Justify all answers. Submit as a single PDF.

1. (5 points) How many strings with seven or more characters can be formed from the letters in `EVERGREEN`?

2. (5 points) Find the coefficient of x^7y^5 in $(2x-3y)^{12}$.

3. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \;=\; 29,$$

where x_i (for $i = 1, 2, 3, 4, 5, 6$) are nonnegative integers such that

   (a) (5 points) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$.

   (b) (5 points) $x_1 \geq 1,\ x_2 \geq 2,\ x_3 \geq 3,\ x_4 \geq 4,\ x_5 > 5,\ x_6 \geq 6$.

   (c) (5 points) $x_1 \leq 5$.

   (d) (5 points) $x_1 < 8$ and $x_2 > 8$.

4. (10 points) How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(4, 3, 5)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited, so that no backtracking is allowed.)

5. (10 points) Give a combinatorial proof of $k\binom{n}{k} = n\binom{n-1}{k-1}$.

6. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

   (a) (10 points) both the balls and boxes are labeled?

   (b) (10 points) the balls are labeled, but the boxes are unlabeled?

   (c) (10 points) the balls are unlabeled, but the boxes are labeled?

   (d) (10 points) both the balls and boxes are unlabeled?

7. Let A and B be finite sets with $|A| = n$ and $|B| = k$.

   (a) (5 points) How many functions $f : A \to B$ are there?

   (b) (5 points) How many injective functions $f : A \to B$ are there? Assuming $n \leq k$.

   (c) (5 points) How many bijections $f : A \to B$ are there? Assuming $n = k$.

   (d) (5 points) Show that the number of surjections $f : A \to B$ is given by $k!S(n, k)$. Assuming $k \leq n$.

8. (10 points) Prove the Multinomial Theorem. If n is a positive integer, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} C(n; n_1, n_2, \ldots, n_m)\, x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!}$$

is called a multinomial coefficient.