Contents

1 The Real Numbers

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) For some $z \in \mathbb{Z}_5$, if we choose $y = 5 - z \pmod 5$ then

$$\begin{aligned} z + y &= (5 - z) + z \pmod 5 \\ &= 0 \pmod 5 \end{aligned}$$

Hence y is an additive identity of z.

(b) This can be done explitly.

z	x
1	1
2	3
3	2
4	4

For example, for $z = 4, x = 4$:

$$\begin{aligned} xz &= 4 \times 4 \pmod 5 \\ &= 16 \pmod 5 \\ &= 1 \pmod 5 \end{aligned}$$

(c) For $\mathbb{Z}_4$, then the construction in (a) will always be valid for a additive inverse. Hence an additive inverse exists for $\mathbb{Z}_n \,\forall\, n \in \mathbb{N}$.

Finding a multiplictive inverse is not always possible for any $z \in \mathbb{Z}_4$. For instance, $z = 2$ does not have a multiplictive inverse since any number multiplied by two will be even. However, we require the product to be $4n + 1$ for some $n \in \mathbb{N}$ which is impossible since the product will always be even.

Hence, we conjecture that $\exists$ a multiplitive inverse $\forall\, z \in \mathbb{Z}_n$ given n is prime so that no element of $\mathbb{Z}_n$ shares a factor with n i.e. the greatest common divisor between any element of the set and n is 1.

Exercise 1.3.2

(a)

Definition 1.3.1. *A real number i is a greatest lower bound for a set $A \subseteq \mathbb{R}$ if it meets the following criteria:*

(i) i is a lower bound for A.

(ii) If b is any lower bound for A, then $b \le i$.

(b)

Lemma 1.3.1. *Assume $i \in \mathbb{R}$ is a lower bound on a set $A \subseteq \mathbb{R}$. Then $i = \inf A$ if and only if $\forall\, \varepsilon > 0 \,\exists\, a \in A$ s.t. $i + \varepsilon > a$.*

Proof. ($\Rightarrow$) Suppose $i = \inf A$, then $\forall\, \epsilon > 0$ $i + \varepsilon > i$ but $i + \varepsilon$ cannot be a greater lower bound so there must exist $a \in A$ s.t. $i + \varepsilon > a$.

($\Leftarrow$) Suppose for some $i \in \mathbb{R}$ $i + \varepsilon > a$ for some $a \in A$ for arbitrary $\varepsilon > 0$ then:

(i) i is a lower bound by assumption.

(ii) Suppose $i \ne b = \inf A$ with $b > i$, then $b = i + \varepsilon$ for $\varepsilon = |b - i|$ but $b = i + \varepsilon > a$ for some $a \in A$ so we have reached a contradition since we assumed b is a lower bound on A. Hence, i is indeed the greatest lower bound.

□

Exercise 1.3.3

(a) *Proof.* (i) Suppose $\inf A$ is not an upper bound on B so $\exists\, b \in B$ s.t. $\inf A \le b$ but b is a lower bound on A so

$$\inf A < b \le a \,\forall a \in A$$

This is a contradition since this would imply b is a greater lower bound on A so we conclude $\inf A$ is indeed an upper bound on B.

(ii) A lower bound on A is given by $\inf A - \varepsilon$ for any $\varepsilon > 0$ so $\inf A - \varepsilon \in B \forall \varepsilon > 0$. But this would imply

$$\inf A - \varepsilon < \inf A - \varepsilon/2 = b \in B$$

for any $\varepsilon > 0$ which implies that $\inf A$ is a least upper bound for B.

Alternatively, $\inf A$ is a lower bound on A then $\inf A \in B$. So since $\inf A$ is an upper bound within B then $\inf A$ is a maximum of B. If a maximum exists then $\sup B = \max B = \inf A$.

Hence $\sup B = \inf A$.

For set A define set $-A = \{-a : a \in A\}$. It is clear that $-A$ is bounded so we only need to prove

$$-\sup(-A) = \inf A$$

□

Exercise 1.3.4

Proof. If $B \subseteq A$ then $\forall b \in B \, \exists a \in A$ s.t. $a \geq b$, this implies

$$\sup A \geq a \geq \sup B$$

□

Exercise 1.3.5

(a) *Proof.* (i) By def$^{\underline{n}}$, $\forall a \in A \sup A \geq a$. Adding c to both sides we get $\forall a \in A \, c + \sup A \geq c + a$. Hence, $c + \sup A$ is an upper bound on $c + A$.

(ii) Now for arbitrary $\varepsilon > 0$, $\sup A - \varepsilon < a$ for some $a \in A$. This implies, $c + \sup A - \varepsilon < c + a \in c + A$ for some $a \in A$.

Hence, $\sup c + A = c + \sup A$. □

(b) *Proof.* (i) By def$^{\underline{n}}$, $\forall a \in A \sup A \geq a$. Multiplying by c on both sides we get $\forall a \in A \, c \sup A \geq ca$. Hence, $c \sup A$ is an upper bound on cA.

(ii) Now for arbitrary $\varepsilon > 0$, $\sup A - \varepsilon < a$ for some $a \in A$. This implies, $c \sup A - \varepsilon < ca \in cA$ for some $a \in A$.

□

(c) We postulate $\sup cA = -c \inf A$. This can be easily seen by imagining the sequence of maniplations from $A \to cA \to -cA$ and observe how the up changes.

Exercise 1.3.6

(a) 3

(b) 1

(c) 1/2

(d) 9

Exercise 1.3.7

Proof. Since is an upper bound by assumption, we only need to check that a is the least upepr bound. However, this is trivial since for arbitrary $\varepsilon > 0$ $a - \varepsilon < a \in A$ so $a = \sup A$. □

Exercise 1.3.8

Proof. Let $\varepsilon = (\sup A + \sup B)/2$. So

$$\sup A < \sup B - \varepsilon < b$$

for some $b \in B$. So $b \in B$ is an upper bound for A. □

Exercise 1.3.9

(a) True

(b) False, consider set

$$A = \left\{ L - \frac{1}{n} : n \in \mathbb{N} \right\}$$

where $\sup A = L$.

(c) False, consider sets

$$A = \left\{ L - \frac{1}{n} : n \in \mathbb{N} \right\}$$
$$B = \left\{ L + \frac{1}{n} : n \in \mathbb{N} \right\}$$

However, $\inf B = \sup A = L$.

(d) True

(e) False, consider sets

$$A = \left\{1 - \frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is even}\right\}$$
$$B = \left\{1 - \frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is odd}\right\}$$

Then, $\sup A = 1 \leq \sup B = 1$ but $\nexists b \in B$ that is an upper bound for A.

This statement can be contradited more easily by noting that the statement is false for two indentical sets with the supremum not in the set. The above example demonstrates this statement is false for disjoint sets also.

1.4 Consequences of Completeness

Exercise 1.4.1

Theorem 1.4.1. *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof. We consider two cases:

- $a < 0, b > 0$: By Th$^{\underline{m}}$ 1.4.3 $\exists$ r s.t.

$$a < 0 \leq r < b$$

- $a < 0, b \leq 0$: By Th$^{\underline{m}}$ 1.4.3 $\exists$ r s.t. $-b < r < -a \Rightarrow a < -r < b$.

$\square$

Exercise 1.4.2

(a) *Proof.* Let $p/q, r/s \in \mathbb{Q}$ with $p, q, r, s \in \mathbb{Z}$, then

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in \mathbb{Q}$$

Similarly,

$$\frac{p}{q}\frac{r}{s} = \frac{pq}{rs} \in \mathbb{Q}$$

$\square$

(b) *Proof.* We proceed by contradiction: Suppose $p, q, r, s \in \mathbb{Z}$ with $a = p/q$, then

$$at = \frac{p}{q}t = \frac{r}{s}$$
$$\Rightarrow t = \frac{rq}{sp} \in \mathbb{Q}$$

$\square$

However, t is irrational.

(c) $\mathbb{I}$ is not closed under addition or multiplication. For instance, consider examples

$$-\sqrt{2} + \sqrt{2} = 0 \in \mathbb{Q}$$
$$\sqrt{2}\sqrt{2} = 2 \in \mathbb{Q}$$

Exercise 1.4.3

Corollary 1.4.1. *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Proof. Using Th$^{\underline{m}}$ 1.4.3 $\exists\, r \in \mathbb{Q}$ s.t.

$$a - \sqrt{2} < r < b - \sqrt{2}$$

given $a < b$. However, if we add $\sqrt{2}$ this implies

$$a < r + \sqrt{2} < b$$

Using **Exercise** 1.4.2 we can conclude that $t = r + \sqrt{2} \in \mathbb{I}$. □

Exercise 1.4.4

Proof. Let $A = \inf\{1/n : n \in \mathbb{N}\}$

(a) Since $1/n \neq 0$ for any $n \in \mathbb{N}$ then 0 is indeed a lower bound for A.

(b) By the Archimedean Property of $\mathbb{R}$ for any $\varepsilon > 0\, \exists n \in \mathbb{N}$ s.t.

$$0 + \varepsilon = \varepsilon > 1/n \in A$$

Hence $\inf A = 0$. □

Exercise 1.4.5

Proof. We proceed by contradiction, suppose $\exists x \in \bigcap_{n=1}^{\infty}$, However, by the Archimedean Property $\exists m \in \mathbb{N}$ s.t. $x > 1/n$ so $x \notin (0, 1/m) \Rightarrow x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$ □

Exercise 1.4.6

(a) Continuing on from the proof of Th$^{\underline{m}}$ 1.4.5: By the Archimedean Property $\exists\, n_0 \in \mathbb{N}$ s.t. $1/n_0 < \frac{\alpha^2+2}{2\alpha}$ which implies

$$\left(\alpha - \frac{1}{n_0}\right) > \alpha^2 - \frac{2\alpha}{n_0} > 2$$

But we assumed α is a least upper bound so we have reached a contradition. Hence, $\alpha^2 = 2 \Rightarrow \alpha = \sqrt{2}$.

(b) We only need to repeat the proof given in Th$^{\underline{m}}$ 1.4.5 using the set

$$S = \{s \in \mathbb{R} : s^2 < b\}$$

We would conclude $\sup S = \sqrt{b}$.

Exercise 1.4.7

Theorem 1.4.2. *If $A \subseteq B$ and B is countable, then A is either countable, finite, or empty.*

Proof. We proceed by induction: We will induct on the domain of $g : \mathbb{N} \to A$.

- Base case ($m = 1$): Let $T_1 = \{n_1 \in \mathbb{N} : f(n) \in A\}$. Let $n_1 = \min T_1$ and $g(1) = f(n_1)$.
- Inductive step: ($m > 1$): Let $T_{m+1} = T_m$ n_m. Let $n_{m+1} = \min T_{m+1}$ and $g(n_{m+1}) = f(n_{m+1})$.

Clearly g will be 1-1 and onto since dom$g$ and ran$g$ are subsets of dom$f$ and ran$f$ respectively which is also 1-1 and onto.

If A is either empty or finite then the indcution stops at some m. □

Exercise 1.4.8

(a) *Proof.* Without loss of generality suppose A_1, B_2 are disjoint. We begin by assuming that both sets are infinite. Since $B_2 \subseteq A_2$ we proved in **Exercise** 1.4.7 that B_2 is countable so $\exists$ bijections

$$f : A_2 \to \mathbb{R}$$
$$g : B_2 \to \mathbb{R}$$

We define new function $h : \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & n \text{ odd} \\ g(\frac{n}{2}) & n \text{ even} \end{cases}$$

h simply returns the elements of A_1 and B_2 in alternating order. Since h is 1-1 (since A_1 and B_2 are disjoint) and onto we can conclude $A_1 \cup B_2$ is countable. However since $A_1 \cup B_2 =$

$A_1 \cup A_2$ then the same conclusion follows for any two infinite countable sets.

If B_2 is finite with m elements then we simply define

$$h(n) = \begin{cases} g(n) & n \leq m \\ f(n-m) & n > m \end{cases}$$

By induction this result can be generalized to an arbitrary number of sets since every induction step only involves the union of two countable sets which we have proven. □

(b) We cannot induct on infinity, what case came before infinity?

(c) Let $f_m : \mathbb{N} \to A_m \, \forall m \in \mathbb{N}$ and $A = \{f_m \in m \in \mathbb{M}\}$.

For each $m \in \mathbb{N}$, we could assign A_m to the m^{th} row of the grid and the n^{th} element of A_m to the n^{th} column. To that end, we define bijective function $k : \mathbb{N} \times \mathbb{N} \to A$ given by

$$k(n, m) = f_n(m)$$

Next let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which takes in a row and a column and returns the corresponding value in the two-dimensional array in illustrated in the preblem. Hence g is bijective so g^{-1} also exists and undoes the mapping.

The composition of bijective functions is bijective so we can define $\ell : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ By

$$\ell = k \circ g^{-1}$$

which would prove $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.4.9

(a) If $A \sim B$ then $\exists$ bijection $f : A \to B$. However, the inverse of a bijection is a bijection so f^{-1} exists and is bijective which implies $B \sim A$.

(b) The result follows trivially using the fact that the composition of bijective functions is a bijection. That is, if $f : A \to B$ and $g : B \to C$ then $h : A \to C$ is a bijection and implies $A \sim C$.

Exercise 1.4.10