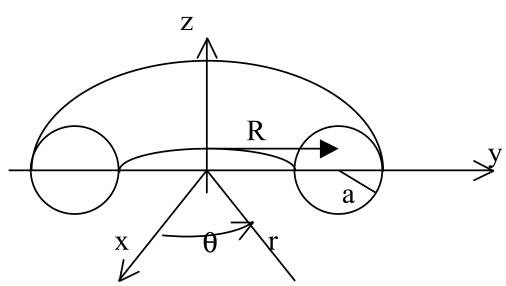
#13: GRADUATE E & M

`PROBLEM:` The figure below shows a cross sectional cut through a torus of magnetic material. The torus is symmetric under rotation around the z-axis and has minor radius a and major radius R. The material carries magnetization $\mathbf{M} = M_0\hat{\theta}$, where M_0 is a constant and θ is the cylindrical coordinate.

(a) Find $\mathbf{B}$ and $\mathbf{H}$ in the torus.

(b) Suppose the torus were cut in half along the plane $x = 0$, opening up a very thin vacuum gap. What force would the two halves exert on one another and would the force be attractive or repulsive?

(c) Suppose the torus were cut in half along the plane $z = 0$. What force would the two halves exert on one another and would the force be attractive or repulsive?



`SOLUTION:`

(a) $\nabla \times H = \frac{4\pi}{c} J_f = 0$

$2\pi r H_\theta = \oint \mathbf{H} \cdot d\boldsymbol{\ell} = 0$

$$B_\theta = \underset{\substack{\parallel \\ 0}}{H_\theta} + 4\pi M_0$$

(b) Normal component of $\mathbf{B}$ is continuous at interface between material and vacuum gap

$(B_\theta)_{gap} = (B_\theta)_{material} = 4\pi M_0$

Force $= 2\pi a^2 \frac{(B_\theta)^2}{8\pi} = 2\pi a^2 \frac{(4\pi M_0)^2}{8\pi}$ attractive

(c) Tangential component of **H** is continuous at the interface between material and vacuum gap

$(B_\theta)_{gap} = (H_\theta)_{gap} = (H_\theta)_{material} = 0$

Therefore Force $= 0$

#14: GRADUATE E & M

`PROBLEM:` An electromagnetic wave,

$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\,\mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$,

is incident on a small dielectric sphere of radius R and dielectric constant ϵ, where $kR \ll 1$. Determine the differential scattering cross section for the scattered radiation.

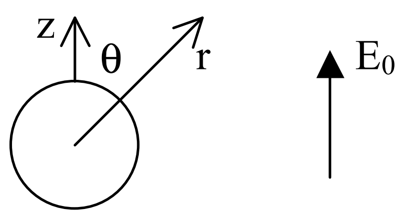
`SOLUTION:`

Can use electrostatics to calculate polarization of sphere since

$kR = \frac{\omega}{c} R \ll 1$

Also, wave field is uniform over scale length of sphere

$\phi_{out} = -E_0 r\, \cos\theta + \frac{A}{r^2}\cos\theta$



$\phi_{in} = Br\, \cos\theta$

$-E_0\cancel{R} + \frac{A}{R^{\cancel{2}3}} = B\,\cancel{R} \qquad \phi_{out}(R,\theta) = \phi_{in}(R,\theta)$

$\epsilon B = -E_0 - \frac{2A}{R^3} \qquad \epsilon\frac{\partial\phi_{in}}{\partial r}|_R = \frac{\partial\phi_{out}}{\partial r}|_R$

$\epsilon B = -E_0 - \frac{2A}{R^3} \qquad \epsilon\frac{\partial\phi_{in}}{\partial r}|_R = \frac{\partial\phi_{out}}{\partial r}|_R$

$\epsilon B + \frac{A}{R^3} = B - \frac{2A}{R^3}$

$B\epsilon + \frac{A}{R^3} = B - \frac{2A}{R^3} - \frac{3A}{R^3} = (\epsilon - 1)B$

$\epsilon B = -E_0 + \frac{2}{3}(\epsilon-1)B\,, \qquad B(\epsilon+2) = -E_0 3$

$\epsilon\mathbf{E} = -\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}\,, \qquad \mathbf{P} = \frac{1}{4\pi}(\epsilon-1)\,\mathbf{E}$

$\mathbf{P} = \frac{3}{4\pi}\frac{(\epsilon-1)}{\epsilon+2}\mathbf{E}_0\,,$

$\mathbf{P} = \frac{4}{3}\pi R^3 \mathbf{P} = \frac{(\epsilon-1)}{(\epsilon+2)} R^3 \mathbf{E}_0$

$\nwarrow$ total dipole

Dipole radiation dominates

$\frac{d\langle P\rangle}{d\Omega} = \frac{c}{8\pi} k^4 |\mathbf{P}|^2 \sin^2\theta$

$\uparrow$ angle between $\mathbf{E}$ and $\mathbf{r}$

$\frac{d\sigma}{d\Omega} = \frac{d\langle P\rangle/d\Omega}{(c/8\pi)|E_0|^2} = k^4 R^6 \frac{(\epsilon-1)^2}{(\epsilon+2)^2} \sin^2\theta$

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a non-relativistic particle of mass m and charge e, in the external fields $\vec{E} = E\hat{z}$ and $\vec{B} = B\hat{y}$, where E and B are constants.

(a) Write the hamiltonian in the gauge where $\vec{A}$ has only $\hat{x}$ component non-zero, and verify that it has translation symmetry in two of the coordinates (which two?). [2 points]

(b) What are the corresponding conserved quantities? [1 point]

(c) Write the corresponding form of the wavefunction $\psi(x,y,z)$ using separation of variables, so they're eigenstates of the above two conserved quantities. Show that this $\psi(x,y,z)$ reduces the Schrodinger equation to that of a 1d problem, in the one coordinate that does not have translation invariance. [2 points]

(d) Use $[H,x]$ to compute $\dot{x}$. Is $p_x = m\dot{x}$? [1 point]

(e) Show that for a cleverly chosen shift in one of the spatial coordinates, you can reduce the remaining 1d Hamiltonian to one what you know very well. In this way, find the quantized energy levels, and in particular the spacing between them. [4 points].

`SOLUTION:`

(a) Taking $\vec{A} = Bz\hat{x}$ and $\phi = -Ez$,

$$H = \frac{1}{2m}((p_x - eBz/c)^2 + p_y^2 + p_z^2) - eEz.$$

This has translation symmetry in x and y.

(b) Correspondingly, p_x and p_y are conserved.

(c) Take $\psi(x,y,z) = e^{i(k_x x + k_y y)}\psi(z)$, which are eigenstates of p_x and p_y, with eigenvalues $p_x = \hbar k_x$ and $p_y = \hbar k_y$ Then the SE becomes

$$\left[\frac{1}{2m}\left((\hbar k_x - \frac{e}{c}Bz)^2 + \hbar^2 k_y^2 + p_z^2\right) - eEz\right]\psi(z) = E\psi(z).$$

(d) $\dot{x} = \frac{1}{i\hbar}[x,H] = \frac{1}{m}(p_x - \frac{e}{c}Bz)$. So $p_x \neq mv_x$. This is not a surprise: $m\vec{v} = \vec{p} - \frac{e}{c}\vec{A}$ is where the original H came from (since $\vec{B}$ does no work, it doesn't contribute to H when writing $K = \frac{1}{2}m\vec{v}^2$).

(e) Shift $z' = z + a$ to get

$$H' = \frac{1}{2m}\left((\hbar k_x - \frac{e}{c}Bz' - \frac{e}{c}Ba)^2 + \hbar^2 k_y^2 + p_z^2\right) - eEz' - eEa.$$

The terms linear in z' can be eliminated by taking

$$a = \frac{c}{eB}\left(\hbar k_x + \frac{Emc}{B}\right).$$

What's left is a 1d SHO, with $\omega = eB/mc$ (the cyclotron frequency). The energy levels are thus quantized like the SHO,

$$E = (n + \tfrac{1}{2})\frac{\hbar eB}{mc} + \frac{\hbar^2 k_y^2}{2m} + \frac{e^2 B^2 a^2}{2mc^2} - eEa.$$

The spacing is $\hbar eB/mc$.

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a (non-relativistic) particle of mass m in the 1d potential

$$V(x) = \begin{cases} \infty & x < 0 \\ cx & x \geq 0. \end{cases} .$$

(a) Use the uncertainty principle to estimate the groundstate energy and characteristic length scale.

(b) Use the variational principle, with xe^{-ax} as the trial function, to estimate the groundstate energy and characteristic length scale. (You'll get a fair amount of partial credit for clearly setting up the calculation.)

`SOLUTION:`

(a) Take $E \approx \frac{\Delta p^2}{2m} + c\Delta x \approx \frac{\hbar^2}{2m\Delta x^2} + c\Delta x$, using $\Delta p \Delta x \sim \hbar$. Minimizing, the characteristic length scale is $\Delta x = (\hbar^2/mc)^{1/3}$ and $E_{min} \approx \frac{3}{2}\left(\frac{\hbar^2 c^2}{m}\right)^{1/3}$.

(b) The trial wavefunction gives

$$\langle E \rangle = \int_0^\infty dx x e^{-ax}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + cx\right) x e^{-ax} \Big/ \int_0^\infty dx x^2 e^{-2ax},$$

$$= \frac{3c}{2a} + \frac{\hbar^2 a^2}{2m}.$$

Minimizing, the characteristic size is

$$a^{-1} = \left(\frac{2\hbar^2}{3cm}\right)^{1/3}$$

and the groundstate energy is bounded below as

$$E_{min} \geq \frac{9}{4}\left(\frac{2\hbar^2 c^2}{3m}\right)^{1/3}.$$

#17: GRADUATE STAT MECH

`PROBLEM:`

For a two-dimensional ideal free Fermi gas, find the temperature T_0 at which the chemical potential of the system is zero, expressed in terms of the Fermi temperature T_F.

Hint: The chemical potential at zero temperature is $k_B T_F$, with k_B = Boltzmann's constant.

`SOLUTION:`

The density of fermions is

$$n = \int d\epsilon g(\epsilon) \tfrac{1}{e^{\beta(\epsilon-\mu)}+1} \tag{13}$$

The density of states (per unit volume) $g(\epsilon)$ of a two-dimensional free Fermi gas is a constant independent of energy:

$$g(\epsilon) = C \tag{14}$$

At zero temperature, calling ϵ_F the chemical potential

$$n = \int_0^{\epsilon_F} d\epsilon g(\epsilon) = C\epsilon_F = Ck_B T_F \tag{15}$$

At the temperature where $\mu = 0$,

$$\begin{aligned} n &= \int_0^\infty d\epsilon g(\epsilon) \tfrac{1}{e^{\beta\epsilon}+1} = \int_0^\infty d\epsilon g(\epsilon) \tfrac{e^{-\beta\epsilon}}{e^{-\beta\epsilon}+1} \\ &= C\tfrac{1}{\beta}[-ln(e^{-\beta\epsilon}+1)]_0^\infty = Cln2k_B T_0 \end{aligned} \tag{16}$$

with $\beta = 1/k_B T_0$, hence

$$Cln2k_B T_0 = Ck_B T_F \tag{17}$$

and

$$T_0 = T_F/ln2 = 1.44T_F \tag{18}$$

#18: GRADUATE STAT MECH

`PROBLEM:` Consider an ultra-relativistic ideal gas of N indistinguishable spin zero particles moving in three dimensions, for which the particle energy is $E = pc$ where p is the magnitude of the momentum.

(a) Calculate the mean energy $U(T, V, N)$ in the classical limit

(b) Find the critical temperature T as a function of particle density N/V for Bose-Einstein condensation

Hint:

$$\int dx x^2 e^{-x} = \Gamma(3), \int dx x^2/(e^x - 1) = \zeta(3)$$

`SOLUTION:`

(a)

$$H(q_i, p_i) = \sum_{i=1}^{N} |\vec{p}_i| c$$

$$Z(T, V, N) = \frac{1}{N!\ h^{3N}} \prod_i \int d^3 q_i d^3 p_i e^{-\beta H(q_i, p_i)} = \frac{1}{N!\ h^{3N}} V^N \left(\int d^3 p\ e^{-\beta |p| c} \right)^N$$

$$\int d^3 p\ e^{-\beta |p| c} = 4\pi \int_0^\infty dp\ p^2 e^{-\beta c p} = \frac{4\pi}{(\beta c)^3} \Gamma(3) = \frac{8\pi}{(\beta c)^3}$$

$$Z(T, V, N) = \frac{1}{N!\ h^{3N}} \left(\frac{8\pi V}{(\beta c)^3} \right)^N = \frac{1}{N!} \left(8\pi V \left(\frac{kT}{hc} \right)^3 \right)^N$$

$$F(T, V, N) = -kT \ln Z = -kT \left[N \ln \left(8\pi V \left(\frac{kT}{hc} \right)^3 \right) - \ln N! \right]$$

$$F(T, V, N) = -NkT \left[1 + \ln \left(\frac{8\pi V}{N} \left(\frac{kT}{hc} \right)^3 \right) \right]$$

$$S = -\left(\frac{\partial F}{\partial T} \right)_{V,N} = Nk \left[4 + \ln \left(\frac{8\pi V}{N} \left(\frac{kT}{hc} \right)^3 \right) \right]$$

$$U(T, V, N) = F + TS = 3NkT$$

or

$$U = -\frac{\partial \ln Z}{\partial \beta} = T^2 \frac{\partial \ln Z}{\partial T}$$

(b)

$$< N >= \sum_{allstates} \frac{1}{e^{\beta(pc+\mu)} - 1}$$

$$= V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\hbar ck+\mu)} - 1} + N_0$$

where N_0 is the number of particles in the ground state ($k = 0$). Scaling k by $\beta\hbar c$ we write this as

$$N = V \frac{4\pi}{(2\pi)^3} \frac{T^3}{(\hbar c)^3} \int dx x^2 \frac{1}{e^{x+\beta\mu} - 1} + N_0$$

The integral has a maximum possible valueof $\zeta(3)$ at $\mu = 0$ implying a transition temperature of

$$T = \hbar c (\frac{2\pi^2 N}{(\zeta(3)V)})^{1/3}$$

#19: GRADUATE MATH

PROBLEM:

Consider a heat equation in a two-dimensional strip

$$\left[\frac{\partial}{\partial t} - D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right] T(t,x) = Q(t,x)\,, \quad 0 < z < d\,,\ -\infty < x < \infty\,.$$

The two sides $z = 0, d$ of the strip are maintained at zero temperature. An instantaneous heat pulse $Q(x,t) = \delta(t)\delta(x)\delta(z - d/2)$ is delivered to the center of the strip at $t = 0$. An observer at some large distance x_* sees the temperature rise and fall as a function of time. Find the time t_* at which the maximum temperature is observed. *Hint*: t_* is determined by the least rapidly decaying eigenfunction of the heat equation.

SOLUTION:

$\left[\frac{\partial}{\partial t} - D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial z^2}\right)\right] T = \delta(t)\delta(x)\delta(z - \frac{d}{2})$

write solution as $T = \sum_{n=1}^{\infty} \int \frac{dk}{2\pi} f_n(t,k)\ \sin\frac{n\pi z}{d}\ e^{ikx}$

$\Rightarrow \quad \frac{\partial f_n}{\partial t} + D\left(k^2 + \frac{n^2\pi^2}{d^2}\right)\ f_n = \delta(t)\frac{2}{d}\sin\frac{n\pi}{2}$

Solution is

$f_n = \frac{2}{d}\sin\frac{n\pi}{2}\ e^{-D(k^2+n^2\pi^2/d^2)t}$

so $T(x,z,t) = \sum_{n=1}^{\infty} \frac{2}{d}\sin\frac{n\pi z}{d}\sin\frac{n\pi}{2}\int\frac{dk}{2\pi}\ e^{-D(k^2+n^2\pi^2/d^2)t+ikx}$

$\swarrow$

$$\frac{1}{2\sqrt{\pi D t}}\ e^{-D\ (n^2\pi^2 t/d^2)} e^{-x^2/4Dt}$$

Time dependence at large times, large x is dominated by $n = 1$ exponential term

$e^{-(D\pi^2 t/d^2) - x^2/4Dt}.$

This has a maximum at $t = \frac{d}{2\pi D}x$.

#20: GRADUATE GENERAL PHYSICS

`PROBLEM:`

Use simple kinetic theory to approximate the specific heat capacity of a basic substance of your choosing—solid, liquid, or gas (just don't use the ideal gas law in the latter case). Assume the nuclei in the substance have mass number, A. Evaluate your expression numerically to produce a **number** with units of J $\text{kg}^{-1}\text{K}^{-1}$. We're only after factor-of-two accuracy here.

Hint: If you don't know the numerical value of Boltzmann's constant, k, you can likely think of a number of physics factoids that can lead you to a reasonable value.

`SOLUTION:`

Forgetting small factors that may differ in different phases, the thermal energy per particle will be about $\frac{3}{2}kT$. Thus the heat capacity per particle is just $\frac{3}{2}k$.

The only remaining step is to figure out the number of particles per kilogram. This can be determined either as $N_A/(1000 \cdot A)$, where N_A is Avogadro's number, 6×10^{23}, and A is the atomic number, or *grams* per mole—thus requiring the factor of 1000. Or one could recall that the atomic mass unit is $a = 1.66 \times 10^{-27}$ kg, so that the number of particles per kilogram is $(Aa)^{-1}$. Note that $a = 1000/N_A$.

The specific heat capacity, in J $\text{kg}^{-1}\text{K}^{-1}$, is then $c_p \approx \frac{3}{2}k/Aa$. Using $k = 1.38 \times 10^{-23}$ J/K, we get $c_p \approx 12500/A$ J $\text{kg}^{-1}\text{K}^{-1}$. Picking a relatively light substance with $A = 20$, we get a specific heat capacity around 600 J $\text{kg}^{-1}\text{K}^{-1}$—which is in the correct range for most substances.

As for the hint on other ways to assess k, you can use the fact that $kT \approx \frac{1}{40}$ eV at room temperature. You could back it out of the ideal gas law: $PV = NkT$, making use of STP conditions and the fact that one mole occupies 22.4 ℓ in these conditions. You could make use of the fact that the sound speed in air must be close to $\sqrt{kT/m}$. Lots of other ways as well.