Conformal welding of independent Gaussian multiplicative chaos measures

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Slides available at
`https://michael-mcauley.github.io`

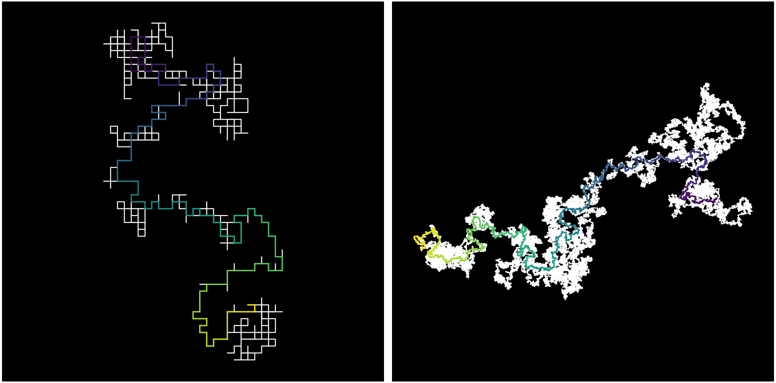


Figure: Random walks with 10^3 and 10^5 steps respectively (white) along with their loop-erasures (colour).

Loop-erased random walk is formed by sequentially removing the loops of a simple random walk.

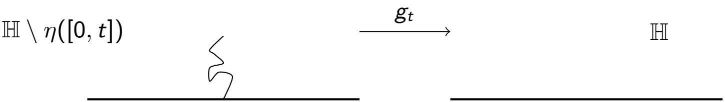
Schramm-Loewner evolution

Definition

- **Loewner theory**: if η is a simple curve (appropriately parameterised) and $g_t : \mathbb{H} \setminus \eta([0,t]) \to \mathbb{H}$ conformal then

$$\begin{cases} \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \\ g_0(z) = z. \end{cases}$$

  for some driving function W_t.



- Schramm [11] identified the possible scaling limit for LERW as having a Brownian motion as its driving function. Lawler, Schramm and Werner proved convergence to the scaling limit [6].
- A **chordal Schramm-Loewner evolution** with parameter $\kappa \geq 0$ is such a curve with driving function a Brownian motion of diffusivity κ.

Schramm-Loewner evolution

Subsequent developments

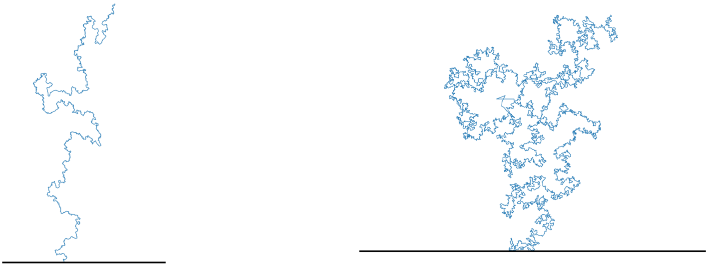
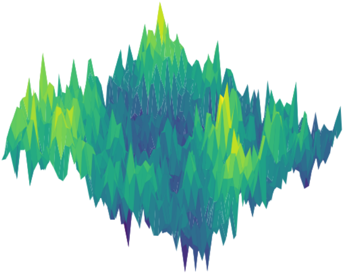


Figure: A simulated SLE path for $\kappa = 2$ (left) and $\kappa = 5$ (right). Source of code for simulations: `https://github.com/james-m-foster/sle-simulation`.

- Many other discrete random models are now known or conjectured to have SLE as their scaling limit.
- Led to new results for these models and made rigorous many arguments (and statements) from the physics literature.
- See [5] for further background and references.

- Given a bounded domain $D \subset \mathbb{C}$, the Gaussian free field h can be thought of as the Gibbs measure for the Dirichlet energy

$$\|f\|_{\nabla}^{2} := \int_D |\nabla f(x)|^2 \, dx.$$



- More precisely we can define $h = \sum_i X_i f_i$ where $X_i \overset{\text{ind}}{\sim} \mathcal{N}(0,1)$ and $(f_i)_i$ is an orthonormal basis with respect to the Dirichlet norm.
- This series is not defined pointwise but converges almost surely in the space of distributions.

- The Gaussian free field has strong motivation from the physics literature.
- It is also natural to study from a mathematical perspective:
  - conformal invariance,
  - scaling limit of discrete models,
  - generalisation of Brownian motion to higher dimensions.
- See [13] or [15] for further background.

- We wish to define a 'random surface' using the Gaussian free field (see [12] Section 1 for motivation).
- A natural way to do so is through the **Gaussian multiplicative chaos** measure
$$\mu(dz) := e^{\gamma h(z)}\, dz$$
where h is a Gaussian free field on D and $\gamma > 0$.
- This definition is problematic, since h is not defined pointwise, but can be made rigorous as
$$\mu(dz) := \lim_{\epsilon \downarrow 0} \exp\left(\gamma h_\epsilon(z) - \frac{\gamma^2}{2}\mathrm{Var}[h_\epsilon(z)]\right)\, dz$$
where h_ϵ is a regularisation of h.
- We interpret (D, μ) as a conformal parameterisation of a **Liouville quantum gravity** surface.

Properties:

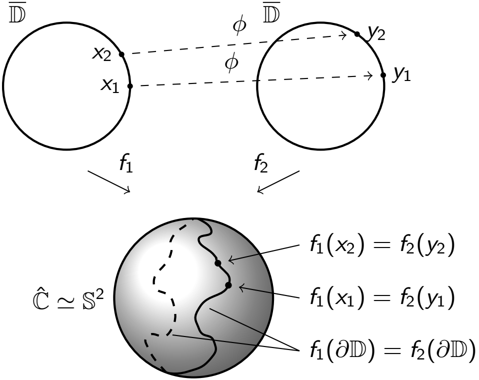
- Conformal invariance of the Gaussian free field implies that Liouville quantum gravity is conformally covariant.
- By choosing $D = \mathbb{H}$, one can define the **quantum boundary length** measure ν of the surface.

Broader context:

- This construction was motivated the work of Polyakov [9, 8] on conformal field theory.
- In the last 15 years, a signficant mathematical literature has been built on this construction making much of the physical analysis rigorous.
- See [3] for further background.

Definition (Conformal welding)

Suppose that $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ is a homeomorphism. To solve the **conformal welding problem** for ϕ is to find conformal maps $f_1 : \mathbb{D} \to D$ and $f_2 : \mathbb{D} \to \mathbb{C} \setminus D$ (for some domain D) which extend homeomorphically to the boundary such that $f_1|_{\partial\mathbb{D}} = f_2 \circ \phi$.



- Let h be the restriction of the Gaussian free field to $\partial\mathbb{D}$ parameterised by $[0,1]$ and '$\tau(dx) = e^{\gamma h(x)}dx$' where $\gamma \in [0, \sqrt{2})$.
- Let $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ be given by

$$\phi(x) = \frac{\tau([0,x])}{\tau([0,1])}.$$

Conjecture

If one can solve the conformal welding problem for ϕ then the boundary curve should be a (closed loop variant of) Schramm-Loewner evolution.

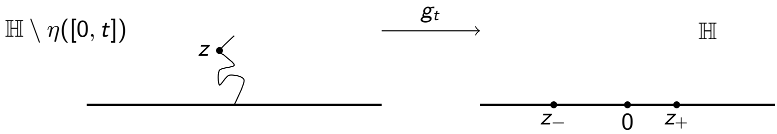
- In [2], it was shown that there is a unique solution to the welding problem for ϕ which varies continuously with $\gamma \in [0, \sqrt{2})$.

Theorem ([12, Theorem 1.3])

There exists a coupling of a Schramm-Loewner evolution η and a Gaussian free field h on $\mathbb{H}$ such that if $\gamma^2 = \kappa$ then for any $z \in \eta([0,t])$

$$\nu_{h,\gamma}([z_-, 0]) = \nu_{h,\gamma}([0, z+])$$

where $g_t^{-1}(z_-) = g_t^{-1}(z_+) = z$ and $z_- \leq 0 \leq z_+$ and $\nu_{h,\gamma}$ is the boundary measure of Liouville quantum gravity.



- Sheffield's result can be viewed as confirming a variation of Jones' conjecture: welding two Gaussian multiplicative chaos measures yields a Schramm-Loewner evolution.
- The result also states that a Schramm-Loewner evolution has a well-defined 'quantum length' with respect to a given Gaussian free field.
- The coupling involves taking an independent Gaussian free field and mapping forward by g_t^{-1}.
- See [12] or [3, Chapter 8] for details of the proof.
- Sheffield's welding has inspired a wealth of subsequent work related to Schramm-Loewner evolutions, Liouville quantum gravity and random planar maps. (See the introduction to [10] for a selection of references).

An alternative approach

Question

Can we derive a relationship between SLE and LQG in the setting of Jones' original conjecture?

Motivation:

- Deeper understanding of relationship,
- Mild differences in statement of result,
- Welding surfaces with different parameter values.

Recall the setting of Jones' conjecture:

- Let h be the restriction of the Gaussian free field to $\partial\mathbb{D}$ parameterised by $[0,1]$ and '$\tau(dx) = e^{\gamma h(x)}dx$' where $\gamma \in [0,\sqrt{2})$.
- Let $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ be given by

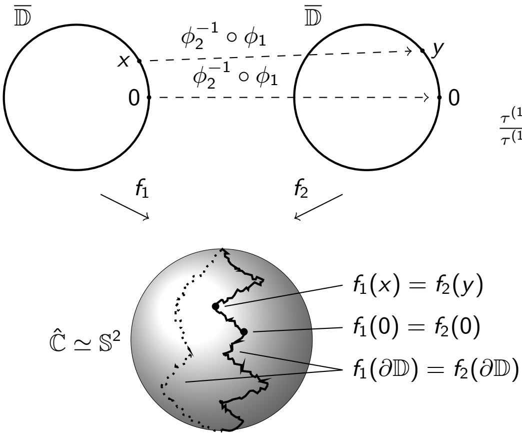
$$\phi(x) = \frac{\tau([0,x])}{\tau([0,1])}.$$

Theorem (Kupiainen-M.-Saksman 23)

Let ϕ_1 and ϕ_2 be independent copies of the above homeomorphism with parameters γ_1 and γ_2. For $\gamma_1, \gamma_2 > 0$ sufficiently small, with probability one there is a solution to the conformal welding problem for $\phi_2^{-1} \circ \phi_1$ which is unique up to Möbius transformations.

An alternative approach

Main result



$$\frac{\tau^{(1)}([0,x])}{\tau^{(1)}([0,1])} = \frac{\tau^{(2)}([0,y])}{\tau^{(2)}([0,1])}$$

Proof

Step 1: Beltrami equation

- We extend ϕ_1 and ϕ_2 to homeomorphisms $\Phi_1 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ and $\Phi_2 : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ via the **Beurling-Ahlfors** extension.
- For suitable functions g, the **complex dilatation** μ_g is defined by $\partial_{\overline{z}} g = \mu_g \partial_z g$.
- To solve the welding problem, it is enough to find a quasiconformal map $F : \mathbb{C} \to \mathbb{C}$ satisfying the **Beltrami equation**

$$\mu_F(z) = \begin{cases} \mu_{\Phi_1^{-1}}(z) & \text{if } z \in \mathbb{D} \\ \mu_{\Phi_2^{-1}}(z) & \text{if } z \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

since $f_1 := F \circ \Phi_1$ and $f_2 := F \circ \Phi_2$ each have zero dilatation and satisfy $f_1 \circ \phi_1^{-1} = f_2 \circ \phi_2^{-1}$.

Proof

Step 1: Beltrami equation

- Classical existence theory for quasiconformal maps states that the Beltrami equation has a solution when the complex dilatation is bounded uniformly away from one (in absolute value).
- This holds when boundary maps are somewhat regular, but fails in our setting.
- We instead consider the sequence of maps F_n satisfying

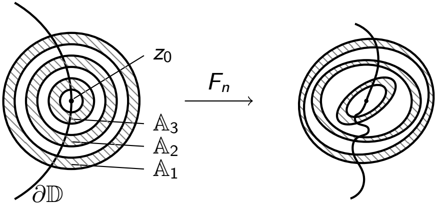
$$\mu_{F_n}(z) = \begin{cases} \frac{n}{n+1}\mu_{\Phi_1^{-1}}(z) & \text{if } z \in \mathbb{D} \\ \frac{n}{n+1}\mu_{\Phi_2^{-1}}(z) & \text{if } z \in \mathbb{C} \setminus \mathbb{D}. \end{cases}$$

- Any subsequential limit of (F_n) would satisfy our original Beltrami equation. Hence if we can prove equicontinuity of (F_n), then by Arzelà-Ascoli we have a solution to the welding problem.
- If we can extend this to uniform Hölder continuity of (F_n), then a conformal removability result will ensure that our solution is unique.

Proof

Step 2: Hölder continuity via undistorted annuli

- We want to translate uniform bounds on the **distortion** $\frac{1+|\mu_{F_n}|}{1-|\mu_{F_n}|}$ into uniform bounds on the modulus of continuity (near $\partial\mathbb{D}$).
- By a conformal modulus argument, Hölder continuity follows if we can find sufficiently many annuli around each point whose images under (F_n) are not too distorted.



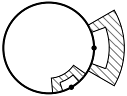
- This would be difficult to do for deterministic annuli $(\mathbb{A}_n)$ since we would need to control the distortion on the random sets $\Phi_1^{-1}(\mathbb{A}_n)$ and $\Phi_2^{-1}(\mathbb{A}_n)$.

Proof

Part 2: Hölder continuity via undistorted annuli

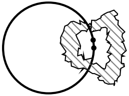
- Instead we consider images under Φ_1 and Φ_2 of deterministic 'half-annuli'. We can estimate the distortion of Φ_1^{-1} and Φ_2^{-1} on such sets which will control their images under (F_n).
- The challenge is to ensure that the images of many half-annuli 'match up' to form a full annulus.

$\mathbb{C} \setminus \partial\mathbb{D}$



$$\xrightarrow{\begin{cases}\Phi_1 & \text{on } \mathbb{D}\\ \Phi_2 & \text{on } \mathbb{C}\setminus\overline{\mathbb{D}}\end{cases}}$$

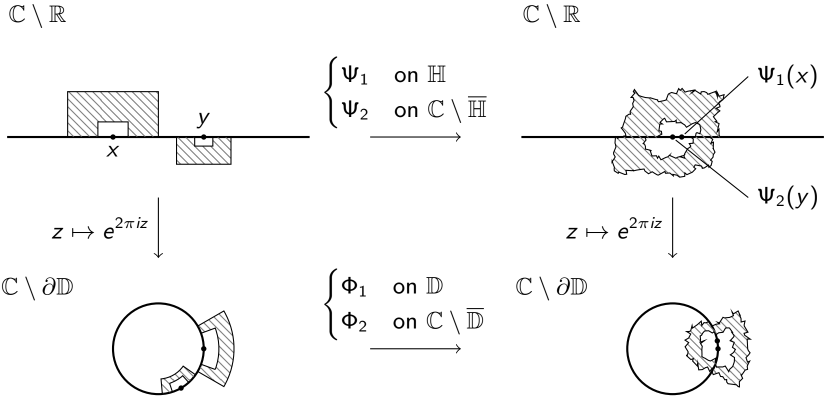
$\mathbb{C} \setminus \partial\mathbb{D}$



Proof

Part 2: Hölder continuity via undistorted annuli

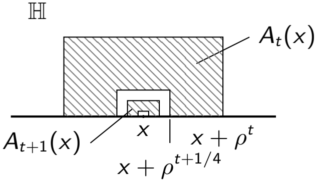
- For notational convenience, we map $\partial\mathbb{D}$ periodically onto $\mathbb{R}$ and use rectangular half-annuli.



Proof

Part 2: Hölder continuity via undistorted annuli

- We define a family of half-annuli $A_t(x) \subset \mathbb{H}$ of size comparable to $\rho^t > 0$ and let $\widetilde{A}_t(x)$ be their reflections in $\mathbb{R}$.



- For each point x in a finely spaced grid of $[0,1]$, we must find $y \in [0,1]$ and two increasing sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ such that with high probability, $\Psi_1(A_{t_n}(x))$ matches with $\Psi_2(\widetilde{A}_{s_n}(y))$ and F_n has bounded distortion on the resulting annulus.
- By a crude union bound argument, we may assume $\Psi_1(x) \approx \Psi_2(y)$.
- The remaining conditions are implied by an intersection of events of the form

$$\frac{\tau^{(1)}(x+\rho^{t_n} I)}{\tau^{(1)}(x+\rho^{t_n} J)} \le c, \quad \frac{\tau^{(2)}(y+\rho^{s_n} I)}{\tau^{(2)}(y+\rho^{s_n} J)} \le c, \quad \frac{\tau^{(1)}(x+[-\rho^{t_n},\rho^{t_n}])}{\tau^{(2)}(y+[-\rho^{s_n},\rho^{s_n}])} \in \left[\frac{1}{C}, C\right]$$

for explicit intervals $I, J \subset [0,1]$ and constants $c, C > 0$.

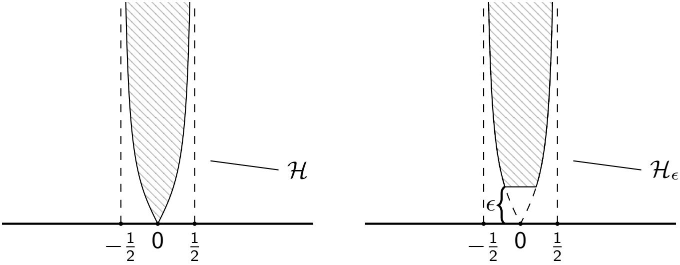
Proof

Part 3: Decoupling via white noise decomposition

- Let W be a white noise for the hyperbolic measure in $\mathbb{H}$.
- If we define $H_\epsilon(x) = W(x + \mathcal{H}_\epsilon)$ where

$$\mathcal{H} = \{|x| \leq 1/2, y \geq (2/\pi)\tan(|\pi x|)\} \quad \text{and} \quad \mathcal{H}_\epsilon = \mathcal{H} \cap \{y \geq \epsilon\}$$

then $H := \lim_{\epsilon \to 0} H_\epsilon$ is a representation of the Gaussian free field trace.



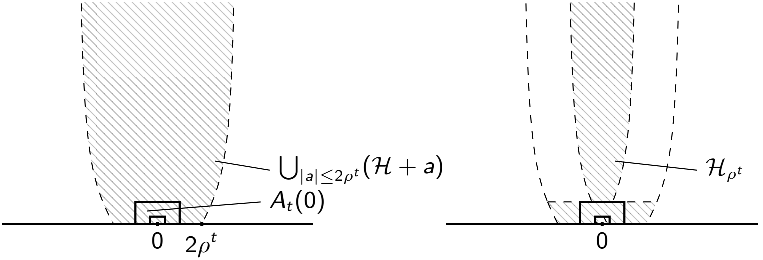
Proof

Part 3: Decoupling via white noise decomposition

- Let $\tau_t^{(1)}$ be the analogue of $\tau^{(1)}$ using the white noise restricted to $\{y \leq \rho^t\}$.
- For sets $A \subset [x - \rho^t, x + \rho^t]$, we use the approximation

$$\tau^{(1)}(A) \approx \exp\left(\gamma_1 H_{\rho^t}(x) - \frac{\gamma_1^2}{2}\mathrm{Var}[H_{\rho^t}(x)]\right) \tau_t^{(1)}(A \setminus [x - \rho^{t+3/4}, x + \rho^{t+3/4}])$$

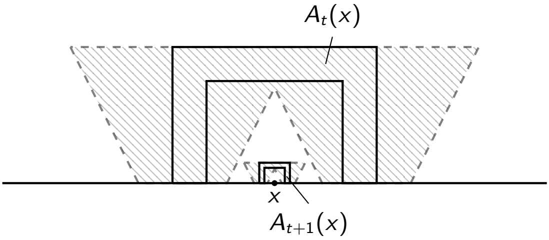
which will be valid for many values of t with high probability.



- Using this approximation, the first type of event we are interested in becomes

$$\frac{\tau^{(1)}(x+\rho^t I)}{\tau^{(1)}(x+\rho^t J)} \approx \frac{\rho^{-t}\tau_t^{(1)}(x+\rho^t I \setminus B_{t+3/4}(x))}{\rho^{-t}\tau_t^{(1)}(x+\rho^t J \setminus B_{t+3/4}(x))} \leq c$$

  where $B_t(x) := [x-\rho^t, x+\rho^t]$.
- These events are independent for $t, t+1, t+2, \ldots$.



- The measures $\rho^{-t}\tau_t^{(1)}(\rho^t \cdot)$ converge in distribution as $t \to \infty$, yielding large deviation bounds for the number of above events which occur.

- Using the previous approximation, the second event of interest can be reduced to
$$\frac{1}{C} \leq \exp(X_{t,s}) \frac{\rho^{-t} \tau_t^{(1)}(B_t(x))}{\rho^{-s} \tau_s^{(1)}(B_s(y))} \leq C$$
where
$$X_{t,s} := \gamma_1 H_{\rho^t}^{(1)}(x) - \gamma_2 H_{\rho^s}^{(1)}(y) - \left(1 + \frac{\gamma_1^2}{2}\right) \log(1/\rho) t + \left(1 + \frac{\gamma_2^2}{2}\right) \log(1/\rho) s.$$
- Our goal is to find sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ with increments in $[1,2]$ (say), such that $|X_{t_n,s_n}| \leq C'$ with high probability for a sufficiently dense subsequence.

Proof

Part 4: Random algorithm for matching half-annuli

- The process $X_{t,s}$ can be thought of as a 'two-parameter biased random-walk':

$$X_{t+u,s+v} - X_{t,s} \sim \mathcal{N}(d_2 v - d_1 u, \sigma_1^2 u + \sigma_2^2 v)$$

independent of $X_{t,s}$ where

$$d_i := \left(1 + \frac{\gamma_i^2}{2}\right)\log(1/\rho) \quad \text{and} \quad \sigma_i^2 := \gamma_i^2 \log(1/\rho).$$

- We therefore choose (t_{n+1}, s_{n+1}) iteratively depending on (t_n, s_n) so that the bias of the increment directs $X_{t,s}$ towards zero.
- The resulting process is an oscillating random walk, for which we can obtain large deviation estimates for the occupation time of $[-C', C']$.

Why do we require small parameter values?

- The measures $\tau^{(1)}$ and $\tau^{(2)}$ are well defined for all $\gamma_1, \gamma_2 \in [0, \sqrt{2})$ however our result only holds for $\gamma_1, \gamma_2 \in [0, \epsilon]$ for some $\epsilon > 0$. Why is this?
- Most statements described above hold for all $\gamma_1, \gamma_2 \in [0, \sqrt{2})$, however two arguments require small values:
  1. Matching half-annuli centres via the union bound
  2. The different events for controlling half-annuli each hold on a subsequence of (t_n, s_n) of constant density. To guarantee the intersection of these events, the density must be close to one which requires γ_1, γ_2 close to zero.

Open questions

- Can this approach be extended to all $\gamma_1, \gamma_2 \in [0, \sqrt{2})$?
  - Progress has been made using a related approach [4].
- Can one characterise the welding curves? Are they related to SLE?
  - This would be of particular interest when $\gamma_1 \neq \gamma_2$.

Thank you for listening!
谢谢

Further reading

- An expository account of Liouville quantum gravity and its relation to other probabilistic objects [14].
- Background on quasi-conformal maps [7] and the conformal welding problem [1].
- Background on the Gaussian free field and Liouville quantum gravity [3].

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