LAURICELLA HYPERGEOMETRIC FUNCTIONS, UNIPOTENT FUNDAMENTAL GROUPS OF THE PUNCTURED RIEMANN SPHERE, AND THEIR MOTIVIC COACTIONS

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Abstract. We prove a recent conjecture arising in the context of scattering amplitudes for a 'motivic' Galois group action on Gauss' ${}_2F_1$ hypergeometric function. More generally, we show on the one hand how the coefficients in a Laurent expansion of a Lauricella hypergeometric function can be promoted, via the theory of motivic fundamental groups, to motivic multiple polylogarithms. The latter admit a 'local' action of the usual motivic Galois group. On the other hand, we define lifts of the full Lauricella functions as matrix coefficients in a Tannakian category of twisted cohomology, which inherit a 'global' action of its Tannaka group. We prove that these two actions are compatible. We also study single-valued versions of these hypergeometric functions, which may be of independent interest.

1. Introduction

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ be distinct points in $\mathbb{C}$, where $\sigma_0 = 0$. In this paper, we study the Lauricella hypergeometric functions with singularities in Σ which are defined by

$$(L_\Sigma)_{ij} \;=\; -s_j \int_0^{\sigma_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \, \frac{dx}{x - \sigma_j} \;, \qquad \text{for } 1 \leq i, j \leq n \;. \tag{1.1}$$

Using Tannakian theory, there are two possible ways in which one might try to define a 'motivic' Galois group acting on these functions, by viewing them in one of the following ways.

(G) Globally, as functions of the exponents $s_i \in \mathbb{C}$. For generic s_i, it is known [DM86] how to interpret (1.1) as periods of the cohomology of $X_\Sigma = \mathbb{A}^1 \backslash \Sigma$ with coefficients in a rank one algebraic vector bundle with integrable connection (or local system). This requires that the $s_i \notin \mathbb{Z}$ for each i and $s_0 + \ldots + s_n \notin \mathbb{Z}$.

(L) Locally, as formal power series in the s_i around the non-generic point $s_0 = \cdots = s_n = 0$. Even though the integral in (1.1) is divergent at that point, the prefactor s_j compensates the pole and (1.1) has a Taylor expansion in the s_i at the origin. Its coefficients are generalised polylogarithms, which can be lifted canonically to motivic periods admitting an action of the usual motivic Galois group.

The most familiar examples are Euler's beta function $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (see §1.2) and Gauss' hypergeometric function, which satisfies the integral formula

$$\beta(b, c-b)\, {}_2F_1(a,b,c;y) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-yx)^{-a} dx$$

whenever it converges. It can be written in the form (1.1) for $\Sigma = \{0, 1, y^{-1}\}$.

The impetus for this work came from a remarkable conjecture [ABD+19] arising in the study of dimensionally-regularised one loop Feynman amplitudes in $4 - 2\varepsilon$ spacetime dimensions, which can be expressed in terms of hypergeometric functions [ABDG17a, ABDG17b, ABD+18]. It was observed that the motivic coaction (L), computed order-by-order in an ε-expansion of ${}_2F_1(n_1 + a_1\varepsilon, n_2 + a_2\varepsilon, n_3 + a_3\varepsilon; y)$, where $n_1, \ldots, n_3, a_1, \ldots, a_3$ are integers, could, at least to low orders in ε, be succinctly packaged into a coaction formula on the hypergeometric function itself with only two terms. In this paper we give a rigorous sense to these statements, and derive, using a very simple Tannakian formalism for cohomology with coefficients, a 'global' coaction on Tannakian lifts of the

Lauricella functions. We then prove that this formula is indeed compatible with Laurent expansion in not just one but all of the s_i. This means the following: the coefficients in the expansion of (1.1) can be interpreted as periods of the mixed Tate motivic fundamental groupoid of the punctured Riemann sphere X_Σ with respect to suitable tangential basepoints, and therefore admit a 'local' coaction dual to the action of the motivic Galois group. Since the interpretations (G) and (L) are quite different, it is not *a priori* obvious that they should coincide.

A large part of this paper is also devoted to defining single-valued versions of the integrals L_Σ and showing that Laurent expansion in the s_i commutes with the single-valued period map. In particular we deduce a double copy formula for the single-valued hypergeometric integral which seems very closely related to constructions in conformal field theory [BPZ84, (E)], [KZ84, §4], [DF85].

1.1. **Contents.** The paper is in two parts, corresponding to the two points of view (G) and (L). In the first part, we interpret the Lauricella function as a matrix coefficient in a Tannakian category of Betti and de Rham realisations of cohomology with coefficients. To make this a little more precise, consider the trivial algebraic vector bundle of rank one on X_Σ with the integrable connection

$$\nabla_{\underline{s}} = d + \sum_{i=0}^{n} s_i \frac{dx}{x-\sigma_i} \ .$$

Let $\mathcal{L}_{\underline{s}}$ be the rank one local system generated by $x^{s_0}\prod_{k=1}^n (1-x\sigma_k^{-1})^{s_k}$, which is a flat section of $\nabla_{-\underline{s}} = \nabla_{\underline{s}}^\vee$. For generic $s_0,\ldots,s_n$, integration defines a canonical pairing between algebraic de Rham cohomology and locally finite homology

$$H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \qquad \text{and} \qquad H_1^{\mathrm{lf}}(X_\Sigma(\mathbb{C}), \mathcal{L}_{\underline{s}})$$

which are both of rank n. The period matrix, with respect to suitable bases, is exactly the $(n\times n)$ matrix (1.1). Its entries can be promoted to equivalence classes

$$(L_\Sigma^{\mathfrak{m}})_{ij} = \left[M_{\Sigma,\underline{s}}\ ,\ \delta_i \otimes x^{s_0}\prod_{k=1}^n (1-x\sigma_k^{-1})^{s_k}\ ,\ -s_j\, d\log(x-\sigma_j)\right]^{\mathfrak{m}}$$

of matrix coefficients, where δ_i is a path from 0 to σ_i, and $M_{\Sigma,\underline{s}}$ is an object of a Tannakian category encoding the data of the Betti and de Rham cohomology together with the integration pairing. They map to (1.1) under the period homomorphism:

$$\mathrm{per}\, L_\Sigma^{\mathfrak{m}} = L_\Sigma \ .$$

We also define de Rham versions of the above as equivalence classes of matrix coefficients as follows. Consider the logarithmic 1-forms:

$$\nu_i = \frac{dz}{z-\sigma_i} - \frac{dz}{z} \qquad \text{for } 1 \leq i \leq n \ ,$$

with residues at $\sigma_i, 0$ only. They are the image of the relative homology class (viewed in cohomology with trivial coefficients) of a path from 0 to σ_i under the map $c_0^\vee$ studied in [BD18]. They define de Rham cohomology classes in the space $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}})$, which is isomorphic, via the de Rham intersection pairing, to the dual $H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})^\vee$. Define

$$\left(L_\Sigma^{\mathfrak{dr}}\right)_{ij} = \left[M_{\Sigma,\underline{s}}\ ,\ [\nu_i],\ -s_j\, d\log(x-\sigma_j)\right]^{\mathfrak{m}} \ .$$

Comultiplication of matrix coefficients immediately implies a global coaction formula which takes the very simple matrix form:

$$\Delta\, L_\Sigma^{\mathfrak{m}} = L_\Sigma^{\mathfrak{m}} \otimes L_\Sigma^{\mathfrak{dr}} \ . \tag{1.2}$$

The de Rham Lauricella functions do not admit a period map on their own, but by passing to a slightly different category (which is equipped with a real Frobenius), and considering slight modifications $\widetilde{L}_\Sigma^{\mathfrak{dr}}$ of the de Rham Lauricella functions (namely, viewing $[\nu_i]$ in $H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{-\underline{s}})^\vee$ instead of $H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})^\vee$) one can define a 'single-valued period' homomorphism $\mathbf{s}$ out of the action of complex conjugation. This second categorical interpretation leads to a very similar coaction formula to (1.2).

Theorem 1.1. *The single-valued periods $L^{\mathbf{s}}_{\Sigma} = \mathbf{s}(\widetilde{L}^{\mathfrak{dr}}_{\Sigma}) \in M_{n\times n}(\mathbb{C})$ satisfy*

$$(L^{\mathbf{s}}_{\Sigma})_{ij} \;=\frac{s_j}{2\pi i}\int_{\mathbb{C}} |z|^{2s_0}\prod_{k=1}^{n}\left|1-z\sigma_k^{-1}\right|^{2s_k}\;\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_j}$$

whenever all the s_i are real, and satisfy suitable conditions for the integral to converge (see Proposition 2.8). Furthermore, there is a double-copy identity

$$L^{\mathbf{s}}_{\Sigma} = \left(L_{\overline{\Sigma}}(-s_0,\ldots,-s_n)\right)^{-1} L_{\Sigma}(s_0,\ldots,s_n) \tag{1.3}$$

where $L_{\Sigma}(s_1,\ldots,s_n)$ denotes the matrix (1.1) with dependence on the s_i made explicit.

The previous theorem yields a single-valued hypergeometric function which may be of independent interest. The double-copy identity (1.3) is reminiscent of other double-copy formulas from the physics literature such as the Kawai–Lewellen–Tye formula [KLT86], which was interpreted in the framework of cohomology with coefficients in [Miz17].

In §3 we explain how to renormalise the integrals (1.1), following a similar procedure to [BD18], to expose their poles in the $s_0,\ldots,s_n$ in a neighbourhood of the origin. These poles are compensated by the prefactors s_j in the definition (1.1), yielding Taylor expansions for both the functions $(L_{\Sigma})_{ij}$ (Proposition 3.2) and their single-valued versions $(L^{\mathbf{s}}_{\Sigma})_{ij}$ (Proposition 3.4).

In the second part of the paper, we compute the periods, single-valued periods and motivic coaction order-by-order in an expansion with respect to the s_i. For this, we can assume that the σ_i lie in a number field $k \subset \mathbb{C}$ and work in the category $\mathcal{MT}(k)$ of mixed Tate motives over k [DG05]. It has a canonical fiber functor ϖ. Alternatively, one can work in the category of mixed Tate motives over the moduli space $\mathcal{M}_{0,n+2}$ of $n+2$ marked points on a Riemann sphere, which leads to identical formulae. The motivic torsor of paths $\pi_1^{\mathrm{mot}}(X_{\Sigma}, t_0, -t_i)$ where t_0 is the tangent vector 1 at 0, and t_i is the tangent vector σ_i at σ_i, is a pro-object of $\mathcal{MT}(k)$. Since it is a torsor over the motivic fundamental group based at t_0, one can define its metabelian (or double-commutator) quotient. It turns out that the periods of the latter are very closely related to a generalised beta-integral of the form (1.1). Indeed, for any formal power series $F \in \mathbb{C}\langle\langle e_0,\ldots,e_n\rangle\rangle$ in non-commuting variables $e_0,\ldots,e_n$ we can consider its *abelianisation* and j^{th} *beta-quotient*

$$\overline{F} \quad \text{and} \quad \overline{F_j} \quad \in \quad \mathbb{C}[[s_0,\ldots,s_n]]$$

where $\overline{S}$ denotes the image of a formal power series S under the abelianisation map $e_i \mapsto s_i$, where the s_i are commuting variables, and

$$F = F_{\varnothing} + F_0 e_0 + \cdots + F_n e_n \ ,$$

where $F_{\varnothing} \in \mathbb{C}$ is simply the constant term of F. The motivic torsor of paths from t_0 to $-t_i$ defines formal power series $\mathcal{Z}^i$, $\mathcal{Z}^{\mathfrak{m},i}$, $\mathcal{Z}^{\varpi,i}$ in the $e_0,\ldots,e_n$ whose coefficients are periods, motivic periods, and canonical de Rham periods respectively, and whose abelianisations and beta-quotients are of interest. We assemble them into a matrix of formal power series:

$$(FL^{\mathfrak{m}}_{\Sigma})_{ij} = \delta_{ij}\overline{\mathcal{Z}^{\mathfrak{m},i}} - s_j\overline{\mathcal{Z}^{\mathfrak{m},i}_j}$$

for $1 \leq i,j \leq n$, (respectively with $\mathfrak{m}$ replaced with ϖ). The first theorem states that these are indeed 'local motivic lifts' of the expanded Lauricella functions (viewed as power series in the s_i).

Theorem 1.2. *For all $s_0,\ldots,s_n$ in a neighbourhood of the origin*

$$\begin{array}{rrcl} (i) & \mathrm{per}\,(FL^{\mathfrak{m}}_{\Sigma}) & = & L_{\Sigma} \\ (ii) & \mathbf{s}\,(FL^{\varpi}_{\Sigma}) & = & L^{\mathbf{s}}_{\Sigma}\ , \end{array}$$

where $\mathbf{s}$ is the single-valued period map (Corollary 5.6 and Theorem 6.4).

In the special case $\Sigma = \{0,1\}$, part (i) of the theorem reduces to Drinfeld's well-known computation of the metabelian quotient of the Drinfeld associator in terms of Euler's beta function.

Having therefore established that $L^{\mathfrak{m}}_{\Sigma}$ and $FL^{\mathfrak{m}}_{\Sigma}$ are appropriate lifts of the Lauricella functions (viewed as functions of s_i, and as power series, respectively), we then prove that the local and global coactions are compatible.

Theorem 1.3. *The motivic coaction on the formal power series $FL^{\mathfrak{m}}_{\Sigma}$ acts by*

$$\Delta FL^{\mathfrak{m}}_{\Sigma}(s_0,\ldots,s_n) = FL^{\mathfrak{m}}_{\Sigma}(\mathbb{L}^{\mathfrak{dr}}s_0,\ldots,\mathbb{L}^{\mathfrak{dr}}s_n)\otimes FL^{\varpi}_{\Sigma}(s_0,\ldots,s_n)\ , \tag{1.4}$$

and similarly with $\mathfrak{m}$ replaced with ϖ, where $\mathbb{L}^{\mathfrak{dr}}$ is the de Rham version of $2\pi i$.

Note that the formula (1.4) is directly comparable to (1.2) (for instance, one can rescale the variables s_i by $(\mathbb{L}^{\mathfrak{m}})^{-1}$ to absorb some of the $\mathbb{L}^{\mathfrak{dr}}$ factors in (1.4)). In conclusion, the periods, single-valued periods and coaction formula for the lifts $L^{\mathfrak{m}}_{\Sigma}, L^{\varpi}_{\Sigma}$ of the functions (1.1) commute with taking Laurent expansions in the s_i. There should be interesting possible generalisations of our results to the elliptic [Mat18] and ℓ-adic [Nak95, IKY87] settings.

1.2. **Example.** Let $n=1$, $\Sigma=\{0,1\}$, $k=\mathbb{Q}$ and $X_\Sigma=\mathbb{P}^1\backslash\{0,1,\infty\}$. Since $k=\mathbb{Q}$, the canonical fiber functor ϖ is simply the de Rham fiber functor.

1.2.1. *Cohomology with coefficients.* Let s_0,s_1 be generic, i.e., $\{s_0,s_1,s_0+s_1\}\cap\mathbb{Z}=\varnothing$.

The algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X;\nabla_{\underline{s}})$ has rank one over $\mathbb{Q}(s_0,s_1)$ and is spanned by the class of $s_1\frac{dx}{1-x}$. The locally finite homology $H^{\mathrm{lf}}_1(X(\mathbb{C});\mathcal{L}_{\underline{s}})$ also has rank one over $\mathbb{Q}(e^{2\pi i s_0},e^{2\pi i s_1})$, and is spanned by the class of $(0,1)\otimes x^{s_0}(1-x)^{s_1}$. The corresponding period matrix is the (1×1) matrix

$$L_{\{0,1\}}=\left(s_1\int_0^1 x^{s_0}(1-x)^{s_1}\frac{dx}{1-x}\right)=\left(\frac{s_0s_1}{s_0+s_1}\beta(s_0,s_1)\right)\ .$$

Note that $L_{\{0,1\}}$ is *a priori* only defined for generic s_0,s_1. It turns out *a posteriori* that it admits a Taylor expansion at the point $(s_0,s_1)=(0,0)$.

The lifted period matrix $L^{\mathfrak{m}}_{\{0,1\}}$ also has a single entry and satisfies

$$\Delta L^{\mathfrak{m}}_{\{0,1\}} = L^{\mathfrak{m}}_{\{0,1\}}\otimes L^{\mathfrak{dr}}_{\{0,1\}}\ . \tag{1.5}$$

This is immediate from the fact that the matrix has rank one.

1.2.2. *Formal series expansion.* Consider the Drinfeld associator

$$\mathcal{Z}=\sum_{w\in\{e_0,e_1\}^\times}\zeta(w)\,w = 1+\zeta(2)(e_0e_1-e_1e_0)+\ldots$$

where $\zeta(w)$ are shuffle regularised multiple zeta values. Its abelianisation satisfies $\overline{\mathcal{Z}}=1$. The (1×1) matrix of formal expansions of Lauricella functions $FL_{\{0,1\}}$ is therefore

$$FL_{\{0,1\}}=\left(1-s_1\overline{\mathcal{Z}_1}\right)\ .$$

Its entry is the formal power series

$$\left(1-s_1\int_{\mathrm{dch}}x^{s_0}(1-x)^{s_1}\frac{dx}{x-1}\right)=\left(\frac{s_0s_1}{s_0+s_1}\beta(s_0,s_1)\right)$$

where dch is the straight line path between tangential base points at 0 and 1, and the second equality follows from Proposition 6.1. It is well-known that

$$\frac{s_0s_1}{s_0+s_1}\beta(s_0,s_1)=\exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}\zeta(n)}{n}\left((s_0+s_1)^n-s_0^n-s_1^n\right)\right)\ . \tag{1.6}$$

The above objects have motivic and de Rham versions $\mathcal{Z}^{\mathfrak{m}},\mathcal{Z}^{\mathfrak{dr}},FL^{\mathfrak{m}}_{\Sigma}$, etc, formally denoted by adding superscripts in the appropriate places. For example, the entry of the matrix $FL^{\mathfrak{m}}_{\Sigma}$ is exactly the right-hand side of (1.6), in which $\zeta(n)$ is replaced by $\zeta^{\mathfrak{m}}(n)$. The coaction satisfies[1]

$$\Delta\, FL^{\mathfrak{m}}_{\{0,1\}}(s_0,s_1)=FL^{\mathfrak{m}}_{\{0,1\}}(\mathbb{L}^{\mathfrak{dr}}s_0,\mathbb{L}^{\mathfrak{dr}}s_1)\otimes FL^{\mathfrak{dr}}_{\{0,1\}}(s_0,s_1)\ .$$

[1] If one writes this in terms of the motivic beta function $\beta^{\mathfrak{m}}(s_0,s_1)$ defined by $(s_0^{-1}+s_1^{-1})$ times the entry of $FL^{\mathfrak{m}}_{\{0,1\}}$, then it takes the form

$$\Delta\beta^{\mathfrak{m}}(s_0,s_1)=\mathbb{L}^{\mathfrak{dr}}\frac{s_0s_1}{s_0+s_1}\beta^{\mathfrak{m}}(\mathbb{L}^{\mathfrak{dr}}s_0,\mathbb{L}^{\mathfrak{dr}}s_1)\otimes\beta^{\mathfrak{dr}}(s_0,s_1)$$

which is equivalent to the equations:

$$\Delta\,\zeta^{\mathfrak{m}}(n) = \zeta^{\mathfrak{m}}(n) \otimes (\mathbb{L}^{\mathfrak{dr}})^n + 1 \otimes \zeta^{\mathfrak{dr}}(n) \tag{1.7}$$

for all $n \geq 2$, using a variant of the well-known fact that in a complete Hopf algebra, an element is group-like if and only if it is the exponential of a primitive element. Since $\zeta^{\mathfrak{dr}}(2n) = 0$ for $n \geq 1$, we retrieve the known coaction formulae on motivic zeta values [Bro14].

Remark 1.4. Equation (1.7) is equivalent to the fact that the odd zeta values are periods of simple extensions in the category $\mathcal{MT}(\mathbb{Q})$. It is curious that this non-trivial statement shows up as the apparently simpler fact (1.5) that L_Σ has rank one.

1.2.3. *Single-valued versions.* For $s_0, s_1 \in \mathbb{R}$, the single-valued beta integral is

$$\frac{-s_1}{2\pi i} \int_{\mathbb{C}} |z|^{2s_0} |1-z|^{2s_1} \left(\frac{d\overline{z}}{\overline{z}-1} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \frac{dz}{1-z} \;\; = \;\; \frac{s_0 s_1}{s_0+s_1} \beta_{\mathbb{C}}(s_0, s_1) \ ,$$

where the 'complex' beta function $\beta_{\mathbb{C}}(s_0, s_1)$ satisfies the 'double-copy' formula

$$\frac{s_0 s_1}{s_0+s_1} \beta_{\mathbb{C}}(s_0, s_1) = -\frac{\beta(s_0, s_1)}{\beta(-s_0, -s_1)} \ .$$

Its expansion can be expressed in the form

$$1 - s_1 \mathbf{s}(\overline{\mathcal{Z}_1^{\mathfrak{dr}}}) = \exp\left(\sum_{n \geq 2} \frac{(-1)^{n-1} \zeta_{\mathrm{sv}}(n)}{n} \left((s_0+s_1)^n - s_0^n - s_1^n \right) \right) \ .$$

where the single-valued zeta $\zeta_{\mathrm{sv}}(n)$ equals $2\,\zeta(n)$ for n odd ≥ 3 and vanishes for even n.

This discussion of the beta function generalises to the case of the moduli spaces of cuves of genus zero. It has been studied in [BD18, VZ18, SS18].

1.3. **Comments.**

(1) This work was motivated by the desire to verify a conjecture in [ABD+19] which had been checked experimentally in low weights. However, our formula (1.2) is not immediately comparable, even after applying confluence relations, to the more specific version stated in [ABD+19] (which involves gamma and exponential factors) since, in our version, the left and right hand sides of the tensor product play different roles.

In any case, our results might be of possible interest insofar as they suggest a meaningful theory of motives associated to twisted cohomology. Since the hypergeometric case is one of the first tests of plausibility of such a theory, we wished to compute all the objects completely in this simple case where they can be compared with existing categories of mixed Tate motives.

(2) It is crucial that the Laurent expansion be taken at maximally non-generic values of the parameters s_i. It is clearly not the case that coaction on cohomology with coefficients should commute with Taylor expansions in general, since the types of motives and periods one obtains by expanding at different rational points in the s_i are completely unrelated to each other. For example, the coefficients of a Taylor expansion of $\beta(s_0, s_1)$ around non-integer values of (s_0, s_1) involve motivic periods which are not multiple zeta values, and whose coaction is quite different. Furthermore, the Laurent expansions at different values of s_i are independent from each other. Indeed, trying to compare expansions at different points quickly leads to infinite identities of the kind:

$$\sum_{n=2}^{\infty} (\zeta(n) - 1) = 1 \ ,$$

for which there is no motivic interpretation. Such an identity is incompatible with any possible action of the motivic Galois group.

(3) The main technical point in this paper is, as usual, dealing with divergences. For cohomology with coefficients, this appears as non-genericity of the parameters s_i. For motivic fundamental groups, it takes the form of tangential base points. The following key example illustrates the point:

Example 1.5. Suppose that $\mathrm{Re}(s) > 0$. Then, viewed as a function of s,

$$I(s) = \int_0^1 x^s \frac{dx}{x} = \frac{1}{s} \ .$$

The renormalised version $I^{\mathrm{ren}}(s)$ of this integral (defined in §3) removes the pole in s, hence $I^{\mathrm{ren}}(s) = 0$. Now consider the integral as a formal power series in s. We perform a Taylor expansion of the integrand and integrate term by term. Since the integrals diverge, they are regularised with respect to a tangent vector of length 1 at the origin, which is equivalent to integrating along the straight line path dch:

$$I^{\mathrm{formal}}(s) = \sum_{n\geq 0} \frac{s^n}{n!} \int_{\mathrm{dch}} \log^n(x) \frac{dx}{x} = 0 \ .$$

Thus $I^{\mathrm{formal}}(s)$ is indeed the Taylor expansion of $I^{\mathrm{ren}}(s)$. In general, our tangential base-points are chosen to be consistent with the renormalisation of divergent integrals.

Throughout this paper we use the following convention: real integrals on $\mathbb{C}$ use the coordinate x, whereas single-valued integrals use the complex coordinate z.

1.3.1. *Higher dimensional generalisations.* There are precursors in the physics literature to coaction formulae on generating series of motivic periods. Indeed, in [SS13] open string amplitudes in genus 0 (which can be computed in terms of associators [BSST14]) were recast in terms of series of motivic multiple zeta values, and some conjectures were formulated about their f-alphabet decomposition to all orders. For four particles, this is equivalent to Example 1.2.

We can make these conjectures precise as a simple application of the framework described here. Let $\mathcal{M}_{0,S}$ denote the moduli space of curves with points marked by a set S with $n+3$ elements, and let $\nabla_{\underline{s}}$, $\mathcal{L}_{\underline{s}}$ the Koba–Nielsen connection and local system considered in [BD18]. The periods and single-valued periods studied in that paper can easily be formalised using the Tannakian categories defined in this paper: in brief, the triple

$$M^S_{\underline{s}} \ = \ \big(H^n_{\mathrm{B}}(\mathcal{M}_{0,S}, \mathcal{L}_{\underline{s}}) \ , \ H^n_{\mathrm{dR}}(\mathcal{M}_{0,S}, \nabla_{\underline{s}}) \ , \ \mathrm{comp}_{\mathrm{B,dR}}\big)$$

defines an object of $\mathcal{T}$ (and has an obvious variant in the category $\mathcal{T}^\infty$). From this one can define 'global' Tannakian lifts of the closed and open superstring amplitudes in genus 0. 'Local' lifts (after expanding in the variables s_{ij}) were worked out in [BD18]. We can define global matrix coefficients

$$I^{\mathfrak{m}}_{\mathcal{T}}(\omega) = [M^S_{\underline{s}}, \gamma, \omega]^{\mathfrak{m}} \qquad \text{and} \qquad I^{\mathfrak{dr}}_{\mathcal{T}}(\nu, \omega) = [M^S_{\underline{s}}, \nu, \omega]^{\mathfrak{dr}}$$

for suitable Betti homology classes γ and de Rham (resp. dual de Rham) classes ω (resp. ν). The period of $I^{\mathfrak{m}}_{\mathcal{T}}(\omega)$ is an open string amplitude, and the single-valued period of (a slight variant of) $I^{\mathfrak{dr}}_{\mathcal{T}}(\nu, \omega)$ is a closed string amplitude. The general coaction formalism (2.17) or [Bro17] yields

$$\Delta I^{\mathfrak{m}}_{\mathcal{T}}(\omega) = \sum_{\eta} I^{\mathfrak{m}}_{\mathcal{T}}(\eta) \otimes I^{\mathfrak{dr}}_{\mathcal{T}}(\eta^\vee, \omega)$$

where η ranges over a basis of $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S}, \nabla_{\underline{s}})$ and $\eta^\vee$ is the dual basis. The objects $I^{\mathfrak{m}}_{\mathcal{T}}(\omega)$ can be viewed as versions of open string amplitudes, the objects $I^{\mathfrak{dr}}_{\mathcal{T}}(\eta^\vee, \omega)$ as versions of closed string amplitudes. In terms of the de Rham intersection pairing, this can equivalently be written

$$\Delta I^{\mathfrak{m}}_{\mathcal{T}}(\omega) = \sum_{\eta, \eta'} \langle \eta, \eta' \rangle^{\mathrm{dR}} \, I^{\mathfrak{m}}_{\mathcal{T}}(\eta) \otimes I^{\mathfrak{dr}}_{\mathcal{T}}(\eta', \omega)$$

where η, η' range over bases of $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S}, \nabla_{\underline{s}})$ and $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S}, \nabla_{-\underline{s}})$ respectively. We proved in [BD18] that the Laurent expansions of open and closed string amplitudes admit (non-canonical) motivic lifts. In the light of the present paper, it is natural to hope that their coactions are compatible with the global formula written above. It would be interesting to see if this is equivalent to the conjectures of Stieberger and Schlotterer mentioned above.

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2. Cohomology with coefficients of a punctured Riemann sphere

We first recall the interpretation of the integrals (1.1) as periods of the cohomology of the punctured Riemann sphere with coefficients in a rank one algebraic vector bundle or local system. Using some simple Tannakian formalism, this enables us to derive a global coaction formula on Tannakian lifts of these periods. We also consider single-valued versions of these integrals.

2.1. **Periods of cohomology with coefficients.**

Let $k \subset \mathbb{C}$ and let $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$ be distinct points in k with $\sigma_0 = 0$.

Write

$$X_\Sigma = \mathbb{A}^1_k \backslash \Sigma \ .$$

We consider a tuple $\underline{s} = (s_0, \ldots, s_n)$ of complex numbers that we shall often assume to be *generic*, meaning that we have:

$$\{s_0, s_1, \ldots, s_n, s_0 + s_1 + \cdots + s_n\} \cap \mathbb{Z} = \varnothing \ . \tag{2.1}$$

An alternative point of view, that we will not develop here, would be to treat the s_i's as formal variables (see §2.4.5).

2.1.1. *Algebraic de Rham cohomology.* Denote two subfields of $\mathbb{C}$ by

$$\mathbb{Q}^{\mathrm{dR}}_{\underline{s}} = \mathbb{Q}(s_0, \ldots, s_n) \qquad \text{and} \qquad k^{\mathrm{dR}}_{\underline{s}} = k(s_0, \ldots, s_n) \ .$$

The algebraic de Rham cohomology groups that we will consider are $k^{\mathrm{dR}}_{\underline{s}}$-vector spaces with a natural $\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$-structure. Define the following logarithmic 1-forms on $\mathbb{P}^1_k$

$$\omega_i = \frac{dz}{z - \sigma_i} \qquad \text{for} \quad i = 0, \ldots, n \ , \tag{2.2}$$

which have residue 0 or 1 at points of Σ and -1 at ∞. They form a basis of the space of global logarithmic forms $\Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k}(\log \Sigma \cup \{\infty\}))$, which maps isomorphically to $H^1_{\mathrm{dR}}(X_\Sigma/k)$.

Definition 2.1. Let $\mathcal{O}_{X_\Sigma}$ denote the trivial rank one bundle on $X_\Sigma \times_k k^{\mathrm{dR}}_{\underline{s}}$, and consider the following logarithmic connection upon it

$$\nabla_{\underline{s}} : \mathcal{O}_{X_\Sigma} \longrightarrow \Omega^1_{X_\Sigma} \qquad \text{given by} \qquad \nabla_{\underline{s}} = d + \sum_{i=0}^n s_i \, \omega_i \ .$$

It is automatically integrable since X_Σ has dimension 1, and is in fact the abelianisation of the canonical connection on the de Rham unipotent fundamental group.

Consider the algebraic de Rham cohomology groups

$$H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) = H^r_{\mathrm{dR}}(X_\Sigma, (\mathcal{O}_{X_\Sigma}, \nabla_{\underline{s}}))$$

which are finite-dimensional $k^{\mathrm{dR}}_{\underline{s}}$-vector spaces. The fact that the s_i are generic implies, by [Del70, Proposition II.3.13], that one has a logarithmic comparison theorem for $(\mathcal{O}_{X_\Sigma}, \nabla_{\underline{s}})$. Since the cohomology of X_Σ is spanned by global logarithmic forms, this implies (see [ESV92]) that the cohomology groups $H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ are computed by the complex of global logarithmic forms

$$0 \longrightarrow k^{\mathrm{dR}}_{\underline{s}} \xrightarrow{\nabla_{\underline{s}}} k^{\mathrm{dR}}_{\underline{s}}\,\omega_0 \oplus \cdots \oplus k^{\mathrm{dR}}_{\underline{s}}\,\omega_n \longrightarrow 0\ ,$$

where $\nabla_{\underline{s}}(1) = \sum_{i=0}^n s_i\omega_i$.

Again by genericity of the s_i, $H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ vanishes for $r \neq 1$, and $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ has dimension n. It is generated by the ω_i subject to the single relation

$$\sum_{i=0}^n s_i\,\omega_i = 0\ . \tag{2.3}$$

Since the forms ω_i have rational residues, they in fact define a natural $\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$-structure on $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ which we shall denote by $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. We therefore have

$$H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \cong \bigoplus_{i=1}^n \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}[\omega_i]\ . \tag{2.4}$$

We shall use the following basis for (2.4):

$$\{-s_i[\omega_i],\ \text{for } i = 1, \ldots, n\}\ . \tag{2.5}$$

2.1.2. *Betti (co)homology.* We introduce the subfield of $\mathbb{C}$ defined by

$$\mathbb{Q}^{\mathrm{B}}_{\underline{s}} = \mathbb{Q}(e^{2\pi i s_0}, \ldots, e^{2\pi i s_n})\ .$$

Definition 2.2. Let $\mathcal{L}_{\underline{s}}$ denote the rank one local system of $\mathbb{Q}^{\mathrm{B}}_{\underline{s}}$-vector spaces on the complex points $X_\Sigma(\mathbb{C}) = \mathbb{C}\backslash\Sigma$ defined by

$$\mathcal{L}_{\underline{s}} = \mathbb{Q}^{\mathrm{B}}_{\underline{s}}\ z^{-s_0}\prod_{k=1}^n (1 - z\sigma_k^{-1})^{-s_k}\ .$$

The local system $\mathcal{L}_{\underline{s}}$ has monodromy $e^{-2\pi i s_j}$ around the point σ_j. After extending scalars to $\mathbb{C}$, it is identified with the horizontal sections of the (analytified) connexion $\nabla_{\underline{s}}$ on the trivial vector bundle of rank one on $X_\Sigma(\mathbb{C})$:

$$\mathcal{L}_{\underline{s}} \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C} \cong \left(\mathcal{O}^{\mathrm{an}}_{X_\Sigma}\right)^{\nabla_{\underline{s}}}\ .$$

We will be interested in its cohomology

$$H^i_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) := H^i(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}})\ \cong\ H_i(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}})^\vee\ ,$$

where $\mathcal{L}^\vee_{\underline{s}}$ is the dual local system

$$\mathcal{L}^\vee_{\underline{s}} = \mathbb{Q}^{\mathrm{B}}_{\underline{s}}\ z^{s_0}\prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k}\ .$$

The genericity assumption (2.1) implies that $\mathcal{L}_{\underline{s}}$ and $\mathcal{L}^\vee_{\underline{s}}$ have non-trivial monodromy around every point of Σ and around ∞, which implies that the natural map

$$H_i(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}}) \longrightarrow H^{\mathrm{lf}}_i(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}})$$

from ordinary homology to locally finite homology is an isomorphism. Its inverse is sometimes called regularisation. An easy computation shows that all homology is concentrated in degree one and $H^{\mathrm{lf}}_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}})$ has rank n. It has a basis consisting of the locally finite chains

$$\delta_i \otimes z^{s_0}\prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k} \tag{2.6}$$

for $i = 1, \ldots, n$, where $\delta_i : (0,1) \to \mathbb{C}\backslash\Sigma$ is a continuous path from 0 to σ_i and $z^{s_0}\prod_{k=1}^n(1 - z\sigma_k^{-1})^{s_k}$ denotes some choice of section of $\mathcal{L}^\vee_{\underline{s}}$ on δ_i. If 0, σ_i, σ_j are not collinear for all i, j, we can simply take δ_i to be the open straight line segment from 0 to σ_i.

2.1.3. *Comparison isomorphism.* There is a canonical isomorphism [Del70, §6]

$$\mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s}) : H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C} \ . \tag{2.7}$$

whose restriction to the $\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$-structure we shall denote by

$$\mathrm{comp}_{\mathrm{B},\varpi}(\underline{s}) : H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C} \ . \tag{2.8}$$

Assuming (2.1), we can identify Betti cohomology $H^i_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}})$ with the dual of locally finite homology, which leads to a bilinear pairing

$$H_1^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}}) \times H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow \mathbb{C} \ .$$

It is well-known that the comparison isomorphism is computed by integration when it makes sense.

Lemma 2.3. *Assuming (2.1), a matrix representative for the comparison isomorphism in the bases (2.5), (2.6) is the $(n\times n)$ period matrix L_Σ with entries*

$$(L_\Sigma)_{ij} = s_j \int_{\delta_i} z^{s_0} \prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k} \frac{dz}{\sigma_j - z} \ ,$$

provided $\mathrm{Re}\, s_0 > -1$ *and* $\mathrm{Re}\, s_i > \begin{cases} -1 & \text{if } i \neq j \ ; \\ 0 & \text{if } i = j \ . \end{cases}$

Proof. One sees that the integrals converge under the assumptions on the real parts of the s_k's. By definition, the pairing that we wish to compute is

$$\langle [\nu_i], \mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s})[\omega_j]\rangle = \int_{\delta_i} z^{s_0} \prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k} \widetilde{\omega}_j \ ,$$

where $\widetilde{\omega}_j$ is a smooth form on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles along $\Sigma \cup \infty$ and compact support on $\mathbb{C}\setminus\Sigma$, representing the cohomology class of ω_j. In other words we have

$$\widetilde{\omega}_j - \omega_j = \nabla_{\underline{s}}\phi = d\phi + \sum_{k=0}^n s_k\, \phi\, d\log(z - \sigma_k) \ ,$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. Since $\widetilde{\omega}_j$ vanishes in the neighbourhood of $\Sigma\cup\infty$, taking residues along points of $\Sigma\cup\infty$ implies that ϕ vanishes at every σ_k, $k\neq i$, including at $\sigma_0 = 0$. Now we only need to prove that the integral

$$\int_{\delta_i} z^{s_0} \prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k}\, \nabla_{\underline{s}}\phi$$

vanishes. Since $z^{s_0}\prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k}\, \nabla_{\underline{s}}\phi = d(z^{s_0}\prod_{k=1}^n(1 - z\sigma_k^{-1})^{s_k}\,\phi)$, this integral is computed by Stokes' theorem. The vanishing properties of ϕ and the assumptions on the s_k's implies that the contributions at 0 and σ_i vanish, and the result follows. □

2.2. **Intersection pairings.** For generic s_i (2.1), the natural map $H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}}) \to H_1^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}})$ is an isomorphism. Poincaré duality gives an isomorphism

$$H_1^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}}) \simeq H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}})^\vee \simeq H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{-\underline{s}})^\vee \ ,$$

where we set $-\underline{s} = (-s_0, \ldots, -s_n)$. By combining these two isomorphisms we get a perfect pairing, called the *Betti intersection pairing* [KY94, §2], [CM95], [MY03, §2]:

$$\langle\ ,\ \rangle_{\mathrm{B}} : H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{-\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{\underline{s}}) \longrightarrow \mathbb{Q}^{\mathrm{B}}_{\underline{s}} \ ,$$

or dually in cohomology:

$$\langle\ ,\ \rangle^{\mathrm{B}} : H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{-\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \longrightarrow \mathbb{Q}^{\mathrm{B}}_{\underline{s}} \ . \tag{2.9}$$

The de Rham counterpart is the *de Rham intersection pairing* [CM95, Mat98]:

$$\langle\ ,\ \rangle^{\mathrm{dR}} : H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow k^{\mathrm{dR}}_{\underline{s}} \ , \tag{2.10}$$

which comes from Poincaré duality and the fact that the natural map

$$H^1_{\mathrm{dR,c}}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \tag{2.11}$$

is an isomorphism if the s_i are generic, where the subscript c denotes compactly supported cohomology. It respects the natural $\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$-structure and induces

$$\langle\ ,\ \rangle^{\varpi} : H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}} H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}\ .$$

The pairings (2.9), (2.10), (2.11) are compatible with the comparison isomorphisms (2.7), (2.8). This implies what are known as the *twisted period relations* [KY94, CM95].

Write

$$\nu_i = \frac{dz}{z-\sigma_i} - \frac{dz}{z} \qquad \text{for} \quad 1 \leq i \leq n \tag{2.12}$$

and view $[\nu_i]$ as a class in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$.

Lemma 2.4. *For all $1 \leq i, j \leq n$, and $s_0, \ldots, s_n$ satisfying (2.1),*

$$\langle [\nu_i]\,, [\omega_j] \rangle^{\mathrm{dR}} = -\frac{1}{s_i}\,\delta_{ij}\ . \tag{2.13}$$

Proof. By definition, the de Rham intersection pairing that we wish to compute is

$$\langle [\nu_i], [\omega_j] \rangle^{\mathrm{dR}} = \frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} \widetilde{\nu}_i \wedge \omega_j\ ,$$

where $\widetilde{\nu}_i$ is a smooth form on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles along $\Sigma \cup \infty$ and compact support on $\mathbb{C}\backslash\Sigma$, representing the cohomology class of ν_i. In other words we have

$$\widetilde{\nu}_i - \nu_i = \nabla_{-\underline{s}}\phi = d\phi - \sum_{k=0}^{n} s_k\, \phi\, d\log(z-\sigma_k)\ ,$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. Since $\widetilde{\nu}_i$ vanishes in the neighbourhood of $\Sigma \cup \infty$, taking residues along points of $\Sigma \cup \infty$ implies that ϕ vanishes at every σ_k, $k \notin \{0, i\}$, and at ∞, and

$$\phi(0) = -\frac{1}{s_0}\ ,\ \phi(\sigma_i) = \frac{1}{s_i}\ .$$

By noticing that $\nu_i \wedge \omega_j = d\log(z-\sigma_k) \wedge \omega_j = 0$, we thus get

$$\langle [\nu_i], [\omega_j] \rangle^{\mathrm{dR}} = \frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} d\phi \wedge \omega_j = \frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} d(\phi\, \omega_j)\ .$$

By Stokes, this last integral can be computed as the limit when ε goes to zero of

$$-\frac{1}{2\pi i} \int_{\partial P_\varepsilon} \phi\, \omega_j$$

where P_ε is the complement in $\mathbb{P}^1(\mathbb{C})$ of ε-disks around the points of $\Sigma \cup \infty$, and the sign comes from the orientation of ∂P_ε. By using the fact that $\phi(\infty) = 0$ and ω_j is regular at every σ_k, $k \neq j$, a local computation (variant of Cauchy's formula) thus gives

$$\langle [\nu_i], [\omega_j] \rangle^{\mathrm{dR}} = -\mathrm{Res}_{\sigma_j}(\phi\, \omega_j) = -\frac{1}{s_i}\delta_{ij}\ .$$

□

This lemma implies that the dual basis to (2.5) is given by the classes of the forms

$$[\nu_i] \quad \in \quad H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \qquad \text{for } i = 1, \ldots, n\ .$$

2.3. **Single-valued periods of cohomology with coefficients.** We can define and compute a period pairing on de Rham cohomology classes by transporting complex conjugation.

2.3.1. *Definition of the single-valued period map.* Let $\overline{k} \subset \mathbb{C}$ denote the complex conjugate of the image of k in $\mathbb{C}$ and let $\overline{\Sigma} = \{\overline{\sigma_0}, \overline{\sigma_1}, \ldots, \overline{\sigma_n}\}$ denote the complex conjugates of the points in Σ. We use the notation $\mathcal{L}_{\underline{s}}$ to denote the rank one local system of $\mathbb{Q}^{\mathrm{B}}_{\underline{s}}$-vector spaces on $\mathbb{C}\backslash\overline{\Sigma}$ with monodromy $e^{-2\pi i s_j}$ around $\overline{\sigma_j}$. We have an isomorphism of complex manifolds

$$\mathrm{conj} : \mathbb{C}\backslash\overline{\Sigma} \longrightarrow \mathbb{C}\backslash\Sigma$$

given by complex conjugation. We note that the induced map $H_1(\mathbb{C}\backslash\overline{\Sigma}) \to H_1(\mathbb{C}\backslash\Sigma)$ sends the class of a positively oriented loop around $\overline{\sigma_j}$ to the class of a negatively oriented loop around σ_j. Since a rank one local system on $\mathbb{C}\backslash\Sigma$ (resp. $\mathbb{C}\backslash\overline{\Sigma}$) is equivalent to a representation of the abelian group $H_1(\mathbb{C}\backslash\Sigma)$ (resp. $H_1(\mathbb{C}\backslash\overline{\Sigma})$) respectively, we see that we have an isomorphism of local systems:

$$\mathrm{conj}^*\mathcal{L}_{\underline{s}} \simeq \mathcal{L}_{-\underline{s}} \ .$$

We thus get a morphism of local systems on $\mathbb{C}\backslash\Sigma$:

$$\mathcal{L}_{\underline{s}} \longrightarrow \mathrm{conj}_*\mathrm{conj}^*\mathcal{L}_{\underline{s}} \simeq \mathrm{conj}_*\mathcal{L}_{-\underline{s}} \ ,$$

which at the level of cohomology induces a morphism of $\mathbb{Q}^{\mathrm{B}}_{\underline{s}}$-vector spaces

$$F_\infty : H^1_{\mathrm{B}}(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}) \longrightarrow H^1_{\mathrm{B}}(\mathbb{C}\backslash\overline{\Sigma}, \mathcal{L}_{-\underline{s}}) \ .$$

We call F_∞ the *real Frobenius* or *Frobenius at the infinite prime*. We will use the notation $F_\infty(\underline{s})$ when we want to make dependence on $\underline{s}$ explicit. One checks that the Frobenius is involutive: $F_\infty(-\underline{s})F_\infty(\underline{s}) = \mathrm{id}$.

Remark 2.5. A morphism similar to F_∞ was considered in [HY99] and leads to similar formulae but has a different definition. Our definition only uses the action of complex conjugation on the complex points of the variety X_Σ relative to two complex embeddings of k, whereas the definition in [*loc. cit.*] conjugates the field of coefficients of the local systems. Note that our definition does not require the s_i to be real.

In the rest of this section, however, we will often assume that the s_i are real. (This is an unnatural assumption and would not be necessary if the s_i were treated as formal variables, see §2.4.5.) In this way, the complex conjugate of the field $k^{\mathrm{dR}}_{\underline{s}}$ inside $\mathbb{C}$ is the field $\overline{k}(s_1, \ldots, s_n)$. We use the notation $(-) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}}$ for the tensor product with $\mathbb{C}$ viewed as a $k^{\mathrm{dR}}_{\underline{s}}$-vector space via this complex conjugate embedding. We thus have an additional $\mathbb{C}$-linear comparison isomorphism:

$$\mathrm{comp}_{\overline{\mathrm{B}},\mathrm{dR}}(\underline{s}) : H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}} \longrightarrow H^1_{\mathrm{B}}(\mathbb{C} \setminus \overline{\Sigma}, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C} \ .$$

Definition 2.6. Assume that the s_i are real. The *single-valued period map* is the $\mathbb{C}$-linear isomorphism

$$\mathbf{s} : H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \mathbb{C} \longrightarrow H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}}$$

defined as the composite

$$\mathbf{s} = \mathrm{comp}^{-1}_{\overline{\mathrm{B}},\mathrm{dR}}(-\underline{s}) \circ (F_\infty \otimes \mathrm{id}) \circ \mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s}) \ .$$

In other words, it is defined by the commutative diagram

$$\begin{tikzcd}
H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \mathbb{C} \arrow[r, "\mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s})"] \arrow[d, "\mathbf{s}"'] & H^1_{\mathrm{B}}(\mathbb{C} \setminus \Sigma, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C} \arrow[d, "F_\infty \otimes \mathrm{id}"] \\
H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}} \arrow[r, "\mathrm{comp}_{\overline{\mathrm{B}},\mathrm{dR}}(-\underline{s})"'] & H^1_{\mathrm{B}}(\mathbb{C} \setminus \overline{\Sigma}, \mathcal{L}_{-\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{B}}_{\underline{s}}} \mathbb{C}
\end{tikzcd}$$

We will use the notation $\mathbf{s}(\underline{s})$ when we want to make dependence on $\underline{s}$ explicit.

The single-valued period map is a transcendental comparison isomorphism that is naturally interpreted at the level of analytic de Rham cohomology via the isomorphisms

$$H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \mathbb{C} \simeq H^1_{\mathrm{dR,an}}(\mathbb{C}\backslash\Sigma, (\mathcal{O}_{\mathbb{C}\backslash\Sigma}, \nabla_{\underline{s}}))$$

and

$$H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}} \simeq H^1_{\mathrm{dR,an}}(\mathbb{C}\backslash\overline{\Sigma}, (\mathcal{O}_{\mathbb{C}\backslash\overline{\Sigma}}, \nabla_{-\underline{s}})) \ .$$

To avoid any confusion we use the coordinate $w = \overline{z}$ on $\mathbb{C}\backslash\overline{\Sigma}$.

Lemma 2.7. *Assume that the s_i are real. In analytic de Rham cohomology, the single-valued period map is induced by the morphism of smooth de Rham complexes*

$$\mathbf{s}_{\mathrm{an}} : (\mathcal{A}^\bullet_{\mathbb{C}\backslash\Sigma}, \nabla_{\underline{s}}) \quad \longrightarrow \quad \mathrm{conj}_*(\mathcal{A}^\bullet_{\mathbb{C}\backslash\overline{\Sigma}}, \nabla_{-\underline{s}})$$

given on the level of sections by

$$\mathcal{A}^\bullet_{\mathbb{C}\backslash\Sigma}(U) \ni \omega \quad \mapsto \quad |w|^{2s_0} \prod_{k=1}^n |1 - w\overline{\sigma}_k^{-1}|^{2s_k} \, \mathrm{conj}^*(\omega) \in \mathcal{A}^\bullet_{\mathbb{C}\backslash\overline{\Sigma}}(\overline{U}) \ .$$

Proof. Let $P = |w|^{2s_0} \prod_{k=1}^n |1 - w\overline{\sigma}_k^{-1}|^{2s_k}$. We first check that $\mathbf{s}_{\mathrm{an}}$ is a morphism of complexes:

$$\begin{aligned}
\nabla_{-\underline{s}}(\mathbf{s}_{\mathrm{an}}(\omega)) &= \nabla_{-\underline{s}}(P\,\mathrm{conj}^*(\omega)) \\
&= P\left(\left(\sum_{k=0}^n s_k\, d\log(w - \overline{\sigma_k}) + \sum_{k=0}^n s_k\, d\log(\overline{w} - \sigma_k)\right) \wedge \mathrm{conj}^*(\omega) + d(\mathrm{conj}^*(\omega))\right) \\
&\qquad - \sum_{k=0}^n s_k\, d\log(w - \overline{\sigma_k}) \wedge (P\,\mathrm{conj}^*(\omega)) \\
&= P\left(\sum_{k=0}^n s_k\, d\log(\overline{w} - \sigma_k) \wedge \mathrm{conj}^*(\omega) + d(\mathrm{conj}^*(\omega))\right) \\
&= P\,\mathrm{conj}^*\left(\sum_{k=0}^n s_k\, d\log(z - \sigma_k) \wedge \omega + d\omega\right) \\
&= \mathbf{s}_{\mathrm{an}}(\nabla_{\underline{s}}(\omega)) \ .
\end{aligned}$$

On the level of horizontal sections, we compute:

$$\begin{aligned}
\mathbf{s}_{\mathrm{an}}\left(z^{-s_0} \prod_{k=1}^n (1 - z\sigma_k^{-1})^{-s_k}\right) &= |w|^{2s_0} \prod_{k=1}^n |1 - w\overline{\sigma}_k^{-1}|^{2s_k}\, \overline{w}^{-s_0} \prod_{k=1}^n (1 - \overline{w}\sigma_k^{-1})^{-s_k} \\
&= w^{s_0} \prod_{k=1}^n (1 - w\overline{\sigma_k}^{-1})^{s_k} \ .
\end{aligned}$$

Thus, $\mathbf{s}_{\mathrm{an}}$ induces the morphism $\mathcal{L}_{\underline{s}} \to \mathrm{conj}_* \mathcal{L}_{-\underline{s}}$ and the result follows. $\square$

2.3.2. *Integral formula for single-valued periods.* One can derive a formula for the single-valued period map $\mathbf{s}$ using the de Rham intersection pairing (2.10).

Proposition 2.8. *Suppose that $\omega, \nu \in \Gamma(X_\Sigma, \Omega^1_{\mathbb{P}^1}(\log \Sigma \cup \infty))$ and $s_0, \ldots, s_n$ are real and generic (2.1). Then $[\omega] \in H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ and the form ν defines a class*

$$[\nu] \quad \in \quad H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \ \cong \ H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}})^\vee$$

via the de Rham duality pairing (2.10). Suppose furthermore that ν has no pole at ∞. Then the single-valued pairing is

$$\langle [\nu], \mathbf{s}[\omega] \rangle = -\frac{1}{2\pi i} \int_{\mathbb{C}} |z|^{2s_0} \prod_{k=1}^n |1 - z\sigma_k^{-1}|^{2s_k}\, \overline{\nu} \wedge \omega \tag{2.14}$$

whenever $\mathrm{Re}(s_i) > 0$ for all $0 \leq i \leq n$, and $2\,\mathrm{Re}(s_0) + \cdots + 2\,\mathrm{Re}(s_n) < 1$.

Proof. Let us first note that the integral in (2.14) converges under the assumptions on ω, ν, and the s_i. To check this pass to local polar coordinates $z = \rho e^{i\theta}$ in the neighbourhood of every point in Σ, and verify that $|z|^{2s} \frac{dz\, d\overline{z}}{z\overline{z}}$ is proportional to $\rho^{2s-1} d\rho\, d\theta$ and is integrable when $\mathrm{Re}(s) > 0$. At ∞ use the fact that ν has no pole obtain a local estimate of the form $\rho^{-2s_0 - \cdots - 2s_n} d\rho\, d\theta$, which is integrable for $2\,\mathrm{Re}(s_0) + \cdots + 2\,\mathrm{Re}(s_n) < 1$.

We use the notation $\omega = \omega_\sigma$ for the smooth form on $\mathbb{C} \setminus \Sigma$ induced by ω, and $\omega_{\overline{\sigma}}$ for the smooth form on $\mathbb{C} \setminus \overline{\Sigma}$ induced by ω. (It is obtained by replacing each occurrence of σ_j in ω by a $\overline{\sigma_j}$.) Then by definition we have $\langle [\nu], \mathbf{s}[\omega]\rangle = \langle [\nu_{\overline{\sigma}}], \mathbf{s}_{\mathrm{an}}[\omega_\sigma]\rangle$. By Lemma 2.7 and the definition of the de Rham intersection pairing this equals the integral

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0} \prod_{k=1}^{n} |1 - w\overline{\sigma_k}^{-1}|^{2s_k}\, \widetilde{\nu_{\overline{\sigma}}} \wedge \mathrm{conj}^*(\omega_\sigma)\ , \tag{2.15}$$

where $\widetilde{\nu_{\overline{\sigma}}}$ is a smooth form on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles along $\Sigma \cup \infty$ and compact support on $\mathbb{C} \setminus \overline{\Sigma}$, representing the cohomology class of $\widetilde{\nu_{\overline{\sigma}}}$. In other words we have

$$\widetilde{\nu_{\overline{\sigma}}} - \nu_{\overline{\sigma}} = \nabla_{\underline{s}}\phi = d\phi - \sum_{k=0}^{n} s_k\, \phi\, d\log(w - \overline{\sigma_k})\ ,$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. The assumption that ν has no pole at ∞ and the fact that $s_0 + \cdots + s_n \neq 0$ imply, by taking residues, that $\phi(\infty) = 0$. We first prove that we may remove the tilde in (2.15), i.e., that the integral

$$\int_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0} \prod_{k=1}^{n} |1 - w\overline{\sigma_k}^{-1}|^{2s_k}\, \nabla_{\underline{s}}\phi \wedge \mathrm{conj}^*(\omega_\sigma) \tag{2.16}$$

vanishes. Its integrand equals

$$d\left(|w|^{2s_0} \prod_{k=1}^{n} |1 - w\overline{\sigma_k}^{-1}|^{2s_k} \phi\, \mathrm{conj}^*(\omega) \right)$$

because $d\overline{w} \wedge \mathrm{conj}^*(\omega_\sigma) = \mathrm{conj}^*(dz \wedge \omega_\sigma) = 0$. By Stokes, the integral (2.16) can be computed as the limit when ε goes to zero of

$$-\int_{\partial P_\varepsilon} |w|^{2s_0} \prod_{k=1}^{n} |1 - w\overline{\sigma_k}^{-1}|^{2s_k} \phi\, \mathrm{conj}^*(\omega)$$

where P_ε is the complement in $\mathbb{P}^1(\mathbb{C})$ of ε-disks around the points of $\Sigma \cup \infty$, and the sign comes from the orientation of ∂P_ε. The contribution of each point of Σ vanishes, as can be seen from a computation in a local coordinate, because of the assumption that $\mathrm{Re}(s_i) > 0$ for all i. The contribution of the point ∞ also vanishes because of the fact that $\phi(\infty) = 0$ and the assumption that $2\,\mathrm{Re}(s_0) + \cdots + 2\,\mathrm{Re}(s_n) < 1$. Thus, we have

$$\begin{aligned}
\langle [\nu],\, \mathbf{s}[\omega]\rangle &= \frac{1}{2\pi i}\int_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0} \prod_{k=1}^{n} |1 - w\overline{\sigma_k}^{-1}|^{2s_k}\, \nu_{\overline{\sigma}} \wedge \mathrm{conj}^*(\omega_\sigma) \\
&= -\frac{1}{2\pi i}\int_{\mathbb{P}^1(\mathbb{C})} |\overline{z}|^{2s_0} \prod_{k=1}^{n} |1 - \overline{z}\overline{\sigma_k}^{-1}|^{2s_k}\, \mathrm{conj}^*(\nu_{\overline{\sigma}}) \wedge \omega_\sigma \\
&= -\frac{1}{2\pi i}\int_{\mathbb{P}^1(\mathbb{C})} |z|^{2s_0} \prod_{k=1}^{n} |1 - z\sigma_k^{-1}|^{2s_k}\, \overline{\nu_\sigma} \wedge \omega_\sigma\ .
\end{aligned}$$

The second equality follows from performing a change of variables via $\mathrm{conj} : \mathbb{C}\setminus\Sigma \to \mathbb{C}\setminus\overline{\Sigma}$, which reverses the orientation of $\mathbb{C}$, hence the minus sign. The third equality relies on $\mathrm{conj}^*(\nu_{\overline{\sigma}}) = \overline{\nu_\sigma}$, which is obvious. The result follows.

□

2.4. **Tannakian interpretations and coaction formulae.**

2.4.1. *Minimalist version.* Let $k_{\mathrm{dR}} \subset \mathbb{C}$ and $\mathbb{Q}_{\mathrm{B}} \subset \mathbb{C}$ be two subfields of $\mathbb{C}$. Consider the $\mathbb{Q}$-linear abelian category $\mathcal{T}$ whose objects consist of triples $V = (V_{\mathrm{B}}, V_{\mathrm{dR}}, c)$ where $V_{\mathrm{B}}, V_{\mathrm{dR}}$ are finite-dimensional vector spaces over $\mathbb{Q}_{\mathrm{B}}$ and k_{dR} respectively, and $c : V_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} V_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}$ is a $\mathbb{C}$-linear isomorphism. The morphisms ϕ in $\mathcal{T}$ are pairs of linear maps ϕ_{B}, ϕ_{dR} compatible with the isomorphisms c. The category $\mathcal{T}$ is Tannakian with two fiber functors

$$\omega_{\mathrm{dR}} : \mathcal{T} \longrightarrow \mathrm{Vec}_{k_{\mathrm{dR}}} \qquad \text{and} \qquad \omega_{\mathrm{B}} : \mathcal{T} \longrightarrow \mathrm{Vec}_{\mathbb{Q}_{\mathrm{B}}} .$$

The ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}))$ is the $(\mathbb{Q}_{\mathrm{B}}, k_{\mathrm{dR}})$-bimodule spanned by equivalence classes of matrix coefficients $[V, \sigma, \omega]^{\mathfrak{m}}$ where $\sigma \in V_{\mathrm{B}}^{\vee}$ and $\omega \in V_{\mathrm{dR}}$. The k_{dR}-algebra $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Aut}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}))$ is spanned by equivalence classes of matrix coefficients $[V, f, \omega]^{\mathfrak{dr}}$ where $f \in V_{\mathrm{dR}}^{\vee}$ and $\omega \in V_{\mathrm{dR}}$. The multiplicative structure is given by tensor products. There is a natural coaction and coproduct

$$\Delta^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \longrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} \qquad \text{and} \qquad \Delta^{\mathfrak{dr}} : \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} \longrightarrow \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$$

which expresses $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ as an algebra comodule over the Hopf algebra $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$. In both cases $\bullet = \mathfrak{m}, \mathfrak{dr}$, they are given by the formula

$$\Delta^{\bullet}[V, s, \omega]^{\bullet} = \sum_i [V, s, e_i]^{\bullet} \otimes [V, e_i^{\vee}, \omega]^{\mathfrak{dr}} \tag{2.17}$$

where the sum ranges over a k_{dR} basis $\{e_i\}$ of V_{dR}, and $e_i^{\vee}$ denotes the dual basis of $V_{\mathrm{dR}}^{\vee}$.

Definition 2.9. For generic complex numbers s_i (2.1), let $k_{\mathrm{dR}} = \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$, $\mathbb{Q}_{\mathrm{B}} = \mathbb{Q}^{\mathrm{B}}_{\underline{s}}$ and define

$$M_{\Sigma,\underline{s}} = \big(\; H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \; , \; H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \; , \; \mathrm{comp}_{\mathrm{B},\varpi}(\underline{s}) \; \big) \quad \in \quad \mathrm{Ob}(\mathcal{T}) \; .$$

Define a matrix $L^{\mathfrak{m}}_{\Sigma,\underline{s}} \in M_{n\times n}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}})$ by

$$(L^{\mathfrak{m}}_{\Sigma,\underline{s}})_{ij} = \left[M_{\Sigma,\underline{s}} \; , \; \Big[\delta_i \otimes z^{s_0} \prod_{k=1}^{n} (1 - z\sigma_k^{-1})^{s_k} \Big] \; , \; -s_j[\omega_j] \right]^{\mathfrak{m}} \; ,$$

where the basis elements are given by (2.5) and (2.6). The image of $L^{\mathfrak{m}}_{\Sigma,\underline{s}}$ under the period homomorphism

$$\begin{array}{rcl} \mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} & \longrightarrow & \mathbb{C} \\ {[(V_{\mathrm{B}}, V_{\mathrm{dR}}, c), \sigma, \omega]^{\mathfrak{m}}} & \mapsto & \langle \sigma \; , \; c(\omega) \rangle \end{array}$$

is precisely the matrix of Lauricella functions (when the integral converges, see Lemma 2.3):

$$\mathrm{per}\left(L^{\mathfrak{m}}_{\Sigma,\underline{s}}\right)_{ij} = -s_j \int_{\delta_i} z^{s_0} \prod_{k=1}^{n} (1 - z\sigma_k^{-1})^{s_k} \frac{dz}{z - \sigma_j} \; .$$

Define a de Rham version

$$(L^{\mathfrak{dr}}_{\Sigma,\underline{s}})_{ij} = \big[M_{\Sigma,\underline{s}} \; , \; [\nu_i] \; , \; -s_j[\omega_j] \big]^{\mathfrak{dr}} \; ,$$

where the forms ν_i, defining classes in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$, were defined in (2.12). Recall that the de Rham pairing (2.11) induces an isomorphism $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \simeq H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})^{\vee}$ and that Lemma 2.4 implies that the basis $\{[\nu_i]\}$ is dual to the basis $\{-s_i[\omega_i]\}$.

Remark 2.10. The class of ν_i is the image of the relative homology class (viewed in homology with trivial coefficients) of the path δ_i under the map $c_0^{\vee}$ studied in [BD18], up to a sign. It would be interesting to know whether this map can be naturally defined at the level of cohomology with coefficients by relying on Hodge theoretic arguments as in [*loc. cit.*].

Example 2.11. For all $n \in \mathbb{Z}$, one has 'Tate' objects

$$\mathbb{Q}_{\underline{s}}(-n) = (\mathbb{Q}^{\mathrm{B}}_{\underline{s}}, \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}, 1 \mapsto (2\pi i)^n) \; .$$

The Betti and de Rham pairings (2.9) and (2.10), along with their compatibility with the comparison map (2.7) (also known as the 'twisted period relations'), can be succinctly encoded as a perfect pairing

$$M_{\Sigma,-\underline{s}} \otimes M_{\Sigma,\underline{s}} \longrightarrow \mathbb{Q}_{\underline{s}}(-1)$$

in the category $\mathcal{T}$. Equivalently, $M^{\vee}_{\Sigma,\underline{s}} \simeq M_{\Sigma,-\underline{s}}(1)$, where (n) is the standard notation for Tate twist, i.e., tensor product with $\mathbb{Q}_{\underline{s}}(n)$.

Lemma 2.12. *The coaction (resp. coproduct) satisfies*

$$\Delta^{\mathfrak{m}} L^{\mathfrak{m}}_{\Sigma,\underline{s}} = L^{\mathfrak{m}}_{\Sigma,\underline{s}} \otimes L^{\mathfrak{dr}}_{\Sigma,\underline{s}} \quad \text{and} \quad \Delta^{\mathfrak{dr}} L^{\mathfrak{dr}}_{\Sigma,\underline{s}} = L^{\mathfrak{dr}}_{\Sigma,\underline{s}} \otimes L^{\mathfrak{dr}}_{\Sigma,\underline{s}} \ .$$

Proof. This is an immediate consequence of the formula (2.17). □

2.4.2. *Version with real Frobenius involutions.* Let $s_i \in \mathbb{R}$ be real numbers satisfying the genericity conditions (2.1). If we wish to incorporate single-valued periods into a tannakian framework, we must enhance the category $\mathcal{T}$ to take into account the real Frobenius involution which does not act on the individual objects $M_{\Sigma,\underline{s}}$. Let $k_{\mathrm{dR}} \subset \mathbb{C}$ and $\mathbb{Q}_{\mathrm{B}} \subset \mathbb{C}$ be two subfields of $\mathbb{C}$. We let $(-)\otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}}$ denote tensor product taken with respect to the complex conjugate embedding. One solution is to consider a category $\mathcal{T}^{\infty}$ defined in a similar manner as $\mathcal{T}$, except that the objects are given by:

(1) A pair of finite-dimensional k_{dR}-vector spaces V^{+}_{dR}, V^{-}_{dR}.
(2) Two pairs of finite-dimensional $\mathbb{Q}_{\mathrm{B}}$-vector spaces V^{+}_{B}, V^{-}_{B} and $V^{+}_{\overline{\mathrm{B}}}$, $V^{-}_{\overline{\mathrm{B}}}$.
(3) Four $\mathbb{C}$-linear comparison isomorphisms

$$\begin{aligned} c^{\pm}_{\mathrm{B},\mathrm{dR}} : V^{\pm}_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} &\xrightarrow{\sim} V^{\pm}_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \ , \\ c^{\pm}_{\overline{\mathrm{B}},\mathrm{dR}} : V^{\pm}_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}} &\xrightarrow{\sim} V^{\pm}_{\overline{\mathrm{B}}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \ . \end{aligned}$$

(4) Four $\mathbb{Q}_B$-linear real Frobenius maps

$$V^{\pm}_{B} \xrightarrow{\sim} V^{\mp}_{\overline{B}} \qquad \text{and} \qquad V^{\pm}_{\overline{B}} \xrightarrow{\sim} V^{\mp}_{B} \ ,$$

which are denoted simply by F_{∞}, since the source and hence target will be clear from context. The composition of any two such maps, when defined, is the identity: $F_{\infty}F_{\infty} = \mathrm{id}$.

The morphisms ϕ between objects are given by the data of two k_{dR}-linear maps $\phi^{\pm}_{\mathrm{dR}}$, and four $\mathbb{Q}_{\mathrm{B}}$-linear maps $\phi^{\pm}_{\mathrm{B}}$, $\phi^{\pm}_{\overline{\mathrm{B}}}$ which are compatible with (3) and (4). One checks that this category is Tannakian, equipped with six fiber functors $\omega^{\pm}_{\mathrm{dR}}, \omega^{\pm}_{\mathrm{B}}, \omega^{\pm}_{\overline{\mathrm{B}}}$ (over $k_{\mathrm{dR}}, \mathbb{Q}_{\mathrm{B}}$ respectively) which are obtained by forgetting all data except one of the vector spaces in (1) or (2).

Remark 2.13. For objects of $\mathcal{T}_{\infty}$ coming from geometry such as the ones that we will shortly define, there are compatibilities between the real Frobenius isomorphisms F_{∞} and the comparison isomorphisms. We do not include such compatibilities in our definition because they will be irrelevant to the computations that we will be performing.

The category $\mathcal{T}_{\infty}$ admits various rings of periods defined in a similar manner as before. Consider the 8 rings of periods:

$$\mathcal{P}^{\mathrm{B}^{\pm},\mathrm{dR}^{\pm}}_{\mathcal{T}_{\infty}} = \mathrm{Isom}^{\otimes}_{\mathcal{T}_{\infty}}(\omega^{\pm}_{\mathrm{dR}}, \omega^{\pm}_{\mathrm{B}}) \qquad \text{and} \qquad \mathcal{P}^{\overline{\mathrm{B}}^{\pm},\mathrm{dR}^{\pm}}_{\mathcal{T}_{\infty}} = \mathrm{Isom}^{\otimes}_{\mathcal{T}_{\infty}}(\omega^{\pm}_{\mathrm{dR}}, \omega^{\pm}_{\overline{\mathrm{B}}}) \ .$$

The four comparison isomorphisms define four period homomorphisms:

$$\mathrm{per} : \mathcal{P}^{\mathrm{B}^{+},\mathrm{dR}^{+}}_{\mathcal{T}_{\infty}} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \mathrm{per} : \mathcal{P}^{\overline{\mathrm{B}}^{+},\mathrm{dR}^{+}}_{\mathcal{T}_{\infty}} \longrightarrow \mathbb{C}$$

and similarly with $+$ replaced with $-$. The Frobenius maps define isomorphisms:

$$F_{\infty} \ : \ \mathcal{P}^{\mathrm{B}^{\pm},\bullet}_{\mathcal{T}_{\infty}} \ \cong \ \mathcal{P}^{\overline{\mathrm{B}}^{\mp},\bullet}_{\mathcal{T}_{\infty}} \ .$$

By composing with the period homomorphism per, one obtains a full set of 8 period homomorphisms for each of our period rings, e.g., $\mathrm{per}\, F_{\infty} : \mathcal{P}^{\mathrm{B}^{+},\mathrm{dR}^{-}}_{\mathcal{T}_{\infty}} \to \mathbb{C}$.

There are also four possible rings of de Rham periods. We shall mainly consider two of them:

$$\mathcal{P}^{\mathrm{dR}^{-},\mathrm{dR}^{+}}_{\mathcal{T}_{\infty}} = \mathrm{Isom}^{\otimes}_{\mathcal{T}_{\infty}}(\omega^{+}_{\mathrm{dR}}, \omega^{-}_{\mathrm{dR}}) \qquad \text{and} \qquad \mathcal{P}^{\mathrm{dR}^{+},\mathrm{dR}^{-}}_{\mathcal{T}_{\infty}} = \mathrm{Isom}^{\otimes}_{\mathcal{T}_{\infty}}(\omega^{-}_{\mathrm{dR}}, \omega^{+}_{\mathrm{dR}}) \ .$$

Duality induces a canonical isomorphism $\mathcal{P}^{\mathrm{dR}^{-},\mathrm{dR}^{+}}_{\mathcal{T}_{\infty}} \cong \mathcal{P}^{\mathrm{dR}^{+},\mathrm{dR}^{-}}_{\mathcal{T}_{\infty}}$ which we shall not make use of here. Both of these rings admit a single-valued period map, which can be used to detect the non-vanishing of de Rham periods.

Definition 2.14. There is a homomorphism 'single-valued period'

$$\mathbf{s}^{-,+} : \mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}^\infty} \longrightarrow \mathbb{C} \otimes_{k^{\mathrm{dR}}} \overline{\mathbb{C}}$$

defined by the composite

$$V^+_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \overset{c^+_{\mathrm{B},\mathrm{dR}}}{\longrightarrow} V^+_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \overset{F_\infty \otimes \mathrm{id}}{\longrightarrow} V^-_{\overline{\mathrm{B}}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \overset{(c^-_{\overline{\mathrm{B}},\mathrm{dR}})^{-1}}{\longrightarrow} V^-_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}} \ .$$

Since $\mathbb{C}$ and $\overline{\mathbb{C}}$ have different k_{dR}-structures, this map defines a point on the torsor of isomorphisms from ω^+_{dR} to ω^-_{dR} only after extending scalars to $\mathbb{C} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}}$. The single-valued map therefore sends $[V, f, \omega]^{\mathfrak{dr}}$ to $\langle f, (c^-_{\overline{\mathrm{B}},\mathrm{dR}})^{-1} F_\infty c^+_{\mathrm{B},\mathrm{dR}}\, \omega\rangle$.

However, in the case when $k^{\mathrm{dR}} \subset \mathbb{R}$, which is the case that we shall mostly consider, we can compose with the multiplication homomorphism $\mathbb{C} \otimes_{k^{\mathrm{dR}}} \overline{\mathbb{C}} \to \mathbb{C}$ to obtain a homomorphism

$$\mathbf{s}^{-,+} : \mathcal{P}^{\mathrm{dR}^+,\mathrm{dR}^-}_{\mathcal{T}^\infty} \longrightarrow \mathbb{C} \ .$$

There is a variant $\mathbf{s}^{-,+}$ obtained by interchanging all $+$'s and $-$'s in the above. When it is clear from the context, we drop the superscripts and simply write $\mathbf{s}$.

Remark 2.15. Alternatively, one can view $\mathbf{s}^{-,+}$ as a morphism of the $(k_{\mathrm{dR}}, k_{\mathrm{dR}})$-bimodule $\mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}^\infty}$ to $\mathbb{C}$, with the bimodule structure on the latter given by $(k_{\mathrm{dR}}, \overline{k}_{\mathrm{dR}})$. In other words, for $\lambda_1, \lambda_2 \in k_{\mathrm{dR}}$ one has $\mathbf{s}^{-,+}(\lambda_1 \xi \lambda_2) = \lambda_1 \mathbf{s}^{-,+}(\xi)\overline{\lambda_2}$. When k is real, then these bimodule structures coincide and we obtain a genuine linear map.

The composition of torsors between fiber functors defines a coaction morphism

$$\Delta : \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty} \longrightarrow \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty} \tag{2.18}$$

and likewise with $+, -$ interchanged. The period homomorphisms defined earlier are compatible with composition of torsors between fiber functions. In particular, one has

$$\mathrm{per}\,(\xi) = (\mathrm{per}\, F_\infty \otimes \mathbf{s})\,(\Delta \xi) \qquad \text{for all} \quad \xi \ \in \ \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty} \ . \tag{2.19}$$

Definition 2.16. For s_i real and generic (2.1), let $k_{\mathrm{dR}} = \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$, $\mathbb{Q}_{\mathrm{B}} = \mathbb{Q}^{\mathrm{B}}_{\underline{s}}$ and define an object of rank n in $\mathcal{T}^\infty$ by

$$\widetilde{M}_{\Sigma,\underline{s}} = \left(V^{\pm}_{\mathrm{dR}}, V^{\pm}_{\mathrm{B}}, V^{\pm}_{\overline{\mathrm{B}}}\right)$$

where

$$V^{\pm}_{\mathrm{dR}} = H^1_{\varpi}(X_\Sigma, \nabla_{\pm\underline{s}}) \quad , \quad V^{\pm}_{\mathrm{B}} = H^1(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\pm\underline{s}}) \quad , \quad V^{\pm}_{\overline{\mathrm{B}}} = H^1(\mathbb{C}\backslash\overline{\Sigma}, \mathcal{L}_{\pm\underline{s}})$$

The Frobenius maps F_∞ are induced on Betti cohomology by complex conjugation conj^* and its inverse. The comparison isomorphisms $c^{\pm}_{\mathrm{B},\mathrm{dR}}, c^{\pm}_{\overline{\mathrm{B}},\mathrm{dR}}$ are defined by

$$\mathrm{comp}_{\mathrm{B},\mathrm{dR}} : H^1_{\varpi}(X_\Sigma, \nabla_{\pm\underline{s}}) \otimes_{k_{\mathrm{dR}}} \mathbb{C} \longrightarrow H^1(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\pm\underline{s}}) \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}$$

$$\mathrm{comp}_{\overline{\mathrm{B}},\mathrm{dR}} : H^1_{\varpi}(X_\Sigma, \nabla_{\pm\underline{s}}) \otimes_{k^{\mathrm{dR}}} \overline{\mathbb{C}} \longrightarrow H^1(\mathbb{C}\backslash\overline{\Sigma}, \mathcal{L}_{\pm\underline{s}}) \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}$$

Since $k_{\mathrm{dR}} = \mathbb{Q}^{\mathrm{dR}}_{\underline{s}} \subset \mathbb{R}$ in this case, the complex conjugate $\overline{\mathbb{C}}$ in the left-hand side of the previous equation can be replaced with $\mathbb{C}$.

The image of $\widetilde{M}_\Sigma$ under the functor $\mathcal{T}^\infty \to \mathcal{T}$ which forgets all data except for $(V^+_{\mathrm{B}}, V^+_{\mathrm{dR}}, c_{\mathrm{B},\mathrm{dR}})$ is the object M_Σ.

Let us define an $(n \times n)$ matrix

$$\widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}} \quad \in \quad M_{n\times n}(\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}^\infty})$$

whose entries are

$$\left(\widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}}\right)_{ij} = \left[\widetilde{M}_{\Sigma,\underline{s}} \ , \ \left[\delta_i \otimes z^{s_0} \prod_{k=1}^{n} (1 - z\sigma_k^{-1})^{s_k}\right] \ , \ -s_j\omega_j\right]^{\mathfrak{m}} \ ,$$

where the Betti class lies in $H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}})^\vee = H_1(\mathbb{C}\backslash\Sigma, \mathcal{L}^\vee_{s})$ and the de Rham class lies in $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. Its image under the natural map $\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}^\infty} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ is $L^{\mathfrak{m}}_{\Sigma,\underline{s}}$ and its period is the Lauricella matrix

$$\mathrm{per}\, \widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}} = L_\Sigma(\underline{s}) \ . \tag{2.20}$$

Now let us define a de Rham motivic Lauricella function

$$\left(\widetilde{L}^{\mathfrak{dr}}_{\Sigma,\underline{s}}\right)_{ij} = \left[\widetilde{M}_{\Sigma,\underline{s}} \ , \ s_i^{-1}\omega_i^\vee \ , \ -s_j\omega_j\right]^{\mathfrak{dr}} \quad \in \quad \mathcal{P}^{\mathrm{dR}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty} \ .$$

The first de Rham class $s_i^{-1}\omega_i^\vee$ is to be viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})^\vee$ but the second is to be viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. Note that this differs from the earlier definition of the de Rham Lauricella matrix in the ring of de Rham periods of the category $\mathcal{T}$ because the de Rham classes reside in different cohomology groups. This is required so that we may speak of its single-valued period.

2.4.3. *Single-valued periods and double copy.*

Theorem 2.17. *The single-valued periods are given by*

$$\mathbf{s}\left(\widetilde{L}^{\mathfrak{dr}}_{\Sigma,\underline{s}}\right)_{ij} = \frac{s_j}{2\pi i}\int_{\mathbb{C}} |z|^{2s_0} \prod_{k=1}^n |1 - z\sigma_k^{-1}|^{2s_k} \left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right) \wedge \omega_j$$

which converge whenever $s_i > 0$ for all $0 \leq i \leq n$, and $2s_0 + \cdots + 2s_k < 1$. It follows from the definition of the single-valued involution, however, that

$$\mathbf{s}\left(\widetilde{L}^{\mathfrak{dr}}_{\Sigma,\underline{s}}\right) = L_{\overline{\Sigma}}(-s_1,\ldots,-s_n)^{-1} L_\Sigma(s_1,\ldots,s_n) \ . \tag{2.21}$$

Equating both expressions gives rise to a 'double copy' formula.

Proof. By (2.13) the dual class $s_i^{-1}\omega_i^\vee$ is represented by the class of the form ν_i (the reason for the change of sign compared to (2.13) is because $\omega_i^\vee$ is now viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})^\vee$). Conclude by applying Proposition 2.8. The second part follows immediately from the definition of $\mathbf{s}$ and the fact that $F^\vee_\infty$ sends the section $w^{-s_0}\prod_{k=1}^n(1 - w\overline{\sigma_k}^{-1})^{-s_k}$ to the section $z^{s_0}\prod_{k=1}^n(1 - z\sigma_k^{-1})^{s_k}$. □

2.4.4. *Coaction.* The coaction (2.18), applied to the motivic Lauricella function, will give rise to a third but closely related quantity, given by the matrix

$$\left(\widetilde{L}^{\mathfrak{m},-}_{\Sigma,\underline{s}}\right)_{ij} = \left[\widetilde{M}_{\Sigma,\underline{s}} \ , \ \left[\delta_i \otimes z^{s_0}\prod_{k=1}^n (1 - z\sigma_k^{-1})^{s_k}\right] \ , \ s_j\omega_j\right]^{\mathfrak{m}} \quad \in \quad \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}^\infty}$$

where the de Rham class $[s_j\omega_j]$ is in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$. It is the image under Frobenius of

$$\left(\widetilde{L}^{\mathfrak{m}}_{\overline{\Sigma},-\underline{s}}\right)_{ij} = \left[\widetilde{M}_{\Sigma,\underline{s}} \ , \ \left[\overline{\delta}_i \otimes \overline{w}^{s_0}\prod_{k=1}^n (1 - \overline{w}\sigma_k^{-1})^{s_k}\right] \ , \ s_j\omega_j\right]^{\mathfrak{m}} \quad \in \quad \mathcal{P}^{\mathrm{B}^-,\mathrm{dR}^-}_{\mathcal{T}^\infty} \ ,$$

where the Betti class is viewed in $H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{-\underline{s}}) = H^1(\mathbb{C}\backslash\overline{\Sigma}, \mathcal{L}_{-\underline{s}})$ and the de Rham class $[s_j\omega_j]$ is viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$ as before. More precisely, we have

$$\widetilde{L}^{\mathfrak{m},-}_{\Sigma,\underline{s}} = F_\infty \widetilde{L}^{\mathfrak{m}}_{\overline{\Sigma},-\underline{\mathbf{s}}}$$

and hence, by definition of the period homomorphism $\mathrm{per}\, F_\infty : \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}^\infty} \to \mathbb{C}$ the period is

$$\mathrm{per}\, F_\infty\left(\widetilde{L}^{\mathfrak{m},-}_{\Sigma,\underline{s}}\right) = L_{\overline{\Sigma},-\underline{s}} \ .$$

Corollary 2.18. *The coaction (2.18) satisfies*

$$\Delta\widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}} = \widetilde{L}^{\mathfrak{m},-}_{\Sigma,\underline{s}} \otimes \widetilde{L}^{\mathfrak{dr}}_{\Sigma,\underline{s}} \ .$$

Proof. This is an immediate consequence of the formula for the coaction on matrix coefficients in a Tannakian category. □

Equation (2.19), applied to the matrix $\widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}}$ reduces to the equation

$$L_\Sigma(\underline{s}) = L_{\overline{\Sigma}}(-\underline{s})\, L^{\mathbf{s}}_\Sigma(\underline{s})$$

which is another way of writing formula (2.21). We can combine the periods together in a different manner as follows. Consider the map

$$\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty} \overset{\Delta}{\longrightarrow} \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty} \overset{\mathrm{per}\, F_\infty \otimes \mathbf{s}^{+,-}}{\longrightarrow} \mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}\ .$$

Then under this map, the elements $(\widetilde{L}^{\mathfrak{m}}_{\Sigma,\underline{s}})_{ij}$ for $1\leq i,j\leq n$ map to

$$\sum_{\ell=1}^n \left(s_\ell \int_{\delta_i} z^{-s_0} \prod_{k=1}^n (1-z\overline{\sigma_k}^{-1})^{-s_k} \frac{dz}{z-\overline{\sigma_\ell}}\right) \otimes \tag{2.22}$$
$$\frac{s_j}{2\pi i}\left(\int_{\mathbb{C}} |z|^{2s_0} \prod_{k=1}^n |1-z\sigma_k^{-1}|^{2s_k} \left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_\ell}} - \frac{d\overline{z}}{\overline{z}}\right) \wedge \frac{dz}{z-\sigma_j}\right)\ .$$

The point of this formula is that both sides of the tensor product admit a Taylor expansion in the s_i, which is the subject of the next paragraph.

2.4.5. *Variant.* We had to assume that the parameters s_i are real in order to get a Tannakian interpretation of the single-valued period homomorphism. This was to ensure that the subfield $\overline{k}(s_0,\ldots,s_n)\subset\mathbb{C}$ is isomorphic to $k(s_0,\ldots,s_n)$ and thus get a comparison map for de Rham cohomology associated to the complex conjugate embedding of k:

$$H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}}) \otimes_{k^{\mathrm{dR}}_{\underline{s}}} \overline{\mathbb{C}} \longrightarrow H^1_{\mathrm{B}}(\mathbb{C}\backslash\overline{\Sigma},\mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}} \mathbb{C}\ .$$

However, such an assumption is unnatural, for instance because the double copy formulas are true for all $s_i\in\mathbb{C}$ for which they make sense. A way to remedy this would be to treat the s_i's as formal parameters and work with modules over the polynomial ring $k[s_0,\ldots,s_n]$. This would also be needed to build a bridge between the framework of cohomology with coefficients that we discussed in this section and the Taylor expansions that we will consider in the rest of this article.

3. Laurent series expansion of periods of cohomology with coefficients

The functions (1.1) are not *a priori* defined for $s_0,\ldots,s_n$ at the origin. We show using a renormalisation procedure that they extend to a neighbourhood of the origin and admit a Taylor expansion there. The reason for the (ab)use of the word 'renormalisation' is explained in [BD18].

3.1. **Renormalisation of forms.** For each $0\leq i\leq n$ there is an inclusion

$$\mathsf{J}_i : X_\Sigma \hookrightarrow \mathbb{P}^1\backslash\{\sigma_i,\infty\} \cong (T_{\sigma_i}\mathbb{P}^1)^\times\ .$$

Let ω denote a smooth section of $\Omega^1_{\mathbb{P}^1}(\log\Sigma\cup\infty)$ and consider

$$\Omega \;=\; x^{s_0}\prod_{k=1}^n (1-x\sigma_k^{-1})^{s_k}\,\omega \qquad , \qquad \Omega^{\mathbf{s}} \;=\; |z|^{2s_0}\prod_{k=1}^n |1-z\sigma_k^{-1}|^{2s_k}\,\omega$$

the sections of $\mathcal{L}_{\underline{s}}\otimes\Omega^1_{\mathbb{P}^1}(\log\Sigma\cup\infty)$ and $\left(\overline{\mathcal{L}_{\underline{s}}}\otimes\mathcal{L}_{\underline{s}}\right)\otimes\Omega^1_{\mathbb{P}^1}(\log\Sigma\cup\infty)$ respectively. Denote their localisations to the tangent space $(T_{\sigma_i}\mathbb{P}^1)^\times$ by R_{σ_i}. Their pull-backs to X_Σ satisfy:

$$\mathsf{J}_i^*\, R_{\sigma_i}\Omega \;=\; \left(\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k}\right) \mathrm{Res}_{\sigma_i}(\omega)\ \left(1-x\sigma_i^{-1}\right)^{s_i}\frac{dx}{x-\sigma_i}$$

$$\mathsf{J}_i^*\, R_{\sigma_i}\Omega^{\mathbf{s}} \;=\; \left(|\sigma_i|^{2s_0}\prod_{k\neq i}\left|1-\sigma_i\sigma_k^{-1}\right|^{2s_k}\right) \mathrm{Res}_{\sigma_i}(\omega)\ \left|1-z\sigma_i^{-1}\right|^{2s_i}\frac{dz}{z-\sigma_i}$$

whenever $i\geq 1$. For $i=0$, we have

$$\mathsf{J}_0^*\, R_0\Omega = \mathrm{Res}_0(\omega)\ x^{s_0}\frac{dx}{x} \qquad , \qquad \mathsf{J}_0^*\, R_0\Omega^{\mathbf{s}} = \mathrm{Res}_0(\omega)\ |z|^{2s_0}\frac{dz}{z}\ .$$

Definition 3.1. Following [BD18], let us define the renormalised version of $\alpha \in \{\Omega, \Omega^{\mathbf{s}}\}$ with respect to the points $\{0, \sigma_i\}$, where $1 \leq i \leq n$, to be

$$\alpha^{\mathrm{ren}_i} = \alpha - (\mathsf{J}_0^* \, R_0 \alpha + \mathsf{J}_i^* \, R_{\sigma_i} \alpha) \ .$$

3.2. **Canonical Laurent expansions.** We extend the region of convergence using the renormalised integrands, first in the case of the ordinary integrals, and then for their single-valued versions. For now assume that no three $\sigma_i \in \Sigma$ are collinear.

Proposition 3.2. *Let Ω be as in (3.1). There is a canonical Laurent expansion:*

$$\int_0^{\sigma_i} \Omega = \frac{R_0}{s_0} - \frac{R_i}{s_i} + \int_0^{\sigma_i} \Omega^{\mathrm{ren}_i} \tag{3.1}$$

where the integration path is the straight line from 0 to σ_i, and

$$R_0 = (\sigma_i^{s_0}) \ \mathrm{Res}_0 \, \omega \quad , \quad R_i = \left(\sigma_i^{s_0} \prod_{k \neq i} (1 - \sigma_i \sigma_k^{-1})^{s_k} \right) \mathrm{Res}_{\sigma_i} \omega \ . \tag{3.2}$$

The integral on the left-hand side of (3.1) converges for $\mathrm{Re}(s_0), \mathrm{Re}(s_i) > 0$. The integral on the right-hand side converges for $\mathrm{Re}(s_0), \mathrm{Re}(s_i) > -1$ and so admits a Taylor expansion at the origin

$$\int_0^{\sigma_i} \Omega^{\mathrm{ren}_i} \quad \in \quad \mathbb{C}[[s_0, \ldots, s_n]] \ .$$

Proof. Write $\Omega = \Omega^{\mathrm{ren}_i} + (\mathsf{J}_0^* \, R_0 \Omega + \mathsf{J}_i^* \, R_{\sigma_i} \Omega)$ and compute

$$\int_0^{\sigma_i} \mathsf{J}_0^* \, R_0 \Omega = \mathrm{Res}_0(\omega) \int_0^{\sigma_i} x^{s_0} \frac{dx}{x} = \mathrm{Res}_0(\omega) \, \frac{\sigma_i^{s_0}}{s_0} \ ,$$

for all $\mathrm{Re}(s_0) > 0$. If furthermore $\mathrm{Re}\,(s_i) > 0$, we have

$$\int_0^{\sigma_i} (1 - x\sigma_i^{-1})^{s_i} \, \frac{dx}{x - \sigma_i} = \left[\frac{1}{s_i} \left(1 - \frac{x}{\sigma_i} \right)^{s_i} \right]_0^{\sigma_i} = -\frac{1}{s_i} \ .$$

It follows that

$$\int_0^{\sigma_i} \mathsf{J}_i^* \, R_i \Omega = -\frac{1}{s_i} \left(\sigma_i^{s_0} \prod_{k \neq i} (1 - \sigma_i \sigma_k^{-1})^{s_k} \right) \mathrm{Res}_{\sigma_i}(\omega) \ .$$

This proves equation (3.1). The integrand Ω^{ren_i} has no poles on the interior of the straight line path between 0 and σ_i. At the boundaries it is asymptotically of the form

$$\begin{array}{lll} \Omega^{\mathrm{ren}_i} & \sim & x^{s_0} dx \qquad\qquad \text{near } x = 0 \ , \\ \Omega^{\mathrm{ren}_i} & \sim & (1 - x\sigma_i^{-1})^{s_i} dx \qquad \text{near } x = \sigma_i \ , \end{array}$$

and is therefore integrable for $\mathrm{Re}(s_0), \mathrm{Re}(s_i) > -1$. Since the region of convergence contains the origin, it admits a Taylor expansion as stated. The Laurent expansion (3.1) is unique by meromorphic continuation in the s_i. □

In the case when $\sigma_i \in \Sigma$ are not in general position, the statement still holds with the domain of integration replaced with a suitable path δ_i. Note that in this case the multi-valued term $\sigma_i^{s_0}$ is defined with respect to analytic continuation along δ_i.

3.2.1. *Single-valued versions.* The following lemma will be used several times in the sequel.

Lemma 3.3. *For any $\sigma, \tau \in \mathbb{C}^\times$ and $\mathrm{Re}(s) > 0$, we have*

$$-\frac{1}{2\pi i} \int_{\mathbb{C}} \left|1 - z\sigma^{-1}\right|^{2s} \left(\frac{d\overline{z}}{\overline{z} - \overline{\sigma} + \overline{\tau}} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \frac{dz}{z - \sigma} = \frac{1}{s} \left(\left| \frac{\tau}{\sigma} \right|^{2s} - 1 \right) \ .$$

Proof. One verifies in local polar coordinates that the integral converges. Let

$$F = -\frac{1}{s}\left|1 - z\sigma^{-1}\right|^{2s}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma}+\overline{\tau}} - \frac{d\overline{z}}{\overline{z}}\right) .$$

Its total derivative is the integrand on the left-hand side. By Stokes' formula applied to the complement of three discs in $\mathbb{P}^1(\mathbb{C})$ centered at $0, \sigma - \tau, \infty$, it suffices to compute

$$-\frac{1}{2\pi i}\int_D F \; = \; \frac{1}{s}\left|1 - z\sigma^{-1}\right|^{2s}\left(\frac{1}{2\pi i}\int_D \frac{d\overline{z}}{\overline{z}-\overline{\sigma}+\overline{\tau}} - \frac{d\overline{z}}{\overline{z}}\right)$$

where D is the union of three small circles winding positively around ∞, 0, and $\sigma - \tau$. The integral around the first vanishes. The integral around the second yields $-s^{-1}$, and the last gives $\frac{1}{s}\left|1-(\sigma-\tau)\sigma^{-1}\right|^{2s} = \frac{1}{s}|\tau\sigma^{-1}|^{2s}$. □

Proposition 3.4. *Let $\Omega^{\mathbf{s}}$ be as in (3.1) and assume that $s_0, \ldots, s_n$ are real. Then there is a canonical Laurent expansion:*

$$(3.3) \qquad -\frac{1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s}} \; = \; \frac{R_0^{\mathbf{s}}}{s_0} - \frac{R_i^{\mathbf{s}}}{s_i} - \frac{1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s},\mathrm{ren}_i}$$

where

$$(3.4) \qquad R_0^{\mathbf{s}} = |\sigma_i|^{2s_0}\,\mathrm{Res}_0\,\omega \quad \textit{and} \quad R_i^{\mathbf{s}} = \left(|\sigma_i|^{2s_0}\prod_{k\neq i}\left|1-\sigma_i\sigma_k^{-1}\right|^{2s_k}\right)\mathrm{Res}_{\sigma_i}\omega \ .$$

The integral on the left-hand side of (3.3) converges for $s_0, s_i > 0$, $2s_k > -1$ for all $k \notin \{0, i\}$, and $2s_0 + \ldots + 2s_n < 1$. The renormalised integral on the right-hand side, however, converges for $2s_0, \ldots, 2s_n > -1$ and $2s_0 + \ldots + 2s_n < 1$, which contains a neighbourhood of the origin. Therefore it admits a Taylor expansion at the origin

$$-\frac{1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s},\mathrm{ren}_i} \quad \in \quad \mathbb{C}[[s_0, \ldots, s_n]] \ .$$

Proof. Similar to proposition 3.4, except that we use the single-valued analogues

$$-\frac{1}{2\pi i}\int_{\mathbb{C}}|1 - z\sigma_i^{-1}|^{2s_i}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_i} = -\frac{1}{s_i} \ ,$$

which follows from lemma 3.3 with $\tau = 0$, and

$$-\frac{1}{2\pi i}\int_{\mathbb{C}}|z|^{2s_0}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z} = \frac{1}{s_0}|\sigma_i|^{2s_0} \ ,$$

which follows from a very similar computation (or by multiplying the statement of lemma 3.3 through by $|\sigma|^{2s}$, setting $\tau = \sigma_i$, and letting $\sigma \to 0$). For the convergence, we only need to check that

$$\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s},\mathrm{ren}_i}$$

has at most simple poles at all points of $\Sigma \cup \infty$. In fact, $\Omega^{\mathbf{s},\mathrm{ren}_i}$ has been defined precisely so that it has no poles at $0, \sigma_i$. Therefore, by passing to local polar coordinates shows that the region of convergence is as stated. □

By uniqueness of Laurent expansions, we can take (3.1) and (3.3) as our definitions of the Laurent expansion of L_Σ and its single-valued versions at the origin. The fact that it was not *a priori* defined at this point, due to condition (2.1), reveals itself as the appearance of poles in the s_i.

4. Notations relating to motivic fundamental groups

In this, the second part of the paper, we study (1.1) from the point of view of the motivic fundamental group of the punctured Riemann sphere. This requires a number of notations and background, mostly from [DG05] and [Bro17], which we recall here. Let $k \subset \mathbb{C}$ be a number field.

4.1. **Categorical and Tannakian.**

(1) Let $\mathcal{MT}(k)$ denote the category of mixed Tate motives over k. One can replace $\mathcal{MT}(k)$ with a category of (Betti and de Rham) realisations, with essentially no change to our arguments. The category $\mathcal{MT}(k)$ has a canonical fiber functor $\varpi : \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}$. Let us set

$$G^{\varpi}_{\mathcal{MT}(k)} = \mathrm{Aut}^{\otimes}_{\varpi}\, \mathcal{MT}(k) \ .$$

It is an affine group scheme over $\mathbb{Q}$. Let us denote by $\omega_{\mathrm{B}} : \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}$ the Betti realisation functor with respect to the given embedding $k \subset \mathbb{C}$.

(2) Let $\mathcal{P}^{\mathfrak{m}} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{MT}(k)}(\varpi, \omega_{\mathrm{B}}))$ denote the $\mathbb{Q}$-algebra of motivic periods on $\mathcal{MT}(k)$ and $\mathcal{P}^{\mathfrak{m},+} \subseteq \mathcal{P}^{\mathfrak{m}}$ the subspace of effective motivic periods. Let

$$\mathcal{P}^{\varpi} = \mathcal{O}(G^{\varpi}_{\mathcal{MT}(k)})$$

denote the $\mathbb{Q}$-algebra of de Rham (more precisely '(ϖ, ϖ)') motivic periods. The latter is a graded Hopf algebra, and the former is a graded algebra comodule over it. Denote the corresponding motivic coactions by

$$\Delta^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi} \qquad \text{and} \qquad \Delta^{\varpi} : \mathcal{P}^{\varpi} \longrightarrow \mathcal{P}^{\varpi} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi}$$

and let $\mathrm{per} : \mathcal{P}^{\mathfrak{m}} \to \mathbb{C}$ be the period homomorphism. We drop the subscripts because it does not matter in which ring of motivic periods we work.

(3) Let $\mathbb{L}^{\mathfrak{m}} = [\mathbb{Q}(-1), 1^{\vee}_{B}, 1_{\varpi}]^{\mathfrak{m}}$ (resp. $\mathbb{L}^{\varpi} = [\mathbb{Q}(-1), 1^{\vee}_{\varpi}, 1_{\varpi}]^{\varpi}$) denote the Lefschetz motivic period, whose period is $2\pi i$ (resp. its de Rham version).

(4) There is a canonical projection homomorphism

$$\pi^{\mathfrak{m},+}_{\varpi} : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\varpi}$$

which, in particular, sends $\mathbb{L}^{\mathfrak{m}}$ to zero. It can be defined by composing the coaction $\Delta^{\mathfrak{m}}$ with projection onto the weight-graded zero piece $W_0\mathcal{P}^{\mathfrak{m},+} \cong \mathbb{Q}$.

4.2. **Geometric.** Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\} \subset \mathbb{A}^1(k)$ where $\sigma_0 = 0$. Suppose for now that the σ_i are in general position, i.e., no three are collinear. Recall $X_\Sigma = \mathbb{A}^1_k \backslash \Sigma$.

(1) Let t_0 denote the tangent vector of length 1 at 0. For each $i \geq 1$, let

$$t_{\sigma_i} \quad \in \quad T_{\sigma_i}\mathbb{A}^1_k$$

be the tangent vector σ_i based at the point σ_i.

(2) For all $0 \leq i, j \leq n$, denote by

$${}_i\Pi^{\bullet}_j = \pi^{\bullet}_1(X_\Sigma, t_i, -t_j) \qquad \text{where} \qquad \bullet \in \{\mathrm{B}, \varpi, \mathrm{mot}\}$$

the Betti, canonical, or motivic fundamental torsor of paths from the tangential base point t_i at σ_i, to $-t_j$ at σ_j. The default (with no superscript) will denote ϖ. As a scheme, it does not depend on i, j, although the action of the (canonical) motivic Galois group upon it depends on i, j. There are maps

$$\gamma \mapsto \gamma^B \ : \ \pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_i, -t_j) \longrightarrow {}_i\Pi^{\mathrm{B}}_j(\mathbb{Q})$$

which are Zariski-dense and compatible with the groupoid structure (composition of paths).

(3) On X_Σ we considered the logarithmic 1-forms ω_i for $i = 0, \ldots, n$ (2.2). Since they have residue 0 or ± 1 at points of Σ, they generate the canonical $\mathbb{Q}$-structure (or ϖ-structure) on the de Rham realisation of $H^1(X_\Sigma) \in \mathrm{Ob}(\mathcal{MT}(k))$, which we shall denote simply by $H^1_{\varpi}(X_\Sigma)$. It is the $\mathbb{Q}$-vector space spanned by $[\omega_i]$.

The affine ring of the de Rham (canonical) torsor of paths is

$$\mathcal{O}({}_i\Pi_j) \cong \bigoplus_{n \geq 0} H^1_{\varpi}(X_\Sigma)^{\otimes n} \ .$$

It is isomorphic to the graded tensor coalgebra on $H^1_{\varpi}(X_\Sigma)$, equipped with the shuffle product ш and deconcatenation coproduct. For any commutative unitary $\mathbb{Q}$-algebra R, the R-points of ${}_i\Pi_j$

$$ {}_i\Pi_j(R) \ \subset \ R\langle\langle e_0, \ldots, e_n\rangle\rangle $$

is the set of group-like formal power series with respect to the continuous coproduct for which the e_i are primitive. They are formal power series

$$ S = \sum_{w\in\{e_0,e_1,\ldots,e_n\}^\times} S(w)\, w \quad \in \quad R\langle\langle e_0, \ldots, e_n\rangle\rangle $$

where $S_\varnothing = 1$ and $w \mapsto S(w)$ (or rather, its linear extension) is a homomorphism with respect to the shuffle product. The letters e_i, for $1 \leq i \leq n$, are dual to the ω_i.

(4) For all $1 \leq i \leq n$, let $\gamma_i \in \pi_1(X_\Sigma(\mathbb{C}), t_0, -t_i)$ denote a path from the tangential base point t_0 at 0 to $-t_i$ at σ_i. For example, if no σ_j lies on the ray between 0 and σ_i one can take the compositum of an infinitely small path from the tangent vector 1 at 0 to the tangent vector σ_i at 0, followed by the straight line path $x \mapsto \sigma_i x : (0,1) \to X_\Sigma(\mathbb{C})$. Any such choice of path γ_i defines a path δ_i considered in the first part of the paper, but not conversely: the topological and homological interpretation of such paths are radically different.

(5) For all $0 \leq i \leq n$, let ${}_0 1_i \in {}_0\Pi_i(\mathbb{Q})$ denote the canonical ϖ-path. It is defined by the augmentation map $\mathcal{O}\left({}_0\Pi_i\right) \to \mathbb{Q}$ onto the degree zero component (or by quotienting by the Hodge filtration F^1). It is the formal power series $1 \in \mathbb{Q}\langle\langle e_0, \ldots, e_n\rangle\rangle$ consisting only of the empty word.

(6) Since the motivic fundamental torsor of paths is a pro-object in the category $\mathcal{MT}(k)$, there is a canonical universal comparison isomorphism of schemes

$$ \mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi} \quad : \quad {}_i\Pi^{\mathrm{B}}_j \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}} \overset{\sim}{\longrightarrow} {}_i\Pi_j \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}} $$

for all $0 \leq i, j \leq n$, compatible with the groupoid structure ${}_a\Pi^{\mathrm{mot}}_b \times {}_b\Pi^{\mathrm{mot}}_c \longrightarrow {}_a\Pi^{\mathrm{mot}}_c$, for all a, b, c in the set of our tangential basepoints (1) (composition of paths). Composing $\mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi}$ with the period homomorphism gives back the canonical comparison isomorphism whose coefficients are regularised interated integrals:

$$ \mathrm{comp}_{\mathrm{B},\varpi} \quad : \quad {}_i\Pi^{\mathrm{B}}_j \times_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} {}_i\Pi_j \times_{\mathbb{Q}} \mathbb{C}\ . $$

5. Torsors of paths for $\mathbb{P}^1\backslash\Sigma$ and their associated beta functions

5.1. Generalised motivic associators.

Definition 5.1. For every $1 \leq i \leq n$, define a formal power series:

$$ \mathcal{Z}^{i,\mathfrak{m}} \ \in \ {}_0\Pi_i(\mathcal{P}^{\mathfrak{m}}) \subset \mathcal{P}^{\mathfrak{m}}\langle\langle e_0, \ldots, e_n\rangle\rangle $$

by $\mathcal{Z}^{i,\mathfrak{m}} = \mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi}(\gamma^{\mathrm{B}}_i)$ where γ^{B}_i is the image of γ_i in ${}_0\Pi^{\mathrm{B}}_i(\mathbb{Q})$. Since $\mathcal{O}({}_0\Pi_i)$ has weights ≥ 0, it follows that $\mathcal{Z}^{i,\mathfrak{m}}$ actually lies in ${}_0\Pi_i(\mathcal{P}^{\mathfrak{m},+})$. Explicitly,

$$ \mathcal{Z}^{i,\mathfrak{m}} = \sum_{w\in\{e_0,\ldots,e_n\}^\times} \left[\mathcal{O}(\pi^{\mathrm{mot}}_1(X_\Sigma, t_0, -t_i)), \gamma^{\mathrm{B}}_i, w\right]^{\mathfrak{m}} w $$

where the sum is over all words w in e_i, which are in turn dual to words in the ω_i (2.2) and hence define an element $w \in \mathcal{O}({}_0\Pi_i)$. The path γ^{B}_i is viewed as an element in $\mathcal{O}({}_0\Pi^{\mathrm{B}}_i)^\vee$. Define

$$ \mathcal{Z}^{i,\varpi} \ \in \ {}_0\Pi_i(\mathcal{P}^{\varpi}) \subset \mathcal{P}^{\varpi}\langle\langle e_0, \ldots, e_n\rangle\rangle $$

to be the canonical element in $\mathrm{Hom}\left(\mathcal{O}({}_0\Pi_i), \mathcal{P}^{\varpi}\right)$ given by the morphism of schemes $G^{\varpi}_{\mathcal{MT}(k)} \to {}_0\Pi_i$ induced by the action $g \mapsto g.{}_0 1_i$ of $G^{\varpi}_{\mathcal{MT}(k)}$ on the canonical ϖ-path ${}_0 1_i$. It is given explicitly by the group-like formal power series

$$ \mathcal{Z}^{i,\varpi} = \sum_{w\in\{e_0,\ldots,e_n\}^\times} \left[\mathcal{O}(\pi^{\mathrm{mot}}_1(X_\Sigma, t_0, -t_i)), {}_0 1_i, w\right]^{\mathfrak{m}} w\ . $$

When we wish to emphasize the dependence on the variables e_i, we shall write $\mathcal{Z}^{i,\mathfrak{m}}(e_0,\ldots,e_n)$ for $\mathcal{Z}^{i,\mathfrak{m}}$, and so on. Since the empty iterated integral along γ_i is 1, it follows that $\mathcal{Z}^{i,\varpi}$ is the image of $\mathcal{Z}^{i,\mathfrak{m}}$ under the coefficient-wise application of the projection $\pi^{\mathfrak{m},+}_{\varpi}$, i.e.,

$$\mathcal{Z}^{i,\varpi}=\pi^{\mathfrak{m},+}_{\varpi}\mathcal{Z}^{i,\mathfrak{m}}\ .$$

5.2. **Generalised associators.** The image $\mathcal{Z}^i=\mathrm{per}\left(\mathcal{Z}^{i,\mathfrak{m}}\right)$ of $\mathcal{Z}^{i,\mathfrak{m}}$ under the period homomorphism is the group-like formal power series

$$\mathcal{Z}^i=\sum_{w\in\{e_0,\ldots,e_n\}^\times}\left(\int_{\gamma_i}w\right)w\quad\in\quad\mathbb{C}\langle\langle e_0,\ldots,e_n\rangle\rangle$$

where the sum is over all words w in $\{e_0,\ldots,e_n\}$, and the integral is the regularised iterated integral (from left to right) of the corresponding word in $\{\omega_0,\ldots,\omega_n\}$.

Example 5.2. Let $\Sigma=\{0,1\}$ and $k=\mathbb{Q}$. Then $\varpi=\omega_{\mathrm{dR}}$ and $\mathcal{Z}^{1,\mathfrak{m}}=\mathcal{Z}^{\mathfrak{m}}$ where

$$\mathcal{Z}^{\mathfrak{m}}=\sum_{w\in\{e_0,e_1\}^\times}\zeta^{\mathfrak{m}}(w)w\quad\in\quad\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Q})}\langle\langle e_0,e_1\rangle\rangle$$

is the motivic Drinfeld associator and $\mathcal{Z}^{1,\varpi}=\mathcal{Z}^{\mathfrak{dr}}$ is the de Rham Drinfeld associator. It is obtained by replacing every motivic multiple zeta value $\zeta^{\mathfrak{m}}$ with its de Rham version $\zeta^{\mathfrak{dr}}$. Drinfeld's associator is $\mathcal{Z}=\mathrm{per}\left(\mathcal{Z}^{\mathfrak{m}}\right)\in\mathbb{R}\langle\langle e_0,e_1\rangle\rangle$.

5.3. **Beta quotients.** Let R be any commutative unitary $\mathbb{Q}$-algebra.

Definition 5.3. For any series $F\in R\langle\langle e_0,\ldots,e_n\rangle\rangle$ let us write

$$F=F_\varnothing+F_0e_0+\cdots+F_ne_n\ .$$

Thus F_i is obtained from F by deleting the last letter from all words ending in e_i. Here $F_\varnothing\in R$ denotes the coefficient of the empty word (leading term of F).

For any two series $A,B\in R\langle\langle e_0,\ldots,e_n\rangle\rangle$ their product satisfies

$$(AB)_i=A\left(B_i\right)+A_iB_\varnothing\ . \tag{5.1}$$

Definition 5.4. Consider the abelianisation map

$$F\mapsto\overline{F}\quad:\quad R\langle\langle e_0,\ldots,e_n\rangle\rangle\longrightarrow R[[s_0,\ldots,s_n]]$$

which sends e_i to s_i, where the s_i are commuting variables. We shall call $\overline{F}$ the *abelianisation* of F, and call $\overline{F_i}$ the i^{th} *beta-quotient* of F.

The $\overline{F_i}$ are very closely related to the image of F in what is known as the metabelian quotient.

We verify that, if F is invertible, then

$$\overline{(F^{-1})_i}=-\frac{1}{F_\varnothing}\frac{\overline{F_i}}{\overline{F}} \tag{5.2}$$

This follows from applying (5.1) to $FF^{-1}=1$ which gives $F(F^{-1})_i+F_iF^{-1}_\varnothing=0$, and then applying the abelianisation map. All series F that we shall consider are group-like (for the continuous coproduct on formal power series for which all letters e_i are primitive) and therefore have constant term $F_\varnothing=1$. Recall that $F(w)$ denotes the coefficient of (a linear combination of words) w in F.

Lemma 5.5. *For a group-like series F, we have:*

$$\overline{F}=\exp\left(F(e_0)s_0+\cdots+F(e_n)s_n\right)=\prod_{i=0}^{n}\exp(F(e_i)s_i)\ .$$

$$\overline{F_j}=\sum_{m_0,\ldots,m_n\geq0}F\left(\left[e_0^{\text{ш}\,m_0}\,\text{ш}\cdots\text{ш}\,e_n^{\text{ш}\,m_n}\right]e_j\right)\frac{s_0^{m_0}}{m_0!}\cdots\frac{s_n^{m_n}}{m_n!}\ ,$$

where $[w]e_j$ denotes the right-concatenation of e_j to any linear combination w of words in the letters $e_0, \ldots, e_n$. The previous expression can also be written

$$\overline{F_j} = \sum_{m_0,\ldots,m_n \geq 0} F\left([e_0^{m_0} ш \cdots ш e_n^{m_n}]\, e_j\right) s_0^{m_0} \cdots s_n^{m_n} \ .$$

Proof. Since F is group-like, $F = \exp(\log F))$, where $\log(F)$ is a Lie series of the form

$$\log(F) = \sum_{i=0}^{n} F(e_i)e_i + \text{commutators}$$

The exponential and logarithm are taken with respect to the concatenation product, and commute with the abelianisation map. Since abelianisation sends all commutators to zero,

$$\overline{F} = \exp(\overline{\log F}) = \exp\left(\sum_{i=0}^{n} F(e_i)s_i\right) \ .$$

For the second part, notice that

$$\overline{F_j} = \sum_{w} \overline{w}\, F(we_j) = \sum_{m_0,\ldots,m_n \geq 0} s_0^{m_0} \cdots s_n^{m_n} \left(\sum_{\overline{w} = s_0^{m_0} \cdots s_n^{m_n}} F(we_j)\right)$$

and substitute in the expression:

$$\left(\sum_{\overline{w} = s_0^{m_0} \cdots s_n^{m_n}} w\right) = \frac{e_0^{ш\, m_0}}{m_0!} ш \cdots ш \frac{e_n^{ш\, m_n}}{m_n!} = e_0^{m_0} ш \cdots ш\, e_n^{m_n} \ .$$

□

Corollary 5.6. *The period of the abelianised generating series is:*

$$\overline{\mathcal{Z}^i} = \operatorname{per}\left(\overline{\mathcal{Z}^{i,\mathfrak{m}}}\right) = \sigma_i^{s_0} \prod_{k \neq i} (1 - \sigma_i \sigma_k^{-1})^{s_k} \ . \tag{5.3}$$

For any indices $i \geq 1$ and all $0 \leq j \leq n$,

$$\overline{\mathcal{Z}^i_j} = \operatorname{per}\left(\overline{\mathcal{Z}^{i,\mathfrak{m}}_j}\right) = \int_{\gamma_i} x^{s_0} \prod_{k=1}^{n} \left(1 - x\, \sigma_k^{-1}\right)^{s_k} \frac{dx}{x - \sigma_j} \ . \tag{5.4}$$

These expressions should be interpreted as formal power series in the s_i by expanding the exponentials as power series. The coefficients of these power series are regularised iterated integrals.

More precisely, the logarithms $\log(1 - x\sigma_k^{-1})$ are to be interpreted as the regularised integral of $\frac{dz}{z - \sigma_k}$ along the path $\gamma_i^x = \gamma_i|_{[0,s]}$ where $x = \gamma_i(s)$ for some $0 \leq s \leq 1$. By the shuffle product formula, any polynomial in such logarithms can be interpreted as an iterated integral along γ_i^x.

Proof. For any $\omega \in \{\omega_i\}$, the definition §4.2 (2) of the path γ_i implies that

$$\int_{\gamma_i} \omega = \int_1^{\sigma_i} (\operatorname{Res}_0 \omega) \frac{dz}{z} + \int_{\gamma_i'} \omega \tag{5.5}$$

where the first integral is viewed in the tangent space of $\mathbb{C}$ at the origin, and γ_i' is the straight line path $x \mapsto \sigma_i x$ from the tangent vector σ_i at 0 to $-t_i$. The first integral vanishes for all i except $i = 0$, in which case it contributes $\log(\sigma_i)$. By a change of variables, the second integral is

$$\int_{\gamma_i'} \omega = \int_{\mathrm{dch}} \sigma_i^* \omega \tag{5.6}$$

where dch is the straight line from $\vec{1}_0$ to $-\vec{1}_1$, the tangent vectors 1 and -1 at $0, 1$ respectively. Since ω_0 is multiplicatively invariant, we have $\sigma_i^* \omega_0 = \omega_0$, and we deduce that

$$\mathcal{Z}^i(e_0) = \int_{\gamma_i} \omega_0 \overset{(5.5)}{=} \log(\sigma_i) + \int_{\gamma_i'} \frac{dz}{z} \overset{(5.6)}{=} \log(\sigma_i) + \int_{\mathrm{dch}} \frac{dz}{z} = \log(\sigma_i) \ .$$

When $\omega = \sigma_i$, we have

$$\mathcal{Z}^i(e_i) = \int_{\gamma_i} \omega_i \overset{(5.5)}{=} 0 + \int_{\gamma_i'} \frac{dz}{z-\sigma_i} \overset{(5.6)}{=} \int_{\mathrm{dch}} \frac{d(\sigma_i z)}{\sigma_i z - \sigma_i} = \int_{\mathrm{dch}} \frac{dz}{z-1} = 0 \ . \tag{5.7}$$

In all other cases $j \neq i$, the tangential basepoints play no role and

$$\mathcal{Z}^i(e_j) = \int_{\gamma_i} \omega_j = \int_0^{\sigma_i} \frac{dz}{z-\sigma_j} = \log\left(\frac{\sigma_j - \sigma_i}{\sigma_j}\right) \ .$$

By invoking the first part of Lemma 5.5, we deduce the formula for $\overline{\mathcal{Z}}^i$.

Now, for any path γ_x from t_0 to $x \in X_\Sigma(\mathbb{C})$, we have

$$\int_{\gamma_x} \frac{dz}{z} = \log(x) \quad \text{ and } \quad \int_{\gamma_x} \frac{dz}{z-\sigma_k} = \log(1 - x\,\sigma_k^{-1}) \ .$$

Let $x = \gamma_i(s)$ for some $0 \leq s \leq 1$, and let γ_i^x denote the restriction of γ_i to the interval $[0, s]$. All branches of the logarithm are canonically defined by the choice of path γ_i. By definition of iterated integrals as iterated line integrals (integrated from left to right), we deduce from Lemma 5.5 that

$$\overline{\mathcal{Z}_j^i} = \sum_{m_0,\ldots,m_n \geq 0} \frac{s_0^{m_0}}{m_0!} \cdots \frac{s_n^{m_n}}{m_n!} \int_{\gamma_i} \log^{m_0}(x) \prod_{k=1}^n \log^{m_n}(1 - x\sigma_k^{-1}) \frac{dx}{x-\sigma_j} \ .$$

This is precisely (5.4). □

Equation (5.4) in the case $\Sigma = \{0, 1\}$ reduces to the Drinfeld's computation of the metabelian quotient of his associator in terms of the usual beta function [Dri90]. See also [Enr06, Li10] for further developments. The following useful lemma can be extracted from the proof.

Lemma 5.7. *For all $m_i \geq 0$,*

$$\int_{\gamma_i} \log^{m_i}(1 - x\sigma_i^{-1}) \frac{dx}{x-\sigma_i} = 0 \ . \tag{5.8}$$

Proof. The integral is proportional to an (m_i+1)-fold iterated integral of $\frac{dx}{x-\sigma_i}$ along γ_i, which, by the shuffle product formula, is in turn proportional to the (m_i+1)th power of the integral of $\frac{dx}{x-\sigma_i}$ along γ_i, which vanishes by (5.7). □

6. Periods and single-valued periods of the beta-quotients

We show that the periods of the beta-quotients $\overline{\mathcal{Z}_j^{i,\mathfrak{m}}}$, which are regularised with respect to tangential base-points, are the coefficients in the expansions of renormalised beta integrals. The same statement holds for their single-valued versions.

6.1. Complex periods without tangential regularisation.

For all $1 \leq j \leq n$ define:

$$\Omega_j = x^{s_0} \prod_{k=1}^n (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x-\sigma_j} \ . \tag{6.1}$$

Let $\Omega_j^{\mathrm{ren}_i}$ denote its renormalised versions with respect to $\{0, \sigma_i\}$ where $1 \leq i \leq n$.

Proposition 6.1. *For all $1 \leq j \leq n$ we have the convergent integral formula:*

$$\overline{\mathcal{Z}_j^i} = \int_0^{\sigma_i} \Omega_j^{\mathrm{ren_i}} \tag{6.2}$$

It amounts to the equations:

$$\overline{\mathcal{Z}_j^i} = \int_0^{\sigma_i} x^{s_0} \prod_{k=1}^n \left(1 - x\,\sigma_k^{-1}\right)^{s_k} \frac{dx}{x-\sigma_j} \qquad \textit{if} \quad j \neq i \ ,$$

$$\overline{\mathcal{Z}_i^i} = \int_0^{\sigma_i} \left(x^{s_0} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} \left(1 - x\,\sigma_k^{-1}\right)^{s_k} - \sigma_i^{s_0} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} \left(1 - \sigma_i\,\sigma_k^{-1}\right)^{s_k} \right) (1 - x\sigma_i^{-1})^{s_i} \frac{dx}{x-\sigma_i} \ .$$

These integrals are to be interpreted as formal power series in the s_i: a Taylor expansion of the integrand leads to a power series in the s_i whose coefficients are convergent integrals.

The proposition holds essentially because iterated integrals regularised with respect to tangential base points and the renormalised forms Ω^{ren_i} are defined via a similar geometric procedure (restriction to the tangent space at a singular point). We give two proofs of the proposition, since they are instructive. First note that the second statement is equivalent to the first by Proposition 3.2 since $R_0\Omega_j = 0$, and

$$\mathsf{J}_i^* R_i \Omega_j = \delta_{ij}\ \sigma_i^{s_0} \prod_{\substack{1\leq k\leq n\\ k\neq i}} \left(1-\sigma_i\,\sigma_k^{-1}\right)^{s_k} (1-x\sigma_i^{-1})^{s_i} \frac{dx}{x-\sigma_i}\ .$$

Thus $\Omega_j^{\mathrm{ren}_i} = \Omega_j$ if $i\neq j$ and equals the integrand in the formula for $\overline{\mathcal{Z}_i^i}$ otherwise.

First proof. For γ a path between (tangential) base-points $x, y \in X_\Sigma$, and ω a closed formal one-form taking values in the Lie algebra of $\mathbb{C}\langle\langle e_0,\ldots,e_n\rangle\rangle$ let

$$I_\gamma(\omega) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots$$

denote the transport of (the connection associated to) ω along γ. It satisfies $I_{\gamma\gamma'}(\omega) = I_\gamma(\omega) I_{\gamma'}(\omega)$. The transport of the formal one-form $\omega_\Sigma = e_0\omega_0 + \cdots + e_n\omega_n$ along the path γ_i equals:

$$\mathcal{Z}^i = I_{\gamma_i}(\omega_\Sigma) = \lim_{x\to\sigma_i} \left(I_{\gamma_x}(\omega_\Sigma) I_{\gamma_x'}(e_i\omega_i)\right)$$

where $x = \gamma_i(t)$ for some $0<t<1$, and γ_x is the restriction of γ_i to $[0,t]$. The second integral is along a path γ_x' from x to $-t_i$ viewed in the tangent space at σ_i, which we identify with $\mathbb{P}^1\backslash\{\sigma_i,\infty\}$. The previous formula follows from the prescription for computing iterated integrals with respect to tangential basepoints: $e_i\omega_i$ is the localisation of ω_Σ to the punctured tangent space and captures the divergent iterated integrals terminating in the letter e_i. It follows from (5.1) that

$$\overline{\mathcal{Z}_j^i} = \lim_{x\to\sigma_i} \left(\overline{I_{\gamma_x}(\omega_\Sigma)}\,\overline{I_{\gamma_x'}(e_i\omega_i)_j} + \overline{I_{\gamma_x}(\omega_\Sigma)_j}\right)\ . \tag{6.3}$$

The tangential integral is an exponential:

$$I_{\gamma_x'}(e_i\omega_i) = \exp(-e_i \log(1-x\sigma_i^{-1}))\ . \tag{6.4}$$

This follows, for example, from

$$\int_x^{-t_i} \frac{dz}{z-\sigma_i} = \left(\int_x^{t_0} + \int_{t_0}^{-t_i}\right)\frac{dz}{z-\sigma_i} \overset{(5.8)}{=} \int_x^0 \frac{dz}{z-\sigma_i} + 0 = -\log(1-x\sigma_i^{-1})\ .$$

From equation (6.4), the expression $\overline{I_{\gamma_x'}(e_i\omega_i)}_j$ vanishes if $j\neq i$, but equals

$$\overline{I_{\gamma_x'}(e_i\omega_i)_i} = \frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{-s_i}\right)$$

otherwise. Thus if $j\neq i$, the first term in (6.3) drops out and we find that

$$\overline{\mathcal{Z}_j^i} = \lim_{x\to\sigma_i}\left(\overline{I_{\gamma_x}(\omega_\Sigma)}_j\right) = \lim_{x\to\sigma_i}\left(\int_0^x x^{s_0}\prod_{k=1}^n \left(1-x\,\sigma_k^{-1}\right)^{s_k} \frac{dx}{x-\sigma_j}\right)\ ,$$

using the computations in the second paragraph of the proof of Corollary 5.6 (one needs only check that one can replace γ_x with an ordinary path from 0 to x, i.e., that the tangential component of γ_x at the origin plays no role since the integral is convergent there). This proves the formula for $j\neq i$. In the case $j=i$, (6.3) and (5.3) give

$$\overline{\mathcal{Z}_i^i} = \lim_{x\to\sigma_i}\left(x^{s_0}\prod_{k=1}^n\left(1-x\,\sigma_k^{-1}\right)^{s_k}\frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{-s_i}\right) + \int_0^x x^{s_0}\prod_{k=1}^n\left(1-x\,\sigma_k^{-1}\right)^{s_k}\frac{dx}{x-\sigma_i}\right)\ .$$

Rewrite this in the form

$$\overline{\mathcal{Z}_i^i} = \lim_{x\to\sigma_i}\left(- \sigma_i^{s_0} \prod_{k\neq i} (1-\sigma_i\,\sigma_k^{-1})^{s_k}\, \frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{s_i}\right) + \int_0^x x^{s_0}\prod_{k=1}^n (1-x\,\sigma_k^{-1})^{s_k}\,\frac{dx}{x-\sigma_i}\right)$$

and substitute in the following identity

$$\int_0^x (1-x\sigma_i^{-1})^{s_i}\,\frac{dx}{x-\sigma_i} = \frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{s_i}\right) \tag{6.5}$$

to deduce the stated formula for $\overline{\mathcal{Z}_i^i}$.

□

Second proof. We consider only the case $\overline{\mathcal{Z}_i^i}$ since the argument for $\overline{\mathcal{Z}_j^i}$ with $j\neq i$ is even simpler. Express $\Omega_j^{\mathrm{ren}_i}$ as a formal power series:

$$\Omega_j^{\mathrm{ren}_i} = \sum_{m_0,\cdots,m_n\geq 0} \Omega_j^{\mathrm{ren}_i}(\underline{m})\,\frac{s_0^{m_0}\cdots s_n^{m_n}}{m_0!\cdots m_n!}$$

where $\underline{m}$ denotes $(m_0,\ldots,m_n)$. In the case $j=i$, the coefficients equal

$$\begin{aligned}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \quad = \quad & \log^{m_i}(1-x\sigma_i^{-1})\frac{dx}{x-\sigma_i}\\ & \times\left(\log^{m_0}(x)\prod_{1\leq k\neq i}\log^{m_k}(1-x\sigma_k^{-1}) - \log^{m_0}(\sigma_i)\prod_{1\leq k\neq i}\log^{m_k}(1-\sigma_i\sigma_k^{-1})\right)\ .\end{aligned}$$

We claim that

$$\int_{\gamma_i}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \ = \ \int_0^{\sigma_i}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \tag{6.6}$$

where the integral on the right converges. To see this, use the fact that regularisation with respect to the tangential basepoint $-t_i$ is equivalent to taking a primitive of $\Omega_i^{\mathrm{ren}_i}(\underline{m})$ in the ring $\mathbb{C}[\log(x-\sigma_i)][[x-\sigma_i]]$, and formally setting all $\log(x-\sigma_i)$ terms to zero, before in turn setting x to σ_i. Since the term in brackets in the above expression for $\Omega_i^{\mathrm{ren}_i}(\underline{m})$ vanishes at $x=\sigma_i$, its primitive actually lies in the subspace $\mathbb{C}[(x-\sigma_i)\log(x-\sigma_i)][[x-\sigma_i]]$, and one can simply take its limit as $x\to\sigma_i$, which is the procedure for computing an ordinary integral (without tangential basepoint regularisation). A simpler argument applies at $x=0$ and proves (6.6).

The formula for $\overline{\mathcal{Z}_i^i}$ follows by applying (5.8) and implies that

$$\int_{\gamma_i}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \ = \int_{\gamma_i}\log^{m_0}(x)\prod_{k=1}^n\log^{m_k}(1-x\sigma_k^{-1})\frac{dx}{x-\sigma_i}\ .$$

This is precisely the coefficient of $\frac{s_0^{m_0}}{m_0!}\cdots\frac{s_n^{m_n}}{m_n!}$ in the Taylor expansion of (5.4). □

6.2. **Single-valued periods.** We repeat a similar analysis for the single-valued periods. The technique in this paragraph can be used more generally to give integral formulae for the single-valued periods of motivic torsors of path between tangential basepoints.

Lemma 6.2. *Suppose that $\sigma,x,z\in\mathbb{C}$ are distinct. Then*

$$-\frac{1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{w}}{\overline{w}-\overline{z}}-\frac{d\overline{w}}{\overline{w}-\overline{x}}\right)\wedge\frac{dw}{w-\sigma} \ = \ \log\left|\frac{z-\sigma}{x-\sigma}\right|^2\ .$$

Proof. This is simply the single-valued version [BD18] of the integral

$$\int_x^z\frac{dw}{w-\sigma} = \log\left(\frac{z-\sigma}{x-\sigma}\right)\ .$$

It can also be proved directly using Stokes' formula (see [BD18, §6.3]). □

Proposition 6.3. *Let $x, y \in X_\Sigma(k)$ be distinct and let*

$$I^{\varpi}(x,y) \;\in\; {}_x\Pi_y(\mathcal{P}^{\varpi})$$

denote the generating series of canonical de Rham periods from x to y. Then

$$\mathbf{s}\left(\overline{I^{\varpi}(x,y)}\right) = \left|\frac{y}{x}\right|^{2s_0} \prod_{k=1}^{n} \left|\frac{1-y\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^{2s_k} \tag{6.7}$$

and the associated single-valued beta function is

$$\mathbf{s}\left(\overline{I^{\varpi}(x,y)_j}\right) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \left|\frac{z}{x}\right|^{2s_0} \prod_{k=1}^{n} \left|\frac{1-z\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^{2s_k} \left(\frac{d\overline{z}}{\overline{z}-\overline{y}} - \frac{d\overline{z}}{\overline{z}-\overline{x}}\right)\wedge\frac{dz}{z-\sigma_j}\ . \tag{6.8}$$

The right-hand side is to be viewed as a formal power series expansion in the $s_0,\ldots,s_n$.

Proof. The following argument is slightly more intuitive using motivic, rather than de Rham periods, so we shall first compute $I^{\mathfrak{m}}_{\gamma}(x,y) \in {}_x\Pi_y(\mathcal{P}^{\mathfrak{m}})$, the image under the universal comparison map of a path $\gamma \in \pi_1(\mathbb{C}\backslash\Sigma, x, y)$, and then use the projection

$$I^{\varpi}(x,y) = \pi^{\mathfrak{m},+}_{\varpi}\left(I^{\mathfrak{m}}_{\gamma}(x,y)\right)$$

to deduce a formula for $I^{\varpi}(x,y)$. Since x,y are ordinary basepoints, the motive underlying the torsor of paths is given by Beilinson's cosimplicial construction [DG05, §3.3] and

$$I^{\mathfrak{m}}(x,y) = \sum_{w\in\{e_0,\ldots,e_n\}^\times, |w|=\ell} [H^\ell(X^\ell_\Sigma, Y^\ell), \gamma_*[\Delta_\ell]\,, w]^{\mathfrak{m}}\, w$$

where $|w|$ denotes the length of a word w, and the divisor $Y^\ell \subset X^\ell_\Sigma$ is

$$Y^\ell = \{z_1 = x\}\cup\{z_1=z_2\}\cup\ldots\cup\{z_{\ell-1}=z_\ell\}\cup\{z_\ell = y\}\ ,$$

and $[\Delta_\ell]$ is the relative homology class of the standard simplex

$$\Delta_\ell = \{0\le t_1\le t_2\le\ldots\le t_\ell<1\}\ .$$

The coordinates $z_1,\ldots,z_\ell$ are the coordinates on X^ℓ_Σ, and $[]^{\mathfrak{m}}$ denote matrix coefficients in a Tannakian category [Bro17]. By lemma 5.5 the coefficient of $\frac{s_0^{m_0}}{m_0!}\ldots\frac{s_n^{m_n}}{m_n!}s_j$ in $\overline{I^{\mathfrak{m}}_{\gamma}(x,y)_j}$ is

$$\left[H^{m+1}(X^{m+1}_\Sigma, Y^{m+1})\ ,\ \gamma_*[\Delta_{m+1}]\ ,\ e_0^{ш\, m_0}\, ш\, \ldots\, ш\, e_n^{ш\, m_n} e_j\right]^{\mathfrak{m}}$$

where $m = m_0+\ldots+m_n$. By permuting the coordinates we can rewrite this as

$$\left[H^{m+1}(X^{m+1}_\Sigma\ ,\ \widetilde{Y}^{m+1})\ ,\ \gamma_*\left[C_{m+1}\right]\ ,\ e_0^{m_0}\ldots e_n^{m_n}e_j\right]^{\mathfrak{m}} \tag{6.9}$$

where

$$\widetilde{Y}^{m+1} = \bigcup \sigma^* Y^{m+1} \qquad\text{and}\qquad C_{m+1} = \bigcup_{\sigma}\sigma\,\Delta_\ell$$

and $\sigma\in\mathfrak{S}_m$ ranges over permutations of all but the last coordinate. Since the union of $m!$ simplices glue together to form a cube, one has

$$C_{m+1} = \{0\le t_1,\ldots,t_m\le t_{m+1}\le 1\}\ .$$

The boundary of $\gamma_*(C_{m+1})$ is contained in the complex points of the divisor $V\subset X^{m+1}_\Sigma$ defined by the union of $\{z_i = x\}$ and $\{z_i = z_{m+1}\}$ for $1\le i\le m$, and $\{z_{m+1}=y\}$. In (6.9), therefore, we can replace $H^{m+1}(X^{m+1}_\Sigma, \widetilde{Y}^{m+1})$ with $H^{m+1}(X^{m+1}_\Sigma, V)$ by equivalence of motivic periods. Now take the image of (6.9) under the projection $\pi^{\mathfrak{m},+}_{\varpi}$. By [BD18], the image of the homology framing under the rational period map $c_0^\vee$ studied in loc. cit. is the differential form (writing $z = z_{m+1}$):

$$\nu = \bigwedge_{i=1}^{m}\left(\frac{dz_i}{z_i - z} - \frac{dz_i}{z_i-x}\right)\wedge\left(\frac{dz}{z-y}-\frac{dz}{z-x}\right)\ .$$

It follows, then, from theorem 3.17 in [BD18] that its single-valued period is

$$\frac{1}{(-2\pi i)^{m+1}}\int_{\mathbb{C}^{m+1}}\overline{\nu}\wedge\frac{dz_1}{z_1-\beta_1}\cdots\frac{dz_m}{z_m-\beta_m}\wedge\frac{dz}{z-\sigma_j}$$

where $(\beta_1,\ldots,\beta_m)$ is $(0^{m_0},\sigma_1^{m_1},\ldots,\sigma_n^{m_n})$ (a sequence of m_0 0's followed by m_1 σ_1's and so on). Now apply lemma 6.2 repeatedly to perform the m integrals:

$$\frac{-1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{z_i}}{\overline{z_i}-\overline{z}}-\frac{d\overline{z_i}}{\overline{z_i}-\overline{x}}\right)\wedge\frac{dz_i}{z_i-\beta_i}=\log\left(\left|\frac{z-\beta_i}{x-\beta_i}\right|\right)$$

for $1\leq i\leq m$ to obtain

$$-\frac{1}{2\pi i}\int_{\mathbb{C}}\log^{m_0}\left(\left|\frac{z}{x}\right|^2\right)\prod_{k=1}^{n}\log^{m_k}\left(\left|\frac{1-z\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^2\right)\left(\frac{d\overline{z}}{\overline{z}-\overline{y}}-\frac{d\overline{z}}{\overline{z}-\overline{x}}\right)\wedge\frac{dz}{z-\sigma_j}\ .$$

This yields (6.8) after expanding the exponential factors as power series in the s_i. □

Theorem 6.4. *The single-valued periods of the beta quotients satisfy*

$$\mathbf{s}(\overline{\mathcal{Z}_j^{i,\varpi}})=-\frac{1}{2\pi i}\int_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega_j^{\mathbf{s},\mathrm{ren}_i}\ .$$

Explicitly, this amounts to the equations

$$\mathbf{s}(\overline{\mathcal{Z}_j^{i,\varpi}})=-\frac{1}{2\pi i}\int_{\mathbb{C}}|z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_j}\tag{6.10}$$

whenever $i\neq j$, and in the case $j=i$ to

$$\mathbf{s}(\overline{\mathcal{Z}_i^{i,\varpi}})=-\frac{1}{2\pi i}\int_{\mathbb{C}}\left|1-z\sigma_i^{-1}\right|^{2s_i}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_i}\ \times\left(|z|^{2s_0}\prod_{\substack{k=1\\k\neq i}}^{n}|1-z\sigma_k^{-1}|^{2s_k}-|\sigma_i|^{2s_0}\prod_{\substack{k=1\\k\neq i}}^{n}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\right)\ .\tag{6.11}$$

The abelianisation satisfies

$$\mathbf{s}(\overline{\mathcal{Z}^{i,\varpi}})=|\sigma_i|^{2s_0}\prod_{k\neq i}\left|1-\sigma_i\sigma_k^{-1}\right|^{2s_k}\ .\tag{6.12}$$

Proof. For any small $0<\varepsilon$ there exists a $\tau\in k^\times$ such that $|\tau|<\varepsilon$. By the composition of paths formula, we have for any such τ,

$$\mathcal{Z}^{i,\varpi}=I^{\varpi}_{t_0,\tau}\,I^{\varpi}_{\tau,\sigma_i-\tau}\,I^{\varpi}_{\sigma_i-\tau,t_i}\tag{6.13}$$

where $I^{\varpi}_{x,y}\in\mathcal{P}^{\varpi}\langle\langle e_0,\ldots,e_n\rangle\rangle$ is the image of the canonical path ${}_x1_y$ from x to y (given by the morphism of schemes $g\mapsto g.{}_x1_y: G^{\varpi}\to{}_x\Pi_y$). We can apply the single-valued period homomorphism $\mathbf{s}$ coefficient-wise to the expression (6.13). Since single-valued periods are by definition quadratic expressions in complex periods and their complex conjugates, we deduce that

$$\mathbf{s}\left(I^{\varpi}_{t_0,\tau}\right)=\exp(e_0\log|\tau|^2)(1+O(\tau,\overline{\tau}))\ .$$

By taking the single-valued version of (6.4) with $x=\sigma_i-\tau$, we have

$$\begin{aligned}\mathbf{s}\left(I^{\varpi}_{\sigma_i-\tau,t_i}\right)&=\exp\left(-e_i\log\left|1-\frac{\sigma_i-\tau}{\sigma_i}\right|^2\right)(1+O(\tau,\overline{\tau}))\\&=\exp(-e_i\log|\tau|^2+e_i\log|\sigma_i|^2)(1+O(\tau,\overline{\tau}))\ .\end{aligned}$$

Since $\mathbf{s}(\mathcal{Z}^{i,\varpi})=\mathbf{s}(I^{\varpi}_{t_0,\tau})\,\mathbf{s}(I^{\varpi}_{\tau,\sigma_i-\tau})\mathbf{s}(I^{\varpi}_{\sigma_i-\tau,t_i})$ is independent of τ, we conclude that

$$\mathbf{s}(\overline{\mathcal{Z}_j^{i,\varpi}})=\mathrm{Reg}_{\tau\to0}\,\overline{\mathbf{s}\left(I^{\varpi}_{\tau,\sigma_i-\tau}\exp(e_i\log|\sigma_i|^2)\right)_j}$$

where $\mathrm{Reg}_{\tau\to0}$ means the following: formally set $\log|\tau|^2$ to zero, and then take the limit as $\tau\to0$. Since the coefficients the formal power series $\mathbf{s}(\mathcal{Z}^{i,\varpi})$ can be expressed as elements in $\mathbb{C}[[\tau,\overline{\tau}]][\log^2|\tau|]$, this operation is well-defined.

First consider the case when $j \neq i$. Then by (5.1) and $\exp(e_i \log|\sigma_i|^2)_j = 0$,

$$\mathbf{s}(\overline{\mathcal{Z}_j^{i,\varpi}}) = \mathrm{Reg}_{\tau \to 0}\, \mathbf{s}\overline{(I^{\varpi}_{\tau,\sigma_i-\tau})_j}\ .$$

By equation (6.8) this equals

$$\mathrm{Reg}_{\tau\to 0}\left(-\frac{1}{2\pi i}\int_{\mathbb{C}}\left|\frac{z}{\tau}\right|^{2s_0}\prod_{k=1}^{n}\left|\frac{1-z\sigma_k^{-1}}{1-\tau\sigma_k^{-1}}\right|^{2s_k}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}-\overline{\tau}}\right)\wedge\frac{dz}{z-\sigma_j}\right)\ .$$

After expanding in the s_i, the term $|\tau|^{-2s_0}$ contributes only powers of $\log|\tau|^2$ which disappear upon regularisation. Then, after taking $\tau \to 0$, we are left precisely with (6.10).

Now consider the case $j = i$. By (5.1)

$$\mathbf{s}(\overline{\mathcal{Z}_i^{i,\varpi}}) = \mathrm{Reg}_{\tau\to 0}\left(\mathbf{s}\overline{(I^{\varpi}_{\tau,\sigma_i-\tau})_i} + \mathbf{s}\overline{(I^{\varpi}_{\tau,\sigma_i-\tau})}\left(|\sigma_i|^{2s_i}\right)_i\right)\ . \tag{6.14}$$

Note that $\left(|\sigma_i|^{2s_i}\right)_i = \frac{1}{s_i}(|\sigma_i|^{2s_i}-1)$. By equation (6.7),

$$\mathrm{Reg}_{\tau\to 0}\left(\mathbf{s}\overline{(I^{\varpi}_{\tau,\sigma_i-\tau})}\left(|\sigma_i|^{2s_i}\right)_i\right) = |\sigma_i|^{2s_0}\prod_{k\neq i}\left|1-\sigma_i\sigma_k^{-1}\right|^{2s_k}\left(\frac{1}{s_i}(1-|\sigma_i|^{-2s_i})\right)\ .$$

Using lemma 3.3, we can interpret the right-most factor as:

$$\left(\frac{1}{s_i}\left(1-|\sigma_i|^{-2s_i}\right)\right) = \mathrm{Reg}_{\tau\to 0}\left(\frac{1}{2\pi i}\int_{\mathbb{C}}\left|1-z\sigma_i^{-1}\right|^{2s_i}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_i}\right)\ .$$

Substitute into (6.14) and taking the regularised limit as $\tau \to 0$ to obtain (6.11).

The formula (6.12) follows from abelianising (6.13) and taking the limit:

$$\mathbf{s}(\overline{\mathcal{Z}^{i,\varpi}}) = \mathrm{Reg}_{\tau\to 0}\left(|\tau|^{2s_0}\mathbf{s}\ \left(\overline{I^{\varpi}_{\tau,\sigma_i-\tau}}\right)\left|1-\frac{(\sigma_i-\tau)}{\tau}\right|^{-2s_i}\right)$$

after substituting in equation (6.7). $\square$

7. Action of the motivic Galois group

We compute the action of the motivic Galois group (or equivalently, the motivic coaction) on the full motivic torsor of paths, and use it to deduce a formula for the coaction on the beta quotients.

7.1. **Formula for the motivic Galois action.** Since the ${}_i\Pi_j$ are realisations of pro-objects in the category $\mathcal{MT}(k)$, they admit an action of the motivic Galois group. Let $\lambda : G^{\varpi}_{\mathcal{MT}(k)} \to \mathbb{G}_m$ be the homomorphism given by the action of $G^{\varpi}_{\mathcal{MT}(k)}$ on $\mathbb{Q}(1)_{\varpi}$.

Proposition 7.1. *For every element $g \in G^{\varpi}_{\mathcal{MT}(k)}$, its action on any $F \in {}_0\Pi_i$ is given by a version of Ihara's formula:*

$$gF(e_0, e_1, \ldots, e_n) = F\left(\lambda_g e_0, G_1\lambda_g e_1 G_1^{-1}, \ldots, G_n\lambda_g e_n G_n^{-1}\right)G_i \tag{7.1}$$

where $G_j \in \mathbb{Q}\langle\langle e_0,\ldots,e_n\rangle\rangle$ is the group-like formal power series $G_j = g\left({}_0 1_j\right)$.

Proof. The argument is a very mild generalisation of the argument given in [DG05, §5] (with the reverse conventions) or [Bro14, Proposition 2.5] so we shall be brief. One first computes the action of g on ${}_0\Pi_0$. It acts on the element $\exp(e_0) \in {}_0\Pi_0$, by scaling $\exp(e_0) \mapsto \exp(\lambda_g e_0)$ since it is in the image of the local monodromy $\pi_1^{\varpi}(\mathbb{G}_m, 1)$ which is isomorphic to $\mathbb{Q}(1)_{\varpi}$. For all $1 \leq i \leq n$, the element $\exp(e_i) \in {}_i\Pi_i$ is in the image of the local monodromy

$$x \mapsto \sigma_i x : (\mathbb{G}_m, 1) \longrightarrow ((T_\sigma \mathbb{A}^1)^{\times}, t_i)$$

and hence, by a similar argument, is also acted upon by g by multiplication by λ_g. Therefore, transporting this action back to ${}_0\Pi_0$ via

$$\left({}_0 1_i\right)\ e_i\ \left({}_0 1_i^{-1}\right) \mapsto e_i \quad : \quad {}_0\Pi_i \times {}_i\Pi_i \times {}_i\Pi_0 \longrightarrow {}_0\Pi_0$$

we deduce that g acts on the element $e_i \in {}_0\Pi_0$ via $e_i \mapsto G_i \lambda_g e_i G_i^{-1}$ for all $1 \leq i \leq n$. Finally, use the torsor structure

$${}_0F_0 \times {}_01_i \mapsto {}_0F_i \quad : \quad {}_0\Pi_0 \times {}_0\Pi_i \longrightarrow {}_0\Pi_i$$

to conclude that the action of g on any $F \in {}_0\Pi_i$ is indeed

$$gF = F\left(\lambda_g e_0, G_1 \lambda_g e_1 G_1^{-1}, \ldots, G_n \lambda_g e_n G_n^{-1}\right) G_i \ .$$

□

Applying this now to the series $F = \mathcal{Z}^{i,\mathfrak{m}}$, we deduce that

$$g\mathcal{Z}^{i,\mathfrak{m}}(e_0, e_1, \ldots, e_n) = \mathcal{Z}^{i,\mathfrak{m}}\left(\lambda_g e_0, G_1 \lambda_g e_1 G_1^{-1}, \ldots, G_n \lambda_g e_n G_n^{-1}\right) G_i \ ,$$

where g acts trivially on coefficients. The same formula holds with $\mathfrak{m}$ replaced by ϖ. This formula can be re-expressed as a universal coaction formula for

$$\Delta^{\mathfrak{m}}\, \mathcal{Z}^{i,\mathfrak{m}} \in (\mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi}) \langle\langle e_0, \ldots, e_n \rangle\rangle$$

where $\Delta^{\mathfrak{m}}$ is applied coefficient-wise to words in the e_i (in other words, it is extended by linearity and continuity to $\mathcal{P}^{\mathfrak{m}}\langle\langle e_0, \ldots, e_n\rangle\rangle$, where it acts trivially on the e_i). The action of g is retrieved by the usual formula $g\, \mathcal{Z}^{i,\mathfrak{m}} = (\mathrm{id} \otimes g) \Delta^{\mathfrak{m}} \mathcal{Z}^{i,\mathfrak{m}}$.

Proposition 7.2. *The following formula holds:*

$$\Delta^{\mathfrak{m}}\left(\mathcal{Z}^{i,\mathfrak{m}}\right) = \mathcal{Z}^{i,\mathfrak{m}}\left(e_0', e_1', \ldots, e_n'\right) \mathcal{Z}^{i,\varpi} \tag{7.2}$$

where $e_0', \ldots, e_n'$ are defined by $e_0' = \mathbb{L}^{\varpi} e_0$ and

$$e_j' = \left(\mathcal{Z}^{j,\varpi}\right) \mathbb{L}^{\varpi} e_j \left(\mathcal{Z}^{j,\varpi}\right)^{-1} \qquad \text{for all} \quad 1 \leq j \leq n$$

Similarly, $\Delta^{\varpi}\left(\mathcal{Z}^{i,\varpi}\right) = \mathcal{Z}^{i,\varpi}\left(e_0', e_1', \ldots, e_n'\right) \mathcal{Z}^{i,\varpi}$.

Proof. For all $g \in G^{\varpi}_{\mathcal{MT}(k)}$, one has $g\left({}_01_i\right) = g\mathcal{Z}^{i,\varpi}$ by definition of $\mathcal{Z}^{i,\varpi}$, where in the right-hand side of this formula, g acts only on the coefficients of $\mathcal{Z}^{i,\varpi}$. Furthermore, any such g acts on the word $e_k \in \mathcal{O}({}_0\Pi_i^{\varpi})$ (dual to the generator e_k of ${}_0\Pi_i^{\varpi}$) via λ_g^{-1}, i.e., $ge_i = (\mathrm{id} \otimes g)(\mathbb{L}^{\varpi} e_i)$. □

All products in the right-hand side of (7.2) are given by concatentation of non-commutative formal power series. The coefficients of these series are viewed in the ring $\mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi}$.

Example 7.3. In the setting of example 5.2, proposition 7.2 yields

$$\Delta^{\mathfrak{m}}\, \mathcal{Z}^{\mathfrak{m}}(e_0, e_1) = \mathcal{Z}^{\mathfrak{m}}\left(\mathbb{L}^{\mathfrak{dr}} e_0, \mathcal{Z}^{\mathfrak{dr}} \mathbb{L}^{\mathfrak{dr}} e_1 \left(\mathcal{Z}^{\mathfrak{dr}}\right)^{-1}\right) \mathcal{Z}^{\mathfrak{dr}}$$

which is a motivic version of Ihara's formula, and expresses the coaction on motivic multiple zeta values. For example, reading off the coefficient of $-e_1 e_0^{n-1}$ yields

$$\Delta^{\mathfrak{m}} \zeta^{\mathfrak{m}}(n) = \zeta^{\mathfrak{m}}(n) \otimes (\mathbb{L}^{\mathfrak{dr}})^n + 1 \otimes \zeta^{\mathfrak{dr}}(n)$$

since $\varpi = \mathfrak{dr}$ in this case. The terms $\mathbb{L}^{\mathfrak{dr}}$ are equivalent to the weight grading.

7.2. **Coaction on the beta quotients.**

Definition 7.4. Consider the $n \times n$ matrix $FL^{\mathfrak{m}}_{\Sigma} \in M_{n \times n}(\mathcal{P}^{\mathfrak{m},+}[[s_0, \ldots, s_n]])$

$$(FL^{\mathfrak{m}}_{\Sigma})_{ij} = \delta_{ij}\, \overline{\mathcal{Z}^{i,\mathfrak{m}}} - s_j \overline{\mathcal{Z}^{i,\mathfrak{m}}_j}$$

where δ_{ik} is the Kronecker delta. Let FL^{ϖ}_{Σ} be the matrix defined in the same way by replacing $\mathfrak{m}$ with ϖ. It is the image of $FL^{\mathfrak{m}}_{\Sigma}$ under $\pi^{\mathfrak{m},+}_{\varpi}$.

Theorem 7.5. *The motivic coaction, applied to the entries of $FL^{\mathfrak{m}}_{\Sigma}$, satisfies:*

$$\Delta^{\mathfrak{m}} FL^{\mathfrak{m}}_{\Sigma}(s_0, \ldots, s_n) = FL^{\mathfrak{m}}_{\Sigma}(\mathbb{L}^{\varpi} s_0, \ldots, \mathbb{L}^{\varpi} s_n) \otimes FL^{\varpi}_{\Sigma}(s_0, \ldots, s_n) \ . \tag{7.3}$$

Similarly, the coproduct satisfies

$$\Delta^{\varpi} FL^{\varpi}_{\Sigma}(s_0, \ldots, s_n) = FL^{\varpi}_{\Sigma}(\mathbb{L}^{\varpi} s_0, \ldots, \mathbb{L}^{\varpi} s_n) \otimes FL^{\varpi}_{\Sigma}(s_0, \ldots, s_n) \ .$$

Proof. It is convenient to compute modulo $\mathbb{L}^{\varpi}=1$ and restore all powers of $\mathbb{L}^{\varpi}$ at the end, since they are uniquely determined by the weight-grading. Using formula (7.2) we have

$$\Delta^{\mathfrak{m}}\left(\overline{\mathcal{Z}_k^{i,\mathfrak{m}}}\right)=\overline{\left(\mathcal{Z}^{i,\mathfrak{m}}\left(e_0',e_1',\ldots,e_n'\right)\mathcal{Z}^{i,\varpi}\right)_k} .$$

The right-hand side reduces via (5.1) to

$$\overline{\mathcal{Z}^{i,\mathfrak{m}}}\,\overline{\mathcal{Z}_k^{i,\varpi}}+\overline{\mathcal{Z}^{i,\mathfrak{m}}\left(e_0',e_1',\ldots,e_n'\right)_k}$$

since e_j' is conjugate to e_j and so they have the same image under abelianisation. The previous expression can in turn be written

$$\overline{\mathcal{Z}^{i,\mathfrak{m}}}\,\overline{\mathcal{Z}_k^{i,\varpi}}+\sum_{j=1}^n\overline{\mathcal{Z}_j^{i,\mathfrak{m}}}\left(\overline{e_j'}\right)_k=\overline{\mathcal{Z}^{i,\mathfrak{m}}}\,\overline{\mathcal{Z}_k^{i,\varpi}}+\sum_{j=1}^n\overline{\mathcal{Z}_j^{i,\mathfrak{m}}}\left(\overline{\mathcal{Z}^{j,\varpi}}\,\overline{e_j}\,\overline{\left(\mathcal{Z}^{j,\varpi}\right)^{-1}}\right)_k .$$

we have

$$\left(\overline{\mathcal{Z}^{j,\varpi}}\,\overline{e_j}\,\overline{\left(\mathcal{Z}^{j,\varpi}\right)^{-1}}\right)_k\overset{(5.1)}{=}\overline{\mathcal{Z}^{j,\varpi}}\overline{e_j}\overline{\left(\mathcal{Z}^{j,\varpi}\right)_k^{-1}}+\overline{\mathcal{Z}^{j,\varpi}}\delta_{jk}\overset{(5.2)}{=}-\overline{e_j}\,\overline{\mathcal{Z}_k^{j,\varpi}}+\delta_{jk}\overline{\mathcal{Z}^{k,\varpi}}$$

Putting the pieces together gives

$$\Delta^{\mathfrak{m}}\left(\overline{\mathcal{Z}_k^{i,\mathfrak{m}}}s_k\right)=\overline{\mathcal{Z}^{i,\mathfrak{m}}}\,\overline{\mathcal{Z}_k^{i,\varpi}}s_k-\sum_{j=1}^n\overline{\mathcal{Z}_j^{i,\mathfrak{m}}}s_j\left(\overline{\mathcal{Z}_k^{j,\varpi}}s_k-\delta_{jk}\overline{\mathcal{Z}^{k,\varpi}}\right)$$

Denote by $\mathsf{Z}^{\mathfrak{m}}$ (resp. Z^{ϖ}) the $n\times n$-matrix whose $(i,k)^{\text{th}}$ entry is $\overline{\mathcal{Z}_k^{i,\mathfrak{m}}}s_k$ (resp. $\overline{\mathcal{Z}_k^{i,\varpi}}s_k$), and $\mathsf{A}^{\mathfrak{m}}$ the $(n\times n)$ diagonal matrix whose $(i,i)^{\text{th}}$ entry is $\overline{\mathcal{Z}^{i,\mathfrak{m}}}$ (and A^{ϖ} likewise). The previous equation, in matrix notation, is exactly

$$\Delta^{\mathfrak{m}}\mathsf{Z}^{\mathfrak{m}}=\mathsf{A}^{\mathfrak{m}}\mathsf{Z}^{\varpi}-\mathsf{Z}^{\mathfrak{m}}\mathsf{Z}^{\varpi}+\mathsf{Z}^{\mathfrak{m}}\mathsf{A}^{\varpi} .$$

It suffices to compute the coaction on A. For this use (7.2) to deduce that

$$\Delta^{\mathfrak{m}}\left(\overline{\mathcal{Z}^{i,\mathfrak{m}}}\right)=\overline{\mathcal{Z}^{i,\mathfrak{m}}\left(e_0,e_1',\ldots,e_n'\right)\mathcal{Z}^{i,\varpi}}=\overline{\mathcal{Z}^{i,\mathfrak{m}}}\,\overline{\mathcal{Z}^{i,\varpi}} .$$

Thus $\Delta^{\mathfrak{m}}(\mathsf{A}^{\mathfrak{m}})=\mathsf{A}^{\mathfrak{m}}\,\mathsf{A}^{\varpi}$. Hence

$$\Delta^{\mathfrak{m}}(\mathsf{Z}^{\mathfrak{m}}-\mathsf{A}^{\mathfrak{m}})=-\mathsf{Z}^{\mathfrak{m}}\mathsf{Z}^{\varpi}+\mathsf{Z}^{\mathfrak{m}}\mathsf{A}^{\varpi}+\mathsf{A}^{\mathfrak{m}}\mathsf{Z}^{\varpi}-\mathsf{A}^{\mathfrak{m}}\,\mathsf{A}^{\varpi}=-(\mathsf{Z}^{\mathfrak{m}}-\mathsf{A}^{\mathfrak{m}})(\mathsf{Z}^{\varpi}-\mathsf{A}^{\varpi}) .$$

Since $FL_{\Sigma}^{\mathfrak{m}}$ is defined to be $FL_{\Sigma}^{\mathfrak{m}}=\mathsf{A}^{\mathfrak{m}}-\mathsf{Z}^{\mathfrak{m}}$, we conclude that $\Delta^{\mathfrak{m}}FL_{\Sigma}^{\mathfrak{m}}=FL_{\Sigma}^{\mathfrak{m}}\otimes FL_{\Sigma}^{\varpi}$. On the other hand, homegeneity in the weight forces the right-hand side of the coaction to have weight equal to the degree in the s_i. This determines the powers of $\mathbb{L}^{\varpi}$ as in equation (7.3). □

8. Comparing $L_{\Sigma}^{\mathfrak{m}}$ and $FL_{\Sigma}^{\mathfrak{m}}$

We prove Theorem 1.2 which compares our two different interpretations of Lauricella functions from the point of view of their periods, and single-valued periods.

8.1. **Complex periods.** Recall $L_{\Sigma}^{\mathfrak{m}}$ from Definition 2.9 and $FL_{\Sigma}^{\mathfrak{m}}$ from Definition 7.4. The periods of the former are functions of $s_0,\ldots,s_n$ satisfying (2.1) and *a priori* have singularities at $s_i=0$. The periods of the latter are formal power series in the s_i.

Theorem 8.1. *The entries of the matrix* $\mathrm{per}(L_{\Sigma,\underline{s}}^{\mathfrak{m}})_{ij}$ *admit an analytic continuation to a neighbourhood of the origin and hence a Taylor expansion at the origin. They satisfy*

$$\mathrm{per}\,(L_{\Sigma}^{\mathfrak{m}})_{ij}=\mathrm{per}\,(FL_{\Sigma}^{\mathfrak{m}})_{ij} \ , \tag{8.1}$$

as an equality of formal power series in $\mathbb{C}[[s_1,\ldots,s_n]]$.

Proof. Recall the definition of Ω_j (6.1). The entries of $(L_{\Sigma})_{ij}$ are

$$(L_{\Sigma})_{ij}=-s_j\int_0^{\sigma_i}\Omega_j$$

where the right-hand side is defined first for s_i satisfying (2.1), and extended via renormalisation by proposition 3.2. Since Ω_j has vanishing residue at $x=0$, its integral from 0 to σ_i has at most a simple pole at $s_j = 0$. It follows that $(L_\Sigma)_{ij}$ is holomorphic at the origin, and therefore admits a convergent Taylor expansion there. More precisely,

$$(L_\Sigma)_{ij} = -s_j \int_0^{\sigma_i} \Omega_j = \delta_{ij} R_i - s_j \int_0^{\sigma_i} \Omega_j^{\mathrm{ren}_i}$$

where R_i is defined by (3.2) and satisfies

$$R_i = \sigma_i^{s_0} \prod_{k \neq i} (1 - \sigma_i^{-1}\sigma_k)^{s_k} = \overline{\mathcal{Z}^i} \ .$$

by equation (5.3). By Proposition 6.1, we have the identity of formal power series:

$$\overline{\mathcal{Z}_j^i} = \int_{\gamma_i} \Omega_j^{\mathrm{ren}_i} = \int_0^{\sigma_i} \Omega_j^{\mathrm{ren}_i} \ .$$

We conclude that $(L_\Sigma)_{ij} = -s_j \overline{\mathcal{Z}_j^i} + \delta_{ij} \overline{\mathcal{Z}^i}$ which is the period of $(FL_\Sigma^{\mathfrak{m}})_{ij}$ by definition 7.4. □

8.2. **Single-valued periods.** Let $(L_\Sigma^{\mathbf{s}})_{ij} = \mathbf{s}(\widetilde{L}_\Sigma^{\varpi})_{ij}$ for $1 \leq i \leq j \leq n$.

Theorem 8.2. *The $(L_\Sigma^{\mathbf{s}})_{ij}$ admit an analytic continuation to a neighbourhood of the origin by Proposition 3.4. Their Taylor expansion at the origin satisfies*

$$(L_\Sigma^{\mathbf{s}})_{ij} = \mathbf{s}\,(FL_\Sigma^{\varpi})_{ij} \ , \tag{8.2}$$

as an equality of formal power series in $\mathbb{C}[[s_1, \ldots, s_n]]$.

Proof. Using the notation (6.1), we have

$$\Omega_j^{\mathbf{s}} = \left(|z|^{2s_0} \prod_{k=1}^{n} |1 - z\sigma_k^{-1}|^{s_k} \right) \frac{dz}{z - \sigma_j} \ .$$

Proposition 2.17 thus states that:

$$(L_\Sigma^{\mathbf{s}})_{ij} = \frac{s_j}{2\pi i} \int_{\mathbb{C}} \left(\frac{d\overline{z}}{\overline{z} - \overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \Omega^{\mathbf{s}} \ .$$

Since $\Omega_j^{\mathbf{s}}$ has no residue at $z = 0$, and no residue at $z = \sigma_i$ unless $i = j$, Proposition 3.4 yields

$$(L_\Sigma^{\mathbf{s}})_{ij} \ = \ \delta_{ij}\, R_i^{\mathbf{s}} + \frac{s_j}{2\pi i} \int_{\mathbb{C}} \left(\frac{d\overline{z}}{\overline{z} - \overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \Omega^{\mathbf{s},\mathrm{ren}_i} \ ,$$

where $R_i^{\mathbf{s}}$ is defined in (3.4) and satisfies

$$R_i^{\mathbf{s}} = \left(|\sigma_i|^{2s_0} \prod_{k \neq i} \left|1 - \sigma_i \sigma_k^{-1}\right|^{2s_k} \right) \overset{(6.12)}{=} \mathbf{s}\, \overline{\mathcal{Z}^{i,\varpi}} \ .$$

In particular, $(L_\Sigma^{\mathbf{s}})_{ij}$ admits a Laurent expansion at the origin. By Theorem 6.4, we conclude that

$$(L_\Sigma^{\mathbf{s}})_{ij} = -s_j\, \mathbf{s}(\overline{\mathcal{Z}_j^{i,\varpi}}) + \delta_{ij}\, \mathbf{s}(\overline{\mathcal{Z}^{i,\varpi}})$$

which is precisely the single-valued period of $(FL_\Sigma^{\varpi})_{ij}$. □

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