Homework Template

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Problem 1

Problem. How many distinct strings with seven or more characters can be formed from the letters in `EVERGREEN`?

Solution. The multiset of letters is $\{E^4,\ R^2,\ G,\ V,\ N\}$ (total of 9 letters, with multiplicities as shown). For length $k \in \{7, 8, 9\}$ choose nonnegative integers $(a_E, a_R, a_G, a_V, a_N)$ satisfying

$$a_E \le 4, \quad a_R \le 2, \quad a_G, a_V, a_N \le 1, \quad a_E + a_R + a_G + a_V + a_N = k.$$

For each feasible choice, the number of distinct strings is $\frac{k!}{a_E!\,a_R!\,a_G!\,a_V!\,a_N!}$. Summing over all feasible 5-tuples yields

$$N_7 = 4515, \qquad N_8 = 7560, \qquad N_9 = 7560.$$

Therefore the total number of strings of length at least 7 is

$$N = N_7 + N_8 + N_9 = \boxed{19635}.$$

Problem 2

Problem. Find the coefficient of x^7y^5 in $(2x - 3y)^{12}$.

Solution. By the binomial theorem, the x^7y^5 term occurs when the x-factor is chosen 7 times and the y-factor 5 times:

$$\binom{12}{7}(2x)^7(-3y)^5 = \binom{12}{7}\,2^7(-3)^5\,x^7y^5.$$

Hence the coefficient is $\binom{12}{7}2^7(-3)^5 = \boxed{-24{,}634{,}368}$.

Problem 3

Problem. How many solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$, where x_i are nonnegative integers and the listed extra conditions hold?

(a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$.
Solution. Let $y_i = x_i - 2 \ge 0$. Then $\sum y_i = 29 - 12 = 17$. By stars and bars, the count is $\binom{17+6-1}{6-1} = \boxed{\binom{22}{5} = 26{,}334}$.

(b) $x_1 \ge 1,\ x_2 \ge 2,\ x_3 \ge 3,\ x_4 \ge 4,\ x_5 > 5,\ x_6 \ge 6.$
Solution. Replace each variable by its minimum plus a slack: $(1, 2, 3, 4, 6, 6)$ sums to 22, so the slack sum is $29 - 22 = 7$. The number of solutions is $\binom{7+6-1}{6-1} = \boxed{\binom{12}{5} = 792}$.

(c) $x_1 \le 5$.

Solution. Unconstrained solutions are $\binom{29+6-1}{6-1} = \binom{34}{5}$. Subtract those with $x_1 \ge 6$. Put $y_1 = x_1 - 6 \ge 0$, then $y_1 + x_2 + \cdots + x_6 = 23$, which has $\binom{23+6-1}{6-1} = \binom{28}{5}$ solutions. Thus the answer is $\boxed{\binom{34}{5} - \binom{28}{5} = 179{,}976}$.

(d) $x_1 < 8$ and $x_2 > 8$.

Solution. Write $x_2 = 9 + y_2$ with $y_2 \ge 0$. Then

$$x_1 + y_2 + x_3 + x_4 + x_5 + x_6 = 20, \qquad 0 \le x_1 \le 7.$$

For a fixed $a = x_1 \in \{0, \ldots, 7\}$ the remaining equation has $\binom{20-a+5-1}{5-1} = \binom{24-a}{4}$ solutions. Hence

$$\sum_{a=0}^{7} \binom{24-a}{4} = \sum_{t=17}^{24} \binom{t}{4} = \binom{25}{5} - \binom{17}{5} = \boxed{46{,}942}.$$

Problem 4

Problem. How many ways are there to travel in xyz-space from the origin $(0, 0, 0)$ to $(4, 3, 5)$ by taking steps of length 1 only in the positive x, y, or z directions (no backtracking)?

Solution. Any such path consists of 4 x-steps, 3 y-steps, and 5 z-steps, in some order. The total number of steps is $4 + 3 + 5 = 12$. Distinct paths correspond to distinct permutations of the multiset $\{x^4, y^3, z^5\}$, hence

$$\#\text{paths} = \frac{12!}{4!\,3!\,5!}.$$

Problem 5

Problem. Give a combinatorial proof of $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Proof. Count the number of ways to form a committee of k people from n and then choose a chair.

Method 1: First choose the committee, then the chair. There are $\binom{n}{k}$ ways to choose the committee and k choices for its chair, for a total of $k\binom{n}{k}$ outcomes.

Method 2: First choose the chair, then the rest of the committee. There are n choices for the chair. After choosing the chair, choose the remaining $k-1$ members from the remaining $n-1$ people, which can be done in $\binom{n-1}{k-1}$ ways. This gives $n\binom{n-1}{k-1}$ outcomes.

Both methods count the same set of outcomes, hence

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Problem 6

Problem. How many ways are there to distribute five balls into seven boxes if each box may contain *at most one* ball?

(a) Balls and boxes labeled. With capacity 1, each labeled ball must go to a distinct labeled box. This is an *injection* from a 5-element set to a 7-element set, counted by permutations:

$$P(7, 5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \frac{7!}{2!} = 2520.$$

(b) Balls labeled, boxes unlabeled. Distinct balls, *identical* boxes, capacity 1. Boxes are indistinguishable and at most one ball per box, so exactly 5 boxes are used and each box contains one ball. This is a partition of the 5 distinct balls into 5 unlabeled singletons, counted by the Stirling number of the second kind $S(5, 5) = 1$. Hence the answer is $\boxed{1}$.

(c) Balls unlabeled, boxes labeled. Identical balls, *distinct* boxes, capacity 1. We simply choose which 5 boxes receive a ball:

$$\binom{7}{5} = 21.$$

(d) Balls unlabeled, boxes unlabeled. Identical balls, *identical* boxes, capacity 1. We only care about how many boxes are occupied. Exactly 5 boxes are occupied with one ball each, which corresponds to the single integer partition $1+1+1+1+1$. Therefore the answer is $\boxed{1}$.

Problem 7

Let A and B be finite sets with $|A| = n$ and $|B| = k$.

(a) How many functions $f : A \to B$ are there?
For each of the n elements of A, there are k choices for its image in B, independently. Hence the number of functions is

$$k^n.$$

(b) How many injective functions $f : A \to B$ are there, assuming $n \le k$?
Choose distinct images in B for the n elements of A:

$$k \cdot (k-1) \cdots (k-n+1) = \frac{k!}{(k-n)!}.$$

(c) How many bijections $f : A \to B$ are there, assuming $n = k$?
When $n = k$, a bijection is a bijective labeling of B by the elements of A. The count is the number of permutations of k elements:

$$k! \quad (\text{equivalently } n!).$$

(d) Show that the number of surjections $f : A \to B$ is $k!\,S(n, k)$ (assume $k \le n$).
Here $S(n, k)$ denotes the Stirling number of the second kind, i.e., the number of ways to partition an n-element set into k nonempty *unlabeled* blocks. A surjection $f : A \to B$ induces a partition of A into the nonempty fibers $f^{-1}(b)$ for $b \in B$. Conversely, given a partition of A into k nonempty blocks, there are $k!$ ways to bijectively assign its k blocks to the k labels in B. Therefore the number of surjections is

$$k!\,S(n, k).$$

References