STATS 305C: Practice Exam

Name:

Problem 1: *Gaussian models*

Consider the following model,

$$
\begin{aligned}
x_{n,d} &\overset{\text{ind}}{\sim} \mathcal{N}(0, \sigma_d^2) && \text{for } n = 1, \ldots, N;\ d = 1, \ldots, D \\
\sigma_d^2 &= \prod_{k=1}^{d} \lambda_k^{-1} && \text{for } d = 1, \ldots, D \\
\lambda_d &\overset{\text{iid}}{\sim} \mathrm{Ga}(\alpha, 1) && \text{for } d = 1, \ldots, D
\end{aligned}
$$

(a) Suppose $\alpha > 1$. Describe how this *multiplicative inverse gamma* prior affects the distribution of the data, $x_{n,d}$. For example, how does the distribution of $x_{n,1}$ generally compare to that of $x_{n,D}$?

(b) Let $\boldsymbol{\lambda} = \{\lambda_k\}_{k=1}^K$ and $\boldsymbol{X} = \{\{x_{n,d}\}d = 1^D\}_{n=1}^N$. Derive a Gibbs sampler for the posterior distribution $p(\boldsymbol{\lambda} \mid \boldsymbol{X}; \alpha)$.

Problem 2: *Hierarchical models.*

Recall the probability density function of the gamma distribution, $p(\lambda; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$, where $\Gamma(\cdot)$ is the gamma function. Now consider the following hierarchical model,

$$\begin{aligned}
\beta_g &\sim \mathrm{Ga}(\alpha_0, \beta_0) \\
\lambda_g &\sim \mathrm{Ga}(\alpha, \beta_g) && \text{for } g = 1, \dots, G \\
x_{g,n} &\sim \mathrm{Po}(\lambda_g) && \text{for } g = 1, \dots, G;\ n = 1, \dots, N.
\end{aligned}$$

Using the Poisson probability mass function $p(x \mid \lambda) = \frac{1}{x!} \lambda^x e^{-\lambda}$, derive a Gibbs sampling algorithm for this hierarchical model. Specifically, derive the conditional distributions,

- $p(\lambda_g \mid \{x_{g,n}\}_{n=1}^N, \beta_g; \alpha)$,
- $p(\beta_g \mid \lambda_g; \alpha_0, \beta_0)$.

Problem 3: *Graphical models.*

(a) Draw the graphical model corresponding to this joint probability distribution,

$$p(\{x_n, y_n\}_{n=1}^N; \alpha, \beta, \gamma) = p(x_1 \mid \alpha)\left[\prod_{n=2}^{N} p(x_n \mid x_{n-1}; \beta)\right]\left[\prod_{n=1}^{N} p(y_n \mid x_n; \gamma)\right].$$

(b) Write the joint distribution corresponding to the graphical model in Figure 1.

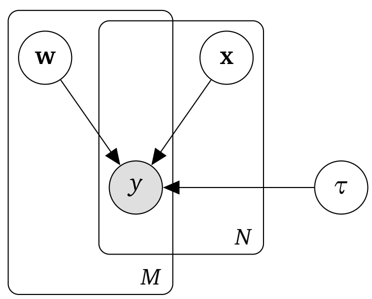


Figure 1

Problem 4: *Continuous latent variable models*

Canonical correlation analysis is a technique for paired datasets $\{(\boldsymbol{x}_n, \boldsymbol{y}_n\})\}_{n=1}^N$ where $\boldsymbol{x}_n \in \mathbb{R}^{D_x}$ and $\boldsymbol{y}_n \in \mathbb{R}^{D_y}$. Like PCA, it can be viewed as a limiting case of a linear Gaussian model latent variable model,

$$\begin{aligned}
\boldsymbol{z}_n &\sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}) \\
\boldsymbol{x}_n &\sim \mathcal{N}(\boldsymbol{W}_x \boldsymbol{z}_n + \boldsymbol{b}_x, \boldsymbol{\Sigma}_x) \\
\boldsymbol{y}_n &\sim \mathcal{N}(\boldsymbol{W}_y \boldsymbol{z}_n + \boldsymbol{b}_y, \boldsymbol{\Sigma}_y).
\end{aligned}$$

Derive the conditional distribution $p(\boldsymbol{y}_n \mid \boldsymbol{x}_n; \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\boldsymbol{W}_x, \boldsymbol{W}_y, \boldsymbol{b}_x, \boldsymbol{b}_y, \boldsymbol{\Sigma}_x, \boldsymbol{\Sigma}_y)$.

Problem 5: *The Bayesian Lasso*

The Lasso problem is an L_1 penalized least squares problem,

$$\mathcal{L}(\boldsymbol{w}) = \sum_{n=1}^{N} \|y_n - \boldsymbol{x}_n^\top \boldsymbol{w}\|_2^2 + \lambda_0 \sum_{d=1}^{D} |w_d|. \tag{1}$$

From a Bayesian perspective, minimizing $\mathcal{L}(\boldsymbol{w})$ is equivalent to *maximum a posteriori* (MAP) estimation in the following Bayesian model,

$$\begin{aligned} w_d &\overset{\text{iid}}{\sim} \text{Lap}(\lambda) \\ y_n &\overset{\text{iid}}{\sim} \mathcal{N}(\boldsymbol{x}_n^\top \boldsymbol{w}, \sigma^2), \end{aligned} \tag{2}$$

where $\text{Lap}(\lambda)$ denotes a Laplace distribution with density $\text{Lap}(w; \lambda) = \frac{\lambda}{2} e^{-\lambda |w|}$.

(a) Find a setting of λ such that the MAP estimate of model (2) is the same as the minimizer of eq. 1. Your solution should be in terms of λ_0 and σ^2.

(b) The Laplace density can also be written as a *scale mixtures of Gaussians*,

$$\text{Lap}(w; \lambda) = \frac{\lambda}{2} e^{-\lambda |w|} = \int_0^\infty \mathcal{N}(w; 0, v) \cdot \text{Exp}\left(v; \tfrac{\lambda^2}{2}\right) \mathrm{d}v = \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{w^2}{2v}} \cdot \frac{\lambda^2}{2} e^{-\frac{\lambda^2 v}{2}} \, \mathrm{d}v$$

Let $\boldsymbol{y} = \{y_n\}_{n=1}^N$ and $\boldsymbol{X} = \{\boldsymbol{x}_n\}_{n=1}^N$. Use the integral representation above to write a joint distribution,

$$p(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{y} \mid \boldsymbol{X}; \lambda, \sigma^2)$$

on an extended space that includes the *augmentation variables* $\boldsymbol{v} = (v_1, \ldots, v_D)$, such that the marginal distribution $p(\boldsymbol{w}, \boldsymbol{y} \mid \boldsymbol{X}; \lambda, \sigma^2)$ matches that of the generative model described in eq. (2).

(c) What algorithm would you use to perform Bayesian inference to approximate the posterior distribution $p(\boldsymbol{w}, \boldsymbol{v} \mid \boldsymbol{X}, \boldsymbol{y}; \lambda, \sigma^2)$? Sketch out the steps involved.

Problem 6: *Mixture Models*

Consider the following *zero-inflated Poisson regression* model where $w, x_n \in \mathbb{R}_+$, $y_n \in \mathbb{N}$, and $z_n \in \{0, 1\}$,

$$\begin{aligned} w \mid \alpha, \beta &\sim \mathrm{Gamma}(\alpha, \beta) \\ z_n \mid \gamma &\overset{\text{iid}}{\sim} \mathrm{Bern}(\gamma) \\ y_n \mid x_n, z_n, w &\overset{\text{iid}}{\sim} \mathrm{Poisson}(w x_n z_n). \end{aligned}$$

(a) Sketch the probability mass function of the marginal distribution $p(y_n \mid x_n, w, \gamma)$ for $\gamma \in \{0, 0.5, 1\}$, assuming $w x_n = 5$. What is $p(y_n = 0 \mid x_n, w, \gamma)$? (Note: $0! = 1$ and $0^0 = 1$.)

(b) Compute the conditional distribution $p(z_n = 1 \mid y_n, x_n, w, \gamma)$.

(c) Compute the expected log probability,

$$\mathscr{L}(w) = \mathbb{E}_{p(z \mid y, x, w', \gamma)}\left[\log p(\{y_n, x_n, z_n\}_{n=1}^N, w \mid \alpha, \beta, \gamma)\right],$$

where w' denotes a fixed weight. For notational simplicity, let $q_n \triangleq p(z_n = 1 \mid y_n, x_n, w', \gamma)$ denote the solution to part (c), and drop terms in $\mathscr{L}(w)$ that are constant with respect to w.

(d) Assume $\alpha > 1$. Solve for $w^\star = \arg\max \mathscr{L}(w)$ using the fact that the mode of the Gamma(a, b) distribution is at $(a - 1)/b$ when $a > 1$.

Problem 7: *Mixed Membership Models*

Latent Dirichlet allocation (LDA) corresponds to the following generative model,

$$
\begin{aligned}
\boldsymbol{\eta}_k &\sim \mathrm{Dir}(\boldsymbol{\phi}) && \text{for } k = 1, \dots, K \\
\boldsymbol{\pi}_n &\sim \mathrm{Dir}(\boldsymbol{\alpha}) && \text{for } n = 1, \dots, N \\
z_{n,\ell} &\sim \boldsymbol{\pi}_n && \text{for } n = 1, \dots, N;\ \ell = 1, \dots, L \\
x_{n,\ell} &\sim \boldsymbol{\eta}_{z_{n,\ell}} && \text{for } n = 1, \dots, N;\ \ell = 1, \dots, L
\end{aligned}
$$

where $\boldsymbol{\eta}_k \in \Delta_V$ are the *topics* (i.e. distributions over words) and $\boldsymbol{\pi}_n \in \Delta_K$ are the *topic proportions* (i.e. distributions over topics).

However, this model fails to capture correlations in the topic proportions; for example, that a "finance" topic and a "government" topic may often co-occur in the same document. *Correlated topic models* address this limitation by replacing the Dirichlet prior on $\boldsymbol{\pi}_n$ with a logistic normal prior,

$$
\begin{aligned}
\boldsymbol{\pi}_n &= \mathrm{softmax}(\boldsymbol{u}_n) = \left[\frac{e^{u_{n1}}}{1 + \sum_{k=1}^{K-1} e^{u_{nk}}}, \dots, \frac{e^{u_{n,K-1}}}{1 + \sum_{k=1}^{K-1} e^{u_{nk}}}, \frac{1}{1 + \sum_{k=1}^{K-1} e^{u_{nk}}} \right]^\top \\
\boldsymbol{u}_n &\sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})
\end{aligned}
$$

where $\boldsymbol{u}_n \in \mathbb{R}^{K-1}$. The correlations in $\boldsymbol{u}_n$ due to the multivariate normal prior induce correlations in $\boldsymbol{\pi}_n$ as well.

(a) Without doing any math, sketch the density of $\boldsymbol{\pi}_n \in \Delta_3$ when $\boldsymbol{\mu} = [0,0]^\top$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Do the same for $\boldsymbol{\mu} = [0,0]^\top$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. Explain your reasoning.

(b) Try to derive CAVI updates for this model. Where do you run into trouble and why?

Problem 8: *Variational autoencoders*

Consider the following *deep mixture model*,

$$\begin{aligned} z_n &\sim \pi \\ \boldsymbol{x}_n &\sim \mathcal{N}(\boldsymbol{\mu}_{z_n}, \boldsymbol{\Sigma}_{z_n}) \\ \boldsymbol{y}_n &\sim \mathcal{N}(f(\boldsymbol{x}_n; \boldsymbol{w}), \sigma^2 \boldsymbol{I}) \end{aligned}$$

where $z_n \in \{1, \ldots, K\}$ is a discrete latent variable, $\boldsymbol{x}_n \in \mathbb{R}^M$ is a continuous latent variable, $\boldsymbol{y}_n \in \mathbb{R}^D$ is an observed data point, and $f : \mathbb{R}^M \mapsto \mathbb{R}^D$ is a neural network with weights $\boldsymbol{w}$. The generative model parameters are $\boldsymbol{\theta} = (\boldsymbol{\pi}, \{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}_{k=1}^K, \boldsymbol{w})$.

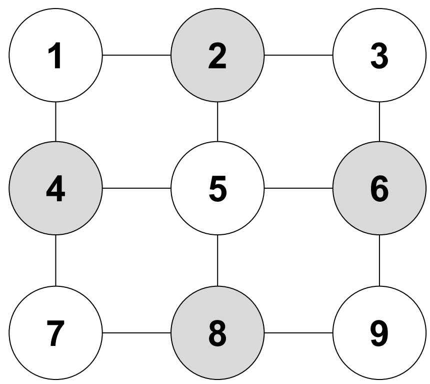
(a) Suppose you wanted to perform fixed form variational inference to approximate the posterior, $p(z_n, \boldsymbol{x}_n \mid \boldsymbol{y}_n; \boldsymbol{\theta}) \approx q(z_n, \boldsymbol{x}_n; \boldsymbol{\phi})$, with variational parameters $\boldsymbol{\phi}$. What challenges might you encounter when trying to maximize the local ELBO, $\mathscr{L}_n(\boldsymbol{\theta}, \boldsymbol{\phi})$, using stochastic gradient ascent and the reparameterization trick (i.e. the pathwise gradient estimator)?

(b) Suggest an alternative to the reparameterization trick that could allow you to fit $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$. What challenges might this alternative present?

(c) Rewrite the generative model by marginalizing over z_n to obtain a collapsed model $p(\boldsymbol{x}_n, \boldsymbol{y}_n; \boldsymbol{\theta})$, and assume a variational posterior $q(\boldsymbol{x}_n; \boldsymbol{\phi})$. Can you use the reparameterization trick now?

Problem 9: *State space models*

In class we studied state space models for sequential data, like hidden Markov models and linear dynamical systems. Here we will consider similar models for 2-dimensional data. Suppose we observe a image $\boldsymbol{y} \in \mathbb{R}^{H \times W}$ which we believe to be a noisy version of an underlying binary image $\boldsymbol{x} \in \{0,1\}^{H \times W}$. Given $\boldsymbol{y}$, we wish to recover the true image $\boldsymbol{x}$ which it was derived from. We formulate this as a probabilistic inference problem. We will assume the image is square and start by constructing a graph which connects neighboring pixels. The graph for $H = W = 3$ is shown below, with the node labels corresponding to the indices in the vectors $\boldsymbol{y}$ and $\boldsymbol{x}$.



Our prior on $\boldsymbol{x}$ will be given as an *Ising model*, which encodes our belief that nearby pixels are likely to be similar:

$$p(\boldsymbol{x}) = \frac{1}{Z(\theta)} \prod_{(ij,kl) \in \mathscr{E}} \psi_\theta(x_{ij}, x_{kl})$$

Here, $\mathscr{E}$ is the edge set of the pixel graph, $Z(\theta)$ is a normalizing constant, and $\psi_\theta : \{0,1\} \times \{0,1\} \to \mathbb{R}_{++}$ is defined by:

$$\psi_\theta(x_{ij}, x_{kl}) = \begin{cases} e^\theta & , x_{ij} = x_{kl} \\ 1 & , x_{ij} \neq x_{kl} \end{cases}$$

where $\theta > 0$ is a hyperparameter. We assume a Gaussian noise model, which gives us a likelihood over $\boldsymbol{y}$ given $\boldsymbol{x}$ as:

$$p(\boldsymbol{y} \mid \boldsymbol{x}) = \prod_{i=1}^{H} \prod_{j=1}^{W} \mathcal{N}(y_{ij} \mid x_{ij}, \sigma^2)$$

where σ^2 is a hyperparameter. Given $\boldsymbol{y}$, we will obtain our denoised image by sampling from the posterior $p(\boldsymbol{x} \mid \boldsymbol{y})$ using Gibbs sampling

(a) Given a pixel (i,j), let $\mathscr{E}(i,j)$ denote its neighbors in the pixel graph. Similarly, given $\boldsymbol{x} \in \{0,1\}^{H \times W}$, let $N_1(i,j,\boldsymbol{x}_{-ij}) = \sum_{(k,l) \in \mathscr{E}(i,j)} x_{k,l}$ denote the number of neighbors of pixel (i,j) set to 1 and let $N_0(i,j,\boldsymbol{x}_{-ij}) = \sum_{(kl) \in \mathscr{E}(i,j)} 1 - x_{kl}$ denote the number of neighbors of pixel (i,j) set to 0.

Show that the complete conditional of x_{ij} is given by:

$$p(x_{ij} = 1 \mid \boldsymbol{x}_{-ij}, \boldsymbol{y}) = \frac{e^{\phi_1}}{e^{\phi_0} + e^{\phi_1}}$$

where

$$\phi_1 = \theta N_1(i, j, \boldsymbol{x}_{-ij}) + \log \mathcal{N}(y_{ij} \mid x_{ij} = 1, \sigma^2)$$
$$\phi_0 = \theta N_0(i, j, \boldsymbol{x}_{-ij}) + \log \mathcal{N}(y_{ij} \mid x_{ij} = 0, \sigma^2)$$

(b) Suppose we also incorporate a prior $p(\theta)$ on θ, e.g. $p(\theta) = \text{Gamma}(\theta; \alpha, \beta)$. It is not possible to derive a closed form for θ's complete conditional $p(\theta \mid \boldsymbol{x}, \boldsymbol{y})$. Explain what we may do instead to approximately sample from this conditional. Why might this be computationally challenging for large images (i.e. when D is large)?

(c) *[Bonus]* Consider the pixel graph, and let $\mathscr{S}$ be a maximal set of nodes such that $\mathscr{E}(i, j) \cap \mathscr{S} = \emptyset$ for all $(i, j) \in \mathscr{S}$. For the example graph, we could use $\mathscr{S}$ as the shaded set of nodes, so $\mathscr{S} = \{2, 4, 6, 8\}$. Explain why we have $\boldsymbol{x}_S \perp\!\!\!\perp \boldsymbol{x}_{-\mathscr{S}} \mid \boldsymbol{y}, \theta$ and how we can exploit this for an efficient parallel block Gibbs update.

Problem 10: *Bayesian nonparametrics*

In class we said that a Dirichlet random variable equal in distribution to a normalized vector of independent gamma random variables,

$$\begin{aligned}
\gamma_k &\overset{\text{ind}}{\sim} \mathrm{Ga}(\alpha_k, 1) \\
\boldsymbol{\pi} &= \left[\frac{\gamma_1}{\sum_{k=1}^K \gamma_k}, \ldots, \frac{\gamma_K}{\sum_{k=1}^K \gamma_k} \right]^\top \\
\Rightarrow \boldsymbol{\pi} &\sim \mathrm{Dir}(\boldsymbol{\alpha}).
\end{aligned}$$

It turns out there are many other useful properties of the gamma distribution, like

$$\gamma_k \overset{\text{ind}}{\sim} \mathrm{Ga}(\alpha_k, 1) \Rightarrow \sum_{k=1}^K \gamma_k \sim \mathrm{Ga}\left(\sum_{k=1}^K \alpha_k, 1 \right).$$

Moreover, the normalized vector of gammas is independent of the sum, $\boldsymbol{\pi} \perp\!\!\!\perp \sum_{k=1}^K \gamma_k$. Finally, the gamma is also related to the beta distribution,

$$\begin{aligned}
\gamma_k &\sim \mathrm{Ga}(\alpha_k, 1); \quad k \in \{0, 1\} \\
\beta &= \frac{\gamma_1}{\gamma_0 + \gamma_1} \\
\Rightarrow \beta &\sim \mathrm{Beta}(\alpha_1, \alpha_0).
\end{aligned}$$

Use these properties to derive a stick breaking procedure for sampling a (finite) Dirichlet distribution with concentration $\boldsymbol{\alpha} \in \mathbb{R}_+^K$.