STATS310A - Lecture 11

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Contents

1 Expectation **1**
1.1 Sums of random variables . . . 1

2 Homework **2**
2.1 Problem (A) . . . 2
2.2 Problem (B) . . . 3

1 Expectation

Definition 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and suppose $X : \Omega \to \mathbb{R}$ is measurable. We define *the expectation of* X to be

$$\mathbb{E}[X] = \int X(\omega)\mathbb{P}(d\omega),$$

whenever the above intergal exists.

Note that if $X = \delta_A$, then $\mathbb{E}[X] = \mathbb{P}(A)$. Thus we can once again reframe our fundamental question of probability:

Given a random variable X, compute or approximate $\mathbb{E}[X]$.

which generalizes our previous versions of this question.

1.1 Sums of random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and suppose X, Y are independent random variables on Ω. Define the following probabilities on $\mathbb{R}$,

$$\mu(A) = \mathbb{P}(X \in A) \text{ and } \nu(A) = \mathbb{P}(Y \in A).$$

Since X, Y are independent, we know that for all measurable $C \subseteq \mathbb{R}^2$,

$$\mathbb{P}((X, Y) \in C) = (\mu \times \nu)(C),$$

where $\mu \times \nu$ is the product measure given by

$$(\mu \times \nu)(C) = \int_{\mathbb{R}} \mu(C_y)\nu(dy) = \int_{\mathbb{R}} \nu(C_x)\mu(dx).$$

Given $D \subseteq \mathbb{R}$ measurable, define $C = \{(x,y) : x+y \in D\} \subseteq \mathbb{R}^2$. Note that $C_y = D - y$ and $C_x = D - x$. Thus

$$\begin{aligned}\mathbb{P}(X+Y \in D) &= \mathbb{P}((X,Y) \in C) \\ &= \int_{\mathbb{R}} \nu(C_x)\mu(dx) \\ &= \int_{\mathbb{R}} \nu(D-x)\mu(dx).\end{aligned}$$

And likewise

$$\mathbb{P}(X+Y \in D) = \int_{\mathbb{R}} \mu(D-y)\nu(dy).$$

The above defines a measure on $\mathbb{R}$ and is called the convolution of μ and ν. It is denoted $\mu * \nu$.

Example 1. Suppose $\mu = \text{Poission}(\theta)$ and $\nu = \text{Possion}(\eta)$ for some $\theta, \eta \geq 0$. That is

$$\mu(A) = \sum_{j \in A} \frac{e^{-\theta}\theta^j}{j!} \text{ and } \nu(B) = \sum_{j \in B} \frac{e^{-\eta}\eta^j}{j!},$$

where $A, B \subseteq \{0, 1, 2, \ldots\}$. Note that for $l = 0, 1, 2, \ldots,$

$$\begin{aligned}(\mu * \nu)(\{l\}) &= \sum_{a=0}^{l} \frac{e^{-\theta}\theta^a}{a!} \frac{e^{-\eta}\eta^{l-a}}{(l-a)!} \\ &= \frac{e^{-(\theta+\eta)}\eta^l}{l!} \sum_{a=0}^{l} \binom{l}{a} \left(\frac{\theta}{\eta}\right)^a \\ &= \frac{e^{-(\theta+\eta)}\eta^l}{l!} \left(1 + \frac{\theta}{\eta}\right)^l \\ &= \frac{e^{-(\theta+\eta)}(\theta+\eta)^l}{l!}.\end{aligned}$$

Thus $\mu * \nu = \text{Poission}(\theta + \eta)$.

Example 2. Similarly if $X \sim \mathcal{N}(\theta_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\theta_2, \sigma_2^2)$ and X, Y are independent, then $X + Y \sim \mathcal{N}(\theta_1 + \theta_2, \sigma_1^2 + \sigma_2^2)$.

2 Homework

Read chapters 21-22 and do problem 20.8, 20.9, 21.3 and 21.6. There are two other problems which we describe now.

2.1 Problem (A)

This problem is about the $\beta - \Gamma$ calculus. For $\alpha > 0$, we say $X \sim \text{Gamma}(\alpha)$ on $[0, \infty)$ if X has density

$$\frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}, \text{ for } x \geq 0.$$

For $\alpha, \beta > 0$, we say that $W \sim \text{Beta}(\alpha, \beta)$ on $[0,1]$ is W has density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \text{ for } 0 \leq x \leq 1.$$

On the homework we will show:

(a) If $X \sim \text{Gamma}(\alpha)$ and $Y \sim \text{Gamma}(\beta)$ and X is independent of Y, then $\frac{X}{X+Y}$ and $X+Y$ are independent, $\frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$ and $X+Y \sim \text{Gamma}(\alpha+\beta)$.

(b) If $X, Y, Z \sim \text{Gamma}(\alpha), \text{Gamma}(\beta), \text{Gamma}(\gamma)$ are independent, then

$$\frac{X}{X+Y}, \frac{X+Y}{X+Y+Z} \text{ and } X+Y+Z,$$

are independent and they have distributions $\text{Beta}(\alpha, \beta)$, $\text{Beta}(\alpha+\beta, \gamma)$ and $\text{Gamma}(\alpha+\beta+\gamma)$ respectively.

2.2 Problem (B)

Before discussing problem (B) on the homework. We will state, prove and apply a theorem.

Theorem 1. *Suppose $X \geq 0$, then*

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)dt = \int_0^\infty \mathbb{P}(X > t)dt.$$

Note that the two integrals are indeed equal since $\mathbb{P}(X = t) > 0$ for at most countably many t. Thus the functions $\mathbb{P}(X \geq t)$ and $\mathbb{P}(X > t)$ agree almost everywhere with respect to Lesbegue measure.

Proof. Suppose that $X = \sum_{j=1}^k x_j \delta_{A_j}$ with $0 \leq x_1 \leq x_2 \leq \ldots \leq x_k$. Then

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{j=1}^k x_j \mathbb{P}(X = x_j) \\
&= \sum_{j=1}^{k-1} x_j \left(\mathbb{P}(X \geq x_j) - \mathbb{P}(X > x_{j+1})\right) + x_k \mathbb{P}(X \geq x_k) \\
&= x_1 \mathbb{P}(X \geq x_1) + \sum_{j=1}^k (x_j - x_{j-1}) \mathbb{P}(X \geq x_j).
\end{aligned}$$

Observe $\mathbb{P}(X \geq t) = \mathbb{P}(X \geq x_1) = 1$ for $t \in [0, x_1]$ and $\mathbb{P}(X \geq t) = 0$ for $t > x_k$. Also note that ff $x_{j-1} < t < x_j$, then

$$\mathbb{P}(X \geq t) = \mathbb{P}(X \geq x_j).$$

Thus the function $t \mapsto \mathbb{P}(X \geq t)$ is a step function and we have

$$\int_0^\infty \mathbb{P}(X \geq t)dt = x_1 + \mathbb{P}(X \geq x_i) + \sum_{j=1}^k x_j - x_{j-1})\mathbb{P}(X \geq x_j) = \mathbb{E}[X].$$

A limiting argument via the monotone convergence theorem allows us to conclude that the result holds for all $X \geq 0$. $\square$