A construction of canonical nonconforming finite element spaces for elliptic equations of any order in any dimension

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Outline

1. High order PDE
2. Morley-Wang-Xu elements ($m \leq n$)
3. New elements for arbituary m, n
4. Unisolvence
5. Numerical experiments

1 High order PDE

2 Morley-Wang-Xu elements ($m \le n$)

3 New elements for arbituary m, n

4 Unisolvence

5 Numerical experiments

$2m$-th-order elliptic PDE

$2m$-th-order elliptic PDE with homogeneous Dirichlet boundary condition:

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \dfrac{\partial^k u}{\partial \nu^k} = 0 & \text{on } \partial\Omega,\ 0 \le k \le m-1. \end{cases} \tag{1}$$

A fundamental question in finite element theory:
How to find u approximately?

- $m = 1$, Poisson equation: $-\Delta u = f$. Lagrange, Crouzeix-Raviart, ...
- $m = 2, n = 2$, plate problem: $\Delta^2 u = f$. Argyris, Morley, ...
- ...

A more universal construction for all $m, n \ge 1$ has to be more intrinsic!

FEMs for high order PDEs

$\boxed{\text{Conforming element}}$: Construction of piecewise polynomial of subspaces of H^m (in $\mathbb{R}^n$) is practically difficult: it requires the following minimal degrees (Hu-Lin-Wu, 2024):

$$(m-1)2^n+1 = \begin{cases} 1 & m=1, n\geq 1 \quad (\text{dof} = n+1) \\ 5 & m=2, n=2 \quad (\text{dof} = 21, \text{Argyris}) \\ 9 & m=3, n=2 \quad (\text{dof} = 55, \text{Bramble \& Zlámal}, 1970, \\ & \qquad\qquad\qquad\qquad \text{Ženíšek, 1970}) \\ 9 & m=2, n=3 \quad (\text{dof} = 220, \text{Zhang, 2009}) \\ 17 & m=3, n=3 \quad (\text{dof} = 1180) \end{cases}$$

- Advantages: $\|u-u_h\|_m \lesssim \inf_{v_h\in V_h} \|u-v_h\|_m$.
- Disadvantages: Computationally impractical.

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Different approach: $\boxed{\text{nonconforming element.}}$

- Advantages: Locally lower-degree polynomial space is more computationally beneficial.
- Disadvantages: Stability and consistency are not inherently guaranteed.

A sufficient condition in the construction

Stability

$\Leftrightarrow$ Inf-sup condition

$\Leftarrow$ Coercivity $\quad a_h(v_h, v_h) \gtrsim \|v_h\|_{m,h}^2$

$\Leftrightarrow$ Poincaré inequ. $\quad |v_h|_{m,h} \gtrsim \|v_h\|_{m,h}$

Consistency

$$
\begin{aligned}
&|u - u_h|_{m,h} \\
\lesssim &\inf_{v_h \in V_h} |u - v_h|_{m,h} \\
+ &\sup_{|\phi_h|_{m,h}=1} |(f, \phi_h) - a_h(u, \phi_h)|.
\end{aligned}
$$

A sufficient condition in the construction

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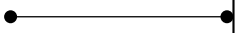
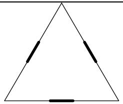
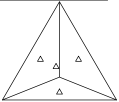
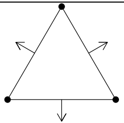
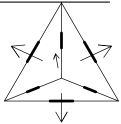
\end{aligned}
$$

Weak continuity For any v_h in V_h, any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$ and any $|\alpha| < m$, $\partial_h^\alpha v_h$ is continuous at a point on F at least.

- Weak continuity $\Rightarrow$ Poincaré inequality $\Rightarrow$ Stability
- Weak continuity + $\int_F [\nabla^{m-1} v_h] = 0$ $\Rightarrow$ Generalized patch test $\Leftrightarrow$ Consistency
- Approximability: $\mathcal{P}_m \subseteq$ local shape function space

Nonconforming elements

$m \backslash n$	1	2	3
1			
2	?		
3	?	?	?
4	?	?	?

$m = 1$: Crouzeix-Raviart (1973)
$m = 2, n = 2$: Morley (1968)
$m = 2, n \geq 2$: Morley-type (2006)

A long-standing question:
How to establish a complete table of nonconforming finite elements?

1 High order PDE

2 Morley-Wang-Xu elements ($m \leq n$)

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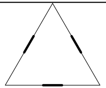
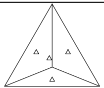
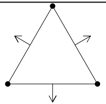
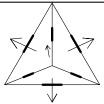
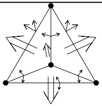
Morley-Wang-Xu Elements ($m \leq n$)

A breakthrough in nonconforming element (Wang & Xu, 2013, MCOM)

- piecewise $\mathcal{P}_m$ space on n-simplex element
- DOF: the integral averages of normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \leq k \leq m$:

$$\#DOF = \sum_{k=1}^{m} C_{m-1}^{m-k} C_{n+1}^{n-k+1} = C_{n+m}^{n} = \dim \mathcal{P}_m$$

$m \backslash n$	1	2	3
1			
2			
3			

An example: $m = 2$, $n = 2$ (Morley element)

How the unisolvence and weak continuity are obtained?

Key observation:

- integral average of tangential derivatives on edges are known.

Unisolvence: If $u \in \mathcal{P}_2(T)$ has all the zero d.o.f., then we claim:

$$\int_{e_{kl}} \partial_i u = 0, \quad \forall i = 1, 2,$$

since

$$\int_{e_{kl}} \frac{\partial u}{\partial \nu} = 0$$

and

$$\int_{e_{kl}} \frac{\partial u}{\partial \tau} = u(a_k) - u(a_l) = 0.$$

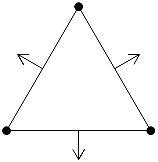
$\Rightarrow \partial_i u \in \mathcal{P}_1(T)$ has zero CR d.o.f...

Weak continuity + $\int_F [\nabla u] = 0$:

$$\int_{e_{kl}} [\partial_\nu u] = 0,$$

$$\int_{e_{kl}} [\partial_\tau u] = [u](a_k) - [u](a_l) = 0,$$

$\Rightarrow \partial_h^\alpha v_h$ is continuous at a point on F.



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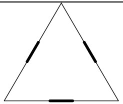
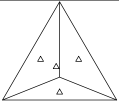
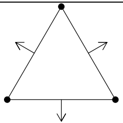
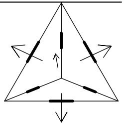
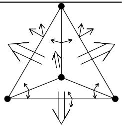
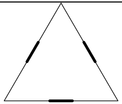
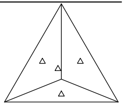
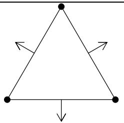
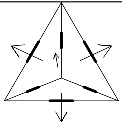
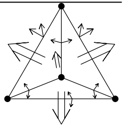
The final piece?

$m \backslash n$	1	2	3
1			
2	?		
3	?	?	
4	?	?	?

Question 1:
How to allocate the DOFs when $m > n$?

The final piece?

$m\backslash n$	1	2	3
1			
2	?		
3	?	?	
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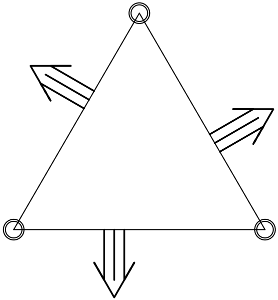
How to allocate the DOFs when $m > n$?

"Multi-layer" structure!

Multi-layer structure

Take 2D $m = 4$ as an example.

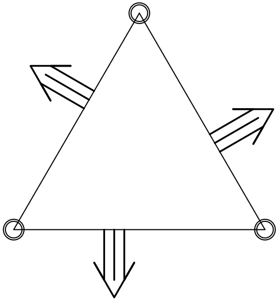


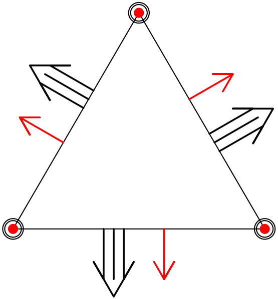
Similar allocation as MWX elements.

What about 1st and 0th order derivatives?

Multi-layer structure

Take 2D $m = 4$ as an example.

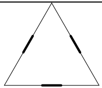
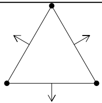
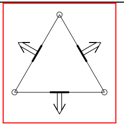
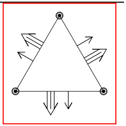
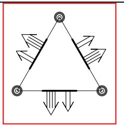




Similar allocation as MWX elements.

What about 1st and 0th order derivatives?

Back to the $(n-1)$-dimensional subsimplex and go on!
Green's formula can be similarly applied.

$m\backslash n$	2
1	
2	
3	
4	
5	

- DOF: each increment in m by n, the degrees of freedom increase by one layer:

$$D_T^{(m,n)} = \bigcup_{\ell=0}^{\lceil \frac{m}{n} \rceil - 1} D_{T,\ell}^{(m,n)}.$$

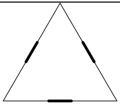
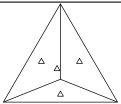
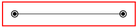
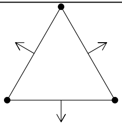
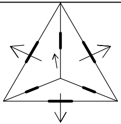
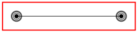
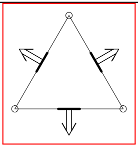
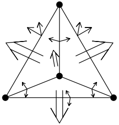
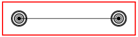
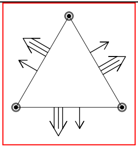
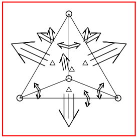
- Shape function space:

$$P_T^{(m,n)} := \sum_{\ell=0}^{\lceil \frac{m}{n} \rceil - 1} \lambda_1^{\ell(n+1)} \mathcal{P}_{m-\ell n}(T).$$

- Dimension counting: $\dim P_T^{(m,n)} = \# D_T^{(m,n)}$:

$$= \dim \mathcal{P}_m(T) + \sum_{\ell=1}^{\lceil \frac{m}{n} \rceil - 1} \left(\dim \mathcal{P}_{m-\ell n}(T) - \dim \mathcal{P}_{m-\ell n-1}(T)\right)$$

Universal construction for arbituary m, n

$m \backslash n$	1	2	3
1			
2			
3			
4			

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Unisolvence theorem (main result)

Theorem (Unisolvence, L.-Wu 2025)

For any $m, n \geq 1$, $D_T^{(m,n)}$ is $P_T^{(m,n)}$-unisolvent.

We will use the special integral-type representation with generator b_l, with an induction argument on m for any fixed n. Outline of the proof:

1. $m - 1$ **not a multiple of** n: the number of layers remains the same.

Take derivative and use induction hypothesis.

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We will use the special integral-type representation with generator b_l, with an induction argument on m for any fixed n. Outline of the proof:

1. $m-1$ **not a multiple of** n:
   the number of layers remains the same.

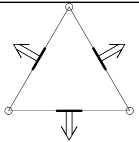
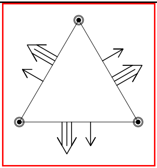
   Take derivative
   and use induction hypothesis.

2. $m-1$ **is a multiple of** n:
   a new layer of space is introduced,
   generated by the nonconforming bubble
   $b_l(\lambda_1)$.

   Make use of properties:
   DOF vanishing,
   single variable,
   nondegeneracy

Case 1: $m-1$ not a multiple of n

Take $m=4, n=2$ as an example

$m\backslash n$	2
3	
4	

$$P_T^3 := \mathcal{P}_3(T) + \lambda_1^3 \mathcal{P}_1(T)$$
$$P_T^4 := \mathcal{P}_4(T) + \lambda_1^3 \mathcal{P}_2(T)$$

Assume $v \in P_T^4$ has vanishing DOF^4.

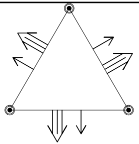
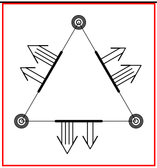
Step 1. $\partial_{\lambda_2} v \in P_T^3$ and $\mathrm{DOF}^3(\partial_{\lambda_2} v) = 0$

$\Rightarrow \partial_{\lambda_2} v = 0,\ v = v(\lambda_1) \in \mathcal{P}_5(\lambda_1)$.

Step 2. $\partial_{\lambda_1} v \in \mathcal{P}_4(\lambda_1) \subseteq P_T^3,\ \mathrm{DOF}^3(\partial_{\lambda_1} v) = 0$

$\Rightarrow \partial_{\lambda_1} v = 0,\ \Rightarrow v = \text{constant}.$

$\mathrm{DOF}^4(v) = 0 \ \Rightarrow\ v = 0.$

$m \backslash n$	2
4	
5	

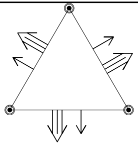
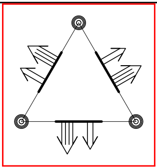
$$P_T^4 := \mathcal{P}_4(T) + \lambda_1^3 \mathcal{P}_2(T)$$

$$P_T^5 := \mathcal{P}_5(T) + \lambda_1^3 \mathcal{P}_3(T) + \underbrace{\left\langle \int_0^{\lambda_1} b_2(t)\,dt, \lambda_2 b_2(\lambda_1) \right\rangle}_{\text{new layer}}$$

where $b_2 \in \mathcal{P}_6(\mathbb{R})$ satisfying $\mathrm{DOF}^4(b_2(\lambda_1)) = 0$.

Assume $v \in P_T^5$ has vanishing DOF^5:

$$v = \tilde{v} + c_1 \int_0^{\lambda_1} b_2(t)\,dt + c_2 \lambda_2 b_2(\lambda_1).$$

$m\backslash n$	2
4	
5	

$$v = \tilde{v} + c_1 \int_0^{\lambda_1} b_2(t)\, dt + c_2 \lambda_2 b_2(\lambda_1).$$

Step 1. $\mathrm{DOF}^4(\partial_{\lambda_2} v) = 0, \partial_{\lambda_2} v = \partial_{\lambda_2}\tilde{v} + c_2 b_2(\lambda_1)$
by DOF vanishing:
$\Rightarrow \mathrm{DOF}^4(\partial_{\lambda_2}\tilde{v}) = 0,\ \tilde{v} = \tilde{v}(\lambda_1) \in \mathcal{P}_6(\lambda_1).$

Step 2. Using DOFs $\int_F v \Rightarrow c_2 = 0,\ v = v(\lambda_1).$

Step 3. $\mathrm{DOF}^4(\partial_{\lambda_1} v) = 0,\ \partial_{\lambda_1} v = \partial_{\lambda_1}\tilde{v} + c_1 b_2(\lambda_1)$
$\Rightarrow \mathrm{DOF}^4(\partial_{\lambda_1}\tilde{v}) = 0,\ \partial_{\lambda_1}\tilde{v} = 0,\ \Rightarrow \tilde{v} = C.$

Step 4. Using DOFs $\int_F v \Rightarrow C = 0, c_1 = 0.$

1 High order PDE

2 Morley-Wang-Xu elements ($m \leq n$)

3 New elements for arbituary m, n

4 Unisolvence

5 Numerical experiments

m-harmonic equation in $\Omega = (0,1)^2$ $m = 3, 4$

$$(-\Delta)^m u = f, \quad \text{Dirichlet B.C.}$$

Exact solution:

$$u = 2^{4m-6}(x-x^2)^m(y-y^2)^m.$$

Table: Example 1 ($m = 3$): Errors and convergence orders.

$1/h$	$\|e_h\|_0/\|u\|_0$	Order	$|e_h|_{1,h}/|u|_1$	Order	$|e_h|_{2,h}/|u|_2$	Order	$|e_h|_{3,h}/|u|_3$	Order
4	4.0363e-1	–	5.0885e-1	–	6.0746e-1	–	1.0750e+0	–
8	3.7355e-1	0.11	3.4528e-1	0.56	2.7339e-1	1.15	6.2572e-1	0.78
16	1.1945e-1	1.64	1.0629e-1	1.70	8.2985e-2	1.72	3.3168e-1	0.91
32	3.1641e-2	1.92	2.7974e-2	1.93	2.1898e-2	1.92	1.6853e-1	0.98
64	8.0192e-3	1.98	7.0815e-3	1.98	5.5512e-3	1.98	8.4609e-2	0.99

m-harmonic equation in $\Omega=(0,1)^2$ $m=3,4$

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Exact solution:

$$u = 2^{4m-6}(x-x^2)^m(y-y^2)^m.$$

Table: Example 1 $(m=4)$: Errors and convergence orders.

$1/h$	$\|e_h\|_0/\|u\|_0$	Order	$\|e_h\|_{1,h}/\|u\|_1$	Order	$\|e_h\|_{2,h}/\|u\|_2$	Order
4	5.7329e-1	–	5.5032e-1	–	6.7387e-1	–
8	3.7467e-1	0.61	3.8495e-1	0.52	3.9303e-1	0.78
16	1.8226e-1	1.04	1.6630e-1	1.21	1.5247e-1	1.37
32	5.2981e-2	1.78	4.7416e-2	1.81	4.2674e-2	1.84
64	1.4476e-2	1.87	1.2968e-2	1.87	1.1667e-2	1.87

$1/h$	$\|e_h\|_{3,h}/\|u\|_3$	Order	$\|e_h\|_{4,h}/\|u\|_4$	Order
4	8.5797e-1	–	1.2082e+0	–
8	3.4855e-1	1.30	7.5549e-1	0.68
16	1.2931e-1	1.43	4.1123e-1	0.88
32	3.6448e-2	1.83	2.1196e-1	0.96
64	9.8660e-3	1.89	1.0694e-1	0.99

Numerical Test: L-shaped domain + singular solution

$|u-u_h|_{m,h} \lesssim h^{\min\{\alpha,1\}}|u|_{m+\alpha}+h^m\|f\|_0$

$\Omega=(-1,1)^2\setminus(0,1)\times(-1,0)$, the exact solution is

$$u=r^{m-1+\beta_m}\sin(m-1+\beta_m)\theta,$$

where β_m denotes the typical singularity exponent for the m-harmonic equation on the L-shaped domain.

Table: Example 2 $(m=3,\ \beta_3\approx 0.510)$: Errors and convergence orders.

$1/h$	$\|e_h\|_0/\|u\|_0$	Order	$\|e_h\|_{1,h}/\|u\|_1$	Order	$\|e_h\|_{2,h}/\|u\|_2$	Order	$\|e_h\|_{3,h}/\|u\|_3$	Order
4	1.0032e-3	–	2.9532e-3	–	1.2864e-2	–	2.4159e-1	–
8	4.6625e-4	1.11	8.2076e-4	1.85	5.0341e-3	1.35	1.6332e-1	0.56
16	2.1185e-4	1.14	3.1971e-4	1.36	1.9423e-3	1.37	1.1491e-1	0.51
32	9.2182e-5	1.20	1.3918e-4	1.20	7.6130e-4	1.35	8.0923e-2	0.51
64	4.0098e-5	1.20	6.0858e-5	1.19	3.0539e-4	1.32	5.6904e-2	0.51

Numerical Test: L-shaped domain + singular solution

$|u-u_h|_{m,h} \lesssim h^{\min\{\alpha,1\}}|u|_{m+\alpha} + h^m\|f\|_0$

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where β_m denotes the typical singularity exponent for the m-harmonic equation on the L-shaped domain.

Table: Example 2 $(m=4,\ \beta_4\approx 0.503)$: Errors and convergence orders.

$1/h$	$\|e_h\|_0/\|u\|_0$	Order	$\|e_h\|_{1,h}/\|u\|_1$	Order	$\|e_h\|_{2,h}/\|u\|_2$	Order
4	5.3081e-4	–	9.1677e-4	–	3.0789e-3	–
8	1.4049e-4	1.92	1.8621e-4	2.30	6.4424e-4	2.26
16	5.7500e-5	1.29	7.5900e-5	1.30	1.8980e-4	1.76
32	2.5200e-5	1.19	3.3300e-5	1.19	7.5200e-5	1.34

$1/h$	$\|e_h\|_{3,h}/\|u\|_3$	Order	$\|e_h\|_{4,h}/\|u\|_4$	Order
4	1.4011e-2	–	2.7929e-1	–
8	5.1145e-3	1.45	1.7319e-1	0.69
16	1.9075e-3	1.42	1.2000e-1	0.53
32	7.1249e-4	1.42	8.4783e-2	0.50

Concluding remarks

1. Universal construction for the polynomial approximation of Sobolev space $H^m(\Omega)$ in $\mathbb{R}^n$, for arbituary m, n.
2. Unisolvence: integral-type representation + nonconforming bubble
3. "Ideal" choice of degrees of freedom:

$$\dim \mathcal{P}_m(T) \leq \dim P_T^{(m,n)} \approx (1 + \frac{1}{n}) \dim \mathcal{P}_m(T).$$

THANK YOU!

Jia Li and Shuonan Wu. A construction of canonical nonconforming finite element spaces for elliptic equations of any order in any dimension. Mathematics of Computation, 2025.