Justify all answers. Submit as a single PDF.

1. The Boolean operation "exclusive or" (XOR), written $\oplus$, is defined by the table

| $x$ | $y$ | $x \oplus y$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

   (a) (10 pts) Prove that $\oplus$ is commutative and associative; i.e., $x \oplus y \equiv y \oplus x$ and $(x \oplus y) \oplus z \equiv x \oplus (y \oplus z)$.

   (b) (10 pts) Show that $x \oplus y \equiv (x \wedge \neg y) \ \vee \ (\neg x \wedge y)$.

2. (20 points) Prove that *every* binary Boolean operation can be expressed using only the basic connectives $\wedge$, $\vee$, $\neg$.

   *Hint:* There are 16 binary Boolean operations. List them all and build each from $\wedge$, $\vee$, $\neg$.

3. (10 points) Show that $x \vee y$ can be written using only $\wedge$ and $\neg$. Conclude that any binary Boolean operation is expressible using just $\{\wedge, \neg\}$.

4. The Boolean operation **nand**, written $\uparrow$, is defined by $x \uparrow y \ = \ \neg(x \wedge y)$.

   (a) (10 pts) Construct the truth table for $x \uparrow y$.

   (b) (10 pts) Determine whether $\uparrow$ is commutative and whether it is associative. Justify.

   (c) (10 pts) Express $x \wedge y$ and $\neg x$ *using only* $\uparrow$. Deduce that $\uparrow$ alone is functionally complete (i.e., can express any binary Boolean operation).

5. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said, "Carlos did it." John said, "I did not do it." Carlos said, "Diana did it." Diana said, "Carlos lied when he said that I did it."

   (a) (10 pts) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

   (b) (10 pts) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

6. (20 points) A village has a barber in it, who shaves all and only the people who do not shave themselves. Who shaves the barber? Justify your answer.