1 Continuous problem

The one-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

To apply a finite-volume method, it is convenient to rewrite this a s first-order sytem.

2 First-order hyperbolic systems

Introduce the auxiliary variables

$$v = \frac{\partial u}{\partial t}, \qquad q = \frac{\partial u}{\partial x},$$

The wave equation becomes

$$\frac{\partial u}{\partial t} = v,$$
$$\frac{\partial v}{\partial t} = c^2 \frac{\partial q}{\partial x},$$
$$\frac{\partial q}{\partial t} = \frac{\partial v}{\partial x}.$$

This is a conservative hyperbolic system.

3 Finite volume formulation

Control volumes

Divide the domain into cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ of width Δx. Define cell averages

$$\bar{u}_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\, dx.$$

and similarly for $\bar{v}_i(t)$ and $\bar{q}_i(t)$.

4 Integral form and discretization

(a) Equation for u

Integrate over the cell: The finite-volume semi-discrete scheme is

$$\frac{d\bar{u}_i}{dt} = \bar{v}_i,$$

(b) Equation for v

$$\frac{d\bar{v}_i}{dt} = \frac{c^2}{\Delta x} \left(q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}} \right),$$

where $q_{i\pm\frac{1}{2}}$ are numerical fluxes at cell interfaces.

(c) Equation for q

$$\frac{d\bar{q}_i}{dt} = \frac{1}{\Delta x}\left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right).$$

5 Numerical fluxes (Using central numerical fluxes)

For a standard second-order FV scheme, interface values are reconstructed using centeral differences:

$$q_{i+\frac{1}{2}} = \frac{1}{2}(\bar{q}_i + \bar{q}_{i+1}), \qquad v_{i+\frac{1}{2}} = \frac{1}{2}(\bar{v}_i + \bar{v}_{i+1}).$$

This yields a second-order accurate spatial discretization.

6 Time discretization (explicit, second order)

$$\frac{\bar{u}_i^{n+1} - 2\bar{u}_i^n + \bar{u}_i^{n-1}}{(\Delta t)^2} = c^2 \frac{\bar{n}_{i+1}^n - 2\bar{u}_i^n + \bar{u}_{i-1}^n}{(\Delta x)^2}$$

This is the finite-volume equivalent of the classical second-order wave scheme, but derived from conservative laws.

7 CFL stability condition

For explicit time integration: the CFL condition is

$$\frac{c\Delta t}{\Delta x} \leq 1.$$