STATS310A - Lecture 19

Persi Diaconis
Scribed by Michael Howes

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Contents

1 Fourier inversion

Theorem 1. *Suppose that μ is a probability on $\mathbb{R}$ with charateristic function ϕ. If $\mu(\{a\}) = \mu(\{b\}) = 0$, then*

$$\mu((a,b)) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt. \tag{1}$$

To prove this we will need several classic calculus facts. We won't prove them here but their derivations are in section 26 of Billingsley.

(a) Let $S(T) = \int_0^T \frac{\sin(t)}{t}dt$ ($S(T)$ is also known as $\text{sinc}(T)$). While $\int_0^\infty \frac{\sin(t)}{t}dt$ does not exist as a Lebesgue integral, a classical calculation shows that

$$\lim_{T\to\infty} S(T) = \frac{\pi}{2}.$$

Thus $S(T)$ is bounded on $[0,\infty)$. Let sgn be the function

$$\text{sgn}(\theta) = \begin{cases} -1 & \text{if } \theta < 0, \\ 0 & \text{if } \theta = 0, \\ 1 & \text{if } \theta > 0. \end{cases}$$

By using a change of variables, one can show that

$$\int_0^T \frac{\sin(t\theta)}{t}dt = \text{sgn}(\theta)S(T|\theta|). \tag{2}$$

(b) Using Taylor's theorem we have for all real t.

i. $\left|e^{ix} - 1\right| \le \min\{|x|, 2\}$.

ii. $\left|e^{ix} - (1 + ix)\right| \le \min\left\{\frac{1}{2}x^2, 2|x|\right\}$.

Using (ii.) we see that the integrand in equation (1) converges to $b - a$ as $t \to 0$. Thus the integrand in (1) is continuous and by (i.) it is bounded. Thus the integral in (1) exists for every T.

With these calculus facts we are ready to prove Theorem 1.

Proof. For $T > 0$, let

$$I_T = \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt.$$

Note that

$$I_T = \frac{1}{2\pi}\int_{-T}^{T}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}e^{itx}\mu(dx)dt.$$

The above integrand is a bounded function and the space $[-T, T] \times \mathbb{R}$ has finite measure with respect to $dt \times \mu(dx)$. Thus we can apply Fubinni's theorem which gives

$$I_T = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}dt\mu(dx).$$

We will now use equation (2) and the fact sin and cos are odd and even respectively. This gives us

$$\begin{aligned} I_T &= \frac{1}{2\pi}\int_{\mathbb{R}} 2\int_{0}^{T} \frac{\sin(t(x-a))}{t} - \frac{\sin(t(x-b))}{t}dt \\ &= \int_{\mathbb{R}}\left[\frac{\operatorname{sgn}(x-a)}{\pi}S(T|x-a|) - \frac{\operatorname{sgn}(x-b)}{\pi}S(T|x-b|)\right]\mu(dx). \end{aligned}$$

The function $f_T(x) = \frac{\operatorname{sgn}(x-a)}{\pi}S(T|x-a|) - \frac{\operatorname{sgn}(x-b)}{\pi}S(T|x-b|)$ is uniformly bounded over x and T. We know that for $x \neq a, b$,

$$\frac{1}{\pi}S(T|x-a|), \frac{1}{\pi}S(T|x-b|) \to \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}.$$

Thus if $\operatorname{sgn}(x-a) = \operatorname{sgn}(x-b)$ and $x \neq a, b$, then $f_T(x) \to 0$. Also if $\operatorname{sgn}(x-a) \neq \operatorname{sgn}(x-b)$ and $x \neq a, b$, then $f_T(x) \to 1$. If $x = a$, then $\operatorname{sgn}(x-b) = -1$ and $f_T(x) \to \frac{1}{2}$. Lastly, if $x = b$, then $\operatorname{sgn}(x-a) = 1$ and $f_T(x) \to \frac{1}{2}$. Summarising this, we can conclude that f_T converges pointwise to the function $\psi_{a,b}$ where

$$\psi_{a,b}(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{1}{2} & \text{if } x = a, \\ 1 & \text{if } a < x < b, \\ \frac{1}{2} & \text{if } x = b, \\ 0 & \text{if } x > b. \end{cases}$$

Since $\mu(\{a, b\}) = 0$, we can conclude that f_T converges to $\delta_{(a,b)}$ μ-almost surely. Thus, by the dominated convergence theorem we have

$$\lim_{T\to\infty} I_T = \lim_{T\to\infty}\int_{\mathbb{R}} f_T(x)\mu(dx) = \int_{\mathbb{R}} \delta_{(a,b)}(x)\mu(dx) = \mu((a, b)).$$ □

Note that this proof also shows that

$$\mu((a, b)) + \frac{1}{2}\mu(\{a, b\}) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt,$$

which was the version of the Fourier inversion theorem stated on Tuesday.

Remark 1. To be honest, while careful, this proof sort of "stink". It doesn't give any feeling for what's going on. We'll come back to this later when we look at a discrete version of characteristic functions. This "lack of transparency" is what led Stein to develop Stein's method. It's also why we studied Lindeberg's proof of the central limit theorem in class.

Remark 2. Note that if F is the cumulative distribution function of μ, then equation (1) can be written as

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)dt.$$

In particular for $c \in \mathbb{R}$ and $h > 0$,

$$\frac{F(c+h) - F(c)}{h} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itc}(1 - e^{-ith})}{ith} \phi(t)dt.$$

If we assume that ϕ is integrable (so that $\int_{\mathbb{R}} |\phi(t)|dt < \infty$), then an analysis of the aboe equation can be used to show that μ has a density f given by

$$f(c) = F'(c) = \int_{\mathbb{R}} e^{-itc} \phi(t)dt.$$

Example 1 (The usual CLT)**.** The following is a typical application. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_1^2] = 1$ and characteristic function ϕ. Let $S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$. Since the first two moments of X_j exist ϕ is twice continuously differentiable with $\phi'(0) = 1$ and $\phi''(0) = -1$.

Since $X_1, X_2, \ldots$, are independent, we know the that

$$\phi_{S_n}(t) = \phi(t/\sqrt{n})^n = \left(1 - \frac{t^2}{2n} + o\left(1/n\right)\right)^n \to e^{-\frac{t^2}{2}} = \phi_{\mathcal{N}}(t),$$

where $\phi_{\mathcal{N}}$ denotes the characteristic function of the standard normal distribution.