The Wiener chaos expansion for non-local functionals of Gaussian fields

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Slides available at
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Outline

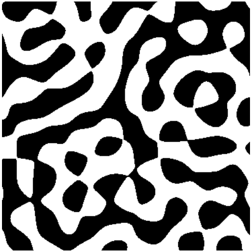
1. Introduction

2. Wiener chaos method for local functionals

3. Wiener chaos method for non-local functionals

Smooth Gaussian fields

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary C^2 Gaussian field with mean zero and variance one.
- The distribution of f is specified by its covariance function $K : \mathbb{R}^d \to \mathbb{R}$ defined as
$$K(x-y) = \mathrm{Cov}[f(x), f(y)] \qquad \forall x, y \in \mathbb{R}^d.$$
- We will consider the geometry/topology of the **excursion sets**
$$\{f \geq \ell\} := \left\{ x \in \mathbb{R}^d \,\middle|\, f(x) \geq \ell \right\} \qquad \text{for } \ell \in \mathbb{R}.$$



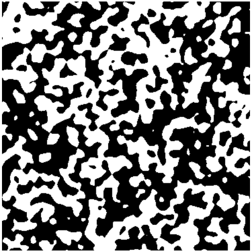


Figure: Excursion sets $\{f \geq 0\}$ in white for the fields on $\mathbb{R}^2$ with $K(x) = J_0(|x|)$, the 0-th Bessel function, (left) and $K(x) = \exp(-|x|^2/2)$ (right).

Motivation: Percolation theory

- **Percolation theory** studies the large scale topological properties of spatial random models.
- **Phase transition**: for a given field, there is a critical level ℓ_c such that, with probability one
  - for $\ell > \ell_c$, $\{f \geq \ell\}$ contains only bounded components,
  - for $\ell < \ell_c$, $\{f \geq \ell\}$ contains a unique unbounded component.

  See [1] for a survey.

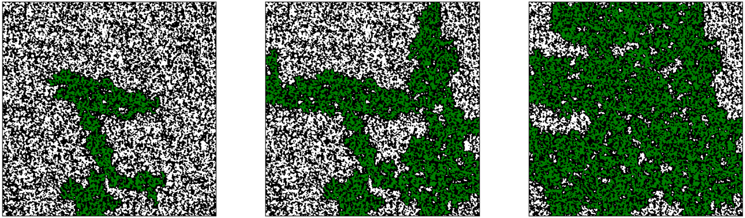


Figure: The excursion sets $\{f \geq \ell\}$ for $\ell = 0.05$ (left), $\ell = 0$ (middle) and $\ell = -0.05$ (right). Largest component highlighted in green.

Local vs non-local functionals

Definition

- A functional of a random field is described as **local** if it is an integral of a pointwise function of the field and its derivatives:

$$\int_D \varphi(f(x), \nabla f(x), \nabla^2 f(x))\, \mu(dx)$$

- Examples

<u>Local functionals</u>

- volume $\int_D \mathbb{1}_{f(x)\geq \ell}\, dx$
- boundary volume $\int_D \mathbb{1}_{f(x)=\ell} \mathcal{H}^{d-1}(dx)$
- Euler characteristic

<u>Non-local functionals</u>

- number of connected components
- Betti numbers
- Volume of the unbounded component

- We focus on non-local functionals which are 'approximately additive' over the domain.

Local vs non-local functionals

What is known?

Local functionals

- Powerful methods available including the **Kac-Rice formula** and the **Wiener chaos expansion**,
- Can typically characterise the mean, variance and asymptotic distribution,
- Results known for a wide variety of covariance structures.

Non-local functionals

- No unifying theory, but many partial results using a variety of methods:

Type of result	Methods
Law of large numbers [10]	Ergodic argument
Variance bounds[11, 4, 3]	Coupling, interpolation formulae
Central limit theorem [2, 8, 7]	Martingale techniques

- Most results are sub-optimal or hold only for a restricted class of covariance functions.

Outline

A classical problem

- Consider a functional of the form

$$F_R = \int_{[-R,R]^d} \varphi(f(x) - \ell)\, dx$$

  for some $\varphi : \mathbb{R} \to \mathbb{R}$.

- For simplicity, assume that

$$K(x) \sim c|x|^{-\alpha} \qquad \text{as } |x| \to \infty$$

  for some $\alpha \in (0, d)$.

Question: Can we describe the asymptotic statistics (mean, variance, distribution) of F_R as $R \to \infty$?

- By Fubini's theorem,

$$\mathbb{E}[F_R] = (2R)^d \mu(\ell)$$

  where $\mu(\ell) := \mathbb{E}[\varphi(f(0) - \ell)]$.

Let $\mathcal{G}$ be a set of centred jointly Gaussian variables. Let $\mathcal{P}_n$ be the space of all polynomials of degree $\leq n$ in $\mathcal{G}$.

The n-th **Wiener chaos** of $\mathcal{G}$ is $\mathcal{G}^{:n:} := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp}$.

Theorem

Let the random variable F be square integrable and $\sigma(\mathcal{G})$-measurable, then

$$F \overset{L^2}{=} \sum_{n=0}^{\infty} Q_n[F]$$

where Q_n denotes projection onto $\mathcal{G}^{:n:}$.

Hermite polynomials

- The Hermite polynomials $(H_n)_{n\geq 0}$ can be defined inductively by setting

$$H_0(x) = 1 \quad \text{and} \quad H_{n+1}(x) = xH_n(x) - H_n'(x).$$

- **Properties:**
  1. If X, Y are jointly normal with mean zero and variance one then

     $$\mathbb{E}[H_n(X)H_m(Y)] = \begin{cases} n!\mathrm{Cov}[X, Y]^n & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

     Hence if $X \in \mathcal{G}$ then $H_n(X) \in \mathcal{G}^{:n:}$.
  2. If $\mathbb{E}[\varphi^2(Z)] < \infty$ for $Z \sim \mathcal{N}(0,1)$ then

     $$\varphi = \sum_{n=0}^{\infty} a_n H_n$$

     where $\sum_n a_n^2 n! < \infty$.

Chaos expansion for a local functional

Variance asymptotics

- Considering the expansion $\varphi(\cdot - \ell) = \sum_n a_n(\ell) H_n$ yields

$$F_R = \sum_{n=0}^{\infty} a_n(\ell) \int_{[-R,R]^d} H_n(f(x))\, dx = \sum_{n=0}^{\infty} Q_n[F_R].$$

- The variance of F_R can be computed by considering

$$\begin{aligned}\mathrm{Var}[Q_n[F_R]] &= a_n(\ell)^2 \iint_{[-R,R]^{2d}} \mathrm{Cov}[H_n(f(x)), H_n(f(y))]\, dxdy \\ &= a_n(\ell)^2 n! \iint_{[-R,R]^{2d}} K(x-y)^n\, dxdy\end{aligned}$$

  for $n \geq 1$.

- Since $K(x) \sim c|x|^{-\alpha}$, for $n \geq 1$

$$\mathrm{Var}[Q_n[F_R]] \sim a_n(\ell)^2 c_{K,n} \times \begin{cases} R^{2d-n\alpha} & \text{if } n\alpha < d, \\ R^d \log R & \text{if } n\alpha = d, \\ R^d & \text{if } n\alpha > d. \end{cases}$$

Chaos expansion for a local functional

Convergence in distribution

- Since $H_1(x) = x$

  $$Q_1[F_R] = a_1(\ell) \int_{[-R,R]^d} f(x)\, dx$$

  which is Gaussian.

- For $n\alpha \geq d$, one can show that

  $$\frac{Q_n[F_R]}{\sqrt{\mathrm{Var}[Q_n[F_R]]}} \xrightarrow{d} \mathcal{N}(0,1)$$

  using the **fourth-moment theorem** (or method of moments) and a **diagram formula** for the moments of Hermite polynomials.

- For $1 < n < d/\alpha$ by expressing Hermite polynomials as **multiple Wiener-Itô integrals** one can show

  $$\frac{Q_n[F_R]}{\sqrt{\mathrm{Var}[Q_n[F_R]]}} \xrightarrow{d} c_{K,n} \int \mathcal{F}[\mathbb{1}_{[-1,1]^d}](\textstyle\sum_{i=1}^n u_i) \displaystyle\prod_{i=1}^n \frac{W(du_i)}{|u_i|^{(d-\alpha)/2}}$$

  where the latter follows a **Hermite distribution**.

Conclusion: Limit theorems for a local functional

Theorem (Breuer-Major/Dobrushin-Major theorem)

Let $n^*(\ell) = \inf\{n : a_n(\ell) \neq 0\}$. If f satisfies some technical conditions, then

$$\mathrm{Var}[F_R] \sim c_{K,\varphi,\ell} \times \begin{cases} R^{2d-n^*\alpha} & \text{if } n^*\alpha < d, \\ R^d \log R & \text{if } n^*\alpha = d, \\ R^d & \text{if } n^*\alpha > d, \end{cases} \qquad \text{as } R \to \infty.$$

Moreover if $n^* = 1$ or $n^*\alpha \geq d$ then

$$\frac{F_R - \mu(\ell)}{\sqrt{\mathrm{Var}[F_R]}} \xrightarrow{d} \mathcal{N}(0,1).$$

For other values of n^*, the limiting distribution is a Hermite distribution.

Remark

- *Typically $n^*(\ell) = 1$ for all but finitely many values of ℓ, which are described as* **anomalous levels**.
- *If φ is regular then $a_n(\ell) = (-1)^n \mu^{(n)}(\ell)/n!$ so that anomalous levels correspond to critical points of μ.*

Outline

In joint work with Stephen Muirhead [9], we adapt the Wiener chaos method to prove limit theorems for a non-local functional.

- Let $f : \mathbb{Z}^d \to \mathbb{R}$ be the Gaussian free field (in $d \geq 3$), so that

$$K(x-y) \sim c_d |x-y|^{-(d-2)}.$$

- The **cluster count** $N_R(f)$ is the number of clusters (i.e. connected components) of the graph $\{f \geq \ell\} \cap [-R, R]^d$.

Proposition

Let $D \subset \mathbb{Z}^d$ be finite and $\Phi \in C^\infty(\mathbb{R}^D)$, then

$$Q_n[\Phi(f)] = \frac{1}{n!} \sum_{x_1,\dots,x_n \in D} \mathbb{E}[\partial_{x_1} \dots \partial_{x_n} \Phi(f)] :f(x_1) \dots f(x_n):$$

where the **Wick polynomial** $:f(x_1) \dots f(x_n):$ *is defined as* $Q_n[f(x_1) \dots f(x_n)]$.

- Proof 1: Elementary argument using Gaussian integration by parts.
- Proof 2: Stroock formula:

$$\begin{aligned} Q_n[\Phi(f)] &= I_n\left(\frac{1}{n!}\mathbb{E}[D^n \Phi(f)]\right) \\ &= \frac{1}{n!} \sum_{x_1,\dots,x_n \in D} \mathbb{E}[\partial_{x_1} \dots \partial_{x_n} \Phi(f)] I_n(e_1 \otimes \dots \otimes e_n). \end{aligned}$$

The **discrete derivative** d_x is defined as

$$d_x N_R(f) = N_R(\{f \geq \ell\} \cup \{x\}) - N_R(\{f \geq \ell\} \setminus \{x\}).$$

Let $\Lambda_R = [-R, R]^d \cap \mathbb{Z}^d$.

Proposition

For $R \geq 1$

$$Q_n[N_R(f)] = \frac{1}{n!} \sum_{x_1,\dots,x_n \in \Lambda_R} P_R(x_1, \dots, x_n) :f(x_1) \dots f(x_n):$$

where the **pivotal intensity** *P_R is defined for distinct points $\underline{x} = (x_1, \dots, x_n)$ as*

$$P_R(\underline{x}) = \mathbb{E}[d_{x_1} \dots d_{x_n} N_R(f) | f(\underline{x}) = \ell] \varphi_{f(\underline{x})}(\ell),$$

and $\varphi_{f(\underline{x})}$ is the density of $f(\underline{x})$.

- Comparing with a local functional:

Local: $$Q_n[F_R] = a_n(\ell) \sum_{x \in \Lambda_R} :f(x)^n:$$

Non-local: $$Q_n[N_R(f)] = \frac{1}{n!} \sum_{x_1,\dots,x_n \in \Lambda_R} P_R(x_1,\dots,x_n) :f(x_1)\dots f(x_n):$$

since $:f(x)^n: = H_n(f(x))$.

- Hence the local case corresponds to $P_R(x_1,\dots,x_n) = n! a_n(\ell) \mathbb{1}_{x_1=\dots=x_n}$.
- One could imagine extending the analysis from the local case if P_R is approximately **stationary** and has **rapid off-diagonal decay**.
- We refer to this as **semi-locality**.

- Recall that for distinct points $\underline{x} = (x_1, \dots, x_n)$

$$P_R(\underline{x}) = \mathbb{E}[d_{x_1} \dots d_{x_n} N_R(f) | f(\underline{x}) = \ell] \varphi_{f(\underline{x})}(\ell).$$

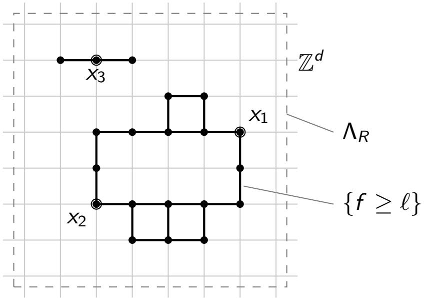


Figure: For this configuration $d_{x_1} d_{x_2} N_R(f) = 1$ but $d_{x_1} d_{x_3} N_R(f) = 0$.

- In general, if $d_{x_1} \dots d_{x_n} N_R(f) \neq 0$ then $x_1, \dots, x_n$ must be joined by bounded clusters of $\{f \geq \ell\}$.

Theorem (Truncated arm decay [6])

Let $f : \mathbb{Z}^d \to \mathbb{R}$ be the Gaussian free field for $d \geq 3$. There exists $\ell_c \in \mathbb{R}$ such that for every $\ell \neq \ell_c$, the probability that 0 is contained in a bounded cluster of $\{f \geq \ell\}$ of diameter at least n is at most e^{-cn^ρ} for some $c, \rho > 0$.

Corollary

For $\ell \neq \ell_c$ there exists $c, C, \rho > 0$ such that

$$P_R(\underline{x}) \leq Ce^{-c\mathrm{diam}(\underline{x})^\rho}$$

where $\mathrm{diam}(\underline{x})$ denotes the diameter of $\underline{x}$.

- Arguments for local functionals (based on the fourth-moment theorem/multiple Wiener-Itô integrals) can be extended to the semi-local case.
- Calculations involving covariance kernels become more involved but are conceptually straightforward:

Local: $$\mathrm{Var}[Q_n[F_R]] = a_n(\ell)^2 \sum_{x,y\in\Lambda_R} K(x-y)^n$$

Non-local: $$\mathrm{Var}[Q_n[N_R(f)]] = \frac{1}{n!} \sum_{\underline{x},\underline{y}\in\Lambda_R^n} P_R(\underline{x}) P_R(\underline{y}) \prod_{i=1}^{n} K(x_i - y_i).$$

- To control the tail of the chaos expansion we use an **interpolation formula** for $\mathrm{Var}[\sum_{n\geq N} Q_n[N_R(f)]]$ in terms of discrete derivatives of order N.

Conclusion: Limit theorems for the cluster count

We define the **mean clusters-per-vertex** as

$$\mu(\ell) := \lim_{R\to\infty} \frac{\mathbb{E}[N_R(f)]}{(2R)^d}.$$

Theorem

Let $f : \mathbb{Z}^3 \to \mathbb{R}$ be the Gaussian free field and $\ell \neq \ell_c$.

$$\mathrm{Var}[N_R(f)] \sim c_\ell \times \begin{cases} R^5 & \text{if } \mu'(\ell) \neq 0 \\ R^4 & \text{if } \mu'(\ell) = 0, \mu''(\ell) \neq 0 \\ R^3 \log R & \text{if } \mu'(\ell) = \mu''(\ell) = 0, \mu'''(\ell) \neq 0 \\ R^3 & \text{otherwise.} \end{cases}$$

In the second case, the (normalised) limiting distribution is a Hermite distribution, in all other cases it is Gaussian.

- Analogous results hold for $d \geq 4$ and other fields but are omitted here for brevity.
- Similar to results in local case, but the requirement that $\ell \neq \ell_c$ is new.

Summary

Open questions:

- Can this approach be extended to smooth fields?
- Does this approach enable the Malliavin-Stein method for non-local functionals?
- What happens at the critical level?

Thank you for listening!

Bibliography I

[1] D. Beliaev. "Smooth Gaussian fields and percolation". In: *Probability Surveys* (2023). URL: https://doi.org/10.1214/23-PS24.

[2] D. Beliaev, M. McAuley, and S. Muirhead. "A central limit theorem for the number of excursion set components of Gaussian fields". In: *Ann. Probab.* (2024). URL: https://doi.org/10.1214/23-AOP1672.

[3] D. Beliaev, M. McAuley, and S. Muirhead. "A covariance formula for the number of excursion set components of Gaussian fields and applications". In: *Ann. Inst. Henri Poincaré Probab. Stat.* (2025). URL: https://doi.org/10.1214/23-aihp1430.

[4] D. Beliaev, M. McAuley, and S. Muirhead. "Fluctuations of the number of excursion sets of planar Gaussian fields". In: *Probab. Math. Phys.* (2022). URL: https://doi.org/10.2140/pmp.2022.3.105.

[5] D. Beliaev, S. Muirhead, and A. Rivera. "A covariance formula for topological events of smooth Gaussian fields". In: *Ann. Probab.* (2020). URL: https://doi.org/10.1214/20-AOP1438.

[6] H. Duminil-Copin et al. "Equality of critical parameters for percolation of Gaussian free field level sets". In: *Duke Math. J.* (2023). URL: https://doi.org/10.1215/00127094-2022-0017.

Bibliography II

[7] C. Hirsch and R. Lachièze-Rey. "Functional central limit theorem for topological functionals of Gaussian critical points". In: *arXiv preprint arXiv:2411.11429* (2024).

[8] M. McAuley. "Three central limit theorems for the unbounded excursion component of a Gaussian field". In: *arXiv preprint arXiv:2403.03033* (2024).

[9] M. McAuley and S. Muirhead. *Limit theorems for the number of sign and level-set clusters of the Gaussian free field*. 2025. URL: https://arxiv.org/abs/2501.14707.

[10] F. Nazarov and M. Sodin. "Asymptotic Laws for the Spatial Distribution and the Number of Connected Components of Zero Sets of Gaussian Random Functions". In: *Journal of Mathematical Physics, Analysis, Geometry* (2016). URL: https://jmag.ilt.kharkiv.ua/index.php/jmag/article/view/jm12-0205e.

[11] F. Nazarov and M. Sodin. "Fluctuations in the number of nodal domains". In: *J. Math. Phys.* (2020). URL: https://doi.org/10.1063/5.0018588.