MODELLING OF NEUTRON STARS

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ABSTRACT

By using a forth order Runge - Kutta algorithm, it is possible to numerically solve the coupled differential equations that form the equation of state and structural model of a Neutron Star. This project specifically deals with Bethe & Johnson equation of state applied to the Tolman – Oppenheimer – Volkoff equations with a rotating perturbation.

INTRODUCTION

A mid-life star is a collection of ionised gases in hydrostatic equilibrium between its gravitational attraction and thermonuclear radiation pressure. Originally the star is composed almost entirely of Hydrogen and Helium, and the fusion reaction follows a proton- proton chain that forms the helium in a highly exothermic reaction. Over the course of a star's lifetime the hydrogen will be used up, which causes a drop in radiation pressure, and the star contracts. This increased nuclear pressure allows for fusion between heavier atoms and the star remains exothermic.

This process continues until the star can no longer sustain a rate of fusion that will produce the thermonuclear pressure required to prevent gravitational collapse. At this point the star's outer layers collapse inwards, which causes a massive instantaneous spike in pressure at the core. The intense pressures overcome Pauli electron exclusion energies and it becomes energetically favourable for the protons and electrons of the core to decay in to neutrons through electron capture. This creates a Bose- Fermi gas which is temperature independent. The spike in energy also rushes to meet the collapsing outer layers causing a massive explosion which blasts much of the original stars mass out in to space – a supernova.

Providing the core is not sufficiently massive to cause further gravitational collapse in to a black hole, the core of the star will remain, a degenerate mass formed entirely of neutrons which is held in a new hydrostatic equilibrium between the neutron degeneracy pressure and the gravitational potential. These objects are only a few tens of kilometres in diameter yet can have a mass greater than our sun. (Hartle J. B., 2003)

The concept of the neutron star dates back to the 1930s, when it was discussed in the context of general relativity by Oppenheimer. His studies found that a dense neutron gas could support itself under gravitational attraction provided the total mass was not greater than ~2 $M_{\odot}$(Solar masses) (Kalogera & Baym, 1996). This idea remained largely academic until the discovery of pulsars in 1967 by J. Burnell et al.

Neutron stars only exist within certain boundaries of density. If a neutron star were larger than the Schwarzschild limit then they will collapse in to a black hole. If a neutron star were less massive than parameters defined by the Chandrasekhar limit then the neutron star would not form at all. Because of the stellar scale of the body the effects of general relativity become significant, and it was necessary to use models based on relativistic models.

The purpose of this project was to use numerical algorithms to solve a number of models for the neutron star coupled with a particle equation of state. By using a number of computational methods it was possible to efficiently feed the runtime data back in to the algorithm and create a convergent solution to the system.

Method

Assume hydrostatic equilibrium

We assume that the neutron star is in hydrostatic equilibrium, that the gravitational force is balanced by the degeneracy pressures. This gives a stable structure based on the equality of the two forces. Firstly to consider the Newtonian potential, the equation for attraction is simply:

$$F = -\frac{Gm(r)}{r^2}\rho(r)$$

Where ρ(r) is the density at each value of r and m(r) is the total mass within the sphere of radius r which can be given by an integral around spherical coordinates:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

Which can be rearranged to give:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

And from the Newtonian potential we get

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Modify model for scale factors

In order to solve the equations we converted the variables to dimensionless form. By applying scaling factors we were also able to lower margin of rounding error on the floating point data types. (Hjorth - Jensen, 1986)

NEUTRON STAR EQUATIONS OF STATE

There exist a number of equations of state for determining the composition of a neutron star, however many are much more involved than the Bethe – Johnson parameterisation used here. A lot of recent theoretical activity has been focused on crust dynamics. Close to the surface the pressure of the gravitational potential is insufficient to create a perfect Bose – Fermi gas, which means the outer layer of the star is not temperature independent, and has some magnetic susceptibility. As such for more accurate simulations of neutron stars it is useful to consider an internal equation of state and a crust equation of state with an extrapolation routine between them. (Lattimer & Prakash, 2001)

These models can be grouped in to three categories, nonrelativistic potential models, relativistic models and relativistic Dirac – Brueckner – Hartree – Fock models. The difference between these models is ultimately derived from the particle interactions they consider. For example the effects of Bose condensates or quark interactions can have additional softening effects on the model, which leads to different model predictions. These equations can also be grouped in to two additional categories. Normal equations that have a pressure that vanishes as density tends to zero, and self-bound equations which have a pressure which vanishes at a certain non-zero density.

Normal equations give zero density at the system surface and can have masses as small as 0.1MO which also give radii up to 100km. i.e. the lighter systems are larger. Self bound stars have no minimum mass but the heavier stars will give largest radii. (Baier, Frohlich, & Bentz, 1998)

Explain link between EoS and Model

The structural model describes the gravitational attraction of the star, however the repulsive force of the neutron degeneracy which prevents the star from collapsing is given by a separate equation of state. The initial model considered is the parameterisation given by (Bethe & Johnson, 1974)

$$P(n) = 363.44 \times n^{2.54}$$

$$\rho(n) = nm_n + 236 \times n^{2.54}$$

This model is only accurate within the density range 95 ≤ρ ≤ 6170 MeV fm-1, which will be applied to all successive data.

Explain RK4

The structural models and the equation of state form a group of coupled partial differential equations that cannot be solved analytically. Instead we find a numerical solution calculated from the Forth Order Runge – Kutta algorithm (RK4) (Abramowitz & Stegun, 1972).

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

This algorithm provides approximate solutions to ordinary differential equations. By incrementing one of the parameters and performing a weighted average of the direct results an approximation is provided with a diverging error of the fifth power of the step size h. After experimenting with a number of step sizes it was found that h = 0.001 provided the best compromise between computational cost and accuracy see Fig.1.

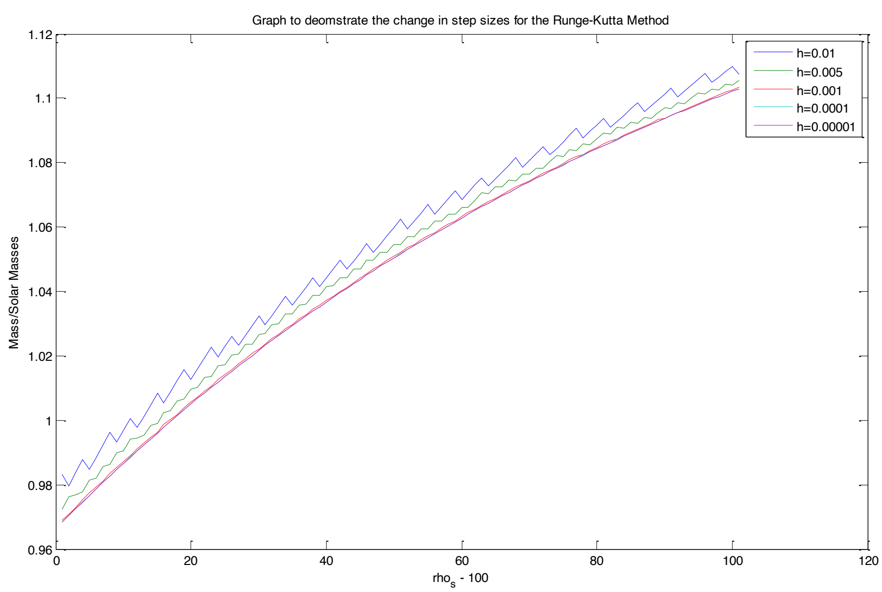


Figure 1 - Comparison of Step Sizes for rho_s = 1000 on Classical Model

Interpolation routine

The Runge - Kutta method is an iterative routine that breaks when the parameters of the model become impossible. By incrementing the radius of the model from the core, the model breaks at the surface when the repulsive pressure becomes negative. Since the final value returned by RK4 will be slightly greater than the surface result the actual mass and radius will be between the final values and the penultimate values. By using a polynomial interpolation routine it is possible to find a much more accurate end value.

Dynamic resolution of RK4

By assuming that the convergence of the results to a global maxima for values of mass or radius are roughly smooth, then it is possible to improve the efficiency of the algorithm by decreasing the RK4 and density step sizes. This means that as the algorithm approaches a globally optimal solution it will increase the resolution of the algorithm, which significantly reduces the error from the RK4 and extrapolation routine. The smoothness assumption can be made by the assumption that any maxima or minima that were not part of a smooth set would be physically unstable, and can be supported post hoc by Figure.1 and Figure.2 that show very smooth variations between models across different densities.

Ideally this method would also be able to reduce computational time by using a lower resolution run where the algorithm was far from a global solution, however above a step size of h = 0.001 it was found that the fluctuations due to RK4 error were greater than the difference between successive mass solutions. This meant that the globally optimum solution would move around for higher step sizes.

This problem was overcome by running the RK4 algorithm over the entire range of number densities, and then repeating the algorithm for a small range around the highest number density. This process can be repeated to provide a much more accurate final result. The trade-off with this method is that while the accuracy of the final solution will be significantly increased the computational runtime will not be reduced, in fact it will be markedly larger (although not nearly as large at running the entire density range at the smaller stepsize).

Explain procession to ToV and then Tov + Rotating

Following the success of the Newtonian model we modified the routine to use a relativistic model. The Tolman – Oppenheimer – Volkoff (TOV) equation (Thorne, 1968) models a spherically symmetric non-rotating mass on the Schwarzschild metric.

$$\frac{dP(r)}{dr} = -\frac{(\epsilon + 1)(4\pi r^3 P + m)}{r^2\left(1 - Y(r)\right)}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \epsilon(r)$$

Following this we attempted to model a rotating relativistic neutron star. However since the TOV equations are based on a Schwarzschild (non-rotating) metric there is no elegant extension to a rotating frame. As such by using the framework given by (Schramm & Zschiesche, 2003) and (Hartle J. B., 1967) it was possible to apply the rotating frame as a perturbation on the TOV model by using a multipole expansion on the metric functions, up until quadrupole order, and ignoring all terms of higher order of OMEGA (and therefore, only considering slowly rotating neutron stars). When dealing with rotation in general relativistic systems, an effect known as frame-dragging comes into play, and is given by the following equation:

$$\frac{d}{dr}\left(j(r)\frac{d\varpi}{dr}+\frac{4}{r}j(r)\varpi(r)\right)=0$$

$$j=e^{-\frac{\nu(r)}{2}}\left[1-\frac{2M(r)}{r}\right]^{\frac{1}{2}}\ and\ \varpi(r)=\Omega-2J/r^3$$

Using these multipole expansions of the metric functions, we can then gain ODE expressions for m_0 and p_0, by inserting the metric (with l = 0 for Legendre polynomials) into the Einstein field equations. They are as follows:

$$\frac{dm_0}{dr}=4\pi r^2\frac{d\epsilon}{dr}(\epsilon+P)p_0+\frac{1}{12}j^2r^4\left(\frac{d\varpi}{dr}\right)^2+\frac{8\pi}{3}r^4j^2\frac{\epsilon+P}{1-Y(r)}\varpi$$

$$\frac{dp_0}{dr}=-\frac{1+8\pi r^2P}{r^2(1-Y)^2}m_0-4\pi r\frac{\varepsilon+P}{1-Y}p_0+\frac{1}{12}\frac{j^2r^3}{1-Y}\left(\frac{d\varpi}{dr}\right)^2+\frac{1}{3}\frac{d}{dr}\left(\frac{r^2j^2\varpi^2}{1-Y}\right)$$

Next we consider the non-spherical deformations of the star, these are also gained by inserting the metric into the Einstein Field equations. These are as follows:

$$\frac{dv_2}{dr}=-2\frac{d\phi}{dr}h_2+\left(\frac{1}{2}+\frac{d\Phi}{dr}\right)\left(-\frac{r^3}{3}\frac{dj^2}{dr}\varpi+\frac{j^2}{6}r^4\left(\frac{d\varpi}{dr}\right)^2\right)$$

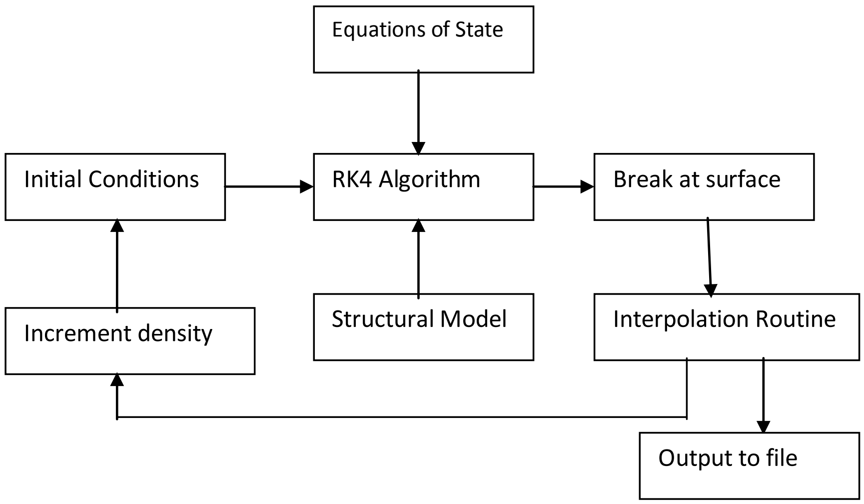
$$\begin{aligned}\frac{dh_2}{dr}=h_2\left[\frac{d\phi}{dr}+\frac{2}{1-Y}\left(2\pi(\epsilon+P)-\frac{m}{r^3}\right)\Big/\left(\frac{d\phi}{dr}\right)\right]-\frac{2}{r^2(1-Y)}\Big/\left(\frac{d\phi}{dr}\right)+\frac{r^3}{6}\left(j\frac{d\varpi}{dr}\right)^2\\-\frac{1}{3}(r\varpi)^2\frac{dj^2}{dr}\left[r\frac{d\phi}{dr}+\left(2r(1-Y)\frac{d\phi}{dr}\right)^{-1}\right]\end{aligned}$$

For this report, there shall only be measurements of maximum mass and radius as with the non-rotating star, though there are of course other points of interest such as the star's shape, and better models when considering fast neutron stars that involve mass shedding. The physical units for this simulation are undetermined due to a lack of time, but the results are put here in geometrised units in Figure. 4

Diagram to link the two

The program written in C++ automates a sequence of algorithms to return an accurate numerical solution of mass and radius of the star. The Runge - Kutta algorithm begins at the core, where the radius is known to be zero, and using the Newton – Raphson method to solve the equation of state to provide an initial number density it was possible to find an initial pressure.

Since the mass, pressure and radius are coupled variables the program increments the radius in small step sizes h, while solving two RK4 problems for mass and pressure at once. When the pressure reaches below zero the algorithm ends, which is physically analogous to integrating from the core to the surface. These results are interpolated to provide more accurate approximation of the total mass and radius of the star, which is stored in a data file. The system then repeats for increasing values of number density.



Results

If the code is implemented to search every integer value of density, then the following results emerge:

Table 1 - Newtonian Model:

R_{max} (*km*)	1.64721 ± 0.000449	$m(R_{max})$ $M_\odot$	11.4287 ± 0.000210	$P(R_{max})$ $MeV\cdot fm^{-1}$	2157
$R(M_{max})$ (*km*)	1.55155 ± 0.000233	M_{max} $M_\odot$	18.9359 ± 0.000102	$P(M_{max})$ $MeV\cdot fm^{-1}$	6170

Table 2 -Tolman – Oppenheimer – Volkoff Model:

R_{max} (*km*)	1.16724 ± 0.000879	$m(R_{max})$ $M_\odot$	1.05672 ± 0.000149	$P(R_{max})$ $MeV\cdot fm^{-1}$	412
$R(M_{max})$ (*km*)	0.971866 ± 0.000474	M_{max} $M_\odot$	$1.8463 \pm$ 8.66e-005	$P(M_{max})$ $MeV\cdot fm^{-1}$	1729

Table 3 - Tolman – Oppenheimer – Volkoff Model with rotating frame perturbation.(no units)

R_{max}	#	0.0133	$m(R_{max})$	#	0.0015	$P(R_{max})$ $MeV\cdot fm^{-1}$	342
$R(M_{max})$	#	0.0118	M_{max}	#	0.0031	$P(M_{max})$ $MeV\cdot fm^{-1}$	1249

It should be noted that the maximum mass for the Newtonian model actually occurs at a density of 19149 $MeV\cdot fm^{-1}$ however since this is well outside of the validity range of the Bethe & Johnson equation of state it has been capped at the maximum validity. Figure.1 and Figure.2 show the range masses and radii across the entire range of validity for the equation of state.

Error analysis

As both the equation of state and the structural model exhibit errors under the RK4 algorithm, the two sources of error in the simulation are this algorithm and the extrapolation routine.

$$\epsilon_{RK4} = \frac{\eta}{h} + h^n \qquad \epsilon_{Numeric} = x^{n+1}$$

Where h is the step size, n is the order of the algorithm and η is the accuracy of double precision floating point arithmetic ($\eta = 2.22 \times 10^{-16}$ for IBM-PC models) (Fitzpatrick, 2006).This is a propagating error which is calculated along each successive iteration of RK4.

Which gives a maximum error of:

$$\sigma = \pm(\frac{\eta}{h} + h^n + x^{n+1})$$

These errors have been included in Table 1, Table 2. However because we were not able to resolve units for the rotating perturbation model, errors have been excluded in Table 3.

Runtime

While not of physical relevance, it should also be noted that the there were significant differences in the number of RK4 iterations required for solutions to emerge across the entire range of densities. As a crude approximation it takes around twice as many RK4 iterations for the classical model to reach solution than the relativistic model.

Newtonian Model: 20.9m Relativistic Model: 11.9m

This would suggest that the Relativistic model converges on a solution much more quickly than the Newtonian model.

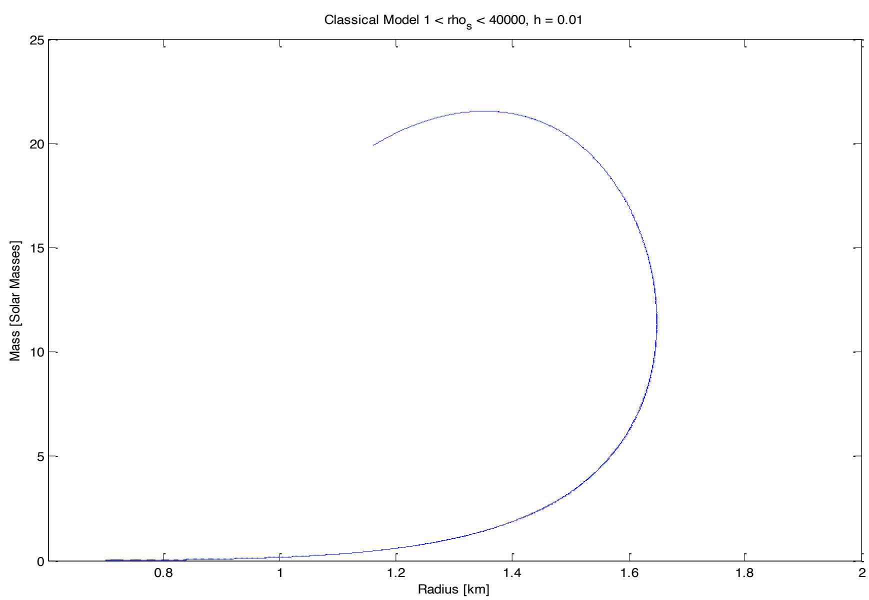


Figure 2 - Classical Model Mass/Radius

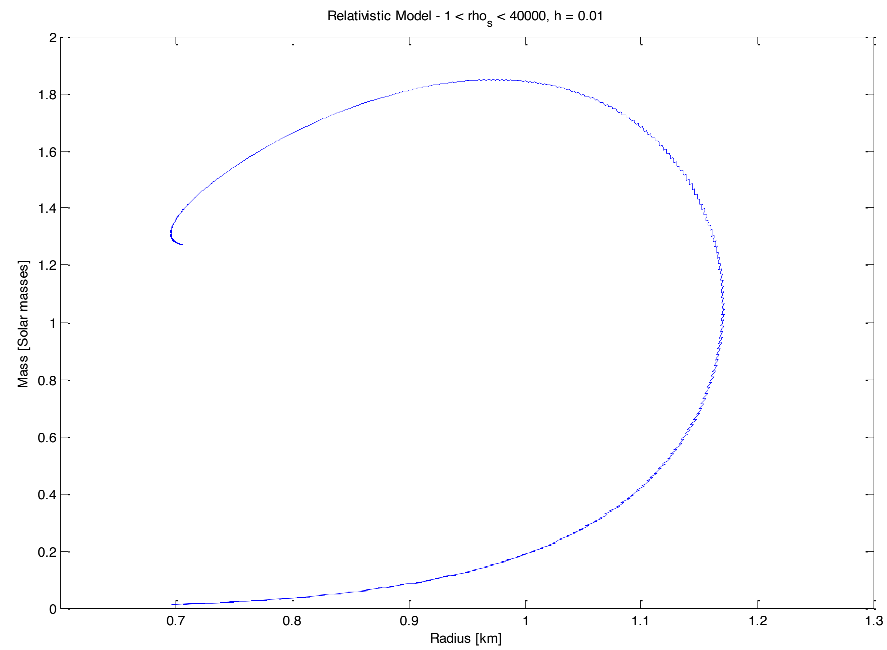


Figure 3 - Relativistic Model Mass/Radius

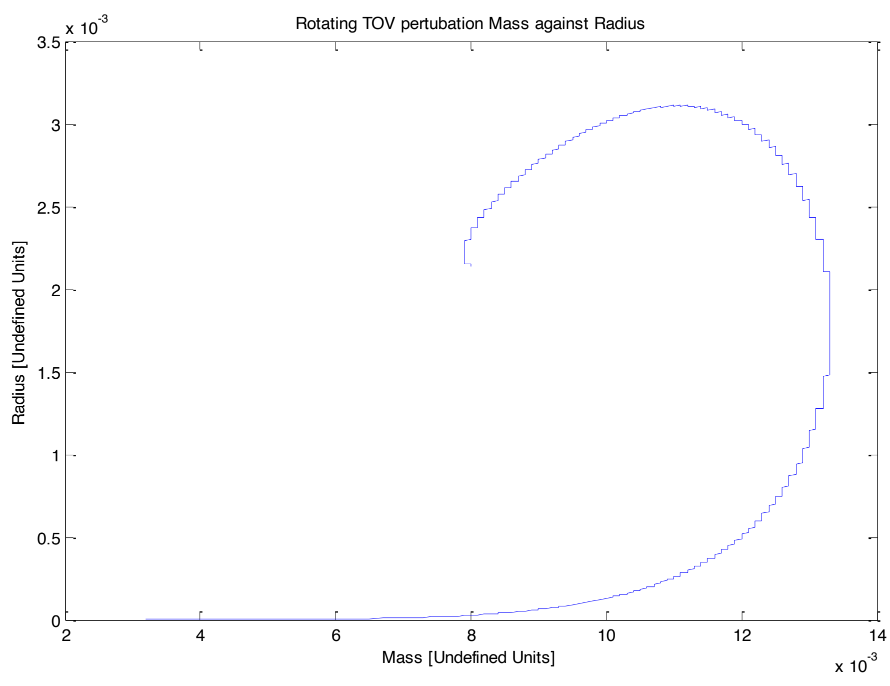


Figure 4 - Rotating relativistic model without units

Conclusion and direction of future research

Apart from the successful implementation of the rotating perturbation model, this project offers several avenues of further study, each of which would have been highly interesting for a greater understanding of neutron star structure. For example this project utilises only the Bethe & Johnson equation of state, which while considering nucleon – nucleon interactions was published before a mature theory of boson exchange was available. A further line of investigation would be to explore the equations of state that arise from considering SU(3) symmetry (Schramm & Zschiesche, 2003), which was not fully available when Bethe and Johnson published their model.

The Bethe and Johnson model assumes that the structure of the neutron star is approximately homogeneous. Later models include not only different particle interactions, but also the existence of different phases of matter within the structure, similar to the various layers of the earth's crust, mantel and core (Lattimer & Prakash, 2001) (Weber, Rodrigo, & Rosenfield, 2007). Investigations of these inhomogeneous structures using extrapolations between the layers are presently the forefront of theoretical research in the field.

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