STATS310B – Lecture 5

Sourav Chatterjee
Scribed by Michael Howes

01/18/22

Contents

1 Sub-martingales and super-martingales

We ended last lecture with the definition of two generalizations of martingales. They were,

Definition 1. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration and let $\{X_n\}_{n\geq 0}$ be an adapted sequence of integrable random variables.

1. The sequence $\{X_n\}_{n\geq 0}$ is a *sub-martingale* if for all n, $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ almost surely.
2. Likewise, the sequence $\{X_n\}_{n\geq 0}$ is a *super-martingale* if for all n, $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$ almost surely.

Clearly $\{X_n\}_{n\geq 0}$ is a martingale if and only if, $\{X_n\}_{n\geq 0}$ is both a sub-martingale and a super-martingale.

1.1 New martingales from old

Jensen's inequality allows us to create many sub-martingales and super-martingales from a martingale.

Proposition 1. *Let $\{X_n\}_{n\geq 0}$ be a martingale and let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $Y_n = \phi(X_n)$ is integrable for all n. Then,*

1. *If ϕ is convex, then $\{Y_n\}_{n\geq 0}$ is sub-martingale.*
2. *If ϕ is concave, then $\{Y_n\}_{n\geq 0}$ is a super-martingale.*

Proof. If ϕ is convex, then

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(\phi(X_{n+1})|\mathcal{F}_n) \geq \phi(\mathbb{E}(x_{n+1}|\mathcal{F}_n)) = \phi(X_n) = Y_n.$$

If ϕ is concave, then $-\phi$ is convex and so $\{-Y_n\}_{n\geq 0}$ is a sub-martingale. This implies that $\{Y_n\}_{n\geq 0}$ is a super-martingale. □

Example 1. If $\{X_n\}_{n\geq 0}$, then $|X_n|$, X_n^2 and $e^{\theta X_n}$ are all sub-martingales (provided the last two are integrable).

We can also get new sub-martingales from a sub-martingale.

Proposition 2. *Let $\{X_n\}_{n\geq 0}$ be a sub-martingale and let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing convex function. If $Y_n = \phi(X_n)$ is integrable from every n, then $\{Y_n\}_{n\geq 0}$ is a sub-martingale.*

Proof. By convexity and Jensen's,

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \geq \phi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)).$$

Also, $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ almost surely. Since ϕ is non-decreasing, this implies

$$\phi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) \geq \phi(X_n) = Y_n.$$

□

Example 2. If $\{X_n\}_{n\geq 0}$ is a sub-martingale, then X_n^+ is a sub-martingale. If $\theta > 0$ and $e^{\theta X_n}$ is integrable for every n, then $e^{\theta X_n}$ is also a sub-martingale. The random variables X_n^2 and $|X_n|$ need not form sub-martingales even if they are integrable.

We can also get a martingale from a sub-martingale.

Proposition 3 (Doob's decomposition)**.** *Let $\{X_n\}_{n\geq 0}$ be a sub-martingale. Then we can write $X_n = X_0 + M_n + A_n$, where $\{M_n\}_{n\geq 0}$ is a martingale and $\{A_n\}_{n\geq 0}$ is a non-decreasing, predictable sequence.*

The definition of a predictable sequence is given below.

Definition 2. Let $\{F_n\}_{n\geq 0}$ be a filtration. An adapted sequence of random variables $\{A_n\}_{n\geq 0}$ is *predictable* if for every $n \geq 1$, A_n is $\mathcal{F}_{n-1}$-measurable.

Proof of Doob's decomposition. Define,

$$M_n = \sum_{k=0}^{n-1} X_{k+1} - X_k - \mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k).$$

Also define,

$$A_n = \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k).$$

Then,

$$M_n + A_n = \sum_{k=0}^{n-1} X_{k+1} - X_k = X_n - X_0.$$

Thus, it remains to show that $\{M_n\}_{n\geq 0}$ is a martingale and that $\{A_n\}_{n\geq 0}$ is predictable and non-decreasing. Note that for every k, $\mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k)$ is $\mathcal{F}_k$-measurable. Thus, A_n is $\mathcal{F}_{n-1}$-measurable. We also have,

$$A_{n+1} - A_n = \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - \mathbb{E}(X_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n \geq 0.$$

Also note that,

$$\mathbb{E}(X_{n+1} - X_n - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)|\mathcal{F}_n) = \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = 0.$$

Thus,

$$\mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1} - X_n - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)|\mathcal{F}_n) = 0.$$

Thus, $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(M_n|\mathcal{F}_n) = M_n$ and so $\{M_n\}_{n\geq 0}$ is a martingale. □

Doob's decomposition result is important because a lot of properties about martingales are well known. Thus, if one can get a handle on the increasing predictable sequence $\{A_n\}_{n\geq 0}$, then the original sub-martingale $\{X_n\}_{n\geq 0}$ can be studied.

1.2 Optional stopping for sub and super-martingales

Proposition 4. *Let $\{X_n\}_{n\geq 0}$ be an integrable sequence adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. Let T and S be bounded stopping times for $\{\mathcal{F}_n\}_{n\geq 0}$. Then,*

1. *If $\{X_n\}_{n\geq 0}$ is a sub-martingale, then $\mathbb{E}(X_T|\mathcal{F}_S) \geq X_S$.*
2. *If $\{X_n\}_{n\geq 0}$ is a super-martingale, then $\mathbb{E}(X_T|\mathcal{F}_S) \leq X_S$.*

The following proposition allows us to rigorously work with sequences $\{X_n\}_{n\geq 0}$ that are (sub/super)-martingales up to a stopping time T.

Proposition 5. *Let $\{X_n\}_{n\geq 0}$ be a sequence of integrable random variables adapted to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Let T be a stopping time with respect to $\{\mathcal{F}_n\}_{n\geq 0}$, such that on the event $\{T > n\}$, we have*

$$X_n \leq \mathbb{E}(X_{n+1}|\mathcal{F}_n) \quad a.s. \tag{1}$$

By which we mean $\mathbb{P}(X_n > \mathbb{E}(X_{n+1}|\mathcal{F}_n), T > n) = 0$. Then the sequence $\{X_{n\wedge T}\}_{n\geq 0}$ is a sub-martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$

Proof. As an exercise, one can show that $X_{n\wedge T}$ is $\mathcal{F}_n$-measurable and integrable. Now note that,

$$\begin{aligned}
\mathbb{E}(X_{(n+1)\wedge T}|\mathcal{F}_n) &= \mathbb{E}\left(\sum_{i=0}^{n} X_{(n+1)\wedge T}\mathbf{1}_{\{T=i\}} + X_{(n+1)\wedge T}\mathbf{1}_{\{T>n\}}|\mathcal{F}_n\right) \\
&= \sum_{i=0}^{n} \mathbb{E}(X_{(n+1)\wedge T}\mathbf{1}_{\{T=i\}}|\mathcal{F}_n) + \mathbb{E}(X_{(n+1)\wedge T}\mathbf{1}_{\{T>n\}}|\mathcal{F}_n) \\
&= \sum_{i=0}^{n} \mathbb{E}(X_i\mathbf{1}_{\{T=i\}}|\mathcal{F}_n) + \mathbb{E}(X_{n+1}\mathbf{1}_{\{T>n\}}|\mathcal{F}_n) \\
&= \sum_{i=0}^{n} X_i\mathbf{1}_{\{T=i\}} + \mathbb{E}(X_{n+1}\mathbf{1}_{\{T>n\}}|\mathcal{F}_n),
\end{aligned}$$

since $X_i\mathbf{1}_{\{T=i\}}$ is $\mathcal{F}_n$-measurable. The event $\{T > n\} = \{T \leq n\}^c$ is in $\mathcal{F}_n$ and thus, by our assumption (1),

$$\begin{aligned}
\mathbb{E}(X_{(n+1)\wedge T}|\mathcal{F}_n) &= \sum_{i=0}^{n} X_i\mathbf{1}_{\{T=i\}} + \mathbf{1}_{\{T>n\}}\mathbb{E}(X_{n+1}|\mathcal{F}_n) \\
&\geq \sum_{i=0}^{n} X_i\mathbf{1}_{\{T=i\}} + \mathbf{1}_{\{T>n\}}X_n \\
&= \sum_{i=0}^{n-1} X_i\mathbf{1}_{\{T=i\}} + X_n\mathbf{1}_{\{T>n-1\}} \\
&= X_{n\wedge T}.
\end{aligned}$$

$\square$

Remark 1. If the inequality in (1) is replaced with an equality, then $\{X_{n\wedge T}\}_{n\geq 0}$ is a martingale. Likewise, if the inequality in (1) is reversed, then $\{X_{n\wedge T}\}_{n\geq 0}$ is a super-martingale. The proofs are analogous.

The idea behind proposition (5) is that even we can ignore what happens after the stopping time T. This is useful in examples when the distribution of X_n changes after T occurs.

Example 3. Suppose a gambler starts with $x > a$ dollars. At each turn the gambler can win or loss a dollar with equal probability. However, will their total is greater a, they have to pay b in tax each turn. We wish to know how long it will take for the gambler to have less than a. Let $\{X_n\}_{n\geq 0}$ be the total the gambler has at each turn and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let $T = \min\{n : X_n \leq a\}$. We wish to bound $\mathbb{E}[T]$. Note that the event $\{T > n\}$ equals $\{X_1 > a, \ldots, X_n > a\}$. Thus, on the event $\{T > n\}$, X_{n+1} is either $X_n + 1 - b$ or $X_n - 1 - b$. Furthermore, $X_{n+1} - X_n$ is independent of $\mathcal{F}_n$. Thus, on $\{T > n\}$,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n + \mathbb{E}[X_{n+1} - X_n] = X_n - b \leq X_n.$$

Now define $Y_n = X_n + nb$. On $\{T > n\}$, we have $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = X_N - b + (n+1)b = Y_n$. Thus, $\{Y_{n\wedge T}\}_{n\geq 0}$ is a martingale. Since $0 \wedge T = 0$, we thus have

$$\mathbb{E}[Y_{T\wedge n}] = \mathbb{E}[Y_0] = \mathbb{E}[X_0] = \mathbb{E}[x] = x.$$

On the other hand,

$$\mathbb{E}[Y_{T\wedge n}] = \mathbb{E}[X_{T\wedge n}] + b\mathbb{E}[T \wedge n] \geq a + b\mathbb{E}[T \wedge n].$$

The inequality holds because if $T > n$, then $X_{T\wedge n} = X_n > a$ and if $T \leq n$, then $X_{T\wedge n} = X_T = a$. The random variables $n \wedge T$ are non-negative and increase up to T. Thus,

$$x \geq a + b\mathbb{E}[T],$$

which implies $\mathbb{E}[T] \leq \frac{x-a}{b}$. This is agrees with our intuition, since the gambler loses b dollars each turn on average and T records the time it takes the gambler's total to go from x dollars to a dollars.

2 Optimal stopping problem

We will now consider a different kind of problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The finite horizon optimal stopping problem is as follows. Suppose we have integrable random variables $\{X_n\}_{n=1}^N$ and a filtration $\{\mathcal{F}_n\}_{n=1}^N$, can we find a stopping time T taking values in $\{1, \ldots, N\}$ that maximizes $\mathbb{E}[X_T]$? Note that we have made no assumptions about $\{X_n\}_{n=1}^N$ beyond integrability. We have not assumed that $\{X_n\}_{n=1}^N$ is adapted to $\{\mathcal{F}_n\}_{n=1}^N$. Nonetheless, this problem has a solution in this very general set up!