

STATS300A - Lecture 11

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1 Announcement

Please arrive at least 5 minutes early to the exam on Wednesday.

2 Admissibility ($d = 1$)

We have seen that unique Bayes estimators are admissible. We wish to boost this result to \bar{X}_n which is a limit of Bayes estimators.

Example 1. Suppose $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where σ^2 is unknown and we are using squared error loss. To show that \bar{X}_n is admissible we will use a limiting Bayes argument. Suppose without loss of generality that $\sigma^2 = 1$ and that \bar{X}_n is inadmissible. Note that the risk of \bar{X}_n is $\frac{1}{n}$ constantly. Thus if \bar{X}_n is inadmissible, then there exists an estimator δ such that $R(\theta, \delta) < 1/n$ for some θ and $R(\theta, \delta) \leq 1/n$ for all θ .

One can use the dominated convergence theorem to show that $\theta \mapsto R(\theta, \delta)$ is continuous. Thus there exists an interval (θ_0, θ_1) such that $\theta_1 - \theta_0 > 0$ and $R(\theta, \delta) \leq 1/n - \varepsilon$ for all $\theta \in (\theta_0, \theta_1)$.

Let r'_τ be the average risk of δ with respect to the prior $\theta \sim \mathcal{N}(0, \tau^2)$. Also let r_τ be the average risk of the Bayes estimator with respect to the prior $\theta \sim \mathcal{N}(0, \tau^2)$. We know that r_τ is the posterior variance of τ and thus

$$r_\tau = \frac{1}{n + 1/\tau^2} = \frac{\tau^2}{n\tau^2 + 1}.$$

Thus r_τ approaches $1/n$ as $\tau \nearrow \infty$. We also know that $r'_\tau \leq 1/n$ for all τ . We will now look at the

ratio $\frac{1/n - r'_\tau}{1/n - r_\tau}$. This is a sort of Taylor's expansion of r_τ and r'_τ about $1/n$. Note that

$$\begin{aligned}
 \frac{1/n - r'_\tau}{1/n - r_\tau} &= \frac{\int_{\mathbb{R}} (1/n - R(\theta, \delta)) \frac{1}{\sqrt{2\pi\tau}} \exp(-1/2\theta^2) d\theta}{\frac{1}{n} - \frac{\tau^2}{n\tau^2+1}} \\
 &= \frac{\int_{\mathbb{R}} (1/n - R(\theta, \delta)) \frac{1}{\sqrt{2\pi\tau}} \exp(-1/2\theta^2) d\theta}{\frac{n\tau^2+1}{n(n\tau^2+1)} - \frac{n\tau^2}{n(n\tau^2+1)}} \\
 &= \frac{\int_{\mathbb{R}} (1/n - R(\theta, \delta)) \frac{1}{\sqrt{2\pi\tau}} \exp(-1/2\theta^2) d\theta}{\frac{1}{n(n\tau^2+1)}} \\
 &= \frac{n(n\tau^2+1)}{\sqrt{2\pi\tau}} \cdot \int_{\mathbb{R}} (1/n - R(\theta, \delta)) \exp(-1/2\theta^2) d\theta \\
 &\geq \frac{n(n\tau^2+1)}{\sqrt{2\pi\tau}} \cdot \int_{\theta_0}^{\theta_1} (1/n - R(\theta, \delta)) \exp(-1/2\theta^2) d\theta \\
 &\geq \frac{n(n\tau^2+1)}{\sqrt{2\pi\tau}} \cdot \varepsilon \int_{\theta_0}^{\theta_1} \exp(-1/2\theta^2) d\theta.
 \end{aligned}$$

As $\tau \rightarrow \infty$, $\frac{n(n\tau^2+1)}{\sqrt{2\pi\tau}} \rightarrow \infty$ and by the dominated convergence theorem $\int_{\theta_0}^{\theta_1} \exp(-1/2\theta^2) d\theta \rightarrow \int_{\theta_0}^{\theta_1} 1 d\theta = \theta_1 - \theta_0 > 0$. Thus we have

$$\lim_{\tau \rightarrow \infty} \frac{1/n - r'_\tau}{1/n - r_\tau} = \infty.$$

In particular there exists $\tau > 0$ such that $\frac{1/n - r'_\tau}{1/n - r_\tau} > 1$. This implies that $r'_\tau < r_\tau$ which is a contradiction.

3 Inadmissibility ($d \geq 3$)

We will now look at an example of simultaneous estimation and look at the James-Stein estimator. The take away will be that minimax estimators and UMRUES need not be admissible.

Suppose $X \in \mathbb{R}^p$ and $X \sim \mathcal{N}(\theta, I_p)$ for some $\theta \in \mathbb{R}^p$. Our goal is to estimate θ under the loss $L(\theta, d) = \sum_{j=1}^p (\theta_j - d_j)^2 = \|\theta - d\|_2^2$. The estimator $\delta(X) = X$ is

- A minimax estimator for θ .
- The UMRUES for θ .
- The MLE for θ , that is $X = \arg \max_{\theta} p(x; \theta)$.

From many perspectives X seems like the best estimator but X is inadmissible for $p \geq 3$. Recall the empirical Bayes estimator for θ

$$\left(1 - \frac{p}{\sum_i X_i^2}\right) X.$$

This came up in the setting $\theta_i \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \tau^2)$ and $X_i | \theta_i \stackrel{\text{ind}}{\sim} \mathcal{N}(\theta_i, 1)$. We will see that a similar estimator will outperform $\delta(X) = X$ uniformly in θ in a frequentist setting.

For intuition one may ask what is the problem with $\delta(X) = X$? The problem is that $\|X\|_2^2$ is normally much larger $\|\theta\|_2^2$ since $\mathbb{E}[\|X\|_2^2] = \sum_{i=1}^p (\theta_i^2 + 1) = \|\theta\|_2^2 + p \gg \|\theta\|_2^2$.

Theorem 1. [TPE 5.5.1] Define the estimator δ^0 by

$$\delta_j^0(X) = \left(1 - \frac{p-2}{\|X\|_2^2}\right) X_j.$$

The estimator δ^0 has strictly smaller risk than $\delta(X) = X$ for all θ . Thus $\delta(X) = X$ is inadmissible.

We call δ^0 a *James-Stein estimator*.

Proof. We know that $R(\theta, \delta) = p$ when $\delta(X) = X$. Now note that

$$\begin{aligned} R(\theta, \delta^0) &= \mathbb{E}_\theta \left[\sum_j \left(\theta_j - \left(1 - \frac{p-2}{\|X\|_2^2}\right) X_j \right)^2 \right] \\ &= \mathbb{E}_\theta \left[\sum_j \left(\theta_j - X_j + \frac{p-2}{\|X\|_2^2} X_j \right)^2 \right] \\ &= \sum_j \mathbb{E}_\theta [(\theta_j - X_j)^2] - 2 \sum_j \mathbb{E}_\theta \left[(X_j - \theta_j) \left(\frac{p-2}{\|X\|_2^2} X_j \right) \right] + \sum_j \mathbb{E}_\theta \left[\frac{(p-2)^2 X_j^2}{\|X\|_2^4} \right] \\ &= p - 2 \sum_j \mathbb{E}_\theta \left[(X_j - \theta_j) \left(\frac{p-2}{\|X\|_2^2} X_j \right) \right] + \sum_j \mathbb{E}_\theta \left[\frac{(p-2)^2 X_j^2}{\|X\|_2^4} \right]. \end{aligned}$$

Recall Stein's identity if $X \sim \mathcal{N}(\mu, \sigma^2)$ we have

$$\mathbb{E}[g(X)(X - \mu)] = \sigma^2 \mathbb{E}[g'(X)].$$

By conditioning this gives

$$\mathbb{E}_\theta \left[(X_j - \theta_j) \left(\frac{p-2}{\|X\|_2^2} X_j \right) \right] = \mathbb{E}_\theta \left[(p-2) \frac{\partial}{\partial X_j} \left(\frac{X_j}{\|X\|_2^2} \right) \right].$$

If we make this substitution we have

$$\begin{aligned} R(\theta, \delta^0) &= p - 2 \sum_j \mathbb{E}_\theta \left[(p-2) \frac{\partial}{\partial X_j} \left(\frac{X_j}{\|X\|_2^2} \right) \right] + \sum_j \mathbb{E}_\theta \left[\frac{(p-2)^2 X_j^2}{\|X\|_2^4} \right] \\ &= p - 2 \sum_j \mathbb{E}_\theta \left[(p-2) \frac{\|X\|_2^2 - 2X_j^2}{\|X\|_2^4} \right] + \sum_j \mathbb{E}_\theta \left[\frac{(p-2)^2 X_j^2}{\|X\|_2^4} \right] \\ &= p - 2(p-2) \mathbb{E}_\theta \left[\sum_j \frac{\|X\|_2^2 - 2X_j^2}{\|X\|_2^4} \right] + (p-2)^2 \mathbb{E}_\theta \left[\sum_j \frac{X_j^2}{\|X\|_2^4} \right] \\ &= p - 2(p-2) \mathbb{E}_\theta \left[\frac{p\|X\|_2^2 - 2\|X\|_2^2}{\|X\|_2^4} \right] + (p-2)^2 \mathbb{E}_\theta \left[\frac{\|X\|_2^2}{\|X\|_2^4} \right] \\ &= p - 2(p-2)^2 \mathbb{E}_\theta \left[\frac{1}{\|X\|_2^2} \right] + (p-2)^2 \mathbb{E}_\theta \left[\frac{1}{\|X\|_2^2} \right] \\ &= p - (p-2)^2 \mathbb{E}_\theta \left[\frac{1}{\|X\|_2^2} \right] \\ &< p. \end{aligned}$$

Thus δ^0 uniformly outperforms δ . □

Remark 1. (a) Why only $p > 2$? The random variable $\frac{1}{\|X\|_2^2}$ is not integrable when $p = 2$. Thus the regularity conditions of Stein's identity are not met.

(b) The above theorem is surprising! We can improve the risk by sharing information across different dimensions. The components $X_{\setminus j}$ are used to estimate θ_j even though X_j are independent and θ_j do not have any restrictions.

(c) Define $\delta_j^v = \left(1 - \frac{p-2}{\|X_v\|_2^2}\right) (X_j - v_j)$ for $v \in \mathbb{R}^p$. Then the above proof shows that δ^v also outperforms δ .

(d) If $\sigma \neq 1$, then we can use $\hat{\theta}_{JS} = \left(1 - \frac{\sigma^2(p-2)}{\|X\|_2^2}\right) X$.

(e) $\hat{\theta}_{JS}$ is inadmissible, we can improve by using

$$\delta'_j(X) = \left(1 - \frac{p-2}{\|X\|_2^2}\right)_+ X_j.$$

(f) A warning: For some j , $\mathbb{E}[(\delta_j^0(X) - \theta_j)^2]$ may be larger than $\mathbb{E}[(X_j - \theta_j)^2]$. An example is "Baseball batting averages - James-Stein estimator"