# Contents



# 1 The Real Numbers

## 1.3 The Axiom of Completeness

**Exercise** 1.3.1

(a) For some $z \in \mathbb{Z}_5$, if we choose $y = 5 - z \pmod 5$ then

$$\begin{aligned} z + y &= (5 - z) + z \pmod 5 \\ &= 0 \pmod 5 \end{aligned}$$

Hence $y$ is an additive identity of $z$.

(b) This can be done explitly.

| $z$ | $x$ |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 2 |
| 4 | 4 |

For example, for $z = 4, x = 4$:

$$\begin{aligned} xz &= 4 \times 4 \pmod 5 \\ &= 16 \pmod 5 \\ &= 1 \pmod 5 \end{aligned}$$

(c) For $\mathbb{Z}_4$, then the construction in (a) will always be valid for a additive inverse. Hence an additive inverse exists for $\mathbb{Z}_n \, \forall \, n \in \mathbb{N}$.

Finding a multiplictive inverse is not always possible for any $z \in \mathbb{Z}_4$. For instance, $z = 2$ does not have a multiplictive inverse since any number multiplied by two will be even. However, we require the product to be $4n + 1$ for some $n \in \mathbb{N}$ which is impossible since the product will always be even.

Hence, we conjecture that $\exists$ a multiplitive inverse $\forall \, z \in \mathbb{Z}_n$ given $n$ is prime so that no element of $\mathbb{Z}_n$ shares a factor with n i.e. the greatest common divisor between any element of the set and n is 1.

**Exercise** 1.3.2

(a)

**Definition 1.3.1.** *A real number $i$ is a greatest lower bound for a set $A \subseteq \mathbb{R}$ if it meets the following criteria:*

*(i) $i$ is a lower bound for $A$.*

*(ii) If $b$ is any lower bound for $A$, then $b \leq i$.*

(b)

**Lemma 1.3.1.** *Assume $i \in \mathbb{R}$ is a lower bound on a set $A \subseteq \mathbb{R}$. Then $i = \inf A$ if and only if $\forall \, \varepsilon > 0 \, \exists \, a \in A$ s.t. $i + \varepsilon > a$.*

*Proof.* ($\Rightarrow$) Suppose $i = \inf A$, then $\forall \, \epsilon > 0$ $i + \varepsilon > i$ but $i + \varepsilon$ cannot be a greater lower bound so there must exist $a \in A$ s.t. $i + \varepsilon > a$.

($\Leftarrow$) Suppose for some $i \in \mathbb{R}$ $i + \varepsilon > a$ for some $a \in A$ for arbitrary $\varepsilon > 0$ then:

(i) $i$ is a lower bound by assumption.

(ii) Suppose $i \neq b = \inf A$ with $b > i$, then $b = i + \varepsilon$ for $\varepsilon = |b - i|$ but $b = i + \varepsilon > a$ for some $a \in A$ so we have reached a contradition since we assumed $b$ is a lower bound on A. Hence, $i$ is indeed the greatest lower bound.

□

**Exercise** 1.3.3

(a) *Proof.* (i) Suppose $\inf A$ is not an upper bound on B so $\exists \, b \in B$ s.t. $\inf A \leq b$ but $b$ is a lower bound on $A$ so

$$\inf A < b \leq a \, \forall a \in A$$

This is a contradition since this would imply $b$ is a greater lower bound on A so we conclude $\inf A$ is indeed an upper bound on B.

(ii) A lower bound on $A$ is given by $\inf A - \varepsilon$ for any $\varepsilon > 0$ so $\inf A - \varepsilon \in B \forall \varepsilon > 0$. But this would imply

$$\inf A - \varepsilon < \inf A - \varepsilon/2 = b \in B$$

for any $\varepsilon > 0$ which implies that $\inf A$ is a least upper bound for $B$.

Alternatively, $\inf A$ is a lower bound on $A$ then $\inf A \in B$. So since $\inf A$ is an upper bound within $B$ then $\inf A$ is a maximum of $B$. If a maximum exists then $\sup B = \max B = \inf A$.

Hence $\sup B = \inf A$.

For set $A$ define set $-A = \{-a : a \in A\}$. It is clear that $-A$ is bounded so we only need to prove

$$-\sup(-A) = \inf A$$

□

**Exercise** 1.3.4

*Proof.* If $B \subseteq A$ then $\forall b \in B \, \exists a \in A$ s.t. $a \geq b$, this implies

$$\sup A \geq a \geq \sup B$$

□

**Exercise** 1.3.5

(a) *Proof.* (i) By def$^{\underline{\text{n}}}$, $\forall a \in A \sup A \geq a$. Adding $c$ to both sides we get $\forall a \in A \, c + \sup A \geq c + a$. Hence, $c + \sup A$ is an upper bound on $c + A$.

(ii) Now for arbitrary $\varepsilon > 0$, $\sup A - \varepsilon < a$ for some $a \in A$. This implies, $c + \sup A - \varepsilon < c + a \in c + A$ for some $a \in A$.

Hence, $\sup c + A = c + \sup A$. □

(b) *Proof.* (i) By def$^{\underline{\text{n}}}$, $\forall a \in A \sup A \geq a$. Multiplying by $c$ on both sides we get $\forall a \in A \, c \sup A \geq ca$. Hence, $c \sup A$ is an upper bound on $cA$.

(ii) Now for arbitrary $\varepsilon > 0$, $\sup A - \varepsilon < a$ for some $a \in A$. This implies, $c \sup A - \varepsilon < ca \in cA$ for some $a \in A$.

□

(c) We postulate $\sup cA = -c \inf A$. This can be easily seen by imagining the sequence of maniplations from $A \to cA \to -cA$ and observe how the up changes.

**Exercise** 1.3.6

(a) 3

(b) 1

(c) 1/2

(d) 9

**Exercise** 1.3.7

*Proof.* Since is an upper bound by assumption, we only need to check that a is the least upepr bound. However, this is trivial since for arbitrary $\varepsilon > 0$ $a - \varepsilon < a \in A$ so $a = \sup A$. □

**Exercise** 1.3.8

*Proof.* Let $\varepsilon = (\sup A + \sup B)/2$. So

$$\sup A < \sup B - \varepsilon < b$$

for some $b \in B$. So $b \in B$ is an upper bound for $A$. □

**Exercise** 1.3.9

(a) True

(b) False, consider set

$$A = \left\{ L - \frac{1}{n} : n \in \mathbb{N} \right\}$$

where $\sup A = L$.

(c) False, consider sets

$$A = \left\{ L - \frac{1}{n} : n \in \mathbb{N} \right\}$$
$$B = \left\{ L + \frac{1}{n} : n \in \mathbb{N} \right\}$$

However, $\inf B = \sup A = L$.

(d) True

(e) False, consider sets

$$A = \left\{1 - \frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is even}\right\}$$
$$B = \left\{1 + \frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is odd}\right\}$$

Then, $\sup A = 1 \leq \sup B = 1$ but $\nexists b \in B$ that is an upper bound for $A$.

This statement can be contradited more easily by noting that the statement is false for two indentical sets with the supremum not in the set.