Justify all answers. Submit as a single PDF.

1. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
   (a) (5 points) How many balls must she select to be sure of having at least three balls of the same color?
   (b) (5 points) How many balls must she select to be sure of having at least three blue balls?
2. (10 points) There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.
3. Use the Principle of Mathematical Induction to prove the following statements.
   (a) (10 points) Let $n \in \mathbb{N}$ be such that $n \geq 2$. Then, $n! < n^n$.
   (b) (10 points) Let $n \in \mathbb{N}$ be such that $n \geq 1$. Then, $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$.
   (c) (10 points) For all $n \in \mathbb{N}$, it holds that $5 \mid (2^{2n+1} + 3^{2n+1})$.
   (d) (10 points) Let $F_k$ be the $k$-th Fibonacci number. Prove that, for every $k \in \mathbb{N}$, it holds that $F_{3k}$ is even.
4. The Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ and initial conditions $L_0 = 2$ and $L_1 = 1$.
   (a) (10 points) Show that $L_n = F_{n-1} + F_{n+1}$ for $n = 2, 3, \ldots$, where $F_n$ is the $n$-th Fibonacci number.
   (b) (10 points) Find an explicit formula for the Lucas numbers.
5. Find the solution to the recurrence relations:
   (a) (5 points) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.
   (b) (5 points) $a_n = 2a_{n-1} + 2^n$, with $a_0 = 2$.
   (c) (5 points) $a_n = 2a_{n-1} + 2n^2$, with $a_1 = 4$.
   (d) (5 points) $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$, with $a_0 = 1$ and $a_1 = 4$.
6. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.
   (a) (5 points) Find a recurrence relation for $\{P_n\}$, where $P_n$ is the amount in the account at the end of $n$ years if no money is ever withdrawn.
   (b) (5 points) How much is in the account after $n$ years if no money has been withdrawn?

7. (20 points) There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

   Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

   One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.

   One evening, he addresses the entire tribe to thank them for their hospitality.

   However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

   What effect, if anything, does this faux pas have on the tribe?

   (Hint: consider the case of having $n = 1$ islanders, $n = 2$ islanders, $n = 3$ islanders, etc. Find the pattern and prove by induction.)