# STATS310A - Lecture 4

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#### 1 The $\pi - \lambda$ Theorem

Let  $(\Omega, \mathcal{F})$  be a measure space. Recall that a collection of sets  $\mathcal{P}$  is a  $\pi$ -system if  $\mathcal{P}$  is closed under finite intersection. A collection of sets L is a  $\lambda$ -system if  $\Omega \in L$ , L is closed under complements and L is closed under countable *disjoint* unions.

**Theorem 1.** If  $\mathcal{P}$  is a  $\pi$ -system, L is a  $\lambda$ -systam and  $\mathcal{P} \subseteq L$ , then  $\sigma(\mathcal{P}) \subseteq L$ .

**Example 1.** Consider the following  $\Omega = \{1, 2, 3, 4\}$ ,  $\mathcal{F} = \{A : A \subseteq \Omega\}$ . Define two probabilities  $P_1, P_2 : \mathcal{F} \to [0, 1]$  by  $P_1(\omega) = \frac{1}{4}$  for all  $\omega \in \Omega$  and  $P_2(\omega) = \frac{1}{2}$  if  $\omega = 2, 4$  and  $P_2(\omega) = 0$  otherwise. The collection  $\mathcal{G} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}\}$  is  $\lambda$ -system but not a  $\sigma$ -algebra (or even an algebra). The two measures  $P_1$  and  $P_2$  agree on  $\mathcal{G}$  but not on  $\sigma(\mathcal{G}) = \mathcal{F}$ . Note that the collection

$$L = \{ A \in \mathcal{F} : P_1(A) = P_2(A) \}$$

is always a  $\lambda$ -system for any probabilities  $P_1, P_2$ . The  $\pi - \lambda$  theorem lets us conclude that L contains  $\sigma(\mathcal{P})$  if  $P_1$  and  $P_2$  agree on  $\mathcal{P}$  and  $\mathcal{P}$  is a  $\pi$ -system.

Note the following facts.

<u>Fact 0</u>: If L is a  $\lambda$  system and  $B_1, B_2 \in L$  with  $B_1 \subseteq B_2$ , then  $B_2 \setminus B_1 \in L$ . This is because  $B_2 \setminus B_1 = B_2 \cap B_1^c = (B_2^c \cup B_1)^c$  and  $B_2^c \cap B_1 = \emptyset$  since  $B_1 \subseteq B_2$ . Thus L is closed under relative complements as well as complements.

<u>Fact 1</u>: If L is both a  $\pi$ -system and a  $\lambda$ -system, then L is a  $\sigma$ -algebra. To see why this is, consider  $(A_i)_{i=1}^{\infty}$  a countable collection of sets in L. Iteratively define  $A'_1 = A_1$  and  $A'_i = A_i \cap (A'_1 \cup \ldots \cup A'_{i-1})$ . Since L is both a  $\pi$ -system and a  $\lambda$ -system,  $A'_i \in L$ . Also  $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i \in L$  since the sets  $(A'_i)_{i=1}^{\infty}$  are disjoint and L is a  $\lambda$ -system.

We will now prove the  $\pi - \lambda$  theorem.

*Proof.* Given  $\mathcal{P}$  a  $\pi$ -system and L a  $\lambda$ -system with  $\mathcal{P} \subseteq L$ , define  $L_0$  to be the  $\lambda$ -system generated by  $\mathcal{P}$ . We will show that  $L_0$  is a  $\pi$ -system which by fact 1 will imply that  $L_0$  is a  $\sigma$ -algebra and hence  $\sigma(\mathcal{P}) = L_0 \subseteq L$ .

For each  $A \in L_0$ , define  $L_A = \{B : B \cap A \in L_0\}$ . We will chose that  $L_A$  is a  $\lambda$ -system. Note first that  $\Omega \in L_A$  since  $A \in L_0$ . Also if  $B \in L_A$ , then  $A \cap B \in L_0$  and  $A \cap B \subseteq A$ . Thus since  $L_0$ 

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is a  $\lambda$ -system and closed under relative complements we have  $A \setminus (A \cap B) = A \cap B^c$  is in  $L_0$ . Thus  $B^c \in L_A$ . Finally if  $(B_i)_{i=1}^{\infty}$  is a countable collection of disjoint element of  $L_A$ , then  $(A \cap B_i)_{i=1}^{\infty}$  is a countable collection of disjoint elements of  $L_0$ . Thus  $\bigcup_{i=1}^{\infty} A \cap B_i = A \cap (\bigcup_{i=1}^{\infty} B_i) \in L_0$  and so  $\bigcup_{i=1}^{\infty} B_i \in L_A$ .

If  $A \in \mathcal{P}$  and  $B \in \mathcal{P}$ , then  $A \cap B \in \mathcal{P} \subseteq L_0$ . Thus  $B \in L_A$ . Since B was arbitrary, this means that  $L_A$  is a  $\lambda$ -system that contains  $\mathcal{P}$  and hence  $L_0 \subseteq L_A$  for every  $A \in \mathcal{P}$ .

Now if  $A \in L_0$  and  $B \in \mathcal{P}$ , then  $L_0 \subseteq L_B$  and hence  $B \cap A \in L_0$  which implies  $B \in L_A$ . Thus  $L_A$  is a  $\lambda$ -system that contains  $\mathcal{P}$  and hence  $L_0 \subseteq L_A$ . But this means that for all  $A, B \in L_0, A \cap B \in L_0$ , thus  $L_0$  is a  $\lambda$ -system.

# 2 Independence

**Definition 1.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. Let  $\mathcal{C}_i \subseteq \mathcal{F}$  for each  $i \in I$  (where I is any index set). The collection  $\{\mathcal{C}_i\}_{i \in I}$  are said to be *independent* if for all finite subsets  $J \subseteq I$  and every choice of  $A_j \in \mathcal{C}_j$  for  $j \in J$ , we have

$$P\left(\bigcap_{j\in J}A_j\right) = \prod_{j\in J}P(A_j).$$

**Theorem 2.** Suppose that  $\{C_i\}_{i\in I}$  are independent  $\pi$ -systems, then  $\{\sigma(C_i)\}_{i\in I}$  are independent.

*Proof.* Since the definition of independence only use finite subsets of I we may assume that  $I = \{1, \ldots, n\}$  for some  $n \in \mathbb{N}$ . Define  $\mathcal{B}_i = \mathcal{C}_i \cup \{\Omega\}$ , the collections  $\mathcal{B}_i$  are still  $\pi$ -systems and  $\sigma(\mathcal{C}_i) = \sigma(\mathcal{B}_i)$  so we will work with  $\mathcal{B}_i$ . Define

$$L = \left\{ B_1 \in \sigma(\mathcal{B}_i) : \prod_{i=1}^n P(B_i) = P\left(\bigcap_{i=1}^n B_i\right) \text{ for all } B_2 \in \mathcal{B}_2, \dots, B_n \in \mathcal{B}_n \right\}.$$

We will show that L is a  $\lambda$ -system. By the  $\pi - \lambda$  theorem we will be able to conclude that  $\sigma(\mathcal{B}_1) \subseteq L$ . First note that  $\Omega \in L$  since  $\{\mathcal{B}_i\}_{i=2}^n$  are independent. Suppose  $B_1 \in L$ , then

$$P(B_2 \cap \ldots \cap B_n) = P(B_1 \cap B_2 \cap \ldots \cap B_n) + P(B_1^c \cap B_2 \cap \ldots \cap B_n).$$

Rearranging we have

$$P(B_1^c \cap B_2 \cap \ldots \cap B_n) = P(B_2 \cap \ldots \cap B_n) - P(B_1 \cap B_2 \cap \ldots \cap B_n).$$

By our assumption that  $\{\mathcal{B}_i\}_{i=2}^n$  are independent and that  $B_1 \in L$ , we have

$$P(B_1^c \cap B_2 \cap ... \cap B_n) = P(B_2 \cap ... \cap B_n) - P(B_1 \cap B_2 \cap ... \cap B_n)$$

$$= \prod_{i=2}^n P(B_i) - \prod_{i=1}^n P(B_i)$$

$$= (1 - P(B_1)) \prod_{i=2}^n P(B_i)$$

$$= P(B_1^c) \prod_{i=2}^n P(B_i).$$

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and thus  $B_1^c \in L$ . Finally if  $(B_{1,j})_{j=1}^{\infty}$  are in L and disjoint, then set  $B_1 = \bigcup_{j=1}^{\infty} B_{1,j}$  and note that

$$P(B_1 \cap B_2 \cap \dots \cap B_n) = \sum_{j=1}^{\infty} P(B_{1,j} \cap B_2 \cap \dots \cap B_n)$$

$$= \sum_{j=1}^{\infty} P(B_{1,j}) \prod_{i=2}^{n} P(B_i)$$

$$= \left(\sum_{j=1}^{\infty} P(B_{1,j})\right) \prod_{i=2}^{n} P(B_i)$$

$$= \prod_{i=1}^{n} P(B_i).$$

Thus  $B_1 \in L$ , L is a  $\lambda$ -system and  $\sigma(\mathcal{B}_1) = L$ . We thus have that  $\{\sigma(\mathcal{B}_1), \mathcal{B}_2, \dots, \mathcal{B}_n\}$  is an independent collection. We can repeat this argument with

$$L_2 = \left\{ B_2 \in \sigma(\mathcal{B}_2) : P\left(\bigcap_{i=1}^n B_2\right) = \prod_{i=1}^n P(B_i) \text{ for all } B_1 \in \sigma(\mathcal{B}_1) \text{ and } B_i \in \mathcal{B}_i, i = 3, \dots, n \right\}.$$

We will then conclude that  $\sigma(\mathcal{B}_2) = L$ . By induction we will prove that  $\{\sigma(\mathcal{B}_i)\}_{i=1}^n$  is an independent collection.

**Example 2.** Consider our coin tossing example  $(\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)$  where  $\mathcal{B}$  is the collection of Borel sets and  $\lambda$  is Lesbegue measure. The digits  $d_i$  are independent in that if we set

$$A_i = \{\Omega, \emptyset, \{\omega : d_i(\omega) = 1\}, \{\omega : d_i(\omega) = 0\}\},\$$

then  $\{A_i\}_{i=1}^{\infty}$  are independent. The collections  $\bigcup_{i=1}^{\infty} A_{2i}$  and  $\bigcup_{i=1}^{\infty} A_{2i-1}$  are also independent and thus from  $\omega$  we can get two uniform (0,1] samples

$$\omega_1 = \sum_{i=1}^{\infty} 2^{-i} d_{2i}(\omega)$$
 and  $\omega_2 = \sum_{i=1}^{\infty} 2^{-i} d_{2i-1}(\omega)$ .

We can repeat this infinitely often to get a countable sequence of indpendent uniform (0,1] random variables. We could use this to get any other iid sequence of real valued random variables.

### 3 Borel-Cantelli Lemmas

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. For  $(A_i)_{i=1}^{\infty}$  in  $\mathcal{F}$  define

$$A_i \ i.o. = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Thus  $\omega \in A_i$  i.o. if and only if  $\omega \in A_i$  for infinitely many i. That is  $\omega$  is in  $A_i$  infinitely often (i.o.).

**Theorem 3.** [Borel-Cantelli 1] If  $\sum_{i=1}^{\infty} P(A_i) < \infty$ , then  $P(A_i \ i.o.) = 0$ .

*Proof.* For every  $n \in \mathbb{N}$ ,  $A_i$  i.o.  $\subseteq \bigcup_{m=n}^{\infty} A_m$  and thus by countable subadditivity,

$$P(A_i \ i.o.) \le \sum_{m=n}^{\infty} P(A_m) \underset{n \to \infty}{\longrightarrow} 0.$$

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**Example 3.** Let  $(\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)$ , let  $l_n(\omega)$  be the length of the heads run starting at n. That is

$$\{\omega : l_n(\omega) = k\} = \{\omega : d_n(\omega) = d_{n+1}(\omega) = \dots = d_{n+k-1}(\omega) = 1 \text{ and } d_{n+k}(\omega) = 0\}.$$

Thus  $P(\{l_n = k\}) = \frac{1}{2^{k+1}}$ . Let  $r_n$  be a sequence of positive integers and let  $A_n = \{\omega : l_n(\omega) \ge r_n\}$ . Then

$$P(A_n) = \sum_{k=0}^{\infty} \frac{1}{2^{r_n + k + 1}} = \frac{1}{2^{r_n}}.$$

By Borel-Cantelli, if  $\sum_{n=1}^{\infty} \frac{1}{2^{r_n}} < \infty$ , then  $P(A_n \ i.o) = 0$ . For example this is the case if  $r_n = (1+\varepsilon)\log_2(n)$ .

**Theorem 4.** [Borel-Cantelli 2] If  $\sum_{i=1}^{\infty} P(A_i) = \infty$  and  $\{A_i\}_{i=1}^{\infty}$  are independent, then

$$P(A_i \ i.o.) = 1.$$

*Proof.* Note that  $(A_i \ i.o.)^c = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c$ . Thus it suffices to show that for each  $n \in \mathbb{N}$ , we have  $P(\bigcap_{m=n}^{\infty} A_m^c) = 0$ . Note that for all n and N, we have

$$P\left(\bigcap_{m=n}^{\infty} A_m^c\right) \leq P\left(\bigcap_{m=n}^{N} A_m^c\right)$$

$$= \prod_{m=n}^{N} P(A_m^c)$$

$$= \prod_{m=n}^{N} (1 - P(A_m))$$

$$\leq \prod_{m=n}^{N} \exp(P(A_m))$$

$$= \exp\left(\sum_{m=n}^{N} P(A_m)\right)$$

$$\xrightarrow[N \to \infty]{} 0.$$

Thus  $P((A_i \ i.o.)^c) = 0.$ 

**Example 4.** As before let  $l_n$  be the length of the heads run at n. For a sequence of real number  $x_i$ , we have  $\limsup_i x_i = l$  if and only if for all  $\varepsilon > 0$ , we have

- (a) For infinitely many  $n, x_n \ge l \varepsilon$ , and
- (b) For sufficiently large  $n, x_n \leq l + \varepsilon$ .

That is  $\limsup_{i} x_i$  is the largest limit point of the sequence  $x_i$ . Using B.C. one and two, it can be shown that

$$\limsup_{n} \frac{l_n(\omega)}{\log_2(n)} = 1,$$

with probability 1. We have already seen from B.C. 1, that with probability 1, for every  $\varepsilon$ ,  $l_n(\omega) \leq (1+\varepsilon)\log_2(n)$  for all but finitely many n.