

On the Mayer series of two-dimensional Yukawa gas at inverse temperature in the interval of collapse

Wilhelm Kroschinsky^{*} and Domingos H. U. Marchetti[†]

Instituto de Física

Universidade de São Paulo

05508-090 São Paulo, SP, Brazil

Abstract

A Theorem on the minimal specific energy for a system with ± 1 charged particles interacting through the Yukawa pair potential v is proved which may be stated as follows. Let v be represented by scale mixtures of d -dimensional Euclid's hat (cutoff at short scale distances) with $d \geq 2$. For any even number of particles n , the interacting energy U_n divided by n , attains an n -independent minimum at a configuration with zero net charge and particle positions collapsed altogether to a point. For any odd number of particles n , the ratio $U_n/(n-1)$ attains its minimum value, the same of the previous cases, at the configuration with ± 1 net charge and particle positions collapsed to a point. This Theorem is then used to resolve an obstructive remark of an unpublished paper (Remark 7.5 of [GM]) which, whether the standard decomposition of the Yukawa potential into scales were adopted, would impede a direct proof of the convergence of the Mayer series of the two-dimensional Yukawa gas for the inverse temperature in the whole interval $[4\pi, 8\pi)$ of collapse. In the present paper, it is proven convergence up to the second threshold 6π and its given explanations on the mechanism that allow it to be extended up to 8π . The paper distinguishes the matters concerning stability from those related to convergence of the Mayer series. In respect to the latter the paper dedicates to the Cauchy majorante method applied to the density function of Yukawa gas in the interval of collapses. It also dedicates to the proof of the main Theorem and estimates of the modified Bessel functions of second kind involved in both representations of two-dimensional Yukawa potential: standard and scale mixture of the Euclid's hat function.

MSC: 60J45, 42A82, 97K10, 82B21, 82B28

Keywords: two-dimensional Yukawa potential, minimal specific energy, scale mixtures of Euclid's hat, stability, Mayer series, collapse interval

^{*}Email: wilhelm.kroschinsky@usp.br

[†]Email: marchetti@if.usp.br

1 Introduction and background of tools and methods

The present paper investigates a system of particles with ± 1 charges living in a two dimensional Euclidian space and interacting through the Yukawa pair potential $v(x) = (-\Delta + 1)^{-1}(0, x)$. Because of Yukawa and Coulomb potentials look like the same at short distances, the two-dimensional Yukawa gas inherits the same instabilities of the corresponding Coulomb system when the inverse temperatures β belongs to the interval $[4\pi, 8\pi)$, in which a sequence of collapses of neutral cluster of size $2n$ occurs at the thresholds $\beta_{2n} = 8\pi(1 - 1/2n)$, $n \in \mathbb{N}$. It remains an open problem for this system to establish convergence of the Mayer series in powers of activity z , with the first even terms removed from the series, how many depending on $\beta \in [4\pi, 8\pi)$. It is our purpose to revisit this long standing problem.

Benfatto[Be] and collaborators from the Italian school (see references therein) initiate a program using iterated Mayer series for pressure (and correlation functions) together with ideas from the work of Gopfert-Mack [GoMa] and Imbrie [I]. Brydges and Kennedy [BK] have also considered the Mayer expansion of the two-dimensional Yukawa gas in the context of the Hamilton-Jacobi equation. We adopt in present investigation their continuum scaling renormalization method, adding to that approach a new ingredient. The novelty is related with the (short-range) decomposition of the Yukawa potential into scales. Instead of the standard decomposition $v(x) = \int_{-\infty}^0 \left(d(-\Delta + e^{-s})^{-1}(0, x)/ds \right) ds$ (or the discrete version of it) adopted in the previous work, we use the scale mixture $v(x) = \int_0^1 g(s)h(|x|/s)ds$ of Euclid's hat $h(r)$. Using a concept introduced by Basuev [Ba1], we first prove a theorem that the minimal specific energy $e(v)$ and the constrained (to non zero net charge) modified minimal specific energy $\bar{e}(v)$ are equal.

Our main theorem on specific energy when applied in the investigation of the two-dimensional Yukawa gas resolve an obstructive limitation that has been posed in an unpublished paper by Guidi and one of the authors (see Conjecture 2.3 and Remark 7.5 of [GM]) towards a direct proof of the convergence of the Mayer series on the entire interval $[4\pi, 8\pi)$ whether the standard decomposition of the Yukawa potential were adopted. The limitation value for the 3-particles interacting energy given by a numerical evaluation in [GM] is proven in Proposition 2.2 and Remark 2.3.

In the present paper, the convergence of the Mayer series is proven up to the second threshold $\beta \in [4\pi, 6\pi)$ and it is provided a full explanation of the mechanisms that allow it to be extend up to 8π . We distinguished the issues concerning stability from those related to convergence of the Mayer series. In respect to the latter, the paper dedicates in Section 3 to the Cauchy majorante method applied to the density function of Yukawa gas in the region of collapse. Regarding the former, we considered only the simplest case of collapse prevention of neutral pair of charges due the presence of other charges in the configuration.

We shall now review the tools and methods employed in present investigation. We refer to the

references for detail.

Decomposition of radial positive functions of positive type. Positive definite functions have arisen in many areas of (pure and applied) mathematics and physics (see [S] for an historical survey). A continuous function f defined in \mathbb{R}^d is called positive definite (abbreviated as p. d.) if the $n \times n$ real matrix $[f(x_i - x_j)]_{1 \leq i, j \leq n}$ is positive definite for $n \in \mathbb{N}$ arbitrary elements x_1, \dots, x_n of \mathbb{R}^d :

$$\sum_{1 \leq i, j \leq n} \bar{z}_i z_j f(x_i - x_j) \geq 0, \quad \forall z_1, \dots, z_n \in \mathbb{C}. \quad (1.1)$$

The celebrated work of Bochner (see e.g. [B]) characterizes these functions as follows: f (with $f(0) = 1$) is positive definite if, and only if, is a Fourier-Stieltjes transform $\check{\mu}(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} d\mu(\xi)$ of a probability Borel measure μ on \mathbb{R}^d . Although powerful, Bochner's theorem may be difficult to use in practice: how do we know that a given f satisfies (1.1)? Even when explicit computation of Fourier transform is available, how do we represent f into suitable scale mixture of elementary functions?

Recently (see [G, HS, JMR] and references therein), investigations towards extending Bochner's theorem seek for concrete examples and easy checkable criteria of p. d. function. A particularly interesting subclass of p. d. functions, denoted in [JMR] by Ω_d^+ , is provided by radial continuous functions: $f(x) = \varphi(|x|)$ for some positive continuous function φ of \mathbb{R}_+ . A simple example of these functions that vanishes out of a ball B_s of radius $s > 0$ centered at origin is given by the Euclid's hat ($d = 2$)

$$\frac{4}{\pi s^2} \chi_{s/2} * \chi_{s/2}(x) \equiv h(|x|/s)$$

where $\chi_r(x) = \chi_{B_r}(x)$ is the characteristic function of B_r . In [JMR] Jaming, Matolcsi and Révész have identified certain compactly supported functions, alike this one, as extrema rays of the cone Ω_d^+ , playing the same role as the family $\{e^{i\xi \cdot x}\}$ for the Bochner's theorem. So, if φ is an extremum ray of Ω_d^+ then, by Choquet representation,

$$\int_0^\infty \varphi(|x|/s) d\nu(s) \quad (1.2)$$

is an element of Ω_d^+ for a suitable positive measure ν supported on the family of scales $\{\varphi(|x|/s)\}$ of φ . An open problem is to find all extrema of Ω_d^+ (see [JMR]).

Geiting [G] and Hainzl-Seiringer [HS] give, on the other hand, complete characterizations of the subclass $H_d \subset \Omega_d^+$ that are formed by scaling mixtures of d -dimensional Euclid's hat, extending Polya's criterion on \mathbb{R}^d , for $d \geq 2$. Hainzl-Seiringer's representation however suffices to make our point in the present work. Let us start with the two-dimensional Yukawa potential, which is an element of Ω_2^+ given by the Green's function $v(1/\sqrt{\kappa}, x) = (-\Delta + \kappa)^{-1}(0, x)$ (the resolvent kernel of the Laplacian operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$). Applying Fourier transform yields (with

$v(x) \equiv v(1, x)$ and $v(1/\sqrt{\kappa}, x) = v(1, \sqrt{\kappa}x)$. See e.g. Sec. 7.2 of [GJ])

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\xi \cdot x} \frac{1}{\xi^2 + 1} d\xi = \frac{1}{2\pi} K_0(|x|) \quad (1.3)$$

where K_0 is the modified Bessel function of second kind of order 0. Hainzl-Seiringer's formula for this function reads

$$v(x) = \int_0^\infty h(|x|/s) g(s) ds \quad (1.4)$$

where ($h(0) = 1$)

$$h(w) = \frac{2}{\pi} \left(\arccos w - w\sqrt{1-w^2} \right), \quad \text{if } 0 < w \leq 1 \quad (1.5)$$

$h(w) = 0$ if $w > 1$ and

$$g(s) = \frac{-s}{4\pi} \int_s^\infty K_0'''(r) \frac{r}{\sqrt{r^2 - s^2}} dr. \quad (1.6)$$

The mixture density function $g(s)$ for the Yukawa potential in $d = 1$ and 3 dimensions and the Coulomb potential in d -dimensions have closed forms (see Examples 1 and 2 of [HS]). For the Yukawa function in 2-dimensions, however, $g(s)$ can only be written in term of Meijer G -functions (see [BS] for an introduction): $g(s) = \sqrt{\pi} G_{13}^{30} \left(s^2/4 \middle| \begin{smallmatrix} 1/2 \\ 0, 1, 3 \end{smallmatrix} \right) / (2\pi s)$.

We observe that $h(w)$ is a convex function of $w \in \mathbb{R}_+$ and a mixture of the Euclid's hat (1.4) preserves convexity. This useful property, as we shall see, distinguishes (1.4) from another common decomposition of (1.3) into scales (see Fig. 4): with $v(s, x) = (-\Delta + 1/s^2)^{-1}(0, x) = (-\Delta + 1)^{-1}(0, x/s) = K_0(|x|/s)$, we write

$$v(x) = \int_0^1 \dot{v}(s; x) ds$$

by the fundamental theorem of calculus. Substituting the derivative $\dot{v}(s; x)$ with respect to s , yields

$$v(x) = \frac{1}{2\pi} \int_0^1 \tilde{h}(|x|/s) \frac{ds}{s} \quad (1.7)$$

where $\tilde{h}(w) = -wK_0'(w) = wK_1(w)$, with K_1 the modified Bessel function of second kind of order 1, like h given by (1.5), decreases monotonously to 0 and satisfies $h_1(0) = 1$ but changes from concave to convex as $w \in \mathbb{R}_+$ varies. The mixture density $g(s)$ for both decompositions of v , (1.4) and (1.7), behaves in the neighborhood of $s = 0$ as $(2\pi s)^{-1}$ implying that $v(x)$ behaves as the Coulomb potential $(-1/2\pi) \log |x|$ at short distances.

Gaussian Processes and renormalization group. Positive definite functions plays an important role on renormalization group (RG) methods in statistical physics. Brydges and collaborators [BGM] (see also [BT]) coined a term “finite range decomposition” to the mixture of different scales (1.2) for some compactly supported radial extremal functions φ . They used a probabilistic argument

as follows: breaking up the range of integration into disjoint union of intervals $I_j = [L^{-j}, L^{-j+1})$, $j \geq 1$ for $L > 1$ and $I_0 = [1, \infty)$, (1.4) may be seen as the “finite range” decomposition

$$\phi = \sum_{j \geq 0} \zeta_j \quad (1.8)$$

of a Gaussian process ϕ of mean $\mathbb{E}\phi(x) = 0$ and covariance $\mathbb{E}\phi(x)\phi(y) = v(x-y)$ into a family of independent Gaussian processes $\{\zeta_j\}$ of mean $\mathbb{E}\zeta_j = 0$ and covariance

$$\mathbb{E}\zeta_j(x)\zeta_j(y) = \int_{I_j} g(s)h(|x-y|/s) ds \equiv v_{I_j}(x-y).$$

Since the covariance of a sum of independent Gaussian random variable is the sum of their covariances, $v = \sum_{j \geq 0} v_{I_j}$ equals (1.4). The authors of [BGM, BT] were also capable of applying suitable finite range decomposition to a large class of positive definite functions on \mathbb{R}^d and \mathbb{Z}^d that comprises integral kernels (Green’s functions) of certain elliptic operators, their corresponding finite differences and fractional powers.

When a statistical system is represented by the expectation $\mathbb{E}\mathcal{Z}$ of a functional $\mathcal{Z}(\phi)$, the decomposition (1.8) of the Gaussian field ϕ can be used to integrate out each ζ_j at a time. Let $\mathbb{E}^{(j)}$ denote the expectation with respect the Gaussian field ζ_j . The renormalization group is a method of calculating the expectation $\mathbb{E}\mathcal{Z}$ through the sequence of maps $\mathcal{Z}_j \mapsto \mathcal{Z}_{j+1} = \mathbb{E}^{(j+1)} \mathcal{Z}_j$ starting from $\mathcal{Z}_0 = \mathcal{Z}$. The limit $\lim_{j \rightarrow \infty} \mathcal{Z}_j = \mathbb{E}\mathcal{Z}$, supposing it exists, is obtained provided $\mathcal{Z}_j \mapsto \mathcal{Z}_{j+1}$ is amenable to be analyzed as a dynamical system depending on parameters in the initial condition. For instance, in the decomposition (1.8) of ϕ into finite range fields ζ_j corresponding to the Yukawa potential (1.4), the limit $j \rightarrow \infty$ drives the statistical system into the short scaling limit $s \rightarrow 0$ for which the potential diverges logarithmically. For an infinitely many-particle system with ± 1 charges, the existence of $\lim_{j \rightarrow \infty} \mathcal{Z}_j$ expresses the thermodynamical stability of the system. We shall come back to this issue below.

Hamilton–Jacobi equation and majorant method. Under the Kac–Siebert transformation, [FS, BM] the grand partition function for the two-dimensional Yukawa gas of particles with ± 1 charges can be written as the expectation $\mathbb{E}\mathcal{Z}_0$ (with respect to the Gaussian field ϕ) of

$$\begin{aligned} \mathcal{Z}_0(\phi) &= \exp(\mathcal{V}_0(\phi)) , \\ \mathcal{V}_0(\phi) &= z \int_{\mathbb{R}^2} : \cos \sqrt{\beta} \phi(x) :_v dx \\ &= \sum_{\sigma \in \{-1, 1\}} \int_{\mathbb{R}^2} dx \, z : e^{i\sqrt{\beta} \sigma \phi(x)} :_v \end{aligned} \quad (1.9)$$

where the parameters β and z are, respectively, the inverse temperature and activity and $: \cdot :_v$ indicates Wick ordering with respect to the potential v . In the present work, we shall adopt the

continuum scale decomposition (1.3) instead of (1.8). The induced RG dynamics is thus generated by a Hamilton-Jacobi equation as proposed in [BK] by Brydges and Kennedy. Let us expand these ideas in some detail. A scale-dependent-interaction $v : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is introduced replacing (1.4) by a mixture supported in a finite interval $[t_0, t]$ of scales

$$v(t, x) = \int_{t_0}^t h(|x|/s) g(s) ds \quad (1.10)$$

where $t_0 > 0$ is a cutoff of the short scale distances. Since g and h are continuous, we have $\lim_{t \searrow t_0} v(t, x) \equiv 0$. The renormalization group is now given by a convolution mapping $(t, \phi) \mapsto \mathcal{Z}(t, \phi) = \mathbb{E}^{(t)} \mathcal{Z}_0(\phi + \cdot)$ with initial data $\mathcal{Z}(t_0, \phi) = \mathcal{Z}_0(\phi)$, where $\mathbb{E}^{(t)}$ denotes the expectation with respect the Gaussian field ζ with covariance $v(t, x - y)$. Formally, $\mathcal{Z}(t, \phi)$ satisfies the initial value problem of a “heat equation”

$$\frac{\partial \mathcal{Z}}{\partial t} = \frac{1}{2} \Delta_{\dot{v}} \mathcal{Z} \quad , \quad \lim_{t \searrow t_0} \mathcal{Z}(t, \phi) = \mathcal{Z}_0(\phi)$$

where $\dot{v}(t, x) := \partial v / \partial t(t, x) = g(t) h(|x|/t)$, by the fundamental theorem of calculus, is the weighted Euclid’s hat scaled by t and $\Delta_{\dot{v}}$ is the “Laplacian” operator

$$\Delta_{\dot{v}} \mathcal{Z} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \dot{v}(t, x - y) \frac{\delta^2 \mathcal{Z}}{\delta \phi(x) \delta \phi(y)} \quad . \quad (1.11)$$

Writing $\mathcal{Z}(t, \phi) = \exp(\mathcal{V}(t, \phi))$, the heat equation turns into a nonlinear equation for \mathcal{V} :

$$\frac{\partial \mathcal{V}}{\partial t} = \frac{1}{2} \Delta_{\dot{v}} \mathcal{V} + \frac{1}{2} (\nabla \mathcal{V}, \nabla \mathcal{V})_{\dot{v}} \quad , \quad \lim_{t \searrow t_0} \mathcal{V}(t, \phi) = \mathcal{V}_0(\phi) \quad (1.12)$$

where $\Delta_{\dot{v}}$ acts as in (1.11) and

$$(\nabla \mathcal{V}, \nabla \mathcal{V})_{\dot{v}} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \dot{v}(t, x - y) \frac{\delta \mathcal{V}}{\delta \phi(x)} \frac{\delta \mathcal{V}}{\delta \phi(y)} \quad . \quad (1.13)$$

In [BK], the authors considered the random field ϕ on \mathbb{Z}^d instead, for which the functional derivative $\int dx v(x) \delta / \delta \phi(x) \mathcal{V}(\phi) = \lim_{\varepsilon \rightarrow 0} (\mathcal{V}(\phi + \varepsilon v) - \mathcal{V}(\phi)) / \varepsilon$ becomes partial derivative $\partial / \partial \phi_x$ with respect to the variable $\phi_x \in \mathbb{R}$ at site x . Because of translation invariance, (1.11) and (1.13) diverges even if \mathbb{R}^2 is replaced by \mathbb{Z}^2 but this can be solved by fixing one point x of \mathbb{Z}^2 . Inserting the Taylor expansion (multi-index formula):

$$\mathcal{V}(t, \phi) = \sum_{n \geq 1} \sum_{\alpha: |\alpha|=n} \frac{1}{\alpha!} \frac{\partial^\alpha \mathcal{V}}{\partial \phi^\alpha}(t, 0) \phi^\alpha$$

into an integral equation equivalent to (1.12), a system of equations for derivatives of \mathcal{V} (by collecting order by order terms), together with an appropriate norm, is used to majorize $\mathcal{V}(t, \phi)$ by the solution $\nu(t, \varphi)$ of a first order PDE equation in two independent real variables (t, φ) , φ playing

the role of chemical potential. The local existence and uniqueness of the initial value problem (1.12) are then proved in ref. [BK] (see Theorem 2.2 and Proposition 2.6 therein) for a domain in plane (t, z) with $z = e^\varphi$ (β may be included as well). Quoting the authors, these results are “the precise version of the Mayer expansion” for the pressure or correlations functions of statistical systems.

Brydges and Kennedy have also provided an equivalent system of ordinary differential equations for the Ursell functions (Lemma 3.3 of [BK]) which replaces (1.12) defined on \mathbb{Z}^d and can be used for systems of point particles in \mathbb{R}^d . If $(\Omega, \mathcal{B}, d\varrho(\zeta))$ denotes the finite measure space on $\{-1, 1\} \times \mathbb{R}^2$ corresponding to the possible states of a single particle (we united σ and x into $\zeta = (\sigma, x)$), the solution of (1.12) may be represented formally as

$$\mathcal{V}(t, \phi) = \sum_{n \geq 1} \frac{1}{n!} \int d^n \varrho \psi_n^c(t, \zeta_1, \dots, \zeta_n) : \exp \left(i\sqrt{\beta} \sum_{j=1}^n \sigma_j \phi(x_j) \right) : \quad (1.14)$$

where the Ursell functions $\psi_n^c(t, \zeta_1, \dots, \zeta_n)$ are translational invariant and invariant under the action of the symmetric group \mathbb{S}_n of permutations of the index set $\{1, \dots, n\}$.¹ To make mathematical sense of the above equations (1.12) and (1.14) one can check, at the very end, whether the solution of the system of ODE's for ψ_n^c 's agrees with the statements of [L1, L2] on point processes of infinitely many particles (consult [R] for the definition of n -point correlation and cluster functions and Theorem 5.4 of [Gi] for a hybrid approach combining methods employed for Poisson point process with correlation functions satisfied by the (sine-Gordon) representation (1.9) of the Yukawa gas). The present work will take the system of equations satisfied by the $\psi_n^c(t, \zeta_1, \dots, \zeta_n)$ (see (3.5) and (3.6) below), together with the scale decomposition (1.10) for the Yukawa potential, as the starting point for our analysis.

Stability condition and minimal specific energy. Stability of the interaction v is a condition under which there exist the thermodynamic functions describing an infinitely large statistical system. Let U_n be the total energy potential of the classical charged system of n point particles at positions x_1, \dots, x_n of \mathbb{R}^2 , with respective charges $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$, interacting through a pair Yukawa potential:

$$U_n(\zeta_1, \dots, \zeta_n; v) = \sum_{1 \leq i < j \leq n} \sigma_i v(x_i - x_j) \sigma_j. \quad (1.15)$$

¹ Applying the functional calculus on (1.14) we obtain formally from (1.12) the system of equations (see eq. (3.5)) satisfied by the ψ_n^c 's. For instance, the Laplacian of \mathcal{V} gives

$$\begin{aligned} \Delta_{\dot{v}} \mathcal{V}(t, \phi) &= \lim_{\varepsilon, \eta \rightarrow 0} \frac{1}{\varepsilon \eta} (\mathcal{V}(t, \phi + \varepsilon \dot{v} + \eta \ddot{v}) - \mathcal{V}(t, \phi + \varepsilon \dot{v}) - \mathcal{V}(t, \phi + \eta \ddot{v}) + \mathcal{V}(t, \phi)) \\ &= \sum_{n \geq 1} \frac{1}{n!} \int d^n \varrho \frac{-\beta}{2} \sum_{i \neq j} \sigma_i \sigma_j \dot{v}(t, x_i - x_j) \psi_n^c(t, \zeta_1, \dots, \zeta_n) : \exp \left(i\sqrt{\beta} \sum_{j=1}^n \sigma_j \phi(x_j) \right) : . \end{aligned}$$

An interacting potential v satisfies the *stability condition* if there exists $B > 0$ such that

$$U_n(\zeta_1, \dots, \zeta_n; v) \geq -nB \quad (1.16)$$

holds for all $(\zeta_1, \dots, \zeta_n)$ on the configurations space $\bigcup_n (\{-1, 1\} \times \mathbb{R}^2)^n$ (otherwise the specific energy U_n/n would not be bounded from below).

The standard stability theorem for charged system due to Fisher and Ruelle [FR] (see Theorem I and eq. (III.7) therein) assures that: if $\hat{v}(\xi) = (1/2\pi) \int_{\mathbb{R}^2} v(x) e^{-i\xi \cdot x} dx \geq 0$ and $v(0) = (1/2\pi) \int_{\mathbb{R}^2} \hat{v}(\xi) d\xi < \infty$, then

$$U_n(\zeta_1, \dots, \zeta_n; v) \geq -\frac{1}{2}v(0) \sum_{j=1}^n \sigma_j^2 \quad (1.17)$$

and, since $\sigma_j^2 = 1$, (1.16) is satisfied with $B = v(0)/2$. The proof of (1.17) follows from the “if” direction of Bochner’s theorem. For this, note that adding $1/2$ of each $i = j$ diagonal terms to (1.15) (i.e., (1.17) with the right hand side passed to the left), the quadratic form has to be positive as v is positive definite. It follows from (1.3) that $\hat{v}(\xi) = (2\pi(\xi^2 + 1))^{-1} \geq 0$ is a positive density but $v(x)$, which yields the stability constant B , grows unboundedly at $x = 0$. As the self-energy $v(0)$ diverges logarithmically, the decomposition (1.7) or (1.4) has to be used instead. The thermodynamic functions are defined when the scales smaller than an $s_0 > 0$ are removed from the decomposition of v , but one has to prove that they remain well defined after the cutoff is removed.

Let v be the scale mixtures of Euclid’s hat (1.10), cutoff on the short scales. We introduce the minimal specific energy $e = e(h)$ of h at the scale $s = 1$

$$\begin{aligned} e &= \inf_{n \geq 2} e_n \\ e_n &= \inf_{(\zeta_1, \dots, \zeta_n)} \frac{1}{n} U_n(\zeta_1, \dots, \zeta_n; h) \end{aligned} \quad (1.18)$$

and the modified minimal specific energy $\bar{e} = \bar{e}(h)$,

$$\bar{e} = \inf_{n \geq 2} \inf_{\substack{(\zeta_1, \dots, \zeta_n) \\ \text{non-neutral}}} \frac{1}{n-1} U_n(\zeta_1, \dots, \zeta_n; h) \quad (1.19)$$

where the infimum is now taken over all non-neutral configurations $(\zeta_1, \dots, \zeta_n): (x_1, \dots, x_n) \in \mathbb{R}^{2n}$ and $(\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$ such that $\sum_{j=1}^n \sigma_j \neq 0$. It is clear that minimal specific energy (modified or not) of the scaled Euclid’s hat $h(\cdot/s)$ with $s \neq 1$ have all the same value by homogeneity of the infimum and we have

$$e(v) = \int_{t_0}^t e(h(\cdot/s)) g(s) ds = e(h) \cdot \int_{t_0}^t g(s) ds. \quad (1.20)$$

In the present paper we determine both specific energies e and \bar{e} and characterize the configuration that they are attained for h and, consequently, for (1.10) by (1.20). From definitions (1.16) and

(1.18), we have $-e(h) \leq h(0)/2 = 1/2$. We show that this is in fact an equality and, moreover, $e = \bar{e} = -1/2$. More precisely, we have proven in Sec. 2 an improvement of (1.17)

$$U_n(\zeta_1, \dots, \zeta_n; h) \geq \frac{1}{2} \left(\left| \sum_{j=1}^n \sigma_j \right| - \sum_{j=1}^n \sigma_j^2 \right) \quad (1.21)$$

from which the equality of specific energies follows at once. Observe that the inequality (1.21) turns out to be an equality for certain configurations.

We should mention a short note written by Basuev [Ba1] on the minimal specific energy for classical one-specie system of particles in \mathbb{R}^3 , interacting through a radial two-body potential $\phi(|x - y|)$. The conclusions of this investigation may be stated as follows. Suppose that ϕ satisfies the two conditions that defines what now-a-days is called Basuev potentials:[LPY] there is $a > 0$ such that: **i.** $\phi(|x|) \geq \phi(a) > 0$, for all $|x| \leq a$; and **ii.** $\phi(a) > 2\mu(a)$ where

$$\mu(a) = \sup_{n \geq 2} \sup_{\substack{(x_1, \dots, x_n) \in \mathbb{R}^n: \\ |x_i - x_j| > a}} \sum_{i=1}^n \max(-\phi(|x_i|), 0)$$

is finite (the supremum is taken over all configurations whose distance between any pair exceeds a). Then ϕ and the potential ϕ^a , given by $\phi^a(|x|) = \phi(|x|)$ if $|x| > a$ and $\phi(|x|) = \phi(a)$ if $|x| \leq a$, are stable with stability constant $B = \mu(a)/2$, and their minimal specific energy are equal: $e(\phi) = e(\phi^a)$ and $\bar{e}(\phi) = \bar{e}(\phi^a)$. Colloquially, it says that an increase of the positive part of the potential does not reduce the binding energy of the system. Basuev class includes potentials of Lenard-Jones type introduced by Fisher (see e.g. [FR], [RT] for an overview and [LPY] for a proof of this statement).

Basuev [Ba1] has in addition shown that $e(\phi) \leq \bar{e}(\phi) \leq 13e(\phi)/12$ for potentials ϕ such that $\min_{x \in \mathbb{R}^3} \phi(|x|) = -\lambda < 0$ and has stated that $e(\phi) = \bar{e}(\phi)$ holds for the majority of stable potentials which is useful in applications. Our result on the equality $e(h) = \bar{e}(h)$ differs, however, in many respects. Typical Basuev potentials are bounded from below by a negative constant $-\lambda$, repulsive at short, attractive and integrable at large distances. Equality in this situation occurs when the infimum in n of (1.18) is attained at ∞ .² The Yukawa potential v on the other hand, as scale mixtures of Euclid's hat $h(|x|/s)$ weighted by $g(s)$ where h, g and consequently v are all positive functions, repels (attracts) two particles with the same (opposite) charges in its entire support. The infimum in (1.18) is attained for neutral configurations, when $n_+ = m \in \mathbb{N}$ positive and $n_- = m$ negative charges collapse into one point while the infimum in the modified specific energy $\bar{e}(v)$ is attained when $|n_+ - n_-| = 1$.

One versus iterated Mayer expansion. The Ursell functions can be written by the well known formula introduced by Mayer (see [UF]):

$$\psi_n^c(\zeta_1, \dots, \zeta_n; v) = \sum_{G \text{ connected}} \prod_{\langle ij \rangle \in E(G)} (\exp(-\beta \sigma_i \sigma_j v(|x_i - x_j|)) - 1) , \quad (1.22)$$

² Because the minimal specific energy may be written as $\bar{e} = \inf_{n \geq 2} \frac{n}{n-1} e_n$.

where the sum runs over all connected linear graphs G with vertices in the index set $\{1, \dots, n\}$ and $E(G)$ denotes the set of edges of G . As far as the estimation of pressure and correlation functions are concerned, equation (1.22) is not useful due to the cardinality of its sum. To reduce the sum over connected Mayer graphs to labeled trees, Penrose [P] has exploited cancellations occurring on the formula under proper re-summation and proved that the Mayer series converge provided the potential v is stable, integrable at large distances and has, in addition, a hard core condition which recently has shown [PY] to be unnecessary (see also [BM] for an overview and extensions). The cardinality of labeled trees of order n is n^{n-2} by the famous Cayley theorem, which makes the tree graph identities suitable for the estimation of thermodynamical functions. Among the proposed tree graph formulas now available we indicate the one in Theorem 3.1 of [BK] as the most adequate to our purposes of representing the Ursell functions $\psi_n^c(t, \zeta_1, \dots, \zeta_n)$ defined by (1.22) with the scale-dependent-interaction (1.10) in the place of v . Such Ursell functions satisfy the system of ordinary differential equations (3.5).

One particular tree graph identity due to Basuev [Ba2] is however worth mentioning in the context of the present work. The representation of Basuev works for radial potentials in \mathbb{R}^d of the form $\phi = \phi^a + \delta$ (see definition of Basuev potentials above), where $\phi^a(r) = \phi(r)$ for $r = |x| > a$, $\phi^a(r) = \phi(a)$ for $r \leq a$, is stable and $\delta(r) = \phi(r) - \phi^a(r) > 0$ for $r \leq a$ and $\delta(r) = 0$ for $r > a$, which may include hard-core: $\delta(r) = \infty$, $r \leq a$. To estimate the Ursell functions efficiently, Basuev uses the modified stability condition

$$U_n(\zeta_1, \dots, \zeta_n; v) \geq -(n-1)\bar{B} \quad (1.23)$$

instead of (1.16), where in the majority of cases important for applications \bar{B} is equal or closed to B . It might appear that a slight improvement on the stability bound would not affect the radius of convergence of Mayer series. It turns out, however, that the estimate of the Ursell functions through the Basuev tree graph identity works so well when (1.23) is applied (see particularly equations (15) and (16) of [Ba2]) that expressive improvements on the convergence are reported (at low temperatures) in Basuev paper, as well as in [LPY].

Let us now explain how the estimate on the Ursell functions gets improved by (1.23) in our case. It is known that the Mayer series for the pressure of a two-dimensional Yukawa gas [BK, Be, GM]

$$\beta p(\beta, z) = \sum_{k \geq 1} b_k z^k \quad (1.24)$$

$$b_k = \frac{1}{k!} \int d^{k-1} \varrho \psi_k^c(\zeta_1, \dots, \zeta_k; v)$$

converges if $|z| < (4\pi - \beta)/(4\pi e\beta)$, the radius of convergence being positive provided $\beta < 4\pi$. Since the Yukawa potential (1.3) diverges logarithmically as $|x| \rightarrow 0$, the proof of such statement requires the use of Brydges–Kennedy approach or iterated Mayer expansion (no one-scale tree expansion formula would be able to deal with this issue). The problem at our hand is to extend the stability of

2-dimensional Yukawa gas to the inverse temperature in the range $4\pi \leq \beta < 8\pi$, passing through the sequence of thresholds $\beta_{2r} = 8\pi(1 - 1/2r)$, $r \in \mathbb{N}$. Here, β_{2r} is the inverse temperature in which a clusters with r positive and r negative charges collapse altogether at once, heuristically given by an argument of entropy–energy (there are r^2 and $r(r - 1)$ distinct pairings of opposite, respectively, same charges):

$$C(\delta) = \int_{|x_2| \leq \delta} dx_2 \cdots \int_{|x_{2r}| \leq \delta} dx_{2r} \exp \left(\beta (r^2 - r(r - 1)) \int_{\delta}^1 g(s) ds \right). \quad (1.25)$$

By the second mean value theorem and $g(s) \asymp 1/(2\pi s)$ as $s \rightarrow 0$, the balance expressed by (1.25) is in favor of entropy $S(\delta) = \delta^{2(2r-1)}$ if $\beta < \beta_{2r}$ while the energy contribution $e^{-\beta E(\delta)} \simeq \delta^{-\beta r/(2\pi)}$ dominates if $\beta > \beta_{2r}$ so we have

$$\lim_{\delta \rightarrow 0} C(\delta) = c \lim_{\delta \rightarrow 0} \delta^{2(2r-1) - \beta r/(2\pi)} = \begin{cases} 0 & \text{if } \beta < \beta_{2r} \\ \infty & \text{if } \beta > \beta_{2r} \end{cases},$$

for some constant $c > 0$.

Avoiding the collapse of neutral clusters: a conjecture As a consequence of the alluded collapses, the leading even coefficients b_{2j} , $j = 1, \dots, n$, of the Mayer series (1.24) diverges for $\beta_{2n} \leq \beta < \beta_{2(n+1)}$ when the short scale cutoff t_0 , introduced in (1.10) (or in (1.7)) to make the system conditionally stable, is removed. A conjecture stated as an open problem in [Be] is as follows:

Conjecture 1.1 *If the leading n even coefficients b_{2j} 's are removed from the Mayer series (1.24), the radius of convergence of the corresponding series remains positive for any $\beta \in [\beta_{2n}, \beta_{2(n+1)})$ and, consequently, for any $\beta < \beta_{2(n+1)}$.*

Brydges-Kennedy [BK] have proved convergence of (1.24) with $O(z^2)$ term omitted for $4\pi \leq \beta < 16\pi/3$ and have explained how it would be extended up to the second threshold 6π . It turns out that the claimed improvement on the estimate of the three-particle energy from $U_3(\xi_1, \xi_2, \xi_3; \dot{v}) \geq -3\dot{v}(t, 0)/2$ to $U_3(\xi_1, \xi_2, \xi_3; \dot{v}) \geq -\dot{v}(t, 0)$ does not hold uniformly on $(\{-1, 1\} \times \mathbb{R}^2)^3$ at each scale for the decomposition (1.7) used by the authors. It has been shown by numerical calculation in [GM] that the factor 3 (the number n of charged particles involved) in the lower bound of U_3 may be improved to 2.14..., which is enough to extend the convergence of Mayer series to any $\beta \in [4\pi, 6\pi)$ but insufficient to establish the conjecture beyond a certain threshold (about $\beta_{15} = 112\pi/15$) up to 8π . For the latter, it is indeed necessary to improve the factor from 3 to 2 ($n = 3$ to $n - 1 = 2$). Both statements are proved in the present work. In addition, we have proved that, if the decomposition (1.4) for the Yukawa potential is used instead, then by (1.21) 3 can be replaced by 2 in the stability bound for U_3 and for every odd n (1.16) can be substituted by (1.23) with $B = \bar{B} = v(0)/2$.

The purpose of the present paper is also to provide a majorant candidate for the pressure of the Yukawa gas at $\beta_{2n} \leq \beta < \beta_{2(n+1)}$, uniformly in the cutoff t_0 , when it is extracted from the even leading Mayer coefficients b_{2j} , $j = 1, \dots, n$, their divergent part. Such majorant has been proposed in [GM] but our presentation is neater than the original paper making it more transparent. We adapt to the potential decomposition (1.4) all ingredients and the construction used in that reference through the scale decomposition (1.7).

The majorant construction is based on the idea already present in the early works by Imbrie [I] and [Br], according to which the Mayer series (1.24) (after some combinatorics together with the stability estimate) is dominated by an expansion in powers of $ez \|\beta v\|_1 e^B$, where $\|\beta v\|_1$ is the L^1 -norm of $\beta v(x)$ and $B = \beta v(0)/2$. A one-step Mayer expansion is not suitable to potentials that B is large in the range that $\|\beta v\|_1$ contributes little, as typically occurs for the two-dimensional Yukawa potential v (see [I, GoMa, BK] for other applications). When v is decomposed into a continuum of scales (see (1.7), or alternatively (1.10)), the Mayer series becomes, roughly speaking, an expansion in powers of $ez\tau(t_0, t)$, where

$$\tau(t_0, t) = \int_{t_0}^t \|\beta \dot{v}(s, \cdot)\|_1 e^{\beta \int_s^t \dot{v}(\tau, 0) d\tau} ds \quad (1.26)$$

solves a linear equation (3.19) satisfied by the majorant C_2 of two times the second Mayer coefficient: $2|b_2| \leq C_2$ (see (3.14) and (3.16)). It has been shown that the Mayer expansion (1.24) converges provided $\beta \in [0, 4\pi)$ and $e|z|\tau(t_0, t) < 1$ uniformly in $t_0 > 0$ (see Theorem 4.1 together with pgs. 41-42 of [BK] and Proposition 3.4, Remarks 3.5 and 3.6 below). Inside the first threshold, the domain of convergence is replaced by $(\beta, z) \in \mathbb{R}_+ \times \mathbb{C}$ such that $\beta \in [4\pi, 16\pi/3)$ and $e|z| \int_{t_0}^t \|\beta \dot{v}(s, \cdot)\|_1 e^{(3\beta/2) \int_s^t \dot{v}(\tau, 0) d\tau} ds < 1$ and we shall see that our candidate to majorant series converges provided $\beta \in [\beta_k, \beta_{k+1})$ and the factor $3\beta/2$ in the exponent of this domain is replaced by $(k+1)\beta/k$ for any $k > 1$. If C_n , $1 < n \leq k$, denote the first $k-1$ majorant coefficients: $n|b_n| \leq C_n$, we observe by (3.27) that $(k+1)(n-1)B/k$ multiplies the linear term of the equation satisfied by C_n (after the divergent part of the even $n \leq k$ coefficients have been extracted through a Lagrange multiplier L_k). In particular, for $n = 2$, $\tau_k(t_0, t)$ given by (3.26) generalizes (1.26) and solves the linear equation (3.28) for C_2 . Since the modified stability condition (1.23) applies for every $n > 1$ odd, the coefficient that multiplies the linear term of the equation for C_n , which is given by $(n-1)B < (k+1)(n-1)B/k$, implies that the same equation satisfied by C_n with n even holds for n odd. To understand why the modified stability bound (1.23) is so crucial, we observe that anything large than $(n-1)B$ would prevent the convergence of the majorant series in the whole interval of collapse $[4\pi, 8\pi)$. Recall that, when the standard scale decomposition is used, the interacting energy U_n with $n = 3$ is bounded below by a factor $-2.14B$, instead of $-2B$, preventing the nb_n to be dominated by C_n for $\beta > \beta_k$ with k verifying the inequality $2.14 > 2(k+1)/k$, i. e. $k > 15$.

Outlines of the present work The present paper is organized as follows. Section 2 is dedicated to the proof of the main Theorem 2.8 and Corollary 2.10 together with estimates on the modified Bessel functions of second kind involved in both representations of two-dimensional Yukawa potential: standard (Proposition 2.2) and scale mixtures of Euclid's hat (Proposition 2.11).

The main Theorem is then used to resolve an obstructive remark of an unpublished paper (Remark 7.5 of [GM]) which, whether the standard decomposition of the Yukawa potential into scales were adopted, would impede a direct proof of the convergence of the Mayer series of the two-dimensional Yukawa gas for the inverse temperature up to 8π . We dedicate Section 3 to the Cauchy majorant method applied to the density function of Yukawa gas on the whole interval $[4\pi, 8\pi)$ of collapses. For this system it is proven that the Mayer series converge up to the second threshold $\beta \in [4\pi, 6\pi)$ and its given explanations on the mechanism that allows it to be extended up to 8π . The paper distinguishes the stability issues from those matters related to convergence of the Mayer series. In respect to the former it is proven at the end of Section 3 that dipoles in the presence of other charges are prevented to collapse.

2 Minimal specific energy: main theorem and estimates involving modified Bessel functions

We prove in this section our main theorem (1.21) and the implications of it on the minimal specific energies $e(h)$ and $\bar{e}(h)$ for the Euclid's hat h in \mathbb{R}^2 .

Three particles minimal specific energy To begin with, let U_n be the n -particle total energy (1.15) and let

$$e_n(v) = \frac{1}{n} \inf_{\substack{(\zeta_1, \dots, \zeta_n), \\ \zeta_i = (\sigma_i, x_i) \in \{-1, 1\} \times \mathbb{R}^2}} U_n(\zeta_1, \dots, \zeta_n; v) \quad (2.1)$$

and

$$\bar{e}_n(v) = \frac{1}{n-1} \inf_{\substack{(\zeta_1, \dots, \zeta_n), \zeta_i \in \{-1, 1\} \times \mathbb{R}^2: \\ \sigma_1 + \dots + \sigma_n \neq 0}} U_n(\zeta_1, \dots, \zeta_n; v) \quad (2.2)$$

be the n -particles minimal, and constrained minimal, specific energies. As the particles of our system have either $+1$ or -1 charges, these two quantities are related to each other when n is an odd number as $e_n(v) = \bar{e}_n(v)(n-1)/n$. Let us first consider the case $n = 3$ and let $v(x)$ be given by the two-dimensional Yukawa potential (1.3) under the standard decomposition into scales (1.7), cut-off at short distances $s \leq t_0$: $v(x) = \int_{t_0}^1 \tilde{h}(|x|/s)/(2\pi s) ds$. Since, by (1.20),

$$\bar{e}_n(v) = \int_{t_0}^1 \frac{1}{2\pi s} \bar{e}_n(\tilde{h}(\cdot/s)) ds = \frac{1}{2\pi} \log \frac{1}{t_0} \bar{e}_n(\tilde{h}) ,$$

it is enough to consider the minimal specific energy of 3-particles $\bar{e}_3(\tilde{h})$ for $\tilde{h}(w) = wK_1(w)$, where K_1 is the modified Bessel function of second kind of order 1.

We shall need among other properties some general features of $\tilde{h}(w)$.

Proposition 2.1 *$w \mapsto \tilde{h}(w) = wK_1(w)$ is a regular function at every point $w \in (0, \infty)$. The function $\tilde{h}(w)$ strictly decreases from its maximum $\tilde{h}(0) = 1$, decays to 0 at ∞ exponentially fast and changes its concavity: $\tilde{h}''(w) < 0$ for $w < w_0$ and $\tilde{h}''(w) > 0$ for $w > w_0$ at $1/2 < w_0 < (1 + \sqrt{17})/8$ whose numerical value is $w_0 = 0.5950(\dots)$.*

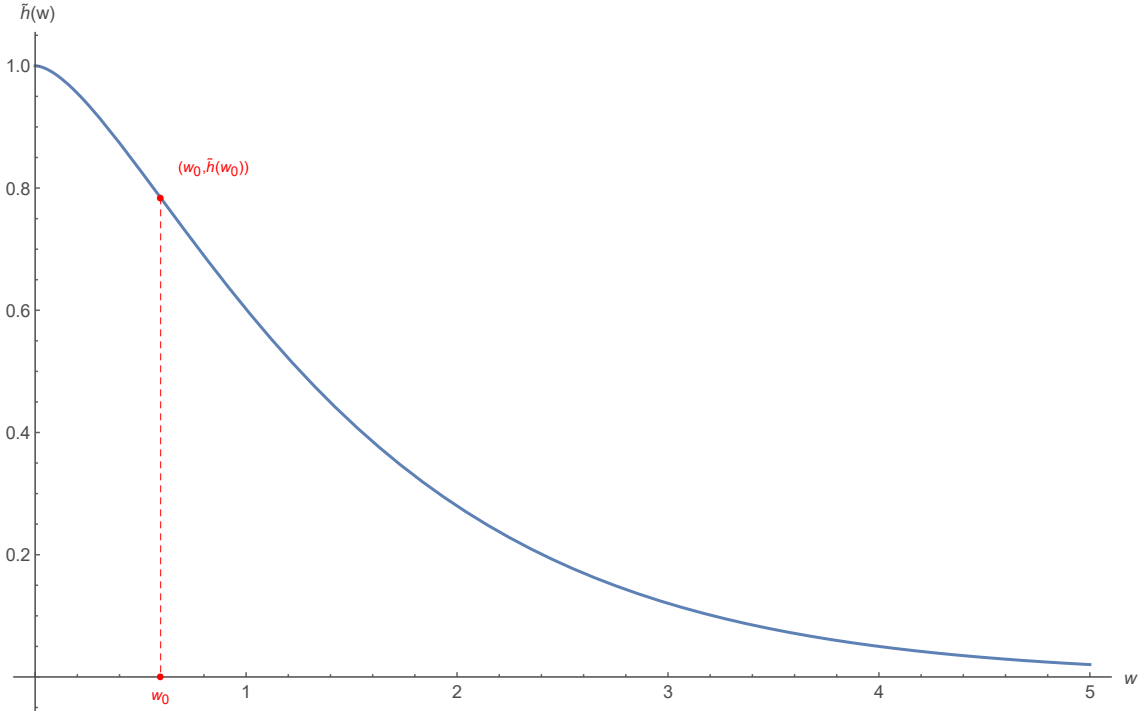


Figure 1: Plot of $\tilde{h}(w)$.

Proof. Regularity and positivity of $K_\nu(x)$ for every $\nu \in \mathbb{R}$ and $x > 0$ are known facts (see e.g. Appendix A of [Ga]). It follows from the equation

$$(x^n K_n(x))' = -x^n K_{n-1}(x) \quad (2.3)$$

with $n = 1$ together with $\lim_{w \rightarrow 0} wK_0(w) = 0$ and $\lim_{w \rightarrow 0} wK_1(w) = 1$ (see [Ga] and Lemma 2.2 of [YC]) that

$$\tilde{h}'(w) = (wK_1(w))' = -wK_0(w) < 0$$

for $w > 0$, proving the strictly decreasing property of \tilde{h} and $\tilde{h}(0) = 1$. The inequalities for $x > 0$:

$$\begin{aligned} \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x+1/2}} &< K_0(x) < \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x}} \\ 1 + \frac{1}{2x+1/2} &< \frac{K_1(x)}{K_0(x)} < 1 + \frac{1}{2x}, \end{aligned} \quad (2.4)$$

find in ref. [YC], imply the exponential decaying of $\tilde{h}(w)$ and together with

$$\begin{aligned}\tilde{h}''(w) &= -(wK_0(w))' \\ &= wK_1(w) - K_0(w) \\ &= K_0(w) \left(\frac{wK_1(w)}{K_0(w)} - 1 \right)\end{aligned}$$

(2.4) yield

$$\tilde{h}''(w) < K_0(w) \left(w - \frac{1}{2} \right) < 0$$

provided $w < 1/2$ and

$$\tilde{h}''(w) > K_0(w) \left(\frac{w}{2w + 1/2} + w - 1 \right) > 0$$

provided $w > (2w + 1/2)(1 - w) = 3w/2 + 1/2 - 2w^2$ or, equivalently, $w > (1 + \sqrt{17})/8 = 0.64039$. The unique solution of $wK_1(w)/K_0(w) - 1 = 0$, whose numerical value is $w_0 = 0.5950(\dots)$, satisfies $1/2 < w_0 < 0.64039$ (see proof of Lemma 2.5). This concludes the proof. \square

Because the particles interact via a pair potential, it is easy to see that the minimum potential energy is due to a system in which two of the three particles have equal signs and the third has charge with the opposite sign. The potential energy (1.15) with $n = 3$ and $\sigma_1 = \sigma_3 = -\sigma_2$ is then given by

$$U_3(\zeta_1, \zeta_2, \zeta_3; \tilde{h}) = -\tilde{h}(|x_1 - x_2|) - \tilde{h}(|x_2 - x_3|) + \tilde{h}(|x_1 - x_3|) .$$

To simplify the expression, we write $r_1 = |x_1 - x_2|$, $r_2 = |x_2 - x_3|$ and $r_3 = |x_1 - x_3|$ can be written, as the particles are located at the vertices of a triangle, by the law of cosine, as

$$r_3(r_1, r_2, \theta) = \sqrt{(r_1 - r_2)^2 + 4r_1r_2 \sin^2 \theta / 2} .$$

Since $\tilde{h}(w)$ is a strictly decreasing function, the minimal specific energy of 3-particles (2.1) thus reads

$$\begin{aligned}\bar{e}_3(\tilde{h}) &= \frac{1}{2} \min_{r_1, r_2 \geq 0, 0 \leq \theta \leq \pi} \left(\tilde{h}(r_3(r_1, r_2, \theta)) - \tilde{h}(r_1) - \tilde{h}(r_2) \right) \\ &= \frac{1}{2} \min_{r_1, r_2 \geq 0} \left(\tilde{h}(r_1 + r_2) - \tilde{h}(r_1) - \tilde{h}(r_2) \right) .\end{aligned}\tag{2.5}$$

The next proposition shows that this quantity does not reach from below the value $-1/2 = (-\sum_{i=1}^3 \sigma_i^2 + |\sum_{i=1}^3 \sigma_i|) / (2 \cdot (3 - 1))$ that one would expected for a convex function h .

Proposition 2.2

$$(K_1(1) - K_1(1/2)) / 2 > \bar{e}_3(\tilde{h}) > -0.535 .\tag{2.6}$$

Remark 2.3 *As the numerical evaluations used in the proofs are sharp up to high decimal order, we may claim that $\bar{e}_3 = -0.530(\dots)$, which is certainly less than $-1/2$ ($-0.527(\dots) > \bar{e}_3 > -0.535$), according to the precision of the machine used to calculate it.*

Proof. To prove (2.6), it is enough by (2.5) to show that

$$\tilde{h}(x+y) - \tilde{h}(x) - \tilde{h}(y) + 1.07 > 0 \quad (2.7)$$

holds for all $x, y \geq 0$. Defining $f(x) = \tilde{h}(x) - 1.07$, equation (2.7) is equivalent to show superadditivity of $f(x)$:

$$f(x+y) > f(x) + f(y) . \quad (2.8)$$

But this is implied by the following

Lemma 2.4 *Let $q(x) = f(x)/x$ be defined for $x > 0$. If $q(x)$ is monotone increasing, then $f(x)$ is superadditive.*

Proof of Lemma 2.4. Suppose that $q(x)$ is monotone increasing function. Then $q(x+y) \geq q(x)$, $q(x+y) > q(y)$ and it follows that

$$\begin{aligned} f(x+y) &= x \frac{f(x+y)}{x+y} + y \frac{f(x+y)}{x+y} \\ &= xq(x+y) + yq(x+y) \\ &> xq(x) + yq(y) \\ &= f(x) + f(y) , \end{aligned}$$

which proves the lemma. □

It remains thus to prove that $q(x) = (\tilde{h}(x) - 1.07)/x = K_1(x) - 1.07/x$ is monotone increasing. From (2.3) with $n = 1$, we deduce

$$K_1(x) + xK_1'(x) = (xK_1(x))' = -xK_0(x)$$

which implies that

$$\begin{aligned} q'(x) &= K_1'(x) + \frac{1.07}{x^2} \\ &= \frac{-1}{x^2} (xK_1(x) + x^2K_0(x) - 1.07) > 0 \end{aligned}$$

for $x > 0$ provided

$$xK_1(x) + x^2K_0(x) < 1.07 .$$

This inequality, however, holds in view of the following:

Lemma 2.5 *The function $x \mapsto p(x) = xK_1(x) + x^2K_0(x)$ defined in \mathbb{R}_+ has a global maximum at x_0 , $1/2 < x_0 < (1 + \sqrt{17})/8$. It strictly increases with $p(0) = 1$ as x varies from 0 to $1/2$ and strictly decreases to 0, exponentially fast, as x varies from $(1 + \sqrt{17})/8$ to ∞ . The second derivative $p''(x)$ of $p(x)$ is negative in the interval $1/2 \leq x_0 \leq (1 + \sqrt{17})/8$. Numerically, $x_0 = 0.5950 (\dots)$ and its (global) maximum values $p(x_0) = 1.061(\dots) < 1.07$.*

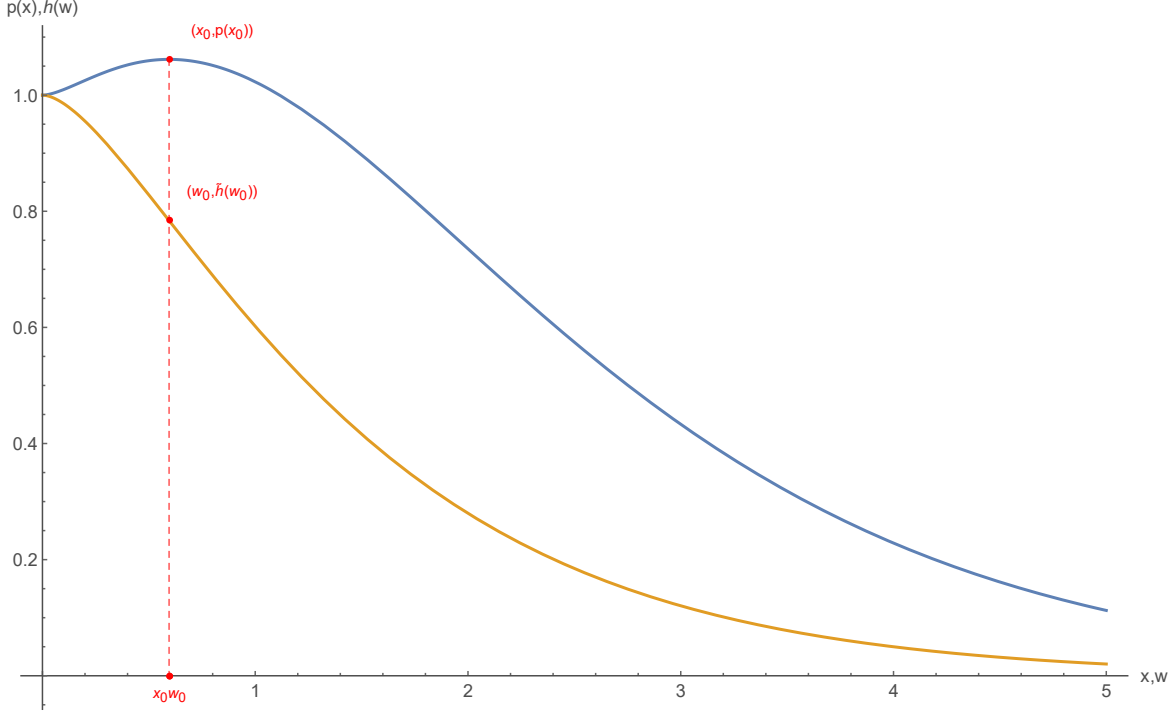


Figure 2: Plot of $p(x)$ and $\tilde{h}(w)$ together.

Proof of Lemma 2.5. Using (2.3) with $n = 1$ together with $K'_0(x) = -K_1(x)$, as in the proof of Proposition 2.1, we have

$$\begin{aligned} p'(x) &= -xK_0(x) + 2xK_0(x) - x^2K_1(x) \\ &= xK_0(x) \left(1 - \frac{xK_1(x)}{K_0(x)} \right) \\ &< xK_0(x) \left(1 - \frac{x}{2x + 1/2} - x \right) < 0 \end{aligned}$$

provided $x > (1 + \sqrt{17})/8$ and

$$p'(x) > xK_0(x) \left(x - \frac{1}{2} \right) > 0$$

provided $x < 1/2$. These prove that $p(x)$ increases in $(0, 1/2)$ and decreases in $((1 + \sqrt{17})/8, \infty)$, exponentially fast in view of (2.4).

$p(x)$ attains its maximum value at the same point at which $\tilde{h}(w)$ changes its concavity. The maximum x_0 of $p(x)$ solves $K_0(x) - xK_1(x) = 0$ and satisfies $1/2 < x_0 < (1 + \sqrt{17})/8 \approx 0.64$, as stated above and showed in Proposition 2.1. To prove that x_0 is the global maximum, it suffices to show that the second derivative of $p(x)$, which may be calculated exactly as before,

$$\begin{aligned}
p''(x) &= (x(K_0(x) - xK_1(x)))' \\
&= K_0(x) - xK_1(x) + x(K_0'(x) - (xK_1(x))') \\
&= (1 + x^2)K_0(x) - 2xK_1(x) \\
&= -2K_0(x) \left(\frac{xK_1(x)}{K_0(x)} - \frac{1 + x^2}{2} \right),
\end{aligned}$$

takes negative values for $x \in [1/2, (1 + \sqrt{17})/8]$. By equation (2.4) and positivity of $K_0(x)$, this is implied by

$$\frac{xK_1(x)}{K_0(x)} - \frac{1 + x^2}{2} > x + \frac{x}{2x + 1/2} - \frac{1 + x^2}{2} > 0.$$

Denoting the function on the right hand side by $l(x) = x + x/(2x + 1/2) - (1 + x^2)/2$, we need to show that $l(x) > 0$ for $x \in [1/2, (1 + \sqrt{17})/8]$. But $l(1/2) = 5/24 \approx 0.20$ and $l((1 + \sqrt{17})/8) = (23 + \sqrt{17})/64 \approx 0.29$ are both positive and the second derivative of $l(x)$,

$$l''(x) = -(17 + 12x + 48x^2 + 64x^3)/(1 + 4x)^3 < 0$$

for all $x > 0$, proving therefore the statement.

We have thus proven that x_0 is a global maximum of $p(x)$, concluding the proof of Lemma 2.5. \square

Returning to the proof of Proposition 2.2, the lower bound stated in (2.6) follow from the superadditivity of $f(x) = \tilde{h}(x) - 1.07$, which is proven in Lemmas 2.4 and 2.5. The numerical estimate for the specific energy \bar{e}_3 stated in Remark 2.3 is obtained when 1.07 is replaced by the maximum values $p(x_0) = 1.061(\dots)$, given in Lemma 2.5, since at this point $f(x_0) = \tilde{h}(x_0) - 1.061(\dots)$ satisfies (2.8) as an equality and consequently, by (2.7), $\tilde{h}(2x_0) - \tilde{h}(x_0) - \tilde{h}(x_0) = -1.061(\dots)$.

By definition (2.5), taking $r_1 = r_2 = 1/2$ in the expression inside the minimum, we have an upper bound

$$\bar{e}_3 < \frac{1}{2} (\tilde{h}(1) - 2\tilde{h}(1/2)) = \frac{1}{2} (K_1(1) - K_1(1/2)) = -0.527(\dots).$$

\square

Remark 2.6 *It does not seem easy to extend the superadditivity method used to estimated the (restricted) minimum specific energy of 3-particles to $(2k + 1)$ -particles with $k > 1$. As we shall see*

in the next section, the result on the minimal specific energy $\bar{e}_3(\tilde{h})$ prevents that the third Mayer coefficient be defined uniformly in the cutoff t_0 in the entire collapse interval $[4\pi, 8\pi]$, although it is enough for concluding convergence of the Mayer series up to the second threshold $[4\pi, 6\pi]$. Numerical calculations performed in [GM] indicate that $\bar{e}_{2k+1}(\tilde{h})$ remains for $k > 1$ strictly smaller than $-1/2$. We should mention that if $\tilde{h}(w)$ were convex, the minimal of (2.5) would be attained at $r_1 = r_2 = 0$, obtaining the expected value $\bar{e}_3 = -1/2$ as it is exactly the case when we use decomposition (1.4) of the Yukawa potential (1.3) instead of (1.7). Since the method based on superadditivity cannot be easily extended to $k > 1$, another method will be employed to obtain $\bar{e}_{2k+1}(h) = -1/2$ with h the Euclid's hat function (1.5).

The main theorem We shall now turn to the representation of Yukawa potential (1.3) given by $v(x) = v_{(0,\infty)}(x) = K_0(|x|)/(2\pi)$ where (see (1.4)):

$$v_{(t_0,t)}(x) = \int_{t_0}^t h(|x|/s)g(s)ds, \quad (2.9)$$

is a scale mixtures of Euclid's hat. Here, for $x \in \mathbb{R}^2$ and $s \in \mathbb{R}_+$,

$$h(|x|/s) = \frac{4}{\pi s^2} \chi_{[0,s/2]} * \chi_{[0,s/2]}(x) \quad (2.10)$$

is the self convolution of indicator function $\chi_{[0,s/2]}(x) := \theta(s/2 - |x|)$ of the 2-dimensional ball (disc) $B_r \equiv B_r(0)$ of radius $r = s/2$ centered at origin and $g(s)$ is the scale mixtures density given by Hainzl–Seiringer: [HS]

$$g(s) = \frac{-s}{4\pi} \int_s^\infty K_0'''(r) \frac{r}{\sqrt{r^2 - s^2}} dr. \quad (2.11)$$

We observe that (2.11) differs from the $g(s)$ in equation (11) of [HS] by a pre-factor $\pi (s/2)^2$ that we have used in (2.10) in order to normalize h at origin: $h(0) = 1$. This normalization is suitable when the radial function $\varphi(|x|) = v(x)$ is the characteristic function of a spherically symmetric probability distribution in \mathbb{R}^d or the covariance of a stationary and isotropic random field on d -dimensional Euclidean space. The latter is the point of view of the present paper, while the former were the focus of Gneiting paper [G], for which the classes H_d of radial positive definite functions generated by scale mixtures of d -dimensional Euclid's hat $h_d(|x|)$ played an important role in the proof of an analogue of Pólya's criterion for $d > 1$. We observe however that the scale mixture used in [G] is of the form $\varphi(t) = \int_0^\infty h_d(rt) dG(r)$, where $G(r)$ is a probability distribution function in $(0, \infty)$ with $G(0+) = c \in [0, 1]$. In order to compare with our $g(s)$ given by (2.11) (by (2.9) $s = 1/r$), which behaves as s goes to 0 as $1/(2\pi s)$, in the case that $dG(r)$ is absolutely continuous and finite positive measure in \mathbb{R}_+ , we write $dG(r) = f(r)dr = -f(1/s)ds/s^2 = -\tilde{f}(s)ds$. We see that $\tilde{f}(s) \sim 1/s$ would lead to a nonintegrable mixture density $f(r) \sim 1/r$ at infinity and, consequently, $\varphi(t)$ with such a density would not belong to the class H_2 considered in that paper.

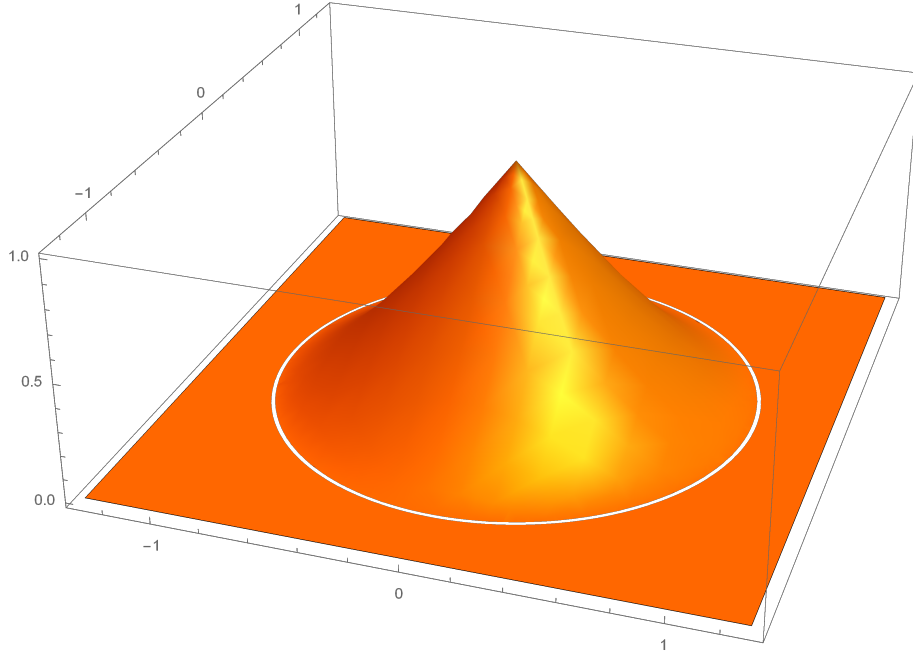


Figure 3: Euclid's hat function.

Equation (2.11) can be written in terms of a Meijer G -functions that is regular at $s = 0$ as

$$2\pi s g(s) = \sqrt{\pi} G_{13}^{30} \left(s^2/4 \middle| \begin{matrix} 1/2 \\ 0, 1, 2 \end{matrix} \right) \quad (2.12)$$

as one can check using Mathematica program together with the shift property: $t^2 G_{13}^{30} \left(t \middle| \begin{matrix} -3/2 \\ -2, -1, 0 \end{matrix} \right) = G_{13}^{30} \left(t \middle| \begin{matrix} 1/2 \\ 0, 1, 2 \end{matrix} \right)$.

We begin by describing the general features of $h(w)$. We shall state and prove our main theorem afterwards and return to the asymptotic properties of (2.12) required for the next section.

Proposition 2.7 $w \mapsto h(w)$ defined by (2.10) is regular at every point $w \in (0, 1)$, convex and non increasing function in $(0, \infty)$. Moreover, it can be written as

$$h(w) = \frac{2}{\pi} \left(\arccos w - w\sqrt{1-w^2} \right), \quad \text{if } 0 \leq w \leq 1 \quad (2.13)$$

$h(w) = 0$ if $w > 1$ so, writing $\varphi(x) = h(|x|)$ we have $\varphi(0) = h(0) = 1$ and $\hat{\varphi}(0) = \int_{\mathbb{R}^2} h(|x|) dx = \pi/4$ is its Fourier transform $\hat{\varphi}(\xi)$ at $\xi = 0$.

Proof. We shall deduce (2.13) from (2.10) by means of a geometric representation of the convolution integral

$$\frac{\pi s^2}{4} h(w) = \int_{\mathbb{R}^2} \chi_{[0, s/2]}(x-y) \chi_{[0, s/2]}(y) dy \quad (2.14)$$

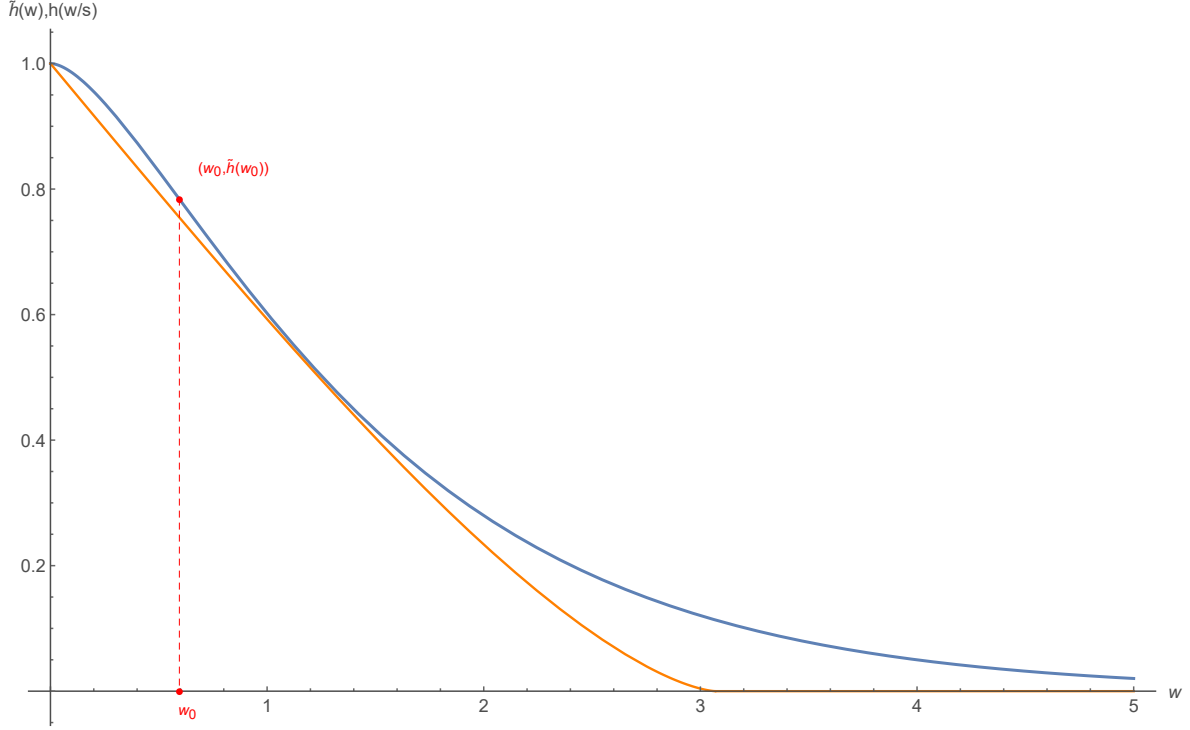


Figure 4: Plot of $h(w/s)$ scaled by $s = 3.07$ and $\tilde{h}(w)$ together.

(see e.g Sec. 2 of [G]). The product of indicator functions does not vanish if their support, the discs $B_{s/2}(x)$ and $B_{s/2}(0)$ centered at x and 0, intersect and this occurs when the distance $|x|$ between their centers is less than their diameter s . Writing $w = |x|/s$, we have

$$h(w) \neq 0 \iff 0 \leq w < 1 .$$

From this point of view, the convolution integral (2.14) is given by the area $A(\theta)$ of two "caps", of common bases, made of a sector of opening angle θ and radius $s/2$ with the triangular region inside removed (see Fig. 5):

$$\frac{\pi s^2}{4} h(w) = A(\theta) = 2 \times \left(\frac{1}{2} \left(\frac{s}{2} \right)^2 \theta - \frac{1}{2} \left(\frac{s}{2} \right)^2 \sin \theta \right) , \quad (2.15)$$

where, with b the length of the caps common bases,

$$\begin{aligned} |x| &= s \cos \theta / 2 \\ b &= s \sin \theta / 2 . \end{aligned} \quad (2.16)$$

By $|x|^2 + b^2 = s^2$, we deduce $b = s\sqrt{1-w^2}$. Solving equations (2.16) for θ and $\sin \theta$:

$$\begin{aligned} \theta &= 2 \arccos w \\ \sin \theta &= 2 \sin \theta / 2 \cos \theta / 2 = 2w\sqrt{1-w^2} , \end{aligned}$$

together with (2.15), yields

$$A(\theta) = \frac{s^2}{4} (\theta - \sin \theta) = \frac{s^2}{2} \left(\arccos w - w\sqrt{1-w^2} \right) . \quad (2.17)$$

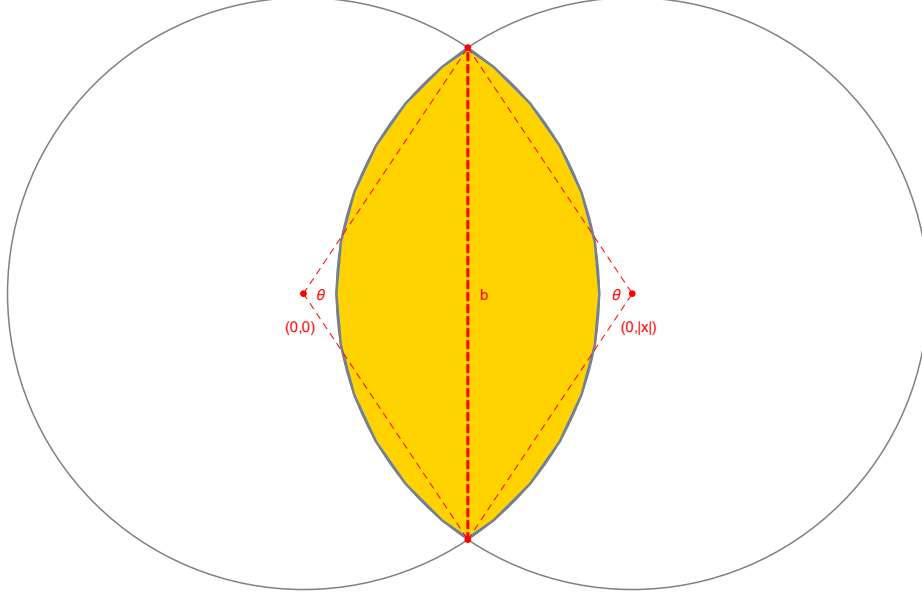


Figure 5: Geometric interpretation of the Euclid's hat function.

Equation (2.13) follows from (2.14), (2.15) and (2.17). The regularity of $h(w)$ in $(0, 1)$ follows from this representation. Since

$$h'(w) = -2\sqrt{1-w^2} < 0 ,$$

$$h''(w) = \frac{2w}{\sqrt{1-w^2}} > 0 ,$$

for any $w \in (0, 1)$, we conclude that $h(w)$ is such that $h(0) = 1$, by definition, is strictly decreasing in $(0, 1)$ monotone non increasing and convex in $(0, \infty)$, concluding the proof.

□

Before we state and prove our main theorem, we use (2.10) to write

$$\sum_{1 \leq i, j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) = \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j \chi_{[0, s/2]}(x_i - x_j - y) \chi_{[0, s/2]}(y) dy .$$

Changing the integration variables for each term of the sum to $z = y + x_j$ yields

$$\begin{aligned} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) &= \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j \chi_{[0, s/2]}(x_i - z) \chi_{[0, s/2]}(z - x_j) dz \\ &= \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \left(\sum_{j=1}^n \sigma_j \chi_{[0, s/2]}(z - x_j) \right)^2 dz \end{aligned} \quad (2.18)$$

since the function $\chi_{[0,s/2]}(x)$ is even.

Theorem 2.8 *For any integer $n \geq 2$, any configuration of n -particle $(\zeta_1, \dots, \zeta_n)$, $\zeta_j = (\sigma_j, x_j) \in \{-1, 1\} \times \mathbb{R}^2$ any $s \in \mathbb{R}_+$, the total energy with interacting potential h satisfies*

$$U_n(\zeta_1, \dots, \zeta_n; h(\cdot/s)) = \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) \geq -\frac{1}{2} \left(n - \left| \sum_{j=1}^n \sigma_j \right| \right). \quad (2.19)$$

Proof. Since $h(0) = 1$, we add $n/2$ to the total energy in order to include the $i = j$ terms into the sum in (2.19):

$$\sum_{1 \leq i < j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) = -\frac{1}{2} \sum_{j=1}^n \sigma_j^2 h(0) + \frac{1}{2} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s).$$

So, the result is proven if we show that

$$\sum_{1 \leq i, j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) \geq \left| \sum_{j=1}^n \sigma_j \right|.$$

Using the fact that $\sum_{j=1}^n \sigma_j \chi_{[0,s/2]}(z - x_j)$ is always an integer number, we have

$$\left(\sum_{j=1}^n \sigma_j \chi_{[0,s/2]}(z - x_j) \right)^2 \geq \left| \sum_{j=1}^n \sigma_j \chi_{[0,s/2]}(z - x_j) \right|$$

and this, together with (2.18), implies that

$$\begin{aligned} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j h(|x_i - x_j|/s) &\geq \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \left| \sum_{j=1}^n \sigma_j \chi_{[0,s/2]}(z - x_j) \right| dz \\ &\geq \left| \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \sum_{j=1}^n \sigma_j \chi_{[0,s/2]}(z - x_j) dz \right| \\ &= \left| \sum_{j=1}^n \sigma_j \frac{4}{\pi s^2} \int_{\mathbb{R}^2} \chi_{[0,s/2]}(z - x_j) dz \right| \\ &= \left| \sum_{j=1}^n \sigma_j \right|, \end{aligned}$$

concluding the proof. □

Remark 2.9 *Since the proof does not set any condition on the dimension of the Euclidean space, Theorem 2.8 holds for any $d \geq 2$. In this case, $h(w)$ has to be replaced by the Euclid's hat $h_d(w)$ (see Sec. 2 of [G] for the proof of the statements in Proposition 2.7 for the corresponding d -dimensional Euclid's hat).*

Theorem 2.8 implies the following

Corollary 2.10 *The minimal specific energy $e(h)$ and the minimal constrained specific energy $\bar{e}(h)$, defined by (1.18) and (1.19), are both $-1/2$.*

Proof. This result follows from the definitions (2.1) and (2.2) and the inequality (2.19). The minimal specific energy $e(h) = \inf_{n \geq 2} e_n(h)$ of h is attained for even number of particles n satisfying $\sum_{j=1}^n \sigma_j = 0$ and $\sum_{j=1}^n \sigma_j^2 = n$ when they collapse to a single point since, in this case, the inequality (2.19) becomes an equality. Likewise, the constrained minimal specific energy $\bar{e}(h) = \inf_{n \geq 2} \bar{e}_n(h)$ of h is attained for odd number of particles n satisfying $\left| \sum_{j=1}^n \sigma_j \right| = 1$ and $\sum_{j=1}^n \sigma_j^2 = n$ when they collapse to a single point. Note that, for a calculation similar to the energy in (1.25), the potential energy (1.15) with $n = 2r + 1$, $\sigma_1 = \dots = \sigma_r = -\sigma_{r+1} = \dots = -\sigma_{2r+1}$ and $x_1 = \dots = x_{2r+1} = x_0 \in \mathbb{R}^2$, is given by $(h(0) = 1)$

$$U_n(\zeta_1, \dots, \zeta_n; h) = -(r+1)r + \frac{r(r-1)}{2} + \frac{(r+1)r}{2} = -r = \frac{-1}{2}(n-1) .$$

□

As a consequence of Corollary 2.10, representing the Yukawa potential v as scale mixtures of Euclid's hat (2.9) regularized at short distances, the stability bound (1.16) can be replaced by (1.23) with

$$B = \bar{B} = \frac{1}{2} \int_{t_0}^t g(s) ds .$$

Properties of the mixture function Regarding the mixture function, we have the following

Proposition 2.11 *The function $g : (0, \infty) \longrightarrow (0, \infty)$ given by (2.11) can be written as*

$$g(s) = \frac{1}{2\pi s} m(s)$$

where

$$m(s) = \frac{1}{2} \int_s^\infty y^2 K_1(y) \frac{y}{\sqrt{y^2 - s^2}} dy \quad (2.20)$$

is a regular function such that $m(0) = 1$, increases monotonously in $(0, s_0)$, where $s_0 = 0.812(\dots)$ and $m(s_0) = m_0 = 1.075(\dots)$, then decreases monotonously in (s_0, ∞) to 0, exponentially fast. Globally, it is bounded from above and from below as

$$\frac{\pi}{4} e^{-s}(1 + s + s^2) < m(s) < \frac{\pi}{4} e^{-s}(3 + 3s + s^2) , \quad \forall s \in [0, \infty) . \quad (2.21)$$

In the vicinity of the origin, it satisfies

$$m(s) \leq 1 + \left(a - \frac{1}{4} \log s \right) s^2, \quad s \in [0, 1] \quad (2.22)$$

where $a = (1 - 3\gamma + \log 4 - \psi(-1/2))/8 = 0.07726(\dots)$, being the r.h.s. of (2.22) asymptotic to $m(s)$ at $s = 0$.

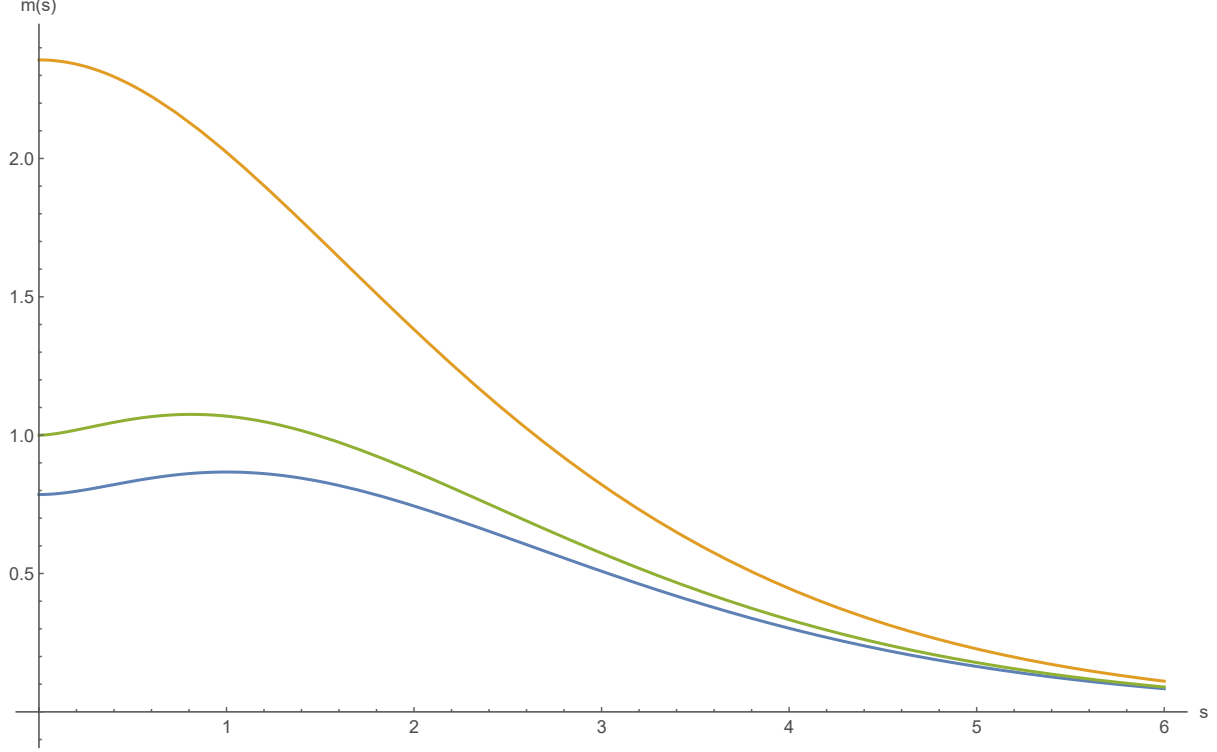


Figure 6: Plot of $m(s)$ together with its upper and lower functions.

Proof. We begin by showing that (2.11) multiplied by $2\pi s$ can be written as (2.20). For this, we use $K_1(w) = -K'_0(w)$ and the representation (see [GJ], Section 7.2)

$$K_0(w) = \int_0^\infty e^{-w\sqrt{k^2+1}} \frac{dk}{\sqrt{k^2+1}}, \quad (2.23)$$

from which we infer that K_0 is regular in $(0, \infty)$. We may thus differentiate (2.23) three times, replace it into (2.11), exchange the integration order and, after multiplying by $2\pi s$ it can be written as

$$m(s) = \int_0^\infty (k^2 + 1) F(s, k) dk \quad (2.24)$$

where

$$\begin{aligned} F(s, k) &= \frac{s^2}{2} \int_s^\infty e^{-r\sqrt{k^2+1}} \frac{r dr}{\sqrt{r^2 - s^2}} \\ &= \frac{s^3}{2} \int_0^\infty e^{-s\sqrt{k^2+1}\sqrt{z^2+1}} dz \\ &= -\frac{s^3}{2} K'_0(s\sqrt{k^2+1}) \\ &= \frac{s^3}{2} K_1(s\sqrt{k^2+1}). \end{aligned} \quad (2.25)$$

We have changed variable $sz = \sqrt{r^2 - s^2}$, so $r = s\sqrt{z^2 + 1}$ and $rdr/\sqrt{r^2 - s^2} = s dz$. Replacing (2.25) back into (2.24), making one more change of variable: $s\sqrt{k^2 + 1} = y$, so that $sk = \sqrt{y^2 - s^2}$ and $sdk = y/\sqrt{y^2 - s^2}$, yields (2.20).

The sequence of operations bringing (2.11) into the form (2.20) will be applied some more times. Let us start by finding a lower bound for (2.20). By monotonicity of the modified Bessel functions with respect to their order (see [C]) and integration by parts, we have

$$\begin{aligned} m(s) &> \frac{1}{2} \int_s^\infty y^2 K_0(y) \frac{y}{\sqrt{y^2 - s^2}} dy \\ &= \frac{-1}{2} \int_s^\infty (y^2 K_0(y))' \sqrt{y^2 - s^2} dy \\ &= L(s) - J(s) \end{aligned} \tag{2.26}$$

where

$$L(s) = \frac{1}{2} \int_s^\infty y^2 K_1(y) \sqrt{y^2 - s^2} dy \tag{2.27}$$

$$J(s) = \int_s^\infty y K_0(y) \sqrt{y^2 - s^2} dy . \tag{2.28}$$

Observe that the boundary term in the partial integration, $y^2 K_0(y) \sqrt{y^2 - s^2} / 2 \Big|_{y=s}^\infty$ vanishes for all $s \in (0, \infty)$ because the exponential decay of $K_0(y)$ and boundedness of $y^2 K_0(y)$.

Lemma 2.12 *Let $I : [0, \infty) \longrightarrow \mathbb{R}$ be defined by*

$$I(s) = \int_s^\infty K_1(y) \sqrt{y^2 - s^2} dy . \tag{2.29}$$

The integral can be written as

$$I(s) = \int_s^\infty K_1(y) \frac{s}{\sqrt{y^2 - s^2}} dy \tag{2.30}$$

and from these we conclude that $I(s) = \pi e^{-s} / 2$.

Proof. Since the integral (2.29) converge uniformly in $[s_0, K)$, for any $s_0 > 0$ and $K < \infty$, the integral (2.30) is minus the derivative of the integral (2.29):

$$I'(s) = \int_s^\infty K_1(y) \frac{-s}{\sqrt{y^2 - s^2}} dy = -I(s) , \quad s > 0 . \tag{2.31}$$

Observe that $K_1(y) \sqrt{y^2 - s^2} \Big|_{y=s} = 0$ for the same reason as before. Since ae^{-s} solves (2.31) for any $a \in \mathbb{R}$, the proof will be completed once we establish that (2.29) implies (2.30) and show $I(0) = \pi/2$. We begin with the latter.

Repeating the operations bringing (2.11) into the form (2.20), it follows from (2.23) and $K'_0(y) = -K_1(y)$ that

$$\begin{aligned}
I(0) &= \int_0^\infty y K_1(y) dy \\
&= \int_0^\infty y \left(\int_0^\infty e^{-y\sqrt{k^2+1}} dk \right) dy \\
&= \int_0^\infty \left(\int_0^\infty y e^{-y\sqrt{k^2+1}} dy \right) dk \\
&= \int_0^\infty \frac{-1}{\sqrt{k^2+1}} \left(y e^{-y\sqrt{k^2+1}} \Big|_{y=0}^\infty - \int_0^\infty e^{-y\sqrt{k^2+1}} dy \right) dk \\
&= \int_0^\infty \frac{1}{k^2+1} dk = \arctan k \Big|_{k=0}^\infty = \frac{\pi}{2}.
\end{aligned} \tag{2.32}$$

Now, we develop (2.29) as

$$\begin{aligned}
\int_s^\infty K_1(y) \sqrt{y^2 - s^2} dy &= \int_s^\infty \left(\int_0^\infty e^{-y\sqrt{k^2+1}} dk \right) \sqrt{y^2 - s^2} dy \\
&= \int_0^\infty \left(\int_s^\infty e^{-y\sqrt{k^2+1}} \sqrt{y^2 - s^2} dy \right) dk \\
&= \int_0^\infty \left(\int_s^\infty e^{-y\sqrt{k^2+1}} \frac{y}{\sqrt{y^2 - s^2}} dy \right) \frac{dk}{\sqrt{k^2+1}} \\
&= s \int_0^\infty \left(\int_0^\infty e^{-s\sqrt{k^2+1}\sqrt{r^2+1}} dr \right) \frac{dk}{\sqrt{k^2+1}} \\
&= s \int_0^\infty K_1 \left(s\sqrt{k^2+1} \right) \frac{dk}{\sqrt{k^2+1}} \\
&= s \int_s^\infty K_1(y) \frac{dy}{\sqrt{y^2 - s^2}}
\end{aligned}$$

In the second equality we exchange the integration order, then we integrate by parts; we change variable $y = s\sqrt{r^2+1}$ in the fourth equality, use (2.23) together with $K'_0(w) = -K_1(w)$ and in the last equality we change again the variable $y = s\sqrt{k^2+1}$. This concludes the proof of the lemma. \square

Returning to the proof of Proposition 2.11, we now deduce an equation for $J(s)$ and $L(s)$ in

terms of $I(s)$. Differentiating (2.28) with respect to s , gives

$$\begin{aligned}
J'(s) &= -yK_0(y)\sqrt{y^2-s^2}\Big|_{y=s} - \int_s^\infty yK_0(y)\frac{s}{\sqrt{y^2-s^2}}dy \\
&= -s \int_s^\infty K_0(y) \left(\sqrt{y^2-s^2}\right)' dy \\
&= -sK_0(y)\sqrt{y^2-s^2}\Big|_{y=s}^\infty - s \int_s^\infty K_1(y)\sqrt{y^2-s^2}dy \\
&= -sI(s) = \frac{-\pi}{2}se^{-s}, \tag{2.33}
\end{aligned}$$

by Lemma 2.12. Analogously, differentiating (2.27) with respect to s , together with (2.3), gives

$$\begin{aligned}
L'(s) &= \frac{-s}{2} \int_s^\infty yK_1(y)\frac{y}{\sqrt{y^2-s^2}}dy \\
&= \frac{s}{2} \int_s^\infty (yK_1)'(y)\sqrt{y^2-s^2}dy \\
&= \frac{-s}{2} \int_s^\infty yK_0(y)\sqrt{y^2-s^2}dy \\
&= -\frac{1}{2}sJ(s). \tag{2.34}
\end{aligned}$$

We need also initial condition to both equations. Performing as in (2.32),

$$\begin{aligned}
J(0) &= \int_0^\infty y^2 K_0(y)dy \\
&= \int_0^\infty y^2 \left(\int_0^\infty e^{-y\sqrt{k^2+1}} \frac{dk}{\sqrt{k^2+1}} \right) dy \\
&= \int_0^\infty \left(\int_0^\infty y^2 e^{-y\sqrt{k^2+1}} dy \right) \frac{dk}{\sqrt{k^2+1}} \\
&= 2 \int_0^\infty \frac{dk}{(k^2+1)^2} \\
&= \left(\arctan k + \frac{k}{k^2+1} \right) \Big|_{k=0}^\infty = \frac{\pi}{2} \tag{2.35}
\end{aligned}$$

and

$$\begin{aligned}
L(0) &= \frac{1}{2} \int_0^\infty y^3 K_1(y) dy \\
&= \frac{1}{2} \int_0^\infty y^3 \left(\int_0^\infty e^{-y\sqrt{k^2+1}} dk \right) dy \\
&= \frac{1}{2} \int_0^\infty \left(\int_0^\infty y^3 e^{-y\sqrt{k^2+1}} dy \right) dk \\
&= 3 \int_0^\infty \frac{dk}{(k^2+1)^2} = \frac{3\pi}{4} .
\end{aligned} \tag{2.36}$$

Integrating (2.33) together with (2.35), yields

$$\begin{aligned}
J(s) &= J(0) - \frac{\pi}{2} \int_0^s t e^{-t} dt \\
&= \frac{\pi}{2} \left(1 + s e^{-s} - \int_0^s e^{-t} dt \right) = \frac{\pi}{2} (1 + s) e^{-s} .
\end{aligned} \tag{2.37}$$

Analogously, integrating (2.34) together with (2.37) and (2.36), yields

$$\begin{aligned}
L(s) &= L(0) - \frac{\pi}{4} \int_0^s (1+t) t e^{-t} dt \\
&= \frac{\pi}{4} \left(3 + (1+s) s e^{-s} - \int_0^s (1+2t) e^{-t} dt \right) \\
&= \frac{\pi}{4} \left(2 + (1+s) s e^{-s} + (1+2s) e^{-s} - 2 \int_0^s e^{-t} dt \right) \\
&= \frac{\pi}{4} (3 + 3s + s^2) e^{-s} .
\end{aligned} \tag{2.38}$$

Equations (2.37) and (2.38) replaced into (2.26) gives the lower bound (2.21).

An upper bound is obtained similarly. By monotonicity of the modified Bessel functions with respect to their order (see [C]) and integration by parts, we have

$$\begin{aligned}
m(s) &< \frac{1}{2} \int_s^\infty y^2 K_2(y) \frac{y}{\sqrt{y^2 - s^2}} dy \\
&= \frac{-1}{2} \int_s^\infty (y^2 K_2)'(y) \sqrt{y^2 - s^2} dy = L(s)
\end{aligned} \tag{2.39}$$

by (2.3), where $L(s)$ is given by (2.27). Equation (2.39) together with (2.38) gives the upper bound (2.21).

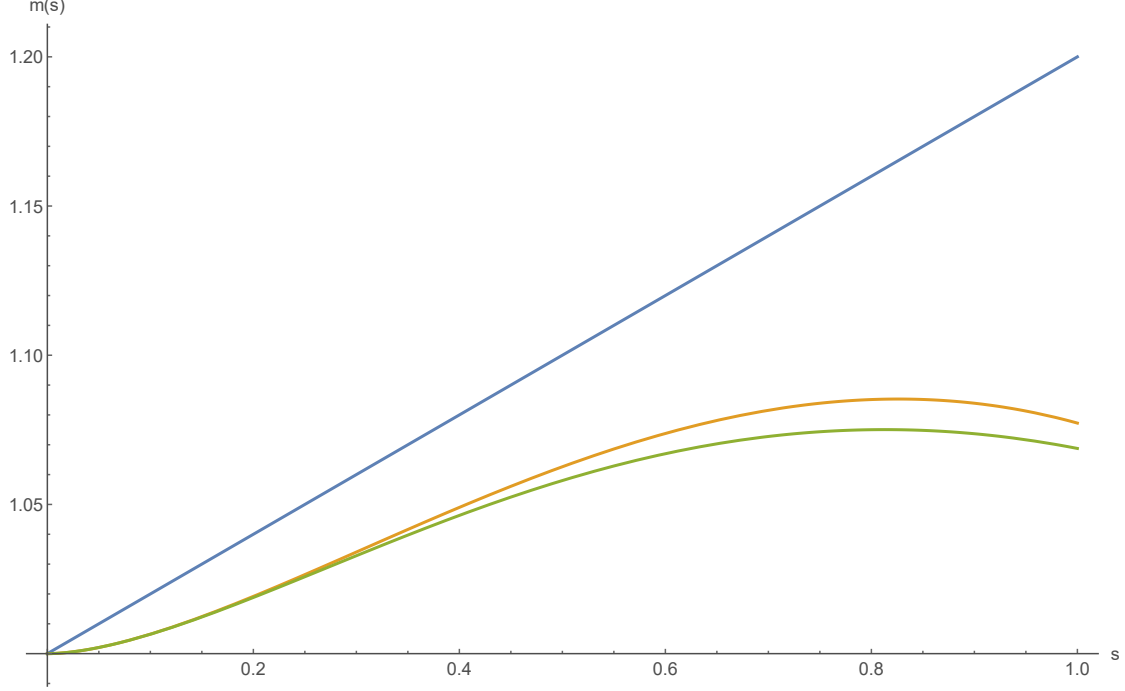


Figure 7: Plot of $m(s)$ together with its best and linear (upper) asymptotes.

The asymptotic behavior (2.22) of $m(s)$ follows from the mean value theorem

$$m(s) - m(0) = \int_0^s m'(t)dt = m'(\tilde{s})s \quad (2.40)$$

for some $\tilde{s} = \tilde{s}(s) \in [0, s]$ depending on s . The value $m(0)$ may be calculated as $I(0)$ in (2.32), using the representation (2.23) for $K_1(y) = -K'_0(y)$ and exchange the integration order:

$$\begin{aligned} m(0) &= \int_0^\infty y^2 K_1(y) dy \\ &= \int_0^\infty \left(\int_0^\infty y^2 e^{-y\sqrt{k^2+1}} dy \right) dk \\ &= - \int_0^\infty \frac{1}{(k^2+1)^{3/2}} dk = \frac{k}{\sqrt{k^2+1}} \Big|_0^\infty = 1 . \end{aligned}$$

To calculate the derivative of $m(s)$ we apply integration by parts twice, before and after the derivative with respect to s :

$$\begin{aligned} m(s) &= \frac{-1}{2} \int_s^\infty (y^2 K_1(y))' \sqrt{y^2 - s^2} dy \\ &= \frac{-1}{2} \int_s^\infty (y K_1(y) - y^2 K_0(y)) \sqrt{y^2 - s^2} dy \end{aligned}$$

by $(y \cdot (yK_1))' = yK_1 + y(yK_1)'$ together with (2.3); by $K_1 + yK_1' = (yK_1)' = -yK_0$ we have $-K_1' = K_0 + K_1/y$ and

$$\begin{aligned} m'(s) &= \frac{s}{2} \int_s^\infty (K_1(y) - yK_0(y)) \frac{y}{\sqrt{y^2 - s^2}} dy \\ &= \frac{-s}{2} \int_s^\infty (K_1(y) - yK_0(y))' \sqrt{y^2 - s^2} dy \\ &= M(s) + \frac{s}{2} N(s) \end{aligned} \quad (2.41)$$

where

$$\begin{aligned} M(s) &= \frac{s}{2} \int_s^\infty K_1(y) \frac{\sqrt{y^2 - s^2}}{y} dy \\ &\leq \frac{s}{2} \int_s^\infty K_1(y) dy = \frac{s}{2} K_0(s) \end{aligned} \quad (2.42)$$

in view of the inequality $\sqrt{y^2 - s^2}/y \leq 1$ for $s \leq y < \infty$, $-K_0'(y) = K_1(y) > 0$ and the fundamental theorem of calculus. Both boundary terms yielded from the partial integrations vanish. We observe that

$$K_0(s) = - \left. \frac{\partial I_\nu(s)}{\partial \nu} \right|_{\nu=0} = -\log(s/2) \sum_{n=0}^\infty \frac{(s/2)^{2n}}{(n!)^2} + \sum_{n=0}^\infty \frac{(s/2)^{2n}}{(n!)^2} \psi(1+n)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and so, $K_0(s) = -\log(s/2) - \gamma + O(s^2)$ where $\gamma = -\psi(1)$ is the Euler-Mascheroni constant. The other term of (2.41) can analogously be bounded by

$$\begin{aligned} N(s) &= \int_s^\infty (2yK_0(y) - y^2K_1(y)) \frac{\sqrt{y^2 - s^2}}{y} dy \\ &\leq \int_s^\infty (2yK_0(y) - y^2K_1(y)) dy \leq 0, \end{aligned} \quad (2.43)$$

provided $s \in [0, \bar{s}]$ where $\bar{s} \geq 3/2$. For this, we used that $\sqrt{y^2 - s^2}/y \leq 1$ and by (2.4),

$$2yK_0(y) - y^2K_1(y) = yK_0(y) \left(2 - \frac{yK_1(y)}{K_0(y)} \right) \geq yK_0(y) \left(2 - x - \frac{1}{2} \right) \geq 0 \quad (2.44)$$

if $x \leq 3/2$. Consequently, for $s \leq 3/2$ the integral of the l.h.s. of (2.44) over $[0, s]$ is positive and, using the representation (2.23) for $K_0(y)$ and for $K_1(y) = -K_0'(y)$ and exchange the integration

order, we have

$$\begin{aligned}
\int_s^\infty (2yK_0(y) - y^2K_1(y)) dy &\leq \int_0^\infty (2yK_0(y) - y^2K_1(y)) dy \\
&= \int_0^\infty \left(\int_0^\infty (2y - y^2\sqrt{k^2+1}) e^{-y\sqrt{k^2+1}} dy \right) \frac{dk}{\sqrt{k^2+1}} \\
&= \int_0^\infty \left(\int_0^\infty (2y - 2y) e^{-y\sqrt{k^2+1}} dy \right) \frac{dk}{\sqrt{k^2+1}} = 0
\end{aligned}$$

by integration by parts. To obtain (2.22) and conclude the proof of Proposition 2.11, we need to optimize the choice of $\tilde{s}(s)$ in (2.40). So far, by (2.40), (2.41), (2.42) and (2.43) we have

$$\begin{aligned}
m(s) &\leq 1 + \frac{1}{2} \int_0^s tK_0(t) ds \\
&= 1 + \frac{1}{2}(1 - sK_1(s))
\end{aligned} \tag{2.45}$$

by Proposition 2.1 and this upper bound is asymptotic as s tends to 0: $m(s) = 1 + O(s^2)$ the s^2 order term in the upper bound is $(1 - 2\gamma - 2\log(s/2))/8 = 0.1539(\dots) - (\log s)/4$. The best upper bound up to $O(s^2)$ term is, however, stated in Proposition 2.11, given by the asymptotic expansion of (2.12), calculated algebraically by the software Mathematica.

□

3 Majorant of the density function

Set up and ingredients Let $(\Omega, \mathcal{B}, \varrho)$ denote the (translational invariant) σ -finite measure space on $\{-1, 1\} \times \mathbb{R}^2$; the set Ω corresponds to the possible configurations of a single particle (we united σ and x into $\zeta = (\sigma, x)$) and $\int d\varrho(\zeta) \cdot = 1/2 \sum_{\sigma \in \{-1, 1\}} \int_{\mathbb{R}^2} d^2x \cdot$ denotes the integration with respect to ρ . Let

$$\beta p(\beta, z) = \sum_{n \geq 1} \frac{z^n}{n!} \int d\varrho(\zeta_2) \cdots d\varrho(\zeta_n) \psi_n^c(\zeta_1, \dots, \zeta_n; \beta v) \tag{3.1}$$

be the pressure of the Yukawa gas in the infinite volume limit, where v is the Yukawa potential regularized at short distances $s \leq t_0$, given by the scale decomposition (1.10). We observe that, as v decays exponentially fast at infinity and has its singularity at origin removed, the finite volume pressure p_{Λ_i} , defined for any increasing sequence $(\Lambda_i)_{i \geq 1}$ of squares with $\lim_i \Lambda_i = \mathbb{R}^2$,³ converges by standard methods (see e.g. [R]) and translational invariance of v to the expression (3.1).

³Given by (3.1) with the integral over the n -particle configurations restricted to Λ_i divided by $|\Lambda_i|$: $\beta p_{\Lambda_i} = \sum_{n \geq 1} \frac{z^n}{n!} \frac{1}{|\Lambda_i|} \int_{\Lambda_i^n} d\varrho(\zeta_1) \cdots d\varrho(\zeta_n) \psi_n^c$

The density function $\rho(\beta, z) = z\partial p/\partial z(\beta, z)$ is another thermodynamical function which will be convenient to write as an Mayer series (1.24) in power of the activity z :

$$\frac{\beta}{z}\rho(\beta, z) = \sum_{n \geq 1} n b_n z^{n-1} \quad (3.2)$$

where $b_1 = 1$ and, for $n > 1$,

$$b_n = \frac{1}{n!} \int d\varrho(\zeta_2) \cdots d\varrho(\zeta_n) \psi_n^c(\zeta_1, \dots, \zeta_n; \beta v)$$

is the so called n -th Mayer coefficient in the infinite volume limit. Note that $\beta\rho(\beta, z)/z = 1$ is the equation of state of an ideal gas and due the interaction of the charged particles through the Yukawa pair potential, the series (3.2) provides corrections to all order about it expressed in terms of the Ursell (cluster) functions ψ_n^c . A formal power series in z

$$\Theta^*(z) = \sum_{n \geq 1} C_n^* z^{n-1}$$

is a majorant of $\beta\rho(\beta, z)/z$ if the C_n^* are nonnegative and

$$n |b_n| \leq C_n^*$$

holds for all $n \in \mathbb{N}$. It follows that, if the $\Theta^*(z)$ series converges on the open disc $D(r) := \{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$, then $\rho(\beta, z)$ is holomorphic function of z on the same disc. The largest r provides an lower bound on the radius of convergence of the Mayer series (3.2) and (1.24).

For the problem at our hand, the most efficient method of constructing majorants combines (multi)scale decomposition of v together with some basic ingredients. Beginning with the latter, the following elementary lemmas are useful.

Lemma 3.1 *If a, b, c and d are positive numbers such that $a - c$ and $b - d$ are positive, then $ab - cd$ is also positive.*

Proof. Writing

$$\begin{aligned} ab - cd &= ab - \frac{1}{2}(ad + bc) - \left(cd - \frac{1}{2}(ad + bc) \right) \\ &= \frac{1}{2}(a(b - d) + (a - c)b - (c - a)d - c(d - b)) \\ &= \frac{1}{2}((a + c)(b - d) + (a - c)(b + d)) > 0, \end{aligned} \quad (3.3)$$

concluding the proof. □

Lemma 3.2 Let $a = (a_n)_{n \geq 1}$, $b = (b_n)_{n \geq 1}$, $\tilde{a} = (\tilde{a}_n)_{n \geq 1}$ and $\tilde{b} = (\tilde{b}_n)_{n \geq 1}$ be positive numerical sequences (a, b, \tilde{a} and $\tilde{b} > 0$) such that $\tilde{a} - a$ and $\tilde{b} - b$ are both positive sequences (i.e., $\tilde{a}_n - a_n > 0$ and $\tilde{b}_n - b_n > 0$ hold for all $n \geq 1$). Let the convolution product $e = c * d$ and the pointwise product $f = c \cdot d$ of two sequences $c = (c_n)_{n \geq 1}$ and $d = (d_n)_{n \geq 1}$ be defined by the sequences $e = (e_n)_{n \geq 1}$ and $f = (f_n)_{n \geq 1}$ where $e_1 = 0$ and

$$e_n = \sum_{k=1}^{n-1} c_k d_{n-k}, \quad n \geq 2$$

and

$$f_n = c_n d_n, \quad n \geq 1.$$

Then, **(i)** $\tilde{a} \cdot \tilde{b} - a \cdot b > 0$; **(ii)** $\tilde{a} * \tilde{b} - a * b > 0$; in particular **(iii)** $\tilde{a} * \tilde{a} - a * a > 0$ and $\tilde{b} * \tilde{b} - b * b > 0$ hold.

Proof. The conclusions **(i)**, **(ii)** and **(iii)** follow immediately from Lemma 3.1: for **(i)** each element of the sequence is of the form (3.3); for **(ii)** and **(iii)** each element of the sequence is a sum of terms of the form (3.3). □

Remark 3.3 The statements of Lemmas 3.1 and 3.2 hold true if the assumption of positivity is replaced by nonnegativity.

The scale decomposition (1.10) becomes effective when the Ursell function in (3.1) is defined by a scaling limit

$$\psi_n^c(\zeta_1, \dots, \zeta_n; \beta v) = \lim_{t \rightarrow \infty} \psi_n^c(t, \zeta_1, \dots, \zeta_n; \beta v(t, \cdot)) \quad (3.4)$$

where $\psi_n^c(t, \zeta_1, \dots, \zeta_n; \beta v(t, \cdot)) \equiv \psi^c(t, \zeta_{\{1, \dots, n\}})$ is the unique solution of the infinite system of ordinary differential equations for $f_I = f_I(t) \equiv f(t, \zeta_I)$, where $\zeta_I = (\zeta_{i_1}, \dots, \zeta_{i_k})$ is the set of variables indexed by $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $n \in \mathbb{N}$: (see Lemma 3.3 of [BK])

$$\dot{f}_I = - \sum_{i, j \in I, i < j} \beta \dot{v}_{ij}(t) f_I - \frac{1}{2} \sum_{J \subset I} \sum_{i \in J, j \in I \setminus J} \beta \dot{v}_{ij}(t) f_J f_{I \setminus J} \quad (3.5)$$

with (ideal gas) initial condition⁴

$$f_I(t_0) = \begin{cases} 1 & \text{if } |I| = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Here $\dot{v}_{ij}(t) \equiv \dot{v}(t, \zeta_i, \zeta_j) = \sigma_i \sigma_j g(t) h(|x_i - x_j|/t)$ so, as $\dot{v}(t, \zeta_i, \zeta_j)$ is a measurable and translational invariant function on the 2-particle configuration space, $\psi^c(t, \zeta_I)$ is a measurable and translational invariant function on the k -particle configuration space $(\{-1, 1\} \times \mathbb{R}^2)^k$.

⁴By (1.10), the interaction $v(t, x)$ between particles is turned off at $t = t_0$.

By the variation of constants formula the system of equations (3.5) is equivalent to a system of integrable equations: $f_I(t) = 1$ if $|I| = 1$ and

$$f_I(t) = \frac{-1}{2} \int_{t_0}^t \exp \left(- \sum_{i,j \in I, i < j} \int_s^t \beta \dot{v}_{ij}(\tau) d\tau \right) \sum_{J \subset I} \sum_{i \in J, j \in I \setminus J} \beta \dot{v}_{ij}(s) f_J(s) f_{I \setminus J}(s) ds, \quad (3.6)$$

if $|I| > 1$, which will be usefull to our application.

Majorant construction for $\beta < 4\pi$ Using (3.6) two of the authors have proven in [GM] (see Theorem 2.2 and equations (4.10)-(4.12) therein) the following

Proposition 3.4 *Let $\Theta = \Theta(t, z)$ be the classical solution of*

$$\Theta_t = \Gamma(z^2 \Theta^2)_z + B((z\Theta)_z - 1), \quad (t, z) \in (t_0, \infty) \times \mathbb{R}_+ \quad (3.7)$$

with $\Theta(t_0, z) = 1$ for all $z \geq 0$, where by (1.10), (1.5) and explicit calculation, $\Gamma = \Gamma(t) = \|\beta \dot{v}(t, \cdot)\|_1$ and $B = B(t) = |\beta \dot{v}(t, 0)|/2$ are given by

$$\Gamma = \beta g(t) \int_{\mathbb{R}^2} h(|x|/t) d^2x = \frac{\beta \pi}{4} t^2 g(t) \quad (3.8)$$

(the integral is exactly $2\pi t^2$ times $\int_0^\infty \frac{2}{\pi} \left(\arccos w - w\sqrt{1-w^2} \right) w dw = 1/8$) and

$$B = \frac{\beta}{2} g(t).$$

Then, the following majorant relation

$$\frac{\beta}{z} \rho(\beta, z) \leq \Theta(\infty, z) \leq \frac{-1}{\tau(t_0, \infty)z} W(-\tau(t_0, \infty)z) \quad (3.9)$$

holds for all (β, z) satisfying

$$ez\tau(t_0, \infty) < 1 \quad (3.10)$$

where

$$\tau(t_0, t) = \int_{t_0}^t \Gamma(s) \exp \left(2 \int_s^t B(s') ds' \right) ds \quad (3.11)$$

and $W(x)$ denotes the Lambert W -function. [C-K]

Remark 3.5 *The proof of Proposition 3.4 in [GM] uses the scale decomposition (1.7) of v , for which $B = \beta/(4\pi t)$ and $\Gamma = 2\beta t$ can be exactly calculated (for comparison, we have set therein $\kappa(t) = 1/t^2$ for $t \in (0, 1]$). Here v is given by (1.10) whose scaling function $g(t)$ agree with the scaling $1/(2\pi t)$ of (1.7) only asymptotically as $t \rightarrow 0$. Writing $\tau(t_0, \infty) = \tau(t_0, 1) \exp \left(\beta \int_1^\infty g(s') ds' \right) +$*

$\tau(1, \infty)$ together with $0 < g(s) \leq (1 + s/5)/(2\pi s)$ if $0 \leq s \leq 1$ by Proposition 2.11 (see Fig 7), for any $0 < \beta < 4\pi$ the limit

$$\begin{aligned} \lim_{t_0 \rightarrow 0} \tau(t_0, 1) &= \frac{\beta\pi}{4} \int_0^1 s^2 g(s) \exp\left(\beta \int_s^1 g(s') ds'\right) ds \\ &\leq \frac{\beta\pi}{4} e^{\beta/10\pi} \int_0^1 \frac{1}{2\pi} \left(s^{1-\beta/2\pi} + \frac{1}{5} s^{2-\beta/2\pi}\right) ds \\ &= \frac{\beta\pi}{4} e^{\beta/10\pi} \left(\frac{1}{4\pi - \beta} + \frac{1}{5} \frac{1}{6\pi - \beta}\right) \end{aligned} \quad (3.12)$$

exists and $\tau(1, \infty)$ is finite since $g(t)$ decays exponentially fast as $t \rightarrow \infty$.

Remark 3.6 The existence of $\tau = \lim_{t_0 \rightarrow 0} \tau(t_0, \infty)$ implies by (3.9) and (3.10) that the radius of convergence $r = \sup\{|z| : e|z|\tau < 1, z \in \mathbb{C}\}$ of the Mayer series (3.2) remains strictly positive. This fact is already remarkable considering that v , given by (1.10) with $t_0 = 0$, does not satisfies the stability condition (1.16) (see also (1.17)), which is sufficient but not necessary for the density (3.2) be defined in the thermodynamic limit.

Proof of Proposition 3.4. By (3.4), (3.6) and stability (1.17), the sequence $(A_n)_{n \geq 1}$ of positive functions $A_n : [t_0, \infty) \rightarrow \mathbb{R}$, defined by

$$A_n(t) = \frac{1}{n!} \int d\varrho(\zeta_2) \cdots d\varrho(\zeta_n) |\psi_n^c(\zeta_1, \dots, \zeta_n; \beta v(t, \cdot))|$$

satisfies a system of integral inequality equations

$$nA_n(t) \leq \frac{n}{2} \int_{t_0}^t ds \exp\left(n \int_s^t B(s') ds'\right) \Gamma(s) \sum_{k=1}^{n-1} k A_k(s) (n-k) A_{n-k}(s), \quad n > 1 \quad (3.13)$$

with $A_1(t) \equiv 1$. Hence, the Mayer coefficients of the series (3.2) are majorized by

$$n |b_n| \leq n A_n(\infty). \quad (3.14)$$

Let $\Theta(t, z)$ be defined by the series

$$\Theta(t, z) = \sum_{n \geq 1} C_n(t) z^{n-1} = 1 + \sum_{n \geq 2} C_n(t) z^{n-1} \quad (3.15)$$

where the sequence $(C_n)_{n \geq 1}$ of positive functions $[t_0, \infty) \ni t \mapsto C_n(t) \in \mathbb{R}_+$ satisfies equations (3.13) for $(nA_n)_{n \geq 1}$ as **an equality** and, consequently,

$$nA_n(t) \leq C_n(t), \quad n \geq 1 \text{ and } t \geq t_0. \quad (3.16)$$

It can be shown (see Sec. 4 of [GM]) that (3.15) satisfies the quasi-linear first order PDE (3.7). So, the first inequality of (3.9) holds and all one needs is to determine a domain in $(t_0, \infty) \times \mathbb{R}_+$ for which the classical solution of (3.7) exists. Observe that (3.7) can be written as a system of first order differential equations for the coefficients $(C_n)_{n \geq 1}$. For this, by (3.15), we have

$$\begin{aligned}\Theta_t &= \sum_{n \geq 1} \dot{C}_n z^{n-1} \\ (z\Theta)_z &= \sum_{n \geq 1} n C_n z^{n-1} \\ (z^2\Theta^2)_z &= \sum_{n \geq 2} n \left(\sum_{k=1}^{n-1} C_k C_{n-k} \right) z^{n-1} .\end{aligned}\tag{3.17}$$

Substituting these series back into the equation, yields

$$\dot{C}_n = n B C_n + \frac{n \Gamma}{2} \sum_{k=1}^{n-1} C_k C_{n-k} , \quad n > 1\tag{3.18}$$

with $C_1(t) \equiv 1$, $t \in [t_0, \infty)$, and initial data $C_n(t_0) = 0$ for all $n \geq 2$.

The first non-trivial equation for $n = 2$,

$$\dot{C}_2 = 2 B C_2 + \Gamma\tag{3.19}$$

with $C_2(t_0) = 0$, has a unique solution $\tau(t_0, t)$ given by (3.11), which can be written as

$$C_2(t) = f_1(t) \int_{t_0}^t \Gamma_1(s) ds$$

where $\Gamma_1(s) = \Gamma(s)/f_1(s)$ and

$$f_1(t) = \exp \left(2 \int_{t_0}^t B(s') ds' \right)$$

is an integrating factor of (3.19). As we shall see $\tau(t_0, t) = C_2(t)$ determines the radius of convergence of the series (3.15) for Θ :

$$e |z| \tau(t_0, t) < 1,\tag{3.20}$$

uniformly in t_0 and t for $\beta < \beta_2$, $0 < t_0 < t < \infty$, where $\beta_2 = 4\pi$ is the first threshold (see Remark 3.5). For this, let $\left(C_n^{(1)}(t) \right)_{n \geq 1}$ be a sequence of positive functions defined by

$$\Psi(t, w) = \Theta(t, w/f_1(t)) = 1 + \sum_{n \geq 2} C_n^{(1)} w^{n-1} .\tag{3.21}$$

Since $C_n^{(1)} = C_n/f_1^{n-1}$ and

$$\begin{aligned}\dot{C}_n^{(1)} &= \frac{\dot{C}_n}{f_1^{n-1}} - (n-1) \frac{\dot{f}_1}{f_1} \frac{C_n}{f_1^{n-1}} \\ &= \frac{\dot{C}_n}{f_1^{n-1}} - 2(n-1) B \frac{C_n}{f_1^{n-1}}\end{aligned}$$

equation (3.18) in terms of the new $C_n^{(1)}$'s reads

$$\dot{C}_n^{(1)} = -(n-2)BC_n^{(1)} + \frac{n\Gamma_1}{2} \sum_{k=1}^{n-1} C_k^{(1)} C_{n-k}^{(1)}, \quad n > 1 \quad (3.22)$$

with $C_1^{(1)}(t) \equiv 1$ and initial data $C_n^{(1)}(t_0) = 0$ for all $n \geq 2$. Since the coefficient $-(n-2)B$ of the linear term is nonpositive for all $n \geq 2$, the solution of the above initial value problem (IVP) can, in turn, be majorized by another sequence $(\tilde{C}_n^{(1)})_{n \geq 1}$:

$$C_n^{(1)}(t) \leq \tilde{C}_n^{(1)}(t) \quad (3.23)$$

which solves the IVP

$$\tilde{C}_n^{(1)} = \frac{n\Gamma_1}{2} \sum_{k=1}^{n-1} \tilde{C}_k^{(1)} \tilde{C}_{n-k}^{(1)}, \quad n > 1$$

with $\tilde{C}_1^{(1)}(t) \equiv 1$ and initial data $\tilde{C}_n^{(1)}(t_0) = 0$ for all $n \geq 2$.

Proof of (3.23). Using the notation introduced in Lemma 3.2, we write $a = (a_n)_{n \geq 1}$ and $b = (b_n)_{n \geq 1}$ with $a_1 = b_1 \equiv 0$, $a_n(t) = (n-2)B(t)$ and $b_n(t) = n\Gamma_1(t)/2$ for $n > 1$. The difference sequence $\Delta = (\Delta_n)_{n \geq 1}$, given by $\Delta_1 \equiv 0$ and $\Delta_n(t) = \tilde{C}_n^{(1)}(t) - C_n^{(1)}(t)$ for $n > 1$, thus satisfies

$$\dot{\Delta} = a \cdot \Delta + b \cdot (\tilde{C}^{(1)} * \tilde{C}^{(1)} - C^{(1)} * C^{(1)}) .$$

Let us assume that (3.23) holds for some $t \geq t_0$. Then, by Lemma 3.2 we have $\dot{\Delta}(t) \geq 0$ which, together with $\Delta(t_0) \equiv 0$, implies that $\Delta(t) \geq 0$. Consequently, (3.23) holds for all $t \geq t_0$. \square

It is shown in Sec. 5 of [GM] that the power series analogous to (3.21): $\tilde{\Psi}(t, w) = 1 + \sum_{n \geq 2} \tilde{C}_n^{(1)} w^{n-1}$ satisfies an equation given by (3.7) setting $B = 0$, $\Gamma = \Gamma_1$ and together with $\tilde{\psi}(t_0, z) \equiv 1$ has by the method of characteristics the classical solution

$$\tilde{\Psi}(t, w) = \frac{-1}{\tilde{\tau}_1(t_0, t)w} W(-\tilde{\tau}_1(t_0, t)w)$$

provided $e|w|\tilde{\tau}_1(t_0, t) < 1$ holds, where $\tilde{\tau}_1(t_0, t) = \int_{t_0}^t \Gamma_1(s)ds$ and $W(x)$ denotes the Lambert W -function defined implicitly by $We^W = x$, whose Taylor series about $x = 0$ (see Lagrange-Bürmann theorem [D]): $W(x) = \sum_{n \geq 1} (-n)^{n-1} x^n / n!$ converges for $|x| < 1/e$, including the branching point at $x = -1/e$ (see e.g. [C-K]).

Joining equations (3.14), (3.16) and (3.23) together, we conclude

$$\Theta(t, z) = \Psi(t, w) \leq \tilde{\Psi}(t, w) = \frac{-1}{\tau_1(t_0, t)z} W(-\tau(t_0, t)z)$$

with $\tau_1(t_0, t)$ given by (3.11), establishing the second inequality of (3.9).

□

Note that, since $f_1(t) > 1$ for any t_0 and t fixed, the radius of convergence of the majorant series $\Theta(t, z) = \Psi(t, w)$ is smaller in z than in w variable. However, in view of Remarks 3.5 and 3.6, it remains strictly positive when the cutoff t_0 is removed provided $\beta < \beta_2$ where $\beta_2 = 4\pi$ is the first threshold.

Majorant construction for β inside the threshold intervals I_n The procedure of finding a majorant series for the density function can be extended for the inverse temperature β in the thresholds interval $I_n = [\beta_{2n}, \beta_{2(n+1)})$, $n \in \mathbb{N}$ where, for convenience, we write $\beta_{k+1} = 8\pi(1 - 1/(k+1)) = 8\pi k/(k+1)$. We shall present an scheme of avoiding neutral cluster collapse which holds for any thresholds interval when the cutoff t_0 is removed. The scheme consists of three stages. Firstly, for $\beta \in I_{(k-1)/2}$ where $k > 1$ is an odd number, we remove from the system any neutral clusters or subclusters of size less than k . Such a removal, which prevents the increasing of the corresponding terms in the Mayer expansion when they collapse, is expressed in terms of the majorant method, by inserting Lagrange multipliers into the equation (3.7), as many as neutral clusters were removed.

The second stage addresses non neutral clusters of size smaller or equal to k in the majorant equation (3.7) which has been overestimated by using the stability bound (1.16) instead of (1.23) (see equation (3.13)). This issue is fixed by replacing the coefficient nB of the linear term of (3.18) by $(n-1)B$. Before we apply the second stage, we shall extract (insirting a Lagrange multiplier L_k) an exact amount from the linear term of (3.18) that allows the solution of (3.7) for $\beta < \beta_{k+1}$ be majorized by a series with positive radius of convergence, uniformly on cutoff t_0 (see Secs. 6 and 7 of [GM]). The Lagrange multiplier L_k of order k is given by the Cesàro mean of the first k Taylor series of $(z\Theta)_z$ around $z = 0$, truncated at order $0 \leq j < k$, and this choice is optimal.

Proposition 3.7 *For any $k \in \mathbb{N}$, let $\Theta = \Theta(t, z)$ be the classical solution of*

$$\Theta_t = \Gamma(z^2\Theta^2)_z + B((z\Theta)_z - L_k) \quad , \quad (t, z) \in (t_0, \infty) \times \mathbb{R}_+ \quad (3.24)$$

with $\Theta(t_0, z) = 1$ for all $z \geq 0$, where $\Gamma = \Gamma(t) = \|\beta\dot{v}(t, \cdot)\|_1$ and $B = B(t) = |\beta\dot{v}(t, 0)|/2$ are given in Proposition 3.4 and

$$L_k = L_k(t) = 1 + \sum_{j=1}^{k-1} \left(1 - \frac{j}{k}\right) \frac{1}{j!} z^j \underbrace{\Theta_z \dots z}_{j\text{-times}}(t, 0)$$

is a Lagrange multiplier. Then, the following majorant relation

$$\Theta(t, z) \leq \frac{-1}{\tau_k(t_0, t)z} W(-\tau_k(t_0, t)z) \quad (3.25)$$

holds for all (β, z) satisfying

$$ez\tau_k(t_0, \infty) < 1$$

where

$$\tau_k(t_0, t) = \int_{t_0}^t \Gamma(s) \exp \left(\frac{k+1}{k} \int_s^t B(s') ds' \right) ds \quad (3.26)$$

and $W(x)$ denotes the Lambert W -function. [C-K]

Remark 3.8 A calculation analogous to (3.12) yields that

$$\begin{aligned} \lim_{t_0 \rightarrow 0} \tau_k(t_0, 1) &= \lim_{t_0 \rightarrow 0} \frac{\beta\pi}{4} \int_{t_0}^1 s^2 g(s) \exp \left(\frac{k+1}{k} \frac{\beta}{2} \int_s^1 g(s') ds' \right) ds \\ &\leq \frac{\beta\pi}{4} e^{2\beta/5\beta_{k+1}} \int_{t_0}^1 \frac{1}{2\pi} \left(s^{1-2\beta/\beta_{k+1}} + \frac{1}{5} s^{2-2\beta/\beta_{k+1}} \right) ds \\ &= \frac{\beta}{16} e^{2\beta/5\beta_{k+1}} \left(\frac{1}{1 - \beta/\beta_{k+1}} + \frac{1}{5} \frac{1}{2 - \beta/\beta_{k+1}} \right), \end{aligned}$$

exists for $\beta < \beta_{k+1}$ and the radius of convergence of the majorant series (3.25) is strictly positive.

Proof of Proposition 3.7. We follow closely the proof of Proposition 3.4. Let $\Theta(t, z)$ be defined by the series (3.15). Observe that, by

$$(z\Theta)_z - L_k = \sum_{n=2}^k \left(n - \frac{k-n+1}{k} \right) C_n z^{n-1} + \sum_{n \geq k+1} n C_n z^{n-1}$$

and the remaining series of (3.17), (3.24) can be written as a system of first order differential equations for $(C_n)_{n \geq 1}$:

$$\dot{C}_n = \frac{k+1}{k} (n-1) B C_n + \frac{n\Gamma}{2} \sum_{k=1}^{n-1} C_k C_{n-k}, \quad 1 < n \leq k \quad (3.27)$$

and (3.18) for $n > k$, with $C_1(t) \equiv 1$ and initial data $C_n(t_0) = 0$ for $n \geq 2$. The equation for $n = 2$

$$\dot{C}_2 = \frac{k+1}{k} B C_2 + \Gamma \quad (3.28)$$

with $C_2(0) = 0$ has a unique solution given by (3.26), which can be written as

$$C_2(t) = f_k(t) \int_{t_0}^t \Gamma_k(s) ds$$

where $\Gamma_k(s) = \Gamma(s)/f_k(s)$ and

$$f_k(t) = \exp \left(\frac{k+1}{k} \int_{t_0}^t B(s') ds' \right)$$

is an integrating factor of (3.28).

Let $\left(C_n^{(k)}(t)\right)_{n \geq 1}$ be a sequence of positive functions defined by

$$\Psi(t, w) = \Theta(t, w/f_k(t)) = 1 + \sum_{n \geq 2} C_n^{(k)}(t) w^{n-1} . \quad (3.29)$$

Since $C_n^{(k)} = C_n/f_k^{n-1}$ and

$$\begin{aligned} \dot{C}_n^{(k)} &= \frac{\dot{C}_n}{f_k^{n-1}} - (n-1) \frac{\dot{f}_k}{f_k} \frac{C_n}{f_k^{n-1}} \\ &= \frac{\dot{C}_n}{f_k^{n-1}} - \frac{k+1}{k} (n-1) B \frac{C_n}{f_k^{n-1}} , \end{aligned}$$

the equations (3.27) for $1 < n \leq k$ and (3.18) for $n > k$ in terms of the new $C_n^{(k)}$'s read

$$\begin{aligned} \dot{C}_n^{(k)} &= \frac{n\Gamma_k}{2} \sum_{j=1}^{n-1} C_j^{(k)} C_{n-j}^{(k)} , \quad 1 < n \leq k \\ \dot{C}_n^{(k)} &= -\left(\frac{n-k-1}{k}\right) B C_n^{(k)} + \frac{n\Gamma_k}{2} \sum_{j=1}^{n-1} C_j^{(k)} C_{n-j}^{(k)} , \quad n > k \end{aligned} \quad (3.30)$$

with $C_1^{(k)}(t) \equiv 1$ and initial data $C_n^{(k)}(t_0) = 0$ for $n \geq 2$. Since the coefficient $-(n-k-1)B/k$ of the linear term of (3.30) is nonpositive for all $n \geq k+1$, the solution of the above IVP can be majorized by another sequence $\left(\tilde{C}_n^{(k)}\right)_{n \geq 1}$:

$$C_n^{(k)}(t) \leq \tilde{C}_n^{(k)}(t) \quad (3.31)$$

which solves the IVP

$$\tilde{C}_n'^{(k)} = \frac{n\Gamma_1}{2} \sum_{j=1}^{n-1} \tilde{C}_j^{(k)} \tilde{C}_{n-j}^{(k)} , \quad n > 1$$

with $\tilde{C}_1^{(k)}(t) \equiv 1$ and initial data $\tilde{C}_n^{(k)}(t_0) = 0$ for $n \geq 2$. For (3.31), one may apply the same proof of (3.23) based in Lemma 3.2. It follows that (see in Sec. 7 of [GM]) the power series $\tilde{\Psi}(t, w) = 1 + \sum_{n \geq 2} \tilde{C}_n^{(k)} w^{n-1}$ satisfies (3.7) setting $B = 0$, $\Gamma = \Gamma_k$ and together with $\tilde{\psi}(t_0, z) \equiv 1$ has the classical solution

$$\tilde{\Psi}(t, w) = \frac{-1}{\tilde{\tau}_k(t_0, t)w} W(-\tilde{\tau}_k(t_0, t)w)$$

provided $e|w|\tilde{\tau}_k(t_0, t) < 1$ holds, where $\tilde{\tau}_k(t_0, t) = \int_{t_0}^t \Gamma_k(s)ds$ and $W(x) = \sum_{n \geq 1} (-n)^{n-1} x^n/n!$ denotes the Lambert W -function.

We thus have

$$\Theta(t, z) = \Psi(t, w) \leq \tilde{\Psi}(t, w) = \frac{-1}{\tau_k(t_0, t)z} W(-\tau_k(t_0, t)z) ,$$

concluding the proof of Proposition 3.7.

□

Returning to the second stage of our scheme, we show that the equation (3.7), under that operation, is replaced by

$$\Theta_t = \Gamma(z^2\Theta^2)_z + Bz\Theta_z. \quad (3.32)$$

For this, let the argument n of the exponential in (3.13) be replaced by $n - 1$. The modified coefficients $(C_n)_{n \geq 1}$ of the power series (3.15) satisfy then a system of integral equations

$$C_n(t) = \frac{n}{2} \int_{t_0}^t ds e^{(n-1)\gamma(s,t)} \Gamma(s) \sum_{k=1}^{n-1} C_k(s) C_{n-k}(s), \quad n > 1 \quad (3.33)$$

with $C_1(t) \equiv 1$ where $\gamma(s, t) = \int_s^t B(s') ds'$. Summing equation (3.33) multiplied by z^{n-1} over n yields an integral equation for Θ :

$$\Theta(t, z) = 1 + \frac{1}{2} \int_{t_0}^t ds e^{-\gamma(s,t)} \Gamma(s) (z^2 e^{2\gamma(s,t)} \Theta^2(s, z e^{\gamma(s,t)}))_z \quad (3.34)$$

and from this we deduce (3.32). Observe that an extra factor $e^{-\gamma(s,t)}$ inside the integration results from the stability improvement (1.23) and the derivative with respect to t applied to this factor produces an additional term $B(\Theta - 1)$ which has to be subtracted (due to the minus sign of the exponent) from the last term on the right hand side of (3.7): $B((z\Theta)_z - 1) - B(\Theta - 1) = Bz\Theta_z$.

The improved equation (3.32) leads to a significant outcome regarding the radius of convergence of the Mayer series (3.2) for β inside each threshold interval $I_n = [\beta_{2n}, \beta_{2(n+1)})$, $n \in \mathbb{N}$.

Proposition 3.9 *Let $\Theta = \Theta(t, z)$ be the classical solution of (3.32) with B and Γ as in Proposition 3.4. Then the following majorant relation*

$$\Theta(t, z) \leq \frac{-1}{\tau_k(t_0, t)z} W(-\tau_k(t_0, t)z)$$

holds for all $k \in \mathbb{N}$ and (β, z) satisfying $ez\tau_k(t_0, t) < 1$, where τ_k is given by (3.26).

Proof. Let $\Theta(t, z)$ be defined by the series (3.15) and observe that, by

$$z\Theta_z = \sum_{n=2}^{\infty} (n-1) C_n z^{n-1},$$

(3.32) can be written as a system of first order differential equations for $(C_n)_{n \geq 1}$:

$$\dot{C}_n = (n-1)BC_n + \frac{n\Gamma}{2} \sum_{k=1}^{n-1} C_k C_{n-k}, \quad 1 < n \leq k \quad (3.35)$$

with $C_1(t) \equiv 1$ and initial data $C_n(t_0) = 0$ for $n \geq 2$. Since the coefficient $(n-1)B$ of the linear term of (3.35) is smaller than $(n-1)(k+1)B/k$ for $2 \leq n \leq k$ and smaller than nB for all $n \geq k+1$,⁵ for any $k \in \mathbb{N}$, the solution of the above IVP can be majorized, in view of Lemma 3.2, by the solution of the IVP in (3.27), which by Proposition 3.7 satisfies (3.25). The proof of Proposition 3.9 is concluded. □

Stability of a neutral pair in the presence of other particles The third and last stage of our scheme deals with neutral subclusters of order smaller than k that are part of a cluster of order larger or equal to $k+1$. So far, we have proved a weak version of the Conjecture 1.1. Let $k > 1$ an odd number and suppose that all neutral clusters and subclusters of order smaller than k have their singularities been removed by hand. Then, the density function (3.2), after the removal, satisfies

$$\frac{\beta}{z} |\rho(\beta, z)| \leq \frac{-1}{\tau_k(t_0, \infty)z} W(-\tau_k(t_0, \infty)z) ,$$

and the majorant series has strictly positive radius of convergence uniformly in the cutoff t_0 for $\beta < \beta_{k+1}$. From the point of view of the Mayer coefficients b_n however, for $n \leq k$ our hypotheses are better than the formulated in the conjecture – instead of removing the coefficients b_n 's entirely we remove the part of these that diverges as $t_0 \rightarrow \infty$. The weakness of our hypotheses is that no assumptions on the coefficients b_n for $n > k$ are made in Conjecture 1.1. The situations here is different from what we have done before. According to Proposition 3.7, when n is larger than k , we don't need improve the stability condition and we actually cannot for neutral clusters of even size n . However, no assumptions mean that neutral subclusters of size smaller than k inside a cluster of order n do not diverges as t_0 tends to 0 and this claim needs to be proven.

To deal with this scenario, instead of a sequence $(A_n)_{n \geq 1}$ defined by (3.13), we introduce a sequence $(\tilde{A}_m)_{m \geq 1}$ of appended at ζ_0 analogous quantities

$$\tilde{A}_m(s, \sigma_1, \dots, \sigma_m) = \frac{1}{m!} \int_{\mathbb{R}^2 \times \dots \times \mathbb{R}^2} dx_1 \cdots dx_m \left| \sum_{j=1}^m \sigma_0 g(s) h(|x_j - x_0|/s) \sigma_j \psi_m^c(s, \zeta_1, \dots, \zeta_m) \right| \quad (3.36)$$

which are independent of $\zeta_0 = (x_0, \sigma_0)$ by translational invariance of variable x_0 and $|\sigma_0| = 1$. In the next paper we shall study in particular the recursion relations satisfied by theses quantities together with their majorant equations as well as a systematic majorant approach for the correlation function. In the present paper, we shall restrict ourselves to the simplest case of $m = 2$ of (3.36).

Let us consider a n -particle cluster containing a neutral pair subcluster. Referring to the formula (3.6), let I be an index set of a cluster with $|I| = n$ and let J be the index set of a pair

⁵This part would not be necessary for keeping positive the radius of convergence, uniformly in t_0 , at $\beta < \beta_{k+1}$.

$|J| = 2$ of particles with opposite charges: $\sigma_1\sigma_2 = -1$ located at x_1 and x_2 , whose Ursell function at scale s is simply given by

$$\psi_2^c(s, \zeta_1, \zeta_2) = \beta \int_{t_0}^s g(\tilde{s}) h(r/\tilde{s}) \exp \left(\beta \int_{\tilde{s}}^s g(\tau) h(r/\tau) d\tau \right) d\tilde{s} . \quad (3.37)$$

where $r = |x_2 - x_1|$. We refer to Sec. 6.3 of [GM] for detail. Let $\Delta = \Delta(s, \tilde{s}, x_0, x_1, x_2)$ be the h part of (3.36) including (3.21) given by

$$\Delta = (h(|x_0 - x_1|/s) - h(|x_0 - x_2|/s)) h(|x_1 - x_2|/\tilde{s}) \quad (3.38)$$

with $t_0 \leq \tilde{s} < s$. Using the convolution form (2.14) of the Euclid's hat $h(w)$ together with its geometric interpretation as the area of "caps" (see proof of Proposition 2.7), we shall find an upper bound for the integral over x_1 and x_2 of this quantity. For this, we write

$$\Delta = \frac{4}{\pi s^2} \int_{\mathbb{R}^2} dz \cdot \frac{4}{\pi \tilde{s}^2} \int_{\mathbb{R}^2} d\tilde{z} \chi_{s/2}(x_0 - z) (\chi_{s/2}(z - x_1) - \chi_{s/2}(z - x_2)) \chi_{\tilde{s}/2}(x_1 - \tilde{z}) \chi_{\tilde{s}/2}(\tilde{z} - x_2)$$

and observe that the integrand of Δ differs from 0 if, and only if, either x_1 is inside of the non null intersection $B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})$ and x_2 is inside the complementary region $B_{\tilde{s}/2}(\tilde{z}) \setminus (B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})) \neq \emptyset$ or vice-versa. As a consequence, we have

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} dx_1 dx_2 |\chi_{s/2}(z - x_1) - \chi_{s/2}(z - x_2)| \chi_{\tilde{s}/2}(x_1 - \tilde{z}) \chi_{\tilde{s}/2}(\tilde{z} - x_2) = 2A \cdot B$$

where, denoting by $|D|$ the area of a bounded domain $D \subset \mathbb{R}^2$, $A = |B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})|$ and $B = |B_{\tilde{s}/2}(\tilde{z}) \setminus (B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z}))| = (\pi \tilde{s}^2/4) - A$. Using

$$2A \cdot B = \frac{1}{2} (A + B)^2 - \frac{1}{2} (A - B)^2 \leq \frac{1}{2} (A + B)^2 = \frac{1}{2} \left(\frac{\pi \tilde{s}^2}{4} \right)^2$$

and the fact that A and B are different from 0 if and only if

$$\frac{s - \tilde{s}}{2} < |z - \tilde{z}| < \frac{s + \tilde{s}}{2}$$

we have

$$\begin{aligned} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx_1 dx_2 |\Delta(s, \tilde{s}, x_0, x_1, x_2)| &= \frac{4}{\pi s^2} \int_{\mathbb{R}^2} dz \chi_{s/2}(x_0 - z) \cdot \frac{4}{\pi \tilde{s}^2} \int_{\mathbb{R}^2} d\tilde{z} 2A \cdot B \\ &\leq \frac{1}{\tilde{s}^2} ((s + \tilde{s})^2 - (s - \tilde{s})^2) \cdot \frac{1}{2} \left(\frac{\pi \tilde{s}^2}{4} \right)^2 \\ &= \frac{1}{8} \pi^2 \tilde{s}^3 s , \end{aligned} \quad (3.39)$$

and this implies that $\tilde{A}_2(s)$ given by (3.36) with $m = 2$ and $\sigma_1\sigma_2 = -1$ is bounded uniformly with respect to the cutoff t_0 provided $\beta \in [0, 6\pi)$, i. e., inside the first threshold interval $I_1 = [4\pi, 6\pi)$. For this, observe that by Proposition 2.11 (see (3.12))

$$\begin{aligned}\tilde{A}_2(s) &= \frac{1}{2}g(s) \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx_1 dx_2 \beta \int_{t_0}^s g(\tilde{s}) |\Delta(s, \tilde{s}, x_0, x_1, x_2)| \exp \left(\beta \int_{\tilde{s}}^s g(\tau) h(|x_1 - x_2|/\tau) d\tau \right) d\tilde{s} \\ &\leq \frac{\beta}{64} m(s) \int_{t_0}^s \tilde{s}^2 m(\tilde{s}) \exp \left(\frac{\beta}{2\pi} \int_{\tilde{s}}^s \frac{1}{\tau} m(\tau) d\tau \right) d\tilde{s} \\ &< C \int_{t_0}^1 \tilde{s}^{2-\beta/2\pi} d\tilde{s} < \frac{C}{3-\beta/2\pi} < \infty\end{aligned}\tag{3.40}$$

if $\beta < 6\pi$, uniformly in t_0 .

The integral of (3.38) performed over x_1 and x_2 disregarding the minus sign would be proportional to $\tilde{s}^2 s^2$, by (3.8). The small cluster neutrality condition leads to a rearrangement of the powers in \tilde{s} and s (3.39) in favor of \tilde{s} . Note that $sg(s)$ remains integrable by Proposition 2.11 and a new function $\Gamma(s)$ needs to be redefined accordingly. The rearrangement is not enough to prevent the collapse of the neutral pair inside the high order threshold intervals and we need to be more careful when $|x_1 - x_2|/\tau$ is small. By the first mean value theorem, there exist $\tau^* \in [\tilde{s}, s]$ such that

$$\frac{\beta}{2\pi} \int_{\tilde{s}}^s m(\tau) h(|x_1 - x_2|/\tau) \frac{d\tau}{\tau} = m(\tau^*) h(|x_1 - x_2|/\tau^*) \frac{\beta}{2\pi} \log \frac{s}{\tilde{s}}.$$

For fixed s , let us say $s = 1$, let $\Lambda = \{(r, \tilde{s}) \in \mathbb{R}_+ \times [t_0, 1] : r/\tau^*(r, \tilde{s}) \leq 0.2\}$ and note that $m(\tau^*) h(r/\tau^*) < 3/4$ for (r, \tilde{s}) in the complementary set $(\mathbb{R}_+ \times [t_0, 1]) \setminus \Lambda$, by Propositions 2.7 and 2.11. Under this condition $\tilde{A}_2(s)$ can be bounded by the last integral in (3.40) with the exponent $2 - \beta/2\pi$ of \tilde{s} replaced by $2 - (3/4)\beta/2\pi = 2 - 3\beta/8\pi$, which is finite for $\beta < 8\pi$. On the other hand, one can show that the integral of Δ over $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ in (3.39) restricted to Λ , where $r = |x_1 - x_2|$, is proportional to \tilde{s}^4 (instead of \tilde{s}^3) under the change of variables $x_i = \tilde{s}y_i$, $i = 1, 2$. Observe that $\tau^*(r, \tilde{s})$ tends to \tilde{s} when r tends to 0. As a consequence, $\tilde{A}_2(s)$ can be bounded by the last integral in (3.40) with the exponent $2 - \beta/2\pi$ of \tilde{s} replaced by $3 - \beta/2\pi$ which is finite again for $\beta < 8\pi$.

4 Summary and open question

The main result of the present paper, Theorem 2.8, states that the energy $U_n(\xi; h)$ of a configuration $\xi = (x, \sigma) = (x_1, \dots, x_n, \sigma_1, \dots, \sigma_n)$ of n particles, with $(x_i, \sigma_i) \in \mathbb{R}^2 \times \{+1, -1\}$, interacting through the two-dimensional Euclid's hat pair potential $h(\cdot/s)$ at scale s satisfies (2.19). Since the inequality saturates when the n particles collapses all together to a single point with net charge 0 if n is even and ± 1 if n is odd, a corollary to this (see Corollary 2.10) is that the minimal specific energy $e(h)$

and the minimal constrained specific energy $\bar{e}(h)$, defined by (1.18) and (1.19), are both $-1/2$. The same statement holds to positive radial potentials of positive type in any dimension $d \geq 2$ provided it can be written as scale mixtures of Euclid's hat: $v(x) = \int g(s)h(|x|/s)ds$, $g(s) \geq 0$ and the right hand side of (2.19) is multiplied by $\int g(s)ds$. Consequently, if n is odd the stability bound (1.16) can be replaced by (1.23) for any potential of this class with $B = \bar{B} = \frac{1}{2} \int g(s)ds$.

We have applied the main result to the two-dimensional Yukawa gas with particles activity z at the inverse temperature β in the interval of collapse $[4\pi, 8\pi)$. A Cauchy majorant, proposed in [GM] for the pressure and density function, can be written in terms of the principal branch of the W -Lambert function which is analytic provided its argument $-z\tau_k$, with $\tau_k = \tau_k(t_0, t)$ given by (3.26), satisfies $e|z|\tau_k < 1$, $\beta < \beta_{2n} = 8\pi(1 - 1/2n)$ when the divergent part of the leading even Mayer coefficients up to order $2n \leq k + 1$, $k > 1$, are extracted. It has been assumed in addition that an improved stability condition (see Conjecture 2.3 of [GM]) holds for any odd number of particles $2n - 1 \leq k$. However, the numerical evaluation (see Remark 7.5 of [GM]) of the total energy $U_{2n-1}(\xi; \beta\dot{v})$ for the standard scaling decomposition (1.7) when $n = 2$ and 3 have indicated that it would fail for sufficient large k and Proposition 2.2 now proves that $U_3(\xi; \beta\dot{v})$ does not satisfy the improved stability for $k > 15$. We have in the present paper proved that when the Yukawa potential v is represented as scale mixtures of Euclid's hat it satisfies Conjecture 2.3 of [GM] for any $k > 1$ and, moreover, all the estimates necessary to establish convergence of the majorant series in [GM] holds for this representation of v due to Proposition 2.11. We have reestimated Propositions 3.4, 3.7 and 3.9 accordingly for the reader convenience.

It is important to stress at this point that the classical solution $\Theta_k = \Theta_k(t, z)$ of (3.24) is actually a majorant for the density function (3.2) and the same statement (3.2) holds in Proposition 3.7 as long as $t_0 > 0$. One open question is whether the majorant Θ_k remain faithful when the cutoff t_0 is set to 0. We answer the question affirmatively only for $k = 3$ and argue that this question might be dealt using the superstability of the (two-species) Yukawa potential restricted to configurations in which a neutral subcluster is located in a small volume of linear size $t_0 > 0$ (see [Gi, RT] and references therein).

References

- [Ba1] A. G. Basuev. *Theorem on the minimal specific energy for classical systems*, English translation: Theoret. and Math. Phys. **37**, 923-926 (1978)
- [Ba2] A. G. Basuev. *Representation for Ursell functions and cluster estimates*, English translation: Theoret. and Math. Phys. **39**, 343-350 (1979)

- [Be] G. Benfatto. *An iterated Mayer expansion for the Yukawa gas*, Journ. Stat. Phys. **41**, 671-684 (1985)
- [BS] R. Beals and J. Szmigielski. *Meijer G-functions: A gentle introduction*, Notices of the AMS **60**, 866-872 (2013)
- [B] S. Bochner. *Lectures on Fourier integrals*, translated by N. Tenenbaum and H. Pollard, Annals of Mathematics Studies **42**, Princeton University Press, Princeton, NJ, 1959
- [Br] David C. Brydges. *Convergence of Mayer expansions*, Journ. Stat. Phys. **42**, 425-435 (1986)
- [BGM] D. C. Brydges, G. Guadagni and P. K. Mitter. *Finite range decomposition of Gaussian Processes*, Journ. Stat. Phys. **115**, 415-449 (2004)
- [BK] D. C. Brydges, T. Kennedy. *Mayer expansions and the Hamilton–Jacobi equation*, Journ. Stat. Phys. **48**, 19-49 (1987)
- [BM] D. C. Brydges and Ph. A. Martin. *Coulomb systems at low density: A review*, Journ. Stat. Phys. **96**, 1163-1330 (1999)
- [BT] D. Brydges and A. Talarczyk. *Finite range decomposition of positive-definite functions*, J. Funct. Analys. **236**, 682-711 (2006)
- [C] J. A. Cochran. *The monotonicity of modified Bessel functions with respect to their order*, J. Math. and Phys. **46**, 220-222 (1967)
- [C-K] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth. *On the Lambert W function*, Adv. Comput. Math. **5**, 329-359 (1996)
- [D] Philip J. Davis. *The Schwarz function and its applications*, The Carus Mathematical Monographs **17**, The Mathematical Association of America, 1974
- [E] Lawrence C. Evans. *Partial differential equations*, Second edition. Graduate Studies in Mathematics, **19**. American Mathematical Society, Providence, RI, 2010
- [FR] M. E. Fisher and D. Ruelle. *The stability of many-particle systems*, J. Math. Phys. **7**, 260-270 (1966)
- [FS] J. Fröhlich and T. Spencer. *Phase diagrams and critical properties of (classical) Coulomb systems*, In: G. Velo and A. S. Wightman (eds) “Atomic and Molecular Physics”, 327-370 (1981). NATO Advanced Study Institute Series, vol. **74**, Springer, Boston, MA
- [Ga] R. E. Gaunt. *Inequalities for modified Bessel functions and their integrals*, J. Math. Anal. Appl. **420**, 373-386 (2014)

- [Gi] Roman Gielerak. *Free energy density for continuous systems with and without superstability assumptions*, J. Math. Phys. **30**, 115-133 (1989)
- [GJ] J. Glimm and A. Jaffe. *Quantum Physics. A functional integral point of view*, Springer-Verlag, New York 1981
- [G] T. Gneiting. *Radial positive definite functions generated by Euclid's hat*, J. Multivariate Anal. **69**, 88-119 (1999)
- [GoMa] M. Göpfert and G. Mack. *Iterated Mayer expansion for classical gases at low temperatures*, Commun. Math. Phys. **81**, 97-126 (1981)
- [GM] L. F. Guidi and D. H. U. Marchetti. *Convergence of the Mayer series via Cauchy majorant methods with applications to the Yukawa gas in the region of collapse*, Preprint arXiv:math-ph/0310025 (2004)
- [HS] C. Hainzl and R. Seiringer. *General decomposition of radial functions on \mathbb{R}^n and applications to N -body quantum systems*, Lett. Math. Phys. **61**, 75-84 (2002)
- [JMR] P. Jaming, M. Matolcsi and S. G. Révész. *On the extremal rays of the cone of positive, positive definite functions*, J. Fourier Anal. Appl. **15**, 561-582 (2009)
- [I] J. Z. Imbrie. *Iterated Mayer expansions and their application to Coulomb gases*, in "Scaling and Self-Similarity in Physics. Renormalization in Statistical Mechanics and Dynamics", Birkhäuser, Boston (1983)
- [L1] A. Lenard. *States of classical statistical mechanical system of infinitely many particles I*, Arch. Rational Mech. Anal. **59**, 219-239 (1975)
- [L2] A. Lenard. *States of classical statistical mechanical system of infinitely many particles II*, Arch. Rational Mech. Anal. **59**, 240-256 (1975)
- [LPY] B. N. B. de Lima, A. Procacci and S. Yuhjtman. *On stable pair potentials with attractive tail, remarks on two papers by A. G. Basuev*, Commun. Math. Phys. **343**, 445-476 (2016)
- [P] O. Penrose. *Convergence of fugacity expansion for fluids and lattice gases*, J. Math. Phys. **4**, 1312-1321 (1963) and *Convergence of fugacity expansion for classical systems*, In: A. Bak (ed.) "Statistical mechanics; foundations and applications" Benjamin, New York (1967)
- [PY] A. Procacci and S. Yuhjtman. *Convergence of Mayer and Virial expansions and the Penrose tree-graph identity*, Lett. Math. Phys. **107**, 31-46 (2017)
- [RT] A. L. Rebenko and M. V. Tertychnyi. *On stability, superstability and strong superstability of classical systems of statistical mechanics*, Methods Funct. Anal. Topology **14**, 287-296 (2008)

- [R] David Ruelle. *Statistical Mechanics: Rigorous results*, Addison-Wesley Publ. Co., Inc., Advanced Book Classics 1989
- [S] James Stewart. *Positive definite functions and generalizations, an historical survey*, Rocky Mountain J. Math. **6**, 409-434 (1976)
- [UF] G. E. Uhlenbeck and G. W. Ford. *Lectures in Statistical Mechanics*, Lectures in Applied Mathematics vol. **1**, American Mathematical Society, Providence, Rhode Island 1963
- [YC] Z-H. Yang and Yu-M. Chu. *On approximating the modified Bessel function of second kind, J. Inequal.. Appl., paper n. 41*, 8 pp. (2017)

----- READERSOURCING 2.0 ANNOTATION STARTS HERE -----

Express your rating

----- READERSOURCING 2.0 ANNOTATION ENDS HERE -----