STATS310A - Lecture 8

Persi Diaconis
Scribed by Michael Howes

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Contents

1 Integration

1.1 Definition

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to [-\infty, \infty]$ be measurable. We wish to define

$$\int f d\mu = \int_\Omega f d\mu = \int_\Omega f(\omega)\mu(d\omega).$$

Definition 1. The function f is a *simple function* if there exists a finite partition of Ω $\{A_i\}_{i=1}^n$ such that $A_i \in \mathcal{F}$ and values $x_i \in [-\infty, \infty]$ such that

$$f(\omega) = \sum_{i=1}^n x_i \delta_{A_i}(\omega),$$

where we use the conventions $0 \cdot \infty = \infty \cdot 0 = 0$ and $x \cdot \infty = \infty \cdot x = \infty$ if $x \in (0, \infty]$.

Lemma 1. *If $f \geq 0$, then there exists a sequence of simple functions $f_n \geq 0$ such that $f_n(\omega) \nearrow f(\omega)$ for all $\omega \in \Omega$.*

Proof. Fix n and define

$$f_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \text{ for some } k = 1, \ldots, n2^n, \\ n & \text{if } f(\omega) \geq n. \end{cases}$$ □

Definition 2. Suppose $f(\omega) \geq 0$ for all ω. Define

$$\int_\Omega f(\omega)\mu(d\omega) = \sup\left(\sum_{i=1}^n \nu_i \mu(A_i)\right),$$

where the supremum is over all measurable partitions of Ω and $\nu_i = \inf_{\omega \in A_i} f(\omega)$. For a general f define

$$f_+(\omega) = \max\{f(\omega), 0\} \text{ and } f_-(\omega) = \max\{-f(\omega), 0\}.$$

Note that $f_+, f_- \geq 0$ and $|f| = f_+ + f_-$ and $f = f_+ - f_-$. We define $\int f d\mu$ based on cases.

- If $\int f_+ d\mu < \infty$ and,$\int f_- d\mu < \infty$, then define $\int f = \int f_+ d\mu - \int f_- d\mu$.
- If $\int f_+ d\mu = \infty$ and $\int f_- d\mu < \infty$, then define $\int f d\mu = \infty$.
- If $\int f_+ d\mu < \infty$ and $\int f_- d\mu = \infty$, then define $\int f d\mu = -\infty$.
- If $\int f_+ d\mu = \infty$ and $\int f_- d\mu = \infty$, then $\int f d\mu$ is not defined.

1.2 Properties

Proposition 1. *Suppose $f, g, f_n \geq 0$.*

(a) If $f(\omega) = \sum_{i=1}^n x_i \delta_{A_i}(\omega)$ is simple, then $\int f d\mu = \sum_{i=1}^n x_i \mu(A_i)$.

(b) If $f(\omega) \leq g(\omega)$ for all ω, then $\int f d\mu \leq \int g d\mu$.

(c) If $f_n(\omega) \nearrow f(\omega)$ for all n and ω, then $\int f_n d\mu \nearrow \int f d\mu$. (Monotone convergence theorem).

(d) If $\alpha, \beta \geq 0$, then $\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$.

Proof. (a) Fix a partition $\{B_i\}_{i=1}^m$ and let $\beta_j = \inf_{\omega \in B_j} f(\omega)$. If $\omega \in A_I \cap B_j$, then $\beta_j \leq f(\omega) = x_i$. Thus

$$\begin{aligned}
\sum_{j=1}^m \beta_j \mu(B_j) &= \sum_{j=1}^m \beta_j \sum_{i=1}^n \mu(A_i \cap B_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m \beta_j \mu(A_i \cap B_j) \\
&\leq \sum_{i=1}^n \sum_{j=1}^m x_i \mu(A_i \cap B_j) \\
&= \sum_{i=1}^n x_i \sum_{j=1}^m \mu(A_i \cap B_j) \\
&= \sum_{i=1}^n x_i \mu(A_i).
\end{aligned}$$

Thus $\int f d\mu \leq \sum_{i=1}^n x_i \mu(A_i)$. The other direction we get for free since $\{A_i\}_{i=1}^n$ is a partition and f equals x_i on A_i.

(b) This follows from the definition since

$$\inf\{f(\omega) : \omega \in A_i\} \leq \inf\{g(\omega) : \omega \in A_i\},$$

for any measurable set A_i.

(c) By (b) we know that $\int f_n d\mu$ is an increasing sequence and that $\int f_n d\mu$ is bounded above by $\int f d\mu$. Thus $\lim_n \int f_n d\mu$ exists and $\lim_n \int f_n d\mu \leq \int f d\mu$. It remains to prove that $\int f d\mu \leq \lim_n \int f_n d\mu$. Thus we must show that for every partition $\{A_i\}_{i=1}^m$, we have

$$\sum_{i=1}^m \nu_i \mu(A_i) \leq \lim_n \int f_n d\mu,$$

where $\nu_i = \inf\{f(\omega) : \omega \in A_i\}$. Let $S = \sum_{i=1}^m \nu_i \mu(A_i)$. We will consider different cases.

i. Supppose that S is finite and $0 < \nu_i < \infty$ and $0 < \mu(A_i) < \infty$ for all i. Choose ε such that $\varepsilon < \mu(A_i)$ for all i. Define

$$A_{n,i} = \{\omega \in A_i : f_n(\omega) \geq \nu_i - \varepsilon\}.$$

Since $f_n \nearrow f$, we know that $A_{n,i} \nearrow A_i$ as $n \to \infty$. Thus $\mu(A_{n,i}) \nearrow \mu(A_i)$. Note that

$$\int f_n d\mu \geq \sum_{i=1}^{m} (\nu_i - \varepsilon)\mu(A_{n,i}).$$

Thus

$$\lim_n \int f_n d\mu \geq \sum_{i=1}^{n} (\nu_i - \varepsilon)\mu(A_i) = \sum_{i=1}^{m} \nu_i \mu(A_i) - \varepsilon \sum_{i=1}^{m} \mu(A_i).$$

Letting $\varepsilon \to 0$, we see that $\lim_n \int f_n d\mu \geq \sum_{i=1}^n \nu_i \mu(A_i) = S$. Thus $\lim_n \int f_n d\mu \geq \int f d\mu$.

ii. Now suppose that $S < \infty$ and ν_i or $\mu(A_i)$ is equal to 0 or ∞ for some i. By reording we have $0 < \nu_i, \mu(A_i) < \infty$ for $i = 1, \ldots, i_0$ and the rest of the terms are of the form $0 \cdot \infty$, $\infty \cdot 0$ or $0 \cdot 0$. We can then apply case 1 to the partition $\{A_i\}_{i=1}^{i_0} \cup \left\{ \left(\bigcup_{i=1}^{i_0} A_i \right)^c \right\}$.

iii. Suppose that $S = \infty$. Then for some i_0 we have $\nu_{i_0}\mu(A_{i_0}) = \infty$. Thus either $\nu_{i_0} = \infty$ and $\mu(A_{i_0}) > 0$ or $\nu_{i_0} > 0$ and $\mu(A_{i_0}) = \infty$. Choose $x, y > 0$ so that $0 < x < \nu_{i_0}$ and $0 < y < \mu(A_{i_0})$. Let $B_n = \{\omega : f_n(\omega) > x\}$. Then $B_n \nearrow B = \{\omega : f(w) \geq \omega\}$. So $\mu(B_n) \nearrow \mu(B) \geq \mu(A_{i_0}) \geq y$. Thus by using the parition $\{B_n, B_n^c\}$, we see that

$$\int f_n d\mu \geq x \cdot \mu(B_n).$$

Thus $\lim_n \int f_n d\mu \geq xy$. The product xy can be arbitrary large and thus $\int f_n d\mu \geq \infty$.

(d) Suppose f and g are both simple functions $f = \sum_{i=1}^n x_i \delta_{A_i}$ and $g = \sum_{i=1}^m y_i \delta_{B_i}$. Then

$$\alpha f + \beta g = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha x_i + \beta y_j) \delta_{A_i \cap B_j}.$$

Thus

$$\begin{aligned}
\int \alpha f + \beta g d\mu &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha x_i + \beta y_j) \mu(A_i \cap B_j) \\
&= \alpha \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \mu(A_i \cap B_j) + \beta \sum_{j=1}^{m} y_j \sum_{i=1}^{n} \mu(A_i \cap B_j) \\
&= \alpha \sum_{i=1}^{n} x_i \mu(A_i) + \beta \sum_{j=1}^{m} y_j \mu(B_j) \\
&= \alpha \int f d\mu + \beta \int g d\mu.
\end{aligned}$$

Now for general f and g, let $f_n \nearrow f$ and $g_n \nearrow g$ be approximating sequences of simple functions. Then $(\alpha f_n + \beta g_n) \nearrow \alpha f + \beta g$. Thus

$$\begin{aligned}
\int \alpha f + \beta g d\mu &= \int \alpha f_n + \beta g_n d\mu \\
&= \alpha \int f_n d\mu + \beta \int g_n d\mu \\
&= \alpha \int f d\mu + \beta \int g d\mu.
\end{aligned}$$

□

1.3 Remarks

(a) If we have a continuous function $f : [a, b] \to \mathbb{R}$ and μ is Lebesgue measure on $[a, b]$, then $\int f d\mu$ agrees with the Riemann integral.

(b) In the same setting as (a), the function $f = \delta_{\mathbb{Q}\cap[a,b]}$ is not Riemann integrable but $\int f d\mu$ does exist and $\int f d\mu = 0$.

(c) We still need Riemann integrals for improper integrals. For example $\int_0^\infty \frac{\sin(x)}{x} dx$ is not Lebesgue integrable but it does have a convergent improper Riemann integral.

(d) Riemann integration is also needed for calculations, for Brownian motion and many other things.

(e) The Henstock integral generalises both the Riemann and Lebesgue integrals.

Example 1. In general $f_n \to f$, does not imply $\int f_n d\mu \to \int f d\mu$. Consider

$$f_n(\omega) = \begin{cases} n^2 & \text{if } \omega \in (0, 1/n) \\ 0 & \text{else.} \end{cases}$$

Then $f_n(\omega) \to 0$ for all ω but $\int f_n d\mu = n^2 \cdot \frac{1}{n} = n \nearrow \infty$.

The properties (a)-(d) hold if the hyptheses hold *almost surely*. For example for (b) we can prove

$$\mu(\{\omega : f(\omega) > g(\omega)\}) = 0 \Longrightarrow \int f d\mu \leq \int g d\mu.$$

Proof. Let $G = \{\omega : f(\omega) \leq g(\omega)\}$. By hypothesis $\mu(G^c) = 0$. For every partition $\{A_i\}_{i=1}^m$, $\mu(A_i) = \mu(A_i \cap G)$. It follows that

$$\begin{aligned} \sum_{i=1}^n \inf_{A_i} f(\omega)\mu(A_i) &= \sum_{i=1}^m \inf_{A_i} f(\omega)\mu(A_i \cap G) \\ &\leq \sum_{i=1}^m \inf_{A_i\cap G} f(\omega)\mu(A_i \cap G) \\ &\leq \sum_{i=1}^m \inf_{A_i\cap G} g(\omega)\mu(A_i \cap G) \\ &= \sum_{i=1}^m \inf_{A_i\cap G} g(\omega)\mu(A_i \cap G) + \inf_{G^c} g(\omega)\mu(G^c) \\ &\leq \int g d\mu. \end{aligned}$$

□

Definition 3. If $(\Omega, \mathcal{F}, \mu)$ is a probability space and $X : (\Omega, F) \to \mathbb{R}$ is a random variable, then we call $\int f d\mu$ *the expectation of* f and we write

$$\mathbb{E}[X] := \int X d\mu.$$