STATS300A - Lecture 16

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1 Recap

Our current goal is to find uniformally most powerful unbiased (UMPU) tests for testing $H_0 : \theta \in \Omega_0$ against $H_1 : \theta \in \Omega_1$. Recall that a test function ϕ is unbiased at level α if

$$\mathbb{E}_{\theta_0}\phi \leq \alpha \text{ for all } \theta_0 \in \Omega_0,$$

and

$$\mathbb{E}_{\theta_1}\phi \geq \alpha \text{ for all } \theta_1 \in \Omega_1.$$

We also say a test ϕ was α-similar if for all $\theta \in W$ where $W = \overline{\Omega}_0 \cap \overline{\Omega}_1$. We previously proved the following theorem which relates unbiased and α-similar tests.

Theorem 1 (TSH 4.11)**.** *If $\theta \mapsto \mathbb{E}_\theta\phi$ is continuous for all tests ϕ and ϕ is uniformly most powerful among level α α-similar tests, then ϕ is UMPU at level α.*

Today we will find optimal unbiased tests in multiparameter exponential families. Specifically we will derive optimal one sided tests in the presence of nuisance parameters.

2 Multiparameter exponential families

Suppose we have a model $\{P_{\theta,\lambda}\}$ where $(\theta, \lambda) \in \mathbb{R}^{k+1}$ is unknown and $P_{\theta,\lambda}$ has density

$$p_{\theta,\lambda}(x) = h(x)\exp\left\{\theta U(x) + \sum_{i=1}^{k} \lambda_i T_i(x) - A(\theta, \lambda)\right\}.$$

We wish to test $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$. For a fixed θ, the family $\{p_{\theta,\lambda}\}$ is an exponential family with sufficient statistics $T = (T_1, \ldots, T_k)$ and so

$$P_{\theta,\lambda}(X|T) = P_\theta(X|T).$$

In particular we have $P_{\theta,\lambda}(U(X)|T(X)) = P_\theta(U(X)|T(X))$ and so $U(X)|T(X)$ has no λ dependence.

Remark 1. This observation is important. We have shown that conditioning eliminates the nuisance parameters. Thus we can fix $(\theta_0, \lambda_0) \in \Omega_1$ and (θ_1, λ_1) and construct a test based on $P_{\theta_0}(X|T)$ against $P_{\theta_1}(X|T)$ which has no λ dependence. Even better, conditioning on T gives us a one-dimensional exponential family.

Lemma 1. *For each t, $U(X)|T = t$ forms a one-dimensional exponetial family in θ.*

Proof. We will only consider the discrete case. For all u and t let

$$A_{u,t} = \{x \in \mathcal{X} : U(x) = u, T(x) = t\} \quad \text{and} \quad A_t = \{x \in \mathcal{X} : T(x) = t\}.$$

$$\begin{aligned}
P_{\theta,\lambda}(U(X) = u|T(X) = t) &= \frac{P_{\theta,\lambda}(U(X) = u, T(X) = t)}{P_{\theta,\lambda}(T(X) = t)} \\
&= \frac{\sum\limits_{x \in A_{u,t}} p_{\theta,\lambda}(x)}{\sum\limits_{x \in A_t} p_{\theta,\lambda}(x)} \\
&= \frac{\sum\limits_{x \in A_{u,t}} \exp\left\{\theta u + \sum_{i=1}^k \lambda_i t_i\right\} h(x)}{\sum\limits_{x \in A_t} \exp\left\{\theta U(x) + \sum_{i=1}^k \lambda_i t_i\right\} h(x)} \\
&= \underbrace{\exp\{\theta u\}}_{\text{exponential tilt}} \times \underbrace{\sum_{x \in A_{u,t}} h(x)}_{g(t,u) = \text{base measure}} \times \underbrace{\frac{1}{\sum\limits_{x \in A_t} \exp\{\theta U(x)\} h(x)}}_{c(t,\theta) = \text{normalizing constant}}.
\end{aligned}$$

So $U(X)|T(X) = t$ is a one-dimensional exponential family with sufficient statistic U. $\square$

Thus we can apply our previously developed theory to the conditional distribution $U|T$. Our general recipe for one sided testing $\theta \le \theta_0$ against $\theta > \theta_0$ is

(1) Fix an alternative $\theta = \theta_1 > \theta_0$ and $\lambda_1 \in \mathbb{R}^k$.

(2) Condition on T so that $X|T$ does not depend on λ and $U|T$ follows a one dimensional exponential family.

(3) Construct the MP test for the conditional distribution. That is

$$\phi_t(u) = \begin{cases} 1 & \text{if } u > k(t), \\ \rho(t) & \text{if } u = k(t), \\ 0 & \text{if } u < k(t). \end{cases}$$

where $k(t)$ and $\rho(t)$ are determined by the conditional level constraint

$$\mathbb{E}_{\theta_0}[\phi_t|T = t] = \alpha. \tag{1}$$

We will next argue that under some assumptions that test $\phi^*(u, t) = \phi_t(u)$ is the UMPU test for H_0 against H_1. Note that for every test ϕ

$$\mathbb{E}_{\theta,\lambda}\phi = \mathbb{E}_{\theta,\lambda}\left[\mathbb{E}_{\theta,\lambda}[\phi|T]\right] = \mathbb{E}_{\theta,\lambda}\left[\mathbb{E}_{\theta}[\phi|T]\right].$$

In particular if $\theta \le \theta_0$, then

$$\mathbb{E}_{\theta,\lambda}\phi^* = \mathbb{E}_{\theta,\lambda}\left[\mathbb{E}_{\theta}[\phi|T]\right] \le \mathbb{E}_{\theta,\lambda}[\alpha] = \alpha,$$

and we have equality if $\theta = \theta_0$. Thus ϕ^* is level α and α-similar.

By Neyman-Pearson, there is no test that satisfies the constraint (1) and has strictly large power thatn ϕ^* for some fixed t and $\theta_1 > \theta$. Thus ϕ^* is the most powerful test in the class of tests satisfying (1) for any fixed $\theta_1 > \theta$. Since ϕ^* does not depend on θ_1, the test ϕ^* is in fact the UMP test among tests satisfying the constrain (1).

Recall that we are trying to show that ϕ^* is the UMPU test.