Geometric and topological functionals of smooth Gaussian fields

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Slides available at
`https://michael-mcauley.github.io`

Gaussian fields

Motivation: cosmology

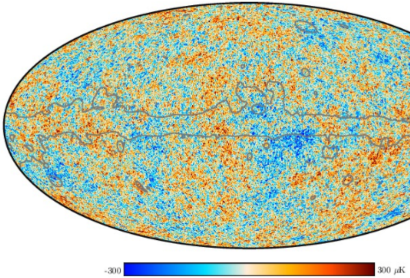

Figure: Fluctuations of the Cosmic Microwave Background Radiation (CMBR) (Source: Planck 2018).

- Physical theory and evidence confirm that the CMBR is well modelled as a realisation of a Gaussian field on the sphere [6].
- Deviations from this model provide insight about the early universe.
- Geometric properties of excursion sets can be used to test for such deviations [7].

Gaussian fields

Motivation: medical imaging

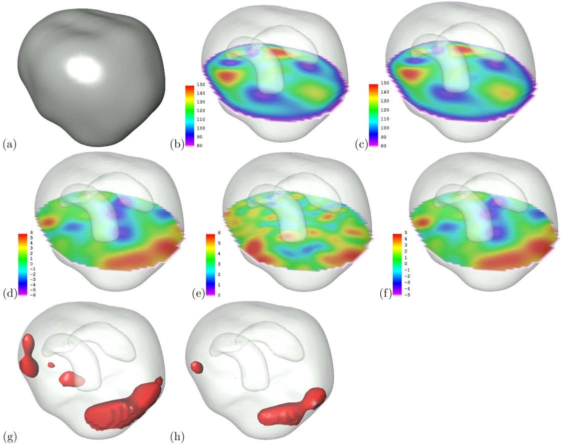

Figure: Measurements from a PET study of brain activity during a reading task. (Source: [14]). See [15] for a technical account.

Gaussian fields
Further applications

- **Quantum chaos**
  It is conjectured that for any Riemannian 2-manifold with 'chaotic' dynamics, the high-energy eigenfunctions of the Laplacian are well modelled by Gaussian random fields [4]. (See [8] for a recent overview.)
- **Atmospheric/climate modelling**
  Time-dependent models of smooth Gaussian fields on the sphere have recently been used to model global temperatures [5] and air pollution [12].

- Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a C^2 Gaussian field with mean zero and variance one (at each point).
- The distribution of the field is specified by its covariance function $K : M^2 \to [-1, 1]$ defined as

$$K(x, y) = \mathbb{E}[f(x)f(y)] \qquad \forall x, y \in M.$$

- We are interested in the geometry of the *excursion sets*

$$\{f \geq \ell\} := \{x \in M \mid f(x) \geq \ell\}$$

for $\ell \in \mathbb{R}$.

Euler characteristic

A rough definition

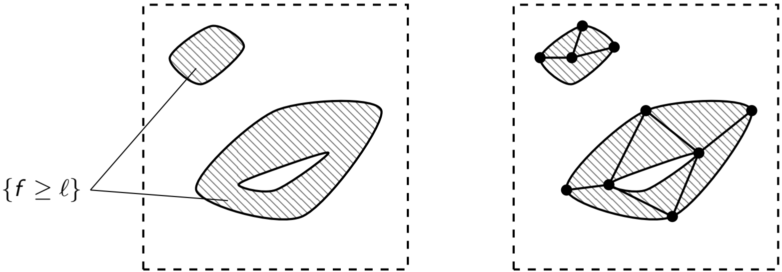

Figure: A simple excursion set in $\mathbb{R}^2$ (left) and a triangulation of the same set (right).

1. The Euler characteristic is an integer valued topological invariant of 'nice' sets in Euclidean space
2. The Euler characteristic of a planar set is the number of components minus the number of 'holes'
3. This coincides with the graphical definition (#Vertices - #Edges + #Faces) for a triangulation of the set

Euler characteristic

Application to Gaussian fields

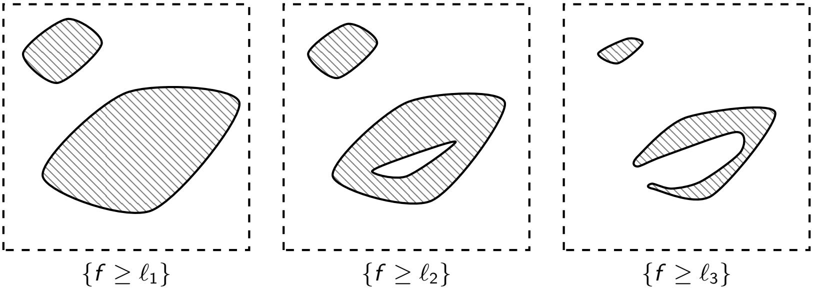

Figure: Excursion sets for a function f above levels $\ell_1 < \ell_2 < \ell_3$.

1. The Euler characteristic of an excursion set for a 'nice' planar function can be decomposed as

$$\text{Euler characteristic} = \#\text{Maxima} - \#\text{Saddles} + \#\text{Minima}.$$

2. The expectation of this quantity for a Gaussian field can be calculated using a generalisation of Kac's counting formula.

Euler characteristic

Application to Gaussian fields

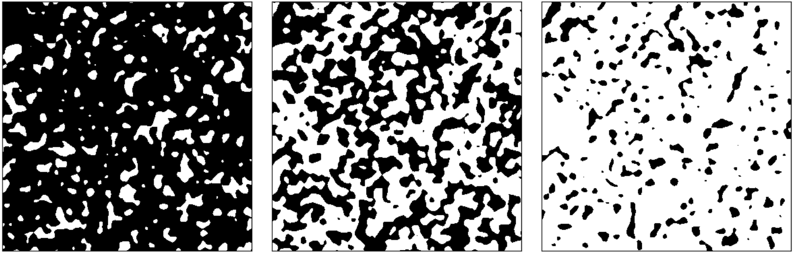

Figure: Excursion sets $\{f \geq \ell\}$ in black for $\ell = -1$ (left), $\ell = 0$ (middle) and $\ell = 1$ (right) where $f : \mathbb{R}^2 \to \mathbb{R}$ has covariance $K(x, y) = \exp(-|x-y|^2/2)$.

For a stationary, planar Gaussian field

$$\mathbb{E}[\mathrm{EC}(\{f \geq \ell\} \cap [-R, R]^2)] = \sqrt{\det \nabla^2 K(0)} \frac{(2R)^2}{(2\pi)^{3/2}} \ell e^{-\ell^2/2} + O(R).$$

Euler characteristic

Cosmological data

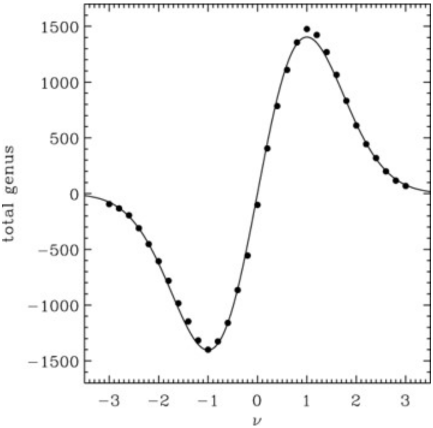

Figure: The observed Euler characteristic of the CMBR restricted to intensities above the level ν (dots) and the expected value for a Gaussian field (solid curve). Source: [7].

Euler characteristic

Medical imaging

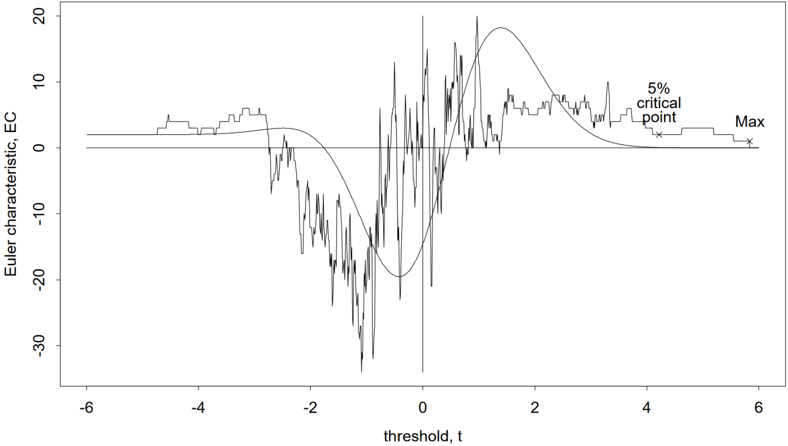

Figure: The observed Euler characteristic for PET data (jagged) and the expected value for a Gaussian field (smooth) at different thresholds. Source: [15].

This type of analysis results from a rich interplay between mathematical theory and applications!

For more details, see

- [14] for a non-technical overview of different applications;
- [1] for theoretical development of the Euler characteristic for Gaussian fields;
- [9] for a mathematical development of Gaussian fields with applications in cosmology.

Geometry vs Topology

A functional of a random field is **local** if it can be represented as an integral of a pointwise function of the field and its derivatives.

Geometric Functionals

- Typically local.
- Examples:
  - Volume of the excursion set
  - Boundary length of the excursion set
  - Euler characteristic of the excursion set
- Statistics of these functionals are well understood.

Topological Functionals

- Typically non-local.
- Examples:
  - Number of components of the excursion set
  - Betti numbers of the excursion set
  - Volume of the unbounded component of the excursion set
- Occur naturally in applications [13] but are much less understood theoretically.

The component count

Law of large numbers

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, centred, smooth Gaussian field.
- Given $\ell \in \mathbb{R}$ and $R > 0$ we let $N_{\mathrm{ES}}(\ell, R)$ be the number of connected components of $\{f \geq \ell\} \cap [-R, R]^d$.

Theorem (Nazarov-Sodin[10])

If f is ergodic, then there exists $c(\ell) \geq 0$ such that

$$\lim_{R\to\infty} \frac{N_{\mathrm{ES}}(\ell, R)}{(2R)^d} = c(\ell)$$

almost surely and in L^1.

- It is straightforward to verify ergodicity using the Fourier transform of the covariance function.
- The result is extremely general: in particular, there is no requirement of fast correlation decay.
- The proof shows that the component count is ‘semi-local’: its value on a macroscopic domain can be well approximated by summing its value on mesoscopic domains.

The component count

Central limit theorem

Assume that $f = q * W$ where W is a Gaussian white noise process on $\mathbb{R}^d$ and q satisfies some regularity conditions, including

$$\sup_{|\alpha|\leq 2} |\partial^\alpha q(x)| \leq c|x|^{-\beta}$$

for some $c > 0$ and $\beta > 9d$ and all $x \in \mathbb{R}^d$.

Theorem (Beliaev-M.-Muirhead[3])

Given $\ell \in \mathbb{R}$, there exists $\sigma^2(\ell) > 0$ such that as $R \to \infty$

$$\frac{\mathrm{Var}[N_{\mathrm{ES}}(\ell, R)]}{(2R)^d} \to \sigma^2(\ell)$$

and

$$\frac{N_{\mathrm{ES}}(\ell, R) - \mathbb{E}[N_{\mathrm{ES}}(\ell, R)]}{(2R)^{d/2}} \xrightarrow{d} \mathcal{N}(0, \sigma^2(\ell)).$$

The component count

Proof of CLT

- The proof adapts a martingale CLT argument from discrete probability [11].
- Let $(\mathcal{F}_v)_{v \in \mathbb{Z}^d}$ be a 'lexicographic' filtration generated by the white noise W and
$$S_n := \frac{N_{\mathrm{ES}}(\ell, n) - \mathbb{E}[N_{\mathrm{ES}}(\ell, n)]}{(2n)^{d/2}}.$$
Then $S_{n,v} := \mathbb{E}[S_n | \mathcal{F}_v]$ defines a 'lexicographic martingale array'.
- A generalisation of the classical martingale CLT states that $S_n \to \mathcal{N}(0, \sigma^2)$ provided that the martingale differences $U_{n,v}$ satisfy certain moment bounds and $\sum_{v \in \mathbb{Z}^d} U_{n,v}^2 \to \sigma^2$ in L^1.
- The latter property follows from an elegant ergodic argument due to Penrose [11].
- The moments bounds follow from relating $U_{n,v}$ to the change in the component count when the white noise W is resampled on a cube of unit length centred at v.

Open questions

- How are the statistics of topological functionals affected by **long-range dependence**?
- Can a similar theory be developed for **non-Gaussian** (e.g. shot-noise) fields?
- There are many related open questions regarding the **percolation** properties of smooth Gaussian fields [2].

Thank you for listening!

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