MATH 223 FALL 2025 - HOMEWORK 6

DUE OCTOBER 28 AT 11:59PM

Each problem is worth 10 points.

Problem 1. The *transpose* of an $m \times n$ matrix A (m rows, n columns) is the $n \times m$ matrix A^T (n rows, m columns) such that the columns of A^T are the rows of A (and so the rows of A^T are the columns of A).

(1) Compute the transpose of $A = \begin{bmatrix} 4 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

Let A be an $m \times n$ matrix and B an $n \times k$ matrix.

(3) Check that matrices $(AB)^T$ and $B^T A^T$ have the same size.

(4) Show that $(AB)^T = B^T A^T$. (Describe how the (i,j)-entry of $(AB)^T$ is obtained from rows/columns of A and B, and similarly describe how the (i,j)-entry of $B^T A^T$ is obtained, and compare these.)

Problem 2. Let A be an $n \times n$ matrix.

(1) Let $A = E_1 \cdots E_\ell$ be the product of elementary matrices. Deduce from Problem 1 that $A^T = E_\ell^T \cdots E_1^T$.

(2) Show that for every elementary matrix E, E^T is also an elementary matrix, and $\det E = \det E^T$.

(3) Prove that for every $n \times n$ matrix A we have $\det A = \det A^T$.

Problem 3. True or False. Justify your answers.

(1)

$$\det(A + B) = \det A + \det B$$

where A, B are $n \times n$ matrices.

(2)

$$\det c \cdot A = c^n \det A$$

where A is an $n \times n$ matrix, $c \in F$ and $c \cdot A$ is obtained from A by multiplying each entry by c.

Problem 4. Compute the determinant.

(1) $A_1 = \begin{bmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix}$

(2) $A_2 = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{bmatrix}$

(3) $A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$