STATS310B – Lecture 9

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1 Uniform integrability

We ended last lecture with the statement of the following proposition,

Proposition 1. *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables. The sequence $\{X_n\}_{n\geq 0}$ is uniformly integrable if and only if the following both hold,*

1. $\sup_n \mathcal{E}[|X_n|] < \infty$.
2. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all A and n, if $\mathbb{P}(A) < \delta$, then $\mathcal{E}[|X_n|\mathbf{1}_A] < \varepsilon$.*

Which we will now prove.

Proof. First suppose that $\{X_n\}_{n\geq 1}$ is uniformly integrable. Fix any $\varepsilon > 0$, then there exists a k such that for all n,

$$\mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \varepsilon.$$

It follows that,

$$\mathcal{E}[|X_n|] = \mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|\leq k\}}] + \mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < k + \varepsilon.$$

Thus, $\sup_n \mathcal{E}[|X_n|] < \infty$. Now let $\varepsilon > 0$ be arbitrary and fix k so that for all n,

$$\mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \varepsilon/2.$$

Set $\delta = \frac{\epsilon}{2k}$ and suppose $\mathbb{P}(A) < \delta$. Then

$$\begin{aligned}
\mathcal{E}[|X_n|\mathbf{1}_A] &= \mathcal{E}[|X_n|\mathbf{1}_{A\cap\{|X_n|>k\}}] + \mathcal{E}[|X_n|\mathbf{1}_{A\cap\{|X_n|\leq k\}}] \\
&\leq \mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] + \mathcal{E}[k\mathbf{1}_A] \\
&< \epsilon/2 + k\mathbb{P}(A) \\
&< \epsilon/2 + k\delta \\
&= \epsilon.
\end{aligned}$$

Now conversely suppose that the two conditions hold and note that,

$$\mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] \geq \mathcal{E}[k\mathbf{1}_{\{|X_n|>k\}}] = k\mathbb{P}(|X_n| > k).$$

Thus, for every n and k, $\mathbb{P}(|X_n| > k) \leq \frac{\sup_m \mathcal{E}[|X_m|]}{k}$. Thus, given $\epsilon > 0$ choose δ so that $\mathbb{P}(A) < \delta$ implies that $\mathcal{E}[|X_n|\mathbf{1}_A] < \epsilon$ for every n. If $k > \frac{\sup_m \mathcal{E}[|X_m|]}{\delta}$, then $\mathbb{P}(|X_n| > k) < \epsilon$ for all n and hence

$$\mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \epsilon.$$ □

The above proposition has the following important corollary.

Corollary 1. *Let X be an integrable random variable. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbb{P}(A) < \delta$, then $\mathcal{E}[X\mathbf{1}_A] < \epsilon$.*

Proof. By proposition (1) it suffices to show that the sequence $X_n = X$ is uniformly integrable. By the dominated convergence theorem

$$\lim_{k\to\infty} \mathcal{E}[|X|\mathbf{1}_{\{|X|>k\}}] = 0,$$

showing that for all ϵ there exists a k such that $\mathcal{E}[|X|\mathbf{1}_{\{|X|>k\}}] < \epsilon$. □

2 Lévy downwards convergence theorem

With these results about uniform integrability we are ready to finish proving L'evy downwards convergence theorem. Recall the theorem's statement,

Theorem 1 (Lévy's downards convergence theorem)**.** *Let X be an integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots$ be a decreasing sequence of sub-σ-algebras of $\mathcal{F}$. Define $\mathcal{F}^* = \bigcap_{n=0}^{\infty} \mathcal{F}_n$, then*

$$\mathcal{E}(X|\mathcal{F}_n) \to \mathcal{E}(X|\mathcal{F}^*),$$

almost surely and in L^1.

Proof. In the previous lecture we use the upcrossing lemma to show that there exists an integrable random variable X^* such that if $X_n = \mathcal{E}(X|\mathcal{F}_n)$, then $X_n \to X^*$ almost surely. It remains to show that $X_n \to X^*$ in L^1 and that $X^* = \mathcal{E}(X|\mathcal{F}^*)$. We will first show that $\{X_n\}_{n\geq 0}$ is uniformly integrable. Thus, fix $\epsilon > 0$ and let δ be such that $\mathbb{P}(A) < \delta$ implies that $\mathcal{E}[|X|\mathbf{1}_A] < \epsilon$. Now choose k so that $\frac{\mathcal{E}[|X|]}{k} < \epsilon$. Let A_n be the event $\{\mathcal{E}(|X||\mathcal{F}_n) > k\}$. Note that, by Markov's inequality,

$$\begin{aligned}\mathbb{P}(A_n) &\leq \frac{\mathcal{E}[\mathcal{E}(|X||\mathcal{F}_n)]}{k} \\ &= \frac{\mathcal{E}[|X|]}{k} \\ &< \delta.\end{aligned}$$

Thus, $\mathcal{E}[|X|\mathbf{1}_{A_n}] < \epsilon$. Furthermore, since $A_n \in \mathcal{F}_n$ we have

$$\mathcal{E}[\mathcal{E}(|X||\mathcal{F}_n)\mathbf{1}_{A_n}] = \mathcal{E}[|X|\mathbf{1}_{A_n}] < \epsilon.$$

By Jensen's inequality $|X_n| = |\mathcal{E}(X|\mathcal{F}_n)| \leq \mathcal{E}(|X||\mathcal{F}_n)$. Thus,$\mathcal{E}[|X_n|\mathbf{1}_{A_n}] < \epsilon$. Furthermore, since $|X_n| < \mathcal{E}(|X||\mathcal{F}_n)$, we have $\{|X_n| > k\} \subseteq A_n$ and so

$$\mathcal{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] \leq \mathcal{E}[|X_n|\mathbf{1}_{A_n}] < \epsilon.$$

Thus, $\{X_n\}_{n\geq 0}$ are uniformly integrable. Since $X_n \to X^*$ almost surely, this implies that $X_n \to X^*$ in L^1. It remains to show that $X^* = \mathcal{E}(X|\mathcal{F}^*)$. First note that X^* is $\mathcal{F}^*$ measurable. This is because for every m, $X_n = \mathcal{E}(X|\mathcal{F}_n)$ is $\mathcal{F}_m$ measurable for every $n \geq m$. Thus, $X^* = \lim_{n\geq m} X_n$ is $\mathcal{F}_m$ measurable. Since this holds for every m we must have that X^* is $\mathcal{F}^* = \bigcap_{m=1}^\infty \mathcal{F}_m$ measurable. Now suppose that $A \in \mathcal{F}^*$. Then $A \in \mathcal{F}_n$ for every n and hence

$$\mathcal{E}[X_n \mathbf{1}_A] = \mathcal{E}[X\mathbf{1}_A],$$

for every m. Furthermore,

$$|\mathcal{E}[X_n\mathbf{1}_A] - \mathcal{E}[X^*\mathbf{1}_A]| \leq \mathcal{E}[|X_n - X^*|\mathbf{1}_A] \leq \mathcal{E}[|X_n - X^*|].$$

Since $X_n \to X^*$ in L^1, this implies that

$$\mathcal{E}[X^*\mathbf{1}_A] = \lim_{n\to\infty} \mathcal{E}[X_n\mathbf{1}_A] = \lim_{n\to\infty}\mathcal{E}[X\mathbf{1}_A] = \mathcal{E}[X\mathbf{1}_A].$$

Thus, $X^* = \mathcal{E}(X|\mathcal{F}^*)$. $\square$

We will now discuss and later prove an application of this convergence theorem.

3 De Finetti's theorem

Definition 1. Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. The sequence $\{X_n\}_{n\geq 1}$ is *exchangeable* if for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ that fixes all but finitely many values $(X_{\pi(1)}, X_{\pi(2)}, \ldots)$ has the same distribution as $(X_1, X_2, \ldots)$.

More precisely, we can think of the sequence $\{X_n\}_{n\geq 1}$ as a single random variable X taking values in $\mathbb{R}^\mathbb{N}$ given by

$$X(\omega) = (X_1(\omega), X_2(\omega), \ldots).$$

The distribution of X is the probability measure in induces on $(\mathbb{R}^\mathbb{N}, \mathcal{F}_{\mathbb{R}^\mathbb{N}})$ where $\mathcal{F}_{\mathbb{R}^\mathbb{N}}$ is the product σ-algebra of countably many copies of the Borel σ-algebra. Thus, the law of X is the probability measure μ_X given by

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)).$$

For a permutation $\pi : \mathbb{N} \to \mathbb{N}$ we can define another random variable $X_\pi = (X_{\pi(1)}, X_{\pi(2)}, \ldots)$. The sequence $\{X_n\}_{n\geq 0}$ is exchangeable if and only if for every π that fixes all but finitely many $i \in \mathbb{N}$, $\mu_{X_\pi} = \mu_X$.

For example, if $\{X_n\}_{n\geq 0}$ is an exchangeable sequence, then

$$\mathbb{P}(X_2 + X_3 + X_5^2 > 20 \text{ and } X_2 < 5) = \mathbb{P}(X_1 + X_3 + X_{10000}^2 > 20 \text{ and } X_1 < 5).$$

De Finetti's theorem is a theorem about exchangeable sequences. A special case of de Finetti's theorem concerns $\{0,1\}$-valued exchangeable sequences.

Theorem 2 (de Finitti's theorem for coin tosses.)**.** *Let $\{X_n\}_{n\geq 0}$ be an exchangeable sequence of $\{0,1\}$ valued random variables. Then there exists a probability measure μ on $[0,1]$ such that for every n and every choice of $x_1, \ldots, x_n \in \{0,1\}$,*

$$\mathbb{P}\left(X_1 = x_1, \ldots, X_n = x_n\right) = \int_0^1 \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}\mu(d\theta).$$

Furthermore, the limit $\Theta = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n X_i$ exists almost surely and the random variable Θ is distributed according to Θ.

Informally, conditional on the random variable Θ, $X_1, \ldots, X_n$ are i.i.d. random variables with $\mathbb{P}(X_i = 1|\Theta = \theta) = \theta$.

3.1 The exchangeable σ-algebra

Let $X_1, X_2, \ldots$ be any sequence of random variables. Recall that the σ-algebra $\mathcal{G} = \sigma(X_1, X_2, \ldots)$ consists of all set of the form $X^{-1}(B)$ where $X = (X_1, X_2, \ldots)$ and B is a measurable subset of $\mathbb{R}^{\mathbb{N}}$.

Definition 2. A set $B \subseteq \mathbb{R}^{\mathbb{N}}$ is *invariant under permutations of the first n coordinates* if for all permutations π of $\{1, \ldots, n\}$, for all $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, if $x \in B$, then $(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots) \in B$.

For example $\{x_1 + x_2 + x_3 > 5\}$ is invariant under permutations of the first 3 coordinates. The set $\{X_1 + x_2 + x_3^2 > 5\}$ is invariant under permutations of the first 2 coordinates but not invariant under permutations of the first 2 coordinates.

Definition 3. Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables and let $\mathcal{G} = \sigma(X_1, X_2, \ldots)$. Define $\mathcal{E}_n$ be the σ-algebra of all $A \in \mathcal{G}$ such that $A = X^{-1}(B)$ for some $B \subseteq \mathbb{R}^{\mathbb{N}}$ that is invariant under permutations of the first n coordinates. Also define $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$. The σ-algebra $\mathcal{E}$ is called the *exchangeable σ-algebra* of $\{X_n\}_{n\geq 0}$.

We are now ready to state the general version of de Finetti's theorem.

Theorem 3 (de Finetti's theorem)**.** *Let $\{X_n\}_{n\geq 0}$ be an exchangeable sequence of random variables and let $\mathcal{E}$ be the exchangable σ-algebra of $\{X_n\}_{n\geq 0}$. Then $\{X_n\}_{n\geq 0}$ are i.i.d. given $\mathcal{E}$ meaning that for every n and every choice of Borel sets $A_1, \ldots, A_n$, we have*

$$\begin{aligned}
\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n | \mathcal{E}) &= \prod_{i=1}^{n} \mathbb{P}(X_i \in A_i | \mathcal{E}) \\
&= \prod_{i=1}^{n} \mathbb{P}(X_1 \in A | \mathcal{E}).
\end{aligned}$$