STATS310A - Lecture 9

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Contents

1 Announcements

- Homework
  - Read sections 16 and 18.
  - Do problems 16.4, 16.7, 18.1, 18.2, 18.4, 18.10.
- Exam on Thursday 5-7 Oct 28th.

2 Convergence theorems

Last time we saw the following. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f : \Omega \to \mathbb{R}$ is measurable, then define

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu,$$

where if $g \geq 0$, we define

$$\int g d\mu = \sup \sum_{i=1}^{n} \nu_i \mu(A_i),$$

where $\nu_i = \inf_{\omega \in A_i} g(\omega)$ and the supremum is over all partitions of Ω by elements of $\mathcal{F}$.

Recall that if $\{x_i\}_{i=1}^{\infty}$ are real, then

$$\liminf_{n \to \infty} x_n = \lim_{n} \inf_{k \geq n} x_k,$$

and

$$\limsup_{n \to \infty} x_n = -\liminf_{n \to \infty} (-x_n).$$

Lemma 1. [Fatou's Lemma] *Let $\{f_n\}$ be any sequence of measurable non-negative functions, then*

$$\int \liminf_n f d\mu \leq \liminf_n \int f_n d\mu.$$

Proof. Define $g_n = \inf_{k \geq n} f_k$. Then $g_n \leq f_n$ for all n and $g_n \nearrow \liminf_n f_n$ by definition. By the monotone convergence theorem and monotonicity:

$$\begin{aligned}\int \liminf_n g_n d\mu &= \int \lim_n g_n d\mu \\ &= \lim_n \int g_n d\mu \\ &= \liminf_n \int g_n d\mu \\ &\leq \liminf_n \int f_n d\mu.\end{aligned}$$

□

Example 1. If $(\Omega, \mathcal{F}, \mu) = ([0,1], \mathcal{B}, \lambda)$ (Lebesgue measure on the Borel subsets of $[0,1]$). Define

$$f_n = \begin{cases} \delta_{[0,1/2)} & \text{if } n \text{ is even,} \\ \delta_{[1/2,1]} & \text{else.} \end{cases}$$

Then $\liminf_n f_n = 0$ and $\int f_n d\mu = 1/2$. Thus $\int \liminf f_n d\mu \leq \liminf_n \int f_n d\mu$ as promised. Note that the sequence f_n is not convergent. In Fatou's lemma the sequence is not required to be convergent. The $\liminf$ of a sequence always exists.

Example 2. Note that we need $f_n \geq 0$. If $f_n = -\frac{1}{n}\delta_{[n,2n]}$, then $f_n \to 0$ uniformly but $\int f_n d\mu = -1$ for all n and $0 \leq -1$ is not true.

Example 3. Let $(r_i)_{i=1}^{\infty}$ be a the usual enumeration of $\mathbb{Q} \cap [0,1]$,

$$(r_i) = (0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6, \ldots).$$

Define

$$f(\omega) = \sum_{i=1}^{\infty} \frac{1}{i^2 |r_i - \omega|^{1/2}}.$$

Note that $f(\omega) = \infty$ if ω is rational. Is $f(\omega)$ finite for any values of ω? Define $f_n(\omega) = \sum_{i=1}^{n} \frac{1}{i^2|r_i - \omega|^{1/2}}$, we have $f_n \to f$ pointwise. We also have

$$\int_0^1 f_n(\omega) d\omega = \sum_{i=1}^{n} \frac{1}{i^2} \int_0^1 \frac{1}{|r_i - \omega|^2} d\omega \leq \sum_{i=1}^{n} \frac{c}{i^2} \leq C.$$

Thus, by Fatou's, $\int f d\omega \leq C$ and thus $f(\omega)$ is finite for almost every $\omega \in [0,1]$. Exercise: find a single ω such that $f(\omega) < \infty$ [hint: take $\omega = 1/\sqrt{2}$].

Theorem 1. [Dominated Convergence Theorem (DCT)] *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, f_n : \Omega \to \mathbb{R}$ be measurable and suppose that $f_n(\omega) \to f(\omega)$ a.e.. Suppose also that there exists a measurable function g such that $|f_n(\omega)| \leq g(\omega)$ for a.e ω and $\int g d\mu < \infty$. Then f is integrable and*

$$\int f d\mu = \lim_n \int f_n d\mu.$$

Proof. Note that $g + f_n, g - f_n \geq 0$. Define $f_* = \liminf_n f_n$ and $f^* = \limsup_n f_n$. We know that $|f_*|, |f^*| \leq g$ and so f_* and f^* are integrable. Futhermore

$$\begin{aligned}
\int g d\mu + \int f_* d\mu &= \int (g + f_*) d\mu \\
&= \int \liminf_n (g + f_n) d\mu \\
&\leq \liminf_n \int g + f_n d\mu \\
&= \int g d\mu + \liminf_n \int f_n d\mu.
\end{aligned}$$

Thus $\int f_* d\mu \leq \liminf_n \int f_n d\mu$. Likewise we have

$$\begin{aligned}
\int g d\mu - \int f^* d\mu &= \int g - f^* d\mu \\
&= \int g - \limsup_n f_n d\mu \\
&= \int g + \liminf_n (-f_n) d\mu \\
&= \int \liminf_n (g - f_n) d\mu \\
&\leq \liminf_n \int g - f_n d\mu \\
&= \int g d\mu + \liminf_n \left(- \int f_n d\mu \right) \\
&= \int g d\mu - \limsup_n \int f_n d\mu.
\end{aligned}$$

And thus $\int f^* d\mu \geq \limsup_n \int f_n d\mu$. But we know that $f_* = f^* = f$. Thus we have

$$\limsup_n \int f_n d\mu \leq \int f^* d\mu \leq \liminf_n \int f_n d\mu.$$

Thus $\int f_n d\mu \to \int f d\mu$. □

Note that we couldn't use Fatou's lemma directly since f_n need not be non-negative. We had to apply Fatou's lemma seperately to $g + f_n$ and $g - f_n$.

3 New measures from old

Definition 1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A *probability density* is a function $f : \Omega \to \mathbb{R}$ s.t f is measure, $f(\omega) \geq 0$ for all ω and $\int f d\mu = 1$.

Proposition 1. *Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space and f is a probability density, then the function $\nu : \mathcal{F} \to \mathbb{R}$ given by*

$$\nu(A) = \int_A f d\mu = \int_\Omega f \delta_A d\mu,$$

is a probability measure.

Proof. Note that $\nu(\emptyset) = \int_\Omega 0 d\mu = 1$ and $\nu(\Omega) = \int_\Omega f d\mu = 1$. Furthermore if $\{A_i\}_{i=1}^\infty$ are disjoint and $A = \bigcup_{i=1}^\infty A_i$, then

$$f\delta_A = \sum_{i=1}^\infty f\delta_{A_i}.$$

Thus the functions $g_n = \sum_{i=1}^n f\delta_{A_i}$ are non-negative and $g_n \nearrow f\delta_A$. Thus by the monotone convergence theorem:

$$\nu(A) = \int f\delta_A d\mu = \lim_n \int \sum_{i=1}^n f\delta_{A_i} d\mu = \lim_n \sum_{i=1}^n \int f\delta_{A_i} d\mu = \lim_n \sum_{i=1}^n \nu(A_i) = \sum_{i=1}^\infty \nu(A_i).$$

Thus ν is countably additive. Furthermore ν is non-negative since f is non-negative. □

4 Change of measure

Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $(\Omega', \mathcal{F}')$ is a measurable space and $T : \Omega \to \Omega'$ is measurable. We have previously seen that $\mu^{T^{-1}}$ given by

$$\mu^{T^{-1}}(B) = \mu(T^{-1}(B)),$$

is a measure on $(\Omega', \mathcal{F}')$. Can we relate integrals on $(\Omega', \mathcal{F}', \mu^{T^{-1}})$ to integrals on $(\Omega, \mathcal{F}, \mu)$? Yes!

Theorem 2. *Suppose that $f : \Omega' \to \mathbb{R}$ is measurable and integrable with respect to $\mu^{T^{-1}}$, then $f \circ T : \Omega \to \mathbb{R}$ is integrable and*

$$\int_\Omega f \circ T d\mu = \int_{\Omega'} f d\mu^{T^{-1}}.$$

Proof. This is the classic (1), (2), (3) argument.

(0) First we write $f = f_+ - f_-$ and then note that $(f \circ T)_+ = f_+ \circ T$ and $(f \circ T)_- = f_- \circ T$. Thus we may assume $f \geq 0$.

(1) Suppose $f = \delta_B$, $B \in \mathcal{F}'$.

(2) Prove it for finite linear combinations.

(3) Monotone convergence + approximation by simple functions.

For (1) note that $\delta_A \circ T = \delta_{T^{-1}(A)}$. Thus

$$\int_{\Omega'} \delta_A d\mu^{T^{-1}} = \mu^{T^{-1}}(A) = \mu(T^{-1}(A)) = \int_\Omega \delta_{T^{-1}(A)} d\mu = \int_\Omega \delta_A \circ T d\mu.$$

For (2) note that both sides are linear in f. Finally for (3) note that if $f_n \nearrow f$, then $f_n \circ T \nearrow f \circ T$. □

I reckon we will see (1), (2), (3) again when we prove Fubinni's theorem.

Example 4. Suppose $\Omega = \{0,1\}^n$, $\mathcal{F} =$ all subsets of Ω, $\mu(A) = \frac{|A|}{2^n}$. Suppose also that $\Omega' = \{0, 1, \ldots, \}$ and $\mathcal{F}'$ is again the discrete σ-algebra. Define $T(\omega) = \sum_{i=1}^n \omega_i$ and thus $\mu^{T^{-1}}(\{j\}) = \frac{1}{2^n}\binom{n}{j}$. If $f' : \Omega' \to \mathbb{R}$, then the change of measure formula states that

$$\sum_{\omega \in \{0,1\}^n} \frac{1}{2^n} f\left(\sum_{i=1}^n \omega_i\right) = \sum_{j=1}^n \frac{1}{2^n}\binom{n}{j} f(j).$$