STATS300A - Lecture 15

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1 Motivation

Today we are going to develop optimal tests in one-dimensional exponential families. A UMP test need not exist in this case. Indeed, in the Gaussian model there is no UMP test for $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. This is because for an alternative $\mu_1 < \mu_0$, the MP test rejects H_0 when our data is small. For $\mu_1 > \mu_0$ the MP test rejects H_0 when our data is large (see picture below).

There are various approaches that can be used when the UMP test does not exist. These include:

(a) Symmetric constraints on test. For example $\phi(cx) = \phi(x)$ for $c > 0$.

(b) Put a probability distribution on Ω_1. Then collapse the risk function and maximize the average power.

(c) Maximize the worst case power (maximin test).

(d) Monotonicity constraints.

(e) Unbiasedness.

2 Unbiased tests

We will use the unbiasedness approach but it is good to know that there are other method available.

Definition 1. A test ϕ is *unbiased* at level α if

(a) For all $\theta_0 \in \Omega_0$, $\mathbb{E}_{\theta_0}\phi \leq \alpha$.

(b) For all $\theta_1 \in \Omega_1$, $\mathbb{E}_{\theta_1}\phi \geq \alpha$.

Remark 1. Some remarks on the above definition

- This is a form of risk unbiasedness. It corresponds to having level α and being risk unbiased under 0-1 loss.
- UMP tests (if they exist) are always unbiased tests.
- Recommending reading: Optimal inference after model selection by Fiftian, Sun and Taylor. This is a recent paper that uses ideas of unbiased tests.

The definition of unbiased gives us a new optimality condition.

Definition 2. A *uniformly most power unbiased (UMPU) test* is an unbiased level α test which is uniformly most powerful among all unbiased level α tests.

Thus we are restricting our set of tests. We have

$$\text{unbiased test} \subsetneq \text{all tests}.$$

A UMPU test is a test in the first class which is uniformly optimal in the first class. Unlike UMP tests, UMPU tests often exist for test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ and for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ in multiple parameter settings.

3 Constructing UMPU tests

3.1 A recipe

Constructing UMPU tests is not a straight forward proceedure. The general recipe that we will use is

(a) Rewrite constraints as weaker constraints. We will replace unbiasedness with a weaker condition.

(b) Fix a simple alternative $\theta_1 \in \Omega_1$.

(c) Find a MP under the constraints from (a). This will require a generalized Neyman Pearson lemma. Call this test ϕ_{θ_1}.

(d) If the test $\phi = \phi_{\theta_1}$ does not depend on θ_1, then ϕ is in fact the UMP under the constraints in (a).

(e) Show that ϕ actuallly satisfies the stronger unbiased constraint and thus ϕ is UMPU.

We will now work on each of the ingredients.

3.2 α-similar tests

Typically Ω_0 and Ω_1 are subsets of $\mathbb{R}^k$ and so we can define

$$W = \overline{\Omega}_0 \cap \overline{\Omega}_1,$$

where $\overline{A}$ is used to denote the closure of $A \subseteq \mathbb{R}^k$.

Examples 1.

- If we have the test $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$, then $W = \{\theta_0\}$.
- If we have the test $H_0 : \theta_1 \leq \widetilde{\theta}$ and $H_1 \theta_1 > \widehat{\theta}$ where $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$, then

$$W = \{\theta \in \mathbb{R}^k : \theta_1 = \widetilde{\theta}\}.$$

Definition 3. A test ϕ is *α-similar* if $\mathbb{E}_\theta \phi = \alpha$ for all $\theta \in W$.

Lemma 1. *Suppose that the function $\beta_\phi(\theta) = \mathbb{E}_\theta \phi$ is continuous on Ω for all test functions ϕ. If a test ϕ_0 is UMP among all α-similar tests, then ϕ_0 is also UMP among all unbiased test.*

Proof. We will first show that ϕ_0 is unbiased. Consider the constant test $\phi = \alpha$. The test ϕ is α-similar. Since ϕ_0 is UMP among all α-similar tests we must have that for all $\theta \in \Omega_1$,

$$\mathbb{E}_{\theta_1}\phi_0 \geq \mathbb{E}_{\theta_1}\phi = \alpha.$$

Thus ϕ_0 is unbiased at level α. To show that ϕ_0 is the UMPU test, it suffices to show that every unbiased test is a level α test. If ϕ is unbiased, then by continuity of β_ϕ we have that

$$\mathbb{E}_\theta \phi = \beta_\phi(\theta) \leq \alpha, \quad \text{for all } \theta \in \overline{\Omega}_0.$$

Likewise,

$$\mathbb{E}_\theta \phi = \beta_\phi(\theta) \geq \alpha, \quad \text{for all } \theta \in \overline{\Omega}_1.$$

Thus $\mathbb{E}_\theta \phi = \alpha$ for all $\theta \in W = \overline{\Omega}_0 \cap \overline{\Omega}_1$. □

The idea behind this proof is that we have shown:

Thus a test with UMP among α-similar tests is UMP among unbiased tests.

3.3 Method of undetermined multipliers

We will now restrict our attention to one-dimensional exponential families so that

$$p_\theta(x) \propto h(a) \exp\left\{\theta T(x)\right\}.$$

We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. Recall that no UMP test exists. One can show by the dominated convergence theorem that the map $\beta_\phi(\theta) = \mathbb{E}_\theta \phi$ is continuous and in fact differentiable for all test functions ϕ.

By our lemma from before we know that if ϕ is unbiased then $\mathbb{E}_{\theta_0}\phi = \alpha$ and $\mathbb{E}_\theta \phi \geq \alpha$ for all θ. Thus every unbiased level α test must statisfy $\mathbb{E}_{\theta_0}\phi = \alpha$ and θ_0 must be a global minimum of $\theta \mapsto \beta_\phi(\theta)$. So our power function looks something like this:

Thus the derivative of the power function at $\theta = \theta_0$ must be zerp. Hence every unbiased test must satisfy the two constraints:

$$\int_{\mathcal{X}} \phi(x) p_\theta(x) d\mu(x) = \alpha \quad \text{and} \quad \partial_\theta \beta_\phi(\theta) \mid_{\theta=\theta_0} = \int_{\mathcal{X}} \phi(x) \partial_\theta p_{\theta_0}(x) d\mu(x) = 0. \tag{1}$$

In the second constrain we used some of the regularity properties of exponential families to exchange integration and differentiation.