A NOTE ON EQUIVARIANT K-STABILITY

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Abstract. We define G-pseudovaluations on a variety with a group action G. By introducing G-pseudovaluations, we are able to give some criteria for G-equivariant K-stability of Fano varieties which are parallel to existing results for usual K-stability.

1. Introduction

We work over the complex number $\mathbb{C}$. A $\mathbb{Q}$-Fano variety is a normal projective variety with klt singularities such that the anti-canonical divisor is ample.

It is conjectured that in order to test K-polystability of a $\mathbb{Q}$-Fano variety it is enough to examine equivariant test configurations with respect to a finite or connected reductive subgroup G of $\mathrm{Aut}(X)$. For the case of Fano manifolds, an analytic proof is given in [DS16]. An algebraic proof is also provided in [LWX18] when G is a torus group.

The purpose of this short note however is to provide another perspective on equivariant K-stability for $\mathbb{Q}$-Fano varieties with arbitrary group action. We give parallel results to some existing theorems on characterizing K-stability by replacing the space of valuations with a special collection of pseudovaluations in terms of the group action. Indeed, for any variety X, let $G \subset \mathrm{Aut}(X)$ denote a group action on X. For any valuation v on X, we define

$$G \cdot v := \inf_{g \in G} g \cdot v,$$

where g acts on the valuation v by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$. We call $G \cdot v$ a G-pseudovaluation and denote all G-pseudovaluations on X by GVal_X. Note that all G-invariant valuations, which we denote by Val_X^G, are contained in GVal_X. For any G-pseudovaluation $G \cdot v$, and a nonnegative real number x, we can define the ideal sheaf $\mathfrak{a}_x(G \cdot v)$ to be

$$\mathfrak{a}_x(G \cdot v) = \bigcap_{g \in v} \mathfrak{a}_x(g \cdot v),$$

where for any valuation w, $\mathfrak{a}_x(w)$ is the ideal sheaf of regular functions with vanishing order no less than x with respect to w. Refer to Section 2 for details about the definition of G-pseudovaluations.

The first theorem is about valuative criteria of equivariant K-stability parallel to the main results in [Fuj16]. Let X be a $\mathbb{Q}$-Fano variety and $G \subset \mathrm{Aut}(X)$ a group action on X. We define the G-equivariant beta invariant of F to be

$$\beta^G(F) := A_X(F)(-K_X)^n - \int_0^{+\infty} \mathrm{vol}_X\left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_F)\right) dx.$$

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We say that F is of finite orbit if the orbit of the valuation ord_F under G-action is finite. We say that F is G-dreamy if F is of finite orbit and moreover the graded ring

$$\bigoplus_{k,j\geq 0} H^0\left(X, \mathcal{O}_X(-kK_X)\otimes \mathfrak{a}_j(G\cdot \mathrm{ord}_F)\right)$$

is finitely generated.

Define

$$\tau^G(F) := \sup\{t>0 |\, \mathrm{vol}_X\left(\mathcal{O}_X(-K_X)\otimes \mathfrak{a}_t(G\cdot \mathrm{ord}_F)\right)>0\}$$

and

$$j^G(F) = \int_0^{\tau^G(F)} \left(\mathrm{vol}_X(-K_X) - \mathrm{vol}_X\left(\mathcal{O}_X(-K_X)\otimes \mathfrak{a}_x(G\cdot \mathrm{ord}_F)\right)\right) dx.$$

Note that for G-invariant divisors over X, the above definitions coincide with the usual ones defined in [Fuj16].

The following theorem gives valuative criteria of K-stability in terms of $\beta^G(F)$:

Theorem A. *Let X be a $\mathbb{Q}$-Fano variety with $G\subset \mathrm{Aut}(X)$ a group action on X.*

(1) *The following are equivalent:*
   (i) *X is uniformly G-equivariantly K-stable;*
   (ii) *there exists $0<\delta<1$, such that $\beta^G(F)\geq \delta j^G(F)$ for any finite-orbit prime divisor F over X;*
   (iii) *there exists $0<\delta<1$, such that $\beta^G(F)\geq \delta j^G(F)$ for any G-dreamy prime divisor F over X.*

(2) *The following are equivalent:*
   (i) *X is G-equivariantly K-semistable;*
   (ii) *$\beta^G(F)\geq 0$ for any finite-orbit prime divisor F over X;*
   (iii) *$\beta^G(F)\geq 0$ for any G-dreamy prime divisor F over X.*

(3) *The following are equivalent:*
   (i) *X is G-equivariantly K-stable;*
   (ii) *$\beta^G(F)>0$ for any G-dreamy prime divisor F over X.*

Remark 1.1. When G is finite, every prime divisor over X is of finite orbit. Moreover, by an argument provided by Yuchen Liu, we can take the quotient of each G-equivariant test configuration and run the process in [LX14] to get a special test configuration. Then by [Fuj16], we know that it is enough to check G-invariant divisors for K-stability for finite G. When G is connected, we know that every finite-orbit divisor is G-invariant. In general, when G is not finite, all the prime divisors induced by G-special test configurations (see Section 2.3 for definition) are still of finite orbit. Therefore, we are not losing any information in terms of test configurations and K-stability by focusing only on divisors of finite orbit.

We can also characterize equivariant K-stability in terms of equivariant normalized volume of G-pseudovaluations. Normalized volume of G-pseudovaluations can be defined similarly as the normalized volume of usual valuations in [Li15] and we will use the same notation. See Section 2 for more details.

Let X be a $\mathbb{Q}$-Fano variety with G-action, denote by $Y=C(X,-K_X)$ the cone over X and $o\in Y$ the vertex of the cone. Suppose $\pi:\ Z=Bl_oY\to Y$ is the blow-up of Y at o. Let E be the exceptional divisor of the blow-up. Denote the divisorial valuation ord_E by v_0. Note that there is a natural G-action induced on the cone Y and the blow-up Z. Since

E is a G-invariant divisor, we know that $v_0 \in \mathrm{Val}^G_{Y,o} \subset \mathrm{GVal}_{Y,o}$, where $\mathrm{Val}^G_{Y,o}$ and $\mathrm{GVal}_{Y,o}$ refer to G-invariant valuations and G-pseudovaluations centered at o respectively.

Under the above notation, we have the following characterization of G-equivariant K-semistability compared to the results in [Li17, LL16, LX16]:

Theorem B. *X is G-equivariantly K-semistable iff the normalized volume function $\widehat{\mathrm{vol}}_{Y,o}$ is minimized at v_0 among all finite-orbit G-pseudovaluations on Y centered at o.*

Remark 1.2. If one can show that the minimizer of $\widehat{\mathrm{vol}}_{Y,o}$ among all valuations on Y centered at o is unique, which is a long existing conjecture first proposed in [Li15], then it is necessarily G-invariant. As it is well known, this would immediately imply the equivalence between G-equivariant K-semistability and usual K-semistability by a similar argument as in the proof of Theorem E in [LX16]. In particular, it would follow that it is enough to consider only G-invariant divisors and G-invariant valuations to check K-semistability.

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2. Pseudovaluations, normalized volumes and Equivariant K-stability

We include in this section relevant equivariant version of notions about valuations and K-stability for reader's convenience.

2.1. **Valuations and pseudovaluations.** For a variety X with a group action G, we define G-pseudovaluations in the following way:

Definition 2.1. Let G be a group action on X and v a valuation on X. Define

$$G \cdot v := \inf_{g \in G} g \cdot v,$$

where $g \cdot v$ is the valuation given by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$. We call $G \cdot v$ a G-pseudovaluation and denote all G-pseudovaluations on X by GVal_X. The center of $G \cdot v$ is defined to be the union of the centers of $g \cdot v$ for all $g \in G$. We say $G \cdot v$ is of finite orbit if the orbit of v under G-action is finite.

Remark 2.1. In general, G-pseudovaluations are not valuations because they do not satisfy the product property. Indeed, for any $f, g \in \mathbb{C}(X)$, we only have

$$G \cdot v(fg) \geq G \cdot v(f) + G \cdot v(g).$$

If $U \subset X$ is an affine open set containing all the centers of the valuations $g \cdot v$, then $G \cdot v$ induces a pseudovaluation on $\mathcal{O}_X(U)$ in the sense of [dFM15]. When G is finite, we can always find such U. Note that pseudovaluations on an affine variety do not extend to its function field due to the lack of product property. In general there is not a clear way to define pseudovaluations on a projective variety.

For a valuation v on X and a nonnegative real number x, the ideal sheaf $\mathfrak{a}_x(v) \subset \mathcal{O}_X$ is defined as follows. For $U \subset X$ an open affine subset of X, if U contains the center of v, then define

$$\mathfrak{a}_x(v)(U) = \{f \in \mathcal{O}_X(U) | v(f) \geq x\}.$$

If U does not contain the center of v, we set $\mathfrak{a}_x(v)(U) = \mathcal{O}_X(U)$. For a G-pseudovaluation $G \cdot v$, and x a nonnegative real number, we define the ideal sheaf $\mathfrak{a}_x(G \cdot v)$ to be

$$\mathfrak{a}_x(G \cdot v) = \bigcap_{g \in v} \mathfrak{a}_x(g \cdot v).$$

2.2. **Equivariant normalized volume.** Let x be a G-invariant point on X. Denote by $\mathrm{GVal}_{X,x}$ all G-pseudovaluations centered at x. We can define the normalized volume $\widehat{\mathrm{vol}}$ on the $\mathrm{GVal}_{X,x}$ almost the same way as normalized volume of usual valuations. First of all, for any G-pseudovaluation $G \cdot v$, we define the volume

$$\mathrm{vol}(G \cdot v) = \lim_{\lambda \to \infty} \frac{\dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathfrak{a}_\lambda(G \cdot v)}{\lambda^n/n!}.$$

Note that $A_X(g \cdot v) = A_X(v)$ for any $g \in G$, so we define the log discrepancy of $G \cdot v$ to be $A_X(v)$. Then the normalized volume of $G \cdot v$ is defined as

$$\widehat{\mathrm{vol}}(G \cdot v) = A_X(v)^n \,\mathrm{vol}(G \cdot v).$$

2.3. **Equivariant K-stability.** We first give the definition of equivariant test configuration.

Definition 2.2. Let (X, L) be a polarized variety. A (semi-)test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) with exponent r consists of the following data:

(1) a proper flat family $\pi : \mathcal{X} \to \mathbb{A}^1$,
(2) an equivariant $\mathbb{C}^*$-action on $\pi : \mathcal{X} \to \mathbb{A}^1$, where $\mathbb{C}^*$ acts on $\mathbb{A}^1$ by multiplication in the standard way, and
(3) a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$ which is π-relatively (semi-)ample,

such that $(\mathcal{X}, \mathcal{L})|_{\pi^{-1}(\mathbb{A}^1 \backslash \{0\})}$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X \times (\mathbb{A}^1 \backslash \{0\}), L^{\otimes r}_{\mathbb{A}^1 \backslash \{0\}})$, where $L_{\mathbb{A}^1 \backslash \{0\}}$ is the pull back of L from X to $X \times (\mathbb{A}^1 \backslash \{0\})$. In addition, let G be a group action on (X, L). We say $(\mathcal{X}, \mathcal{L})$ is a G-equivariant test configuration if G can be extended to an action on $(\mathcal{X}, \mathcal{L})$ such that it commutes with the $\mathbb{C}^*$ on $(\mathcal{X}, \mathcal{L})$, fixes fibers of $\mathcal{X}$ and restricts to the G-action on all fibers of $\mathcal{X}$ other than $\mathcal{X}_0$.

Next, we will focus on $\mathbb{Q}$-Fano varieties with the polarization to be $-K_X$. By replacing $-K_X$ with a sufficiently divisible multiple of itself, we may assume $-K_X$ is already Cartier.

The definition of Donaldson-Futaki invariant for an equivariant test configuration is the same as the usual one. We include a definition using intersection formula here which will come up in later computation.

Definition 2.3. Let X be a $\mathbb{Q}$-Fano variety of dimension n. Pick a rational number r such that rK_X is Cartier. Let $(\mathcal{X}, \mathcal{L})$ be a normal semi-test configuration of $(X, -rK_X)$. We can compactify the test configuration into a flat family $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ over $\mathbb{P}^1$, such that over $\mathbb{P}^1 \backslash \{0\}$, the family $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{P}^1 \backslash \{0\}$ with trivial $\mathbb{C}^*$-action on the fibers. Then we can define the Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)(-K_X)^n} \left(\frac{n}{r^{n+1}} \bar{\mathcal{L}}^{n+1} + \frac{n+1}{r^n} \bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1} \right) \tag{2.1}$$

We also include the definition of $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ following [Fuj16], which can be viewed as the norm of $(\mathcal{X}, \mathcal{L})$. Let

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \times \mathbb{P}^1 \arrow[rr, dotted] & & \bar{\mathcal{X}}. \end{tikzcd}$$

be a common resolution of $X \times \mathbb{P}^1$ and $\bar{\mathcal{X}}$. We set

$$\lambda_{\max}(\mathcal{X}, \mathcal{L}) := \frac{p^*(-K_{X\times\mathbb{P}^1/\mathbb{P}^1})^n \cdot q^*\bar{\mathcal{L}}}{(-K_X)^n},$$

and define

$$J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \lambda_{\max}(\mathcal{X}, \mathcal{L}) - \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)(-rK_X)^n}$$

Definition 2.4. Let X be a $\mathbb{Q}$-Fano variety with $G \subset \mathrm{Aut}(X)$ a group action on X. We have the following three definitions of K-stability:

(1) $(X, -K_X)$ is said to be G-equivariantly K-semistable if the Donaldson-Futaki invariant is nonnegative for all G-equivariant normal test configurations.
(2) $(X, -K_X)$ is said to be G-equivariantly K-stable if the Donaldson-Futaki invariant is positive for all nontrivial G-equivariant normal test configurations.
(3) $(X, -K_X)$ is said to be uniformly G-equivariantly K-stable if there exists $0 < \delta < 1$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for all G-equivariant normal test configurations.

Following the argument in [LX14], we can get a collection of equivariant test configurations that plays the same role as special test configurations for K-stability.

Theorem 2.5. *For any G-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -K_X)$, there exists a finite morphism $\phi : \mathbb{A}^1 \to \mathbb{A}^1$, a test configuration $(\mathcal{X}^s, \mathcal{L}^s)$ with the central fiber being reduced and G-irreducible and a both $\mathbb{C}^*$- and G-equivariant birational map $\mathcal{X}^s \dashrightarrow \mathcal{X} \times_\phi \mathbb{A}^1$ over $\mathbb{A}^1$, such that for any $0 \leq \delta \leq 1$, we have*

$$\mathrm{DF}(\mathcal{X}^s, \mathcal{L}^s) - \delta J^{NA}(\mathcal{X}^s, \mathcal{L}^s) \leq \deg\phi \left(\mathrm{DF}(\mathcal{X}, \mathcal{L}) - \delta J^{NA}(\mathcal{X}, \mathcal{L})\right).$$

In addition, we can choose $\mathcal{L}^s = -K_{\mathcal{X}^s/\mathbb{A}^1}$.

Proof. By running the G-equivariant version of each steps in the proof of the main theorem in [LX14], we get the G-equivariant test configuration $(\mathcal{X}^s, \mathcal{L}^s)$ and the birational map $\mathcal{X}^s \dashrightarrow \mathcal{X} \times_\phi \mathbb{A}^1$. The computation in [LX14] and [Fuj16] gives us the inequality.

Note that both $\mathcal{L}^s$ and $K_{\mathcal{X}^s/\mathbb{A}^1}$ are G-invariant. Then since $\mathcal{L}^s + K_{\mathcal{X}^s/\mathbb{A}^1}$ supports on the central fiber $\mathcal{X}^s_0$, it can only be a multiple of the whole fiber $\mathcal{X}^s_0$. Then by definition we have $\mathrm{DF}(\mathcal{X}^s, \mathcal{L}^s) = \mathrm{DF}(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$ and $J^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{L}^s) = J^{\mathrm{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$. ☐

We call the resulting test configuration $(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$ in Theorem 2.5 a G-special test configuration. As in the usual K-stability case, we know from Theorem 2.5 that it is enough to check only G-special test configurations for G-equivariant K-stability.

A test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ induces a filtration $\mathcal{F}$ on $V_k = H^0(X, -kK_X)$ in the following way:

$$\mathcal{F}^x V_k = \{s \in V_k | t^{-\lceil x \rceil}\bar{s} \in H^0(\mathcal{X}, k\mathcal{L})\},$$

where $\bar{s}$ is the $\mathbb{C}^*$-invariant section of $k\mathcal{L}$ on $\mathcal{X}\backslash\mathcal{X}_0$ induced by s. Note that $\mathcal{F}$ is decreasing, left-continuous, multiplicative and linearly bounded. Filtrations in this paper will always be assumed to satisfy these four properties.

Conversely, let $\mathcal{F}$ be a filtration on $V_\bullet$ such that $\bigoplus_{k\in\mathbb{Z}_{\geq 0}, j\in\mathbb{Z}} \mathcal{F}^j V_k$ is finitely generated. We may assume it is generated in degree $k=1$. Then we can define a test configuration

$$\left(\mathrm{Proj}_{\mathbb{A}^1} \bigoplus_{k\in\mathbb{Z}_{\geq 0}, j\in\mathbb{Z}} t^{-j}\mathcal{F}^j V_k, \mathcal{O}(1)\right)$$

The following proposition gives the relation between filtrations and test configurations.

Proposition 2.6 (Proposition 2.15, [BHJ15])**.** *The above construction sets up a one-to-one correspondence between test configurations of $(X, -K_X)$ and finitely generated filtrations on $V_\bullet$.*

For any prime divisor F over X, we can construct a G-invariant filtration

$$\mathcal{F}^x V_r = \begin{cases} H^0\left(X, \mathcal{O}_X(-rK_X)\otimes \mathfrak{a}_{\lceil x\rceil}(G\cdot \mathrm{ord}_F)\right), & x\geq 0, \\ V_r, & x<0. \end{cases} \tag{2.2}$$

which induces a G-equivariant test configuration.

To conclude this section, we look at some basic examples that illustrate the difference between G-equivariant K-stability and usual K-stability.

Example 2.7. Consider the projective space $X=\mathbb{P}^n$ with $G=PGL(n+1)$-action. Then the only G-equivariant test configuration of $(\mathbb{P}^n, -K_{\mathbb{P}^n})$ is the trivial test configuration $\mathbb{P}^n\times\mathbb{A}^1$. Therefore by definition we know that $\mathbb{P}^n$ is uniformly G-equivariantly K-stable. Note that for any G-pseudovaluation $G\cdot v$, we have that $\mathfrak{a}_x(G\cdot v)=(0)$ for any $x>0$. Therefore for any prime divisor F over $\mathbb{P}^n$, we know that the corresponding G-invariant filtration

$$\mathcal{F}^x V_r = \begin{cases} 0, & x>0, \\ V_r, & x\leq 0, \end{cases}$$

which of course induces the trivial test configuration $\mathbb{P}^n\times\mathbb{A}^1$.

Example 2.8. Consider $X=\mathbb{P}^1\times\mathbb{P}^1$ with $G=PGL(2)$ acting on the first component. Pick any point $p\in X$. Let E be the exceptional divisor of the blow-up of X at p. Let H be the horizontal line through p, and we know that H is the orbit of p under G-action. Therefore E and H induce the same G-invariant filtration. Note that although E is not of finite orbit, we know H is G-invariant. The compactified test configuration corresponding to the G-invariant filtration is $\pi:\ \mathbb{P}^1\times\mathbb{F}_1\to\mathbb{P}^1$, with G acting on the first component and π induced by the Hirzebruch surface $\mathbb{F}_1\to\mathbb{P}^1$.

Similar examples can also be constructed easily when G is non-compact, e.g. a torus action $(\mathbb{C}^*)^r$ on $\mathbb{P}^n$.

3. Equivariant valuative criteria

We separate the proof of Theorem A into 3 parts. We first prove the following theorem which gives a necessary valuative condition of equivariant uniform K-stability in Theorem A.

Theorem 3.1. *Let X be a $\mathbb{Q}$-Fano variety with $G\subset \mathrm{Aut}(X)$ a group aciton on X. If X is uniformly G-equivariantly K-stable, then there exists $0<\delta<1$, such that $\beta^G(F)\geq \delta j^G(F)$ for any finite-orbit prime divisor F over X.*

Proof. We may assume $-K_X$ is already Cartier. Given any divisor F of finite orbit, let $\pi : Y \to X$ be a G-equivariant resolution such that F is a smooth divisor on Y. Following the notation in (2.2), we consider the G-invariant filtration of $\mathcal{F}^x V_r$ defined by F. Note that $\mathcal{F}$ is saturated. Let $I_{(r,x)} := \operatorname{Im}(\mathcal{F}^x V_r \otimes \mathcal{O}_X(rK_X) \to \mathcal{O}_X)$ be the base ideal of $\mathcal{F}^x V_r$. Suppose $F_1 = F, \ldots, F_N$ form the orbit of F under the G-action. We have

$$I_{(r,x)} \cdot \mathcal{O}_Y \subset \mathcal{O}_Y \left(-\lceil x \rceil \sum_{i=1}^{N} F_i \right).$$

Now the same computation as in the proof of Theorem 4.1 in [Fuj16] will give us $\beta^G(F) \geq \delta j^G(F)$. ☐

Remark 3.1. Note that when F is not of finite orbit, it is not possible to find a G-equivariant resolution $Y \to X$ as in the above proof.

Next we study the relation between Donaldson-Futaki invariants of G-special test configurations and equivariant beta invariants.

Theorem 3.2. *Let $(\mathcal{X}, \mathcal{L})$ be a normal G-equivariant test configuration of $(X, -K_X)$ with $\mathcal{X}_0$ reduced and G-irreducible. Suppose the central fiber of $\mathcal{X}$ can be decomposed into irreducible components $\mathcal{X}_0^1, \ldots, \mathcal{X}_0^N$. Let v_i be the restriction on X of the divisorial valuation $\operatorname{ord}_{\mathcal{X}_0^i}$. Then v_i is G-dreamy. We have $\operatorname{DF}(\mathcal{X}, \mathcal{L}) = \beta^G(v_i)/(-K_X)^n$ and $J^{NA}(\mathcal{X}, \mathcal{L}) = j^G(v_i)/(-K_X)^n$ for any i.*

Proof. First note that by Lemma 2.5, we may assume that $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$. Then we claim that each v_i is a divisorial valuation on X corresponding to distinct divisor F_i over X. Indeed, Suppose $\operatorname{ord}_{\mathcal{X}_0^i}$ and $\operatorname{ord}_{\mathcal{X}_0^j}$ restrict to the same valuation v_i and v_j on X. Then since $\operatorname{ord}_{\mathcal{X}_0^i}(t) = \operatorname{ord}_{\mathcal{X}_0^j}(t) = 1$, we know that the two valuations are the same on $\mathcal{X}$. The G-action permutes all the irreducible components of the central fiber $\mathcal{X}_0$, so we have that $F_1, \ldots, F_N$ forms the orbit of F_1 under the G-action.

Now the same argument as in the proof of Theorem 5.1 in [Fuj16] gives us the conclusion. ☐

The following theorem is an immediate consequence of Theorem 3.2.

Theorem 3.3. *If there exists some $0 < \delta < 1$, such that $\beta^G(F) > 0 (\geq \delta j^G(F))$ for any G-dreamy divisor F over X, then X is (uniformly) G-equivariantly K-stable.*

Note that if we set $\delta = 0$ in Theorem 3.1 and Theorem 3.3, we get the corresponding valuative criterion for K-semistability.

By Proposition 2.6, we have a one-to-one correspondence between filtrations on $V_\bullet$ and test configurations. Then combining Theorem 3.2, we have the following theorem:

Theorem 3.4. *Let X be a $\mathbb{Q}$-Fano variety and F a G-dreamy divisor over X. Define a filtration $\mathcal{F}$ on $V_\bullet$ as in (2.2). Then the test configuration*

$$(\mathcal{X}, \mathcal{L}) = \left(\operatorname{Proj}_{\mathbb{A}^1} \bigoplus_{k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}} t^{-j} \mathcal{F}^j V_k, \mathcal{O}(1) \right)$$

is a G-special test configuration such that $\operatorname{DF}(\mathcal{X}, \mathcal{L}) = \beta^G(F)/(-K_X)^n$ and $J^{NA}(\mathcal{X}, \mathcal{L}) = j^G(F)/(-K_X)^n$.

Proof. We only need to show that $\mathcal{X}_0$ is reduced and G-irreducible. Note that

$$\mathcal{X}_0 = \operatorname{Proj} \bigoplus_{k,j\geq 0} \operatorname{gr}^j_{\mathcal{F}} V_k,$$

where

$$\operatorname{gr}^j_{\mathcal{F}} V_k = \frac{\mathcal{F}^j V_k}{\mathcal{F}^{j+1} V_k}$$

is the graded piece of the filtration $\mathcal{F}$. Apparently $\bigoplus_{k,j\geq 0} \operatorname{gr}^j_{\mathcal{F}} V_k$ is reduced. Now pick any $f_i \in \mathcal{F}^{j_i} V_{k_i} \backslash \mathcal{F}^{j_i+1} V_{k_i}$ for $i = 1, 2$. Let $F_1, \ldots, F_N$ form the orbit of F under G-action. Since the orbit is finite, we can find F_{l_i} such that $\operatorname{ord}_{F_{l_i}}(f_i) = G \cdot \operatorname{ord}_F(f_i) = j_i$ for $i = 1, 2$. Suppose $\operatorname{ord}_{F_{l_2}} = g \cdot \operatorname{ord}_{F_{l_1}}$ for some $g \in G$. Then we have

$$G \cdot \operatorname{ord}_F(g(f_1)f_2) = \operatorname{ord}_{F_{l_2}}(g(f_1)f_2) = j_1 + j_2,$$

and consequently $g(f_1)f_2 \in \mathcal{F}^{j_1+j_2} V_{k_1+k_2} \backslash \mathcal{F}^{j_1+j_2+1} V_{k_1+k_2}$. Therefore we know that $\mathcal{X}_0$ is G-irreducible. □

An immediate consequence is the following theorem:

Theorem 3.5. *If X is G-equivariant K-stable, then $\beta^G(F) > 0$ for any G-dreamy divisor F over X.*

Combining the above results we finish the proof of Theorem A.

4. Equivariant normalized volumes

Let X be an n-dimensional $\mathbb{Q}$-Fano variety with group action G and F a prime divisor over X. Denote by $Y = C(X, -K_X)$ the cone over X with respect to the polarization $-K_X$ and $O \in Y$ the vertex of the cone. Suppose $\pi : Z = Bl_O Y \to Y$ is the blow-up of Y at O. Let E be the exceptional divisor, and $\mathbb{F}$ the pull back of F to Z. Denote the divisorial valuation ord_E by v_0 and $\operatorname{ord}_{\mathbb{F}}$ by v_F. Then for $t > 0$, we have $v_t := v_0 + tv_F$ to be a quasi-monomial valuation centered at O. Note that there is a natural G-action induced on the cone Y and the blow-up Z. Now we consider the G-pseudovaluation $G \cdot v_t$. The following proposition gives a relation between the derivative of the normalized volume $\widehat{\operatorname{vol}}(G \cdot v_t)$ and $\beta^G(F)$.

Proposition 4.1. *Under the above notations, we have*

$$\frac{d}{dt}\widehat{\operatorname{vol}}(v_t)\bigg|_{t=0} = (n+1)\beta^G(F).$$

Proof. First of all, we have $A_Y(\mathbb{F}) = A_X(F)$, and $A_Y(v_0) = 1$. Therefore $A_Y(v_t) = 1 + tA_X(F)$. Next we compute the volume of $G \cdot v_t$. Let $V_m = H^0(X, -mK_X)$ and $V = \oplus V_m$. Note that for $f \in V_m$, we have $v_0(f) = m$. Then

$$\dim V/\mathfrak{a}_\lambda(G \cdot v_t) = \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m/\mathfrak{a}_\lambda(G \cdot v_t) = \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m - \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m \cap \mathfrak{a}_\lambda(G \cdot v_t).$$

By asymptotic Riemann-Roch, we know that

$$\sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m = \frac{(-K_X)^n \lambda^{n+1}}{(n+1)!} + O(\lambda^n).$$

On the other hand, for any $f \in V_m$, we know that $g \cdot v_t(f) \geq \lambda$ is equivalent as $g \cdot v_F(f) \geq \frac{\lambda - m}{t}$. Then we know that

$$V_m \cap \mathfrak{a}_\lambda(G \cdot v_t) = H^0\left(\mathcal{O}_X(-mK_X) \otimes \mathfrak{a}_{\frac{\lambda-m}{t}}(G \cdot \mathrm{ord}_F)\right).$$

According to Lemma 4.5 in [Li17], we know that

$$\begin{aligned}&\sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m \cap \mathfrak{a}_\lambda(G \cdot v_t)\\ =&\frac{\lambda^{n+1}}{n!}\int_0^{+\infty} \mathrm{vol}_X\left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_F)\right)\frac{t}{(1+tx)^{n+2}}\,dx + O(\lambda^n).\end{aligned}$$

Putting the above expressions together, we have

$$\mathrm{vol}(G \cdot v_t) = (-K_X)^n - (n+1)\int_0^{+\infty} \mathrm{vol}_X\left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_F)\right)\frac{t}{(1+tx)^{n+2}}\,dx.$$

Taking the derivative we get

$$\left.\frac{d}{dt}\widehat{\mathrm{vol}}(G \cdot v_t)\right|_{t=0} = (n+1)\beta^G(F).$$

□

An immediate consequence of Proposition 4.1 gives one direction of Theorem B:

Corollary 4.2. *If the normalized volume function $\widehat{\mathrm{vol}}$ is minimized at v_0 among all finite-orbit G-pseudovaluations on Y centered at o, then X is G-equivariantly K-semistable.*

Repeating a similar computation as in the proof of Theorem 4.5 in [LX16] also gives the other direction of Theorem B.

5. Other related results

In this section, we list some other results we can get by introducing G-pseudovaluations.

Let X be a variety with $G \subset \mathrm{Aut}(X)$ a group action on X. By replacing usual valuations with G-pseudovaluations, we can define the G-log canonical threshold of any effective divisor D to be

$$\mathrm{Glct}(D) := \inf_E \frac{A_X(E)}{G \cdot \mathrm{ord}_E(D)}.$$

Next assume in addition that X is $\mathbb{Q}$-Fano. We define the G-equivariant alpha invariant of X to be

$$\alpha_G(X) = \inf\{\mathrm{Glct}(D) | 0 \leq D \sim_{\mathbb{Q}} -K_X\}.$$

Remark 5.1. Note that Tian first defines $\alpha_G(X)$ analytically in [Tia87]. It is then shown in [CS08] that the analytic definition of $\alpha_G(X)$ is the same as the following algebraic one:

$$\alpha_G(X) = \inf_m \left\{\mathrm{lct}\left(X, \frac{1}{m}\Sigma\right) \middle| \Sigma \text{ is a } G\text{-invariant linear subsystem in } |-mK_X|\right\}.$$

The above two algebraic definitions of $\alpha_G(X)$ are in fact the same. Indeed, for any G-invariant linear system $\Sigma \sim -mK_X$, pick any divisor $D \in \Sigma$. Then for any prime divisor E over X and $g \in G$, we have $\mathrm{ord}_E(gD) \geq \mathrm{ord}_E(\Sigma)$. Therefore we know that

$$\mathrm{Glct}\left(\frac{1}{m}D\right) \leq \mathrm{lct}\left(\frac{1}{m}\Sigma\right).$$

Conversely, for any effective divisor $D \in |-mK_X|$, let Σ be the linear subsystem of $|-mK_X|$ spanned by $\{gD|g \in G\}$. Then for any effective divisor $D' \in \Sigma$, we know that $\mathrm{ord}_E(D') \geq G \cdot \mathrm{ord}_E(D)$ and hence $\mathrm{ord}_E(\Sigma) \geq G \cdot \mathrm{ord}_E(D)$. Therefore we know that

$$\mathrm{Glct}\left(\frac{1}{m}D\right) \geq \mathrm{lct}\left(\frac{1}{m}\Sigma\right).$$

Next we will give another proof of the following result in [OS12] which is the G-equivariant version of Tian's criterion.

Theorem 5.1 (Theorem 1.10, [OS12])**.** *Let X be a $\mathbb{Q}$-Fano variety of dimension $n \geq 2$ and $G \subset \mathrm{Aut}(X)$ a group action on X. If $\alpha_G(X) > n/(n+1)$, then X is G-equivariantly K-stable.*

Proof. The idea of the proof follows from [Fuj17]. Take any G-special test configuration $(\mathcal{X}, \mathcal{L})$. Let ord_F be the divisorial valuation on X induced by one of the irreducible components of $\mathcal{X}_0$. Then the orbit of F under the action G is induced by all irreducible components of $\mathcal{X}_0$ and hence finite. Let $\pi : Y \to X$ be a G-equivariant birational morphism such that $F_1 = F \ldots, F_N$ are prime divisors on Y and form the orbit of F under G-action. Set

$$S^G(F) = \frac{1}{(-K_X)^n} \int_0^{+\infty} \mathrm{vol}_Y\left(-\pi^* K_X - x \sum F_i\right) dx.$$

It suffices to show that $A_X(F) > S^G(F)$.

Let

$$\tau^G(F) = \sup\{t > 0 | \mathrm{vol}_Y\left(-\pi^* K_X - t \sum F_i\right) > 0\}.$$

Using integration by parts, we have

$$\int_0^{\tau^G(F)} \left(x - S^G(F)\right) \frac{d}{dx} \mathrm{vol}_Y\left(-\pi^* K_X - x \sum F_i\right) dx = 0.$$

Note that by Theorem A and Theorem B of [BFJ09], we have

$$-\frac{1}{n}\frac{d}{dx} \mathrm{vol}_Y\left(-\pi^* K_X - x \sum F_i\right) = N \,\mathrm{vol}_{Y|F}\left(-\pi^* K_X - x \sum F_i\right).$$

where $\mathrm{vol}_{Y|F}$ denotes the restricted volume (see [ELM+09] for definition). For simplicity, we use $V(x)$ to denote the restricted volume function $\mathrm{vol}_{Y|F}\left(-\pi^* K_X - x \sum F_i\right)$. Then we have

$$\int_0^{\tau^G(F)} \left(x - S^G(F)\right) V(x)\, dx = 0.$$

Using log concavity of restricted volume, we have

$$\left(x - S^G(F)\right) V(x) \leq \left(x - S^G(F)\right) V\left(S^G(F)\right) \left(\frac{x}{S^G(F)}\right)^{n-1}.$$

Therefore we get that

$$S^G(F) \leq \frac{n}{n+1} \tau^G(F).$$

Now suppose $A_X(F) \leq S^G(F)$. Then we know that

$$A_X(F) \leq \frac{n}{n+1} \tau^G(F).$$

For arbitrarily small $\epsilon > 0$, pick $0 \leq D \sim_{\mathbb{Q}} -K_X$ such that $G \cdot \mathrm{ord}_F(D) = \tau^G(F) - \epsilon$. Then we know that

$$\mathrm{Glct}(D) \leq \frac{A_X(F)}{G \cdot \mathrm{ord}_F(D)} \leq \frac{n}{n+1} \frac{\tau^G(F)}{\tau^G(F) - \epsilon},$$

Contradicting to the assumption that $\alpha_G(X) > n/(n+1)$.

□

Using G-log canonical threshold, we can also define the G-delta invariant of X to be

$$G\delta(X) := \limsup_m G\delta_m(X),$$

where

$$G\delta_m(X) := \inf\{\mathrm{Glct}(D) \mid D \text{ is of } m\text{-basis type.}\}.$$

For the definition of usual delta invariant, we refer to [BJ17] for details. We have the following theorem parallel to one of the results in [BJ17]:

Theorem 5.2. *Let X be a $\mathbb{Q}$-Fano variety with $G \subset \mathrm{Aut}(X)$ a group action on X. We have that $G\delta(X) = \lim_m G\delta_m(X)$. Let*

$$S^G(F) := \frac{1}{(-K_X)^n} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_F)\right) dx$$

Then we have

$$G\delta(X) = \inf_F \frac{A_X(F)}{S^G(F)}.$$

Proof. By definition, we have

$$\begin{aligned} G\delta_m(X) &= \inf_D \inf_F \frac{A_X(F)}{G \cdot \mathrm{ord}_F(D)} \\ &= \inf_F \inf_D \frac{A_X(F)}{G \cdot \mathrm{ord}_F(D)} \\ &= \inf_F \frac{A_X(F)}{\sup_D\{G \cdot \mathrm{ord}_F(D)\}}, \end{aligned}$$

where D runs through all m-basis type divisors and F runs through all prime divisors over X.

Let $V_m = H^0(X, -mK_X)$. For any prime divisor F over X, we construct the following filtration

$$\mathcal{F}^x V_m = H^0\left(X, \mathcal{O}_X(-mK_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_F)\right),\ x \geq 0.$$

In order to make $G \cdot \mathrm{ord}_F(D)$ as large as possible, we adapt a basis of V_m to $\mathcal{F}^\bullet V_m$ and get

$$\begin{aligned} \sup_D\{G \cdot \mathrm{ord}_F(D)\} &= \frac{1}{mN_m}\left(\sum_{i=0}^{mT_m-1} i\left(\dim \mathcal{F}^i V_m - \mathcal{F}^{i+1} V_m\right) + mT_m \mathcal{F}^{mT_m} V_m\right) \\ &= \frac{1}{mN_m} \sum_{i=0}^{mT_m} \dim \mathcal{F}^i V_m, \end{aligned}$$

where $N_m = \dim V_m$ and $T_m = \frac{1}{m}\max\{G \cdot \mathrm{ord}_F(s) | s \in V_m\}$. We follow the notation in [BJ17] and denote

$$S_m^G(F) = \frac{1}{mN_m} \sum_{i=0}^{mT_m} \dim \mathcal{F}^i V_m.$$

By Corollary 2.12 of [BJ17] we know that $\lim S_m^G(F) = S^G(F)$, and this finishes the proof. ☐

Theorem A and Theorem 5.2 immediately gives the following corollary:

Corollary 5.3. *Let X be a $\mathbb{Q}$-Fano variety with $G \subset \mathrm{Aut}(X)$ a finite group action on X. Then*

(1) *X is G-equivariantly K-semistable if and only if $G\delta(X) \geq 1$;*
(2) *X is uniformly G-equivariantly K-stable if and only if $G\delta(X) > 1$.*

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