Math in Photography

Simplified Optical Model of a Camera

Because I'm not majoring in optics or physics. So here we'll only talk about the geometry instead of the practical engineering. Before we start, we are supposed to build an ideal model in case of the mathematical calculation.

Lens

The use of the lens is to image the objects to the sensor. Nowadays camera lenses are very complicated. That is because the spherical lens can't focus precisely. This is called spherical aberration. Even if there's a non-spherical lens combination that could avoid spherical aberration, light beams with different frequencies have different refractive indices. So it's very difficult to build an ideal lens.

Imagine we have an ideal lens. The thickness of the lens can be ignored and the radius of the lens is infinity. Light beams that are perpendicular to the lens would focus on the focus. Light beams that point straight at the optical center wouldn't change their directions. Light beams from a point of light would focus on another point at the opposite side of the lens.

Sensor

In many conditions, as the sensor is harder to move, the focusing process is the lens moving along the optical axis so the image would focus on the sensor. But here, to make the model simpler, we move the sensor and fix the lens.

It can be proved that, with the same lens, no matter where the object is, as long as the distance between the object and the plane of the lens doesn't change, the distance between the plane of the lens and the image of the object doesn't change, too.

So we define the "object distance" as the distance between the object and the plane of the lens, and the "image distance" as the distance between the image and the plane of the lens.

Aperture

The aperture is the structure that controls the amount of light that goes into the camera. In real life, the aperture is separated from the lens so the radius of the light that is eventually shot on the sensor when going through the lens (Also known as exit pupil) is different from the actual radius of the aperture.

Here we pin the aperture on the lens so the radius of the aperture is the radius of the exit pupil.

If the focal length of the lens is f, and the exit pupil is r, then we can define the "f-number" N as $\frac{f}{2r}$. (f over 2 r)

Ray Transfer Matrix

If we use a vector to represent a light beam:

$$\begin{bmatrix} x \\ \theta \end{bmatrix}$$

In which x is the distance between the optical center and the point that the light beam goes into the lens, θ is the tangent of the angle from the optical axis and the light beam.

There is a matrix that can transform the incident light into the emergent light.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix}$$

In our lens model, the only parameter of a lens is the focal length f. So as long as f is definite, the matrix can be calculated. There are two methods to calculate the matrix of an ideal thin lens.

First, substitute the vector by the special value, and solve the equation. We choose the vectors with $x = 0$ and $\theta = 0$.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} 0 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ \theta \end{bmatrix}$$
$$D\theta = \theta$$
$$B\theta = 0$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ -\frac{x}{f} \end{bmatrix}$$
$$Ax = x$$
$$Cx = -\frac{x}{f}$$

So the matrix of the lens is:

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Another method is to simulate the imaging process. To simplify the calculation, let the light beam start from the point, whose object distance $d = 2f$. Obviously, the image is symmetric with the object about the optical center.

Define r as the distance from the object to the optical axis. So $r = x - 2\theta f$.

Then we just express the emergent light by x and θ and solve the equations.

$$x' = x$$
$$\theta' = \frac{-r - x}{2f} = \frac{\theta f - x}{f} = \theta - \frac{x}{f}$$
$$x' = Ax + B\theta$$
$$\theta' = Cx + D\theta$$

Then we got the matrix too.

Image Distance

To focus, we should calculate the image distance D with the focal length f and the object distance d.

If the object distance is d, and the distance from the object to the optical axis is r. The two special light beams are:

$$\begin{bmatrix} r \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{r}{d} \end{bmatrix}$$

With the matrix, we can get the emergent lights.

$$\begin{bmatrix} r \\ -\frac{r}{f} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{r}{d} \end{bmatrix}$$

The intersection point of the emergent lights is the image. So there is an equation $r = D(\frac{r}{f} - \frac{r}{d})$. Therefore, $D = \frac{fd}{(d-f)}$.

Besides, the distance from the image to the optical axis R can also calculated. Because $\frac{r}{d} = \frac{R}{D}$, so $R = \frac{rD}{d}$.