Control Review

Room 3116 (November 1, 2021)

This is a review of the course ELEC5650 - INTRODUCTION TO NETWORKED SENSING, ESTIMATION AND CONTROL, 2021-22 FALL.

1 LECTURE 2

1.1 Stability

An autonomous system is given by

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k,\ \mathbf{x}_0 \in \mathbb{R}^n. \tag{1}$$

The system with control inputs

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\ \mathbf{u}_k = \mathbf{F}\mathbf{x}_k \Rightarrow \mathbf{x}_{k+1} = (\mathbf{A} + \mathbf{B}\mathbf{F})\,\mathbf{x}_k,$$

is also autonomous if the inputs are linear state feedback. See section 5.2 for details of optimal control.

Definition 1.1. An autonomous system is stable if

$$\forall \mathbf{x}_0,\ |\mathbf{x}_k| \to \mathbf{0} \text{ as } k \to \infty.$$

The system is stable equals to:

1) $\rho(\mathbf{A}) < 1$ where $\rho(\mathbf{A}) \triangleq \max_i |\lambda_i(\mathbf{A})|$ is the spectrum radius of $\mathbf{A}$. Understanding[1]:
   a) For diagonalizable matrix $\mathbf{A}$, it has an EVD as $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$.
   b) For non-diagonalizable matrix $\mathbf{A}$, it can be written in $\mathbf{A} = \mathbf{M}\mathbf{J}\mathbf{M}^{-1}$where $\mathbf{J}$ is the Jordan form with Jordan blocks on its diagonal. Each block in $\mathbf{J}$ has one eigenvalue λ_i, one eigenvector, and 1‘ s above the diagonal.
   c) The power of $\mathbf{A}$ is $\mathbf{A}^k = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}$ or $\mathbf{A}^k = \mathbf{M}\mathbf{J}^k\mathbf{M}^{-1}$. When $k \to \infty$, $\mathbf{A}^k$ is dominated by the $\max_i |\lambda_i(\mathbf{A})|$.
2) $\exists \mathbf{Q} > 0\ \&\ \mathbf{X} > 0$ such that

$$\mathbf{A}^\intercal\mathbf{X}\mathbf{A} - \mathbf{X} = -\mathbf{Q}, \tag{2}$$

   which is the Lyapunov equation. Understanding: The autonomous system (1) doesn't have observations. From the view of Kalman filter, it only has time update process with Lyapunov operator $h(\cdot)$. (2) is equivalent to

$$h(\mathbf{P}_{k-1}) = \mathbf{P}_k.$$

1. Section 6.6 Similar Matrices of INTRODUCTION TO LINEAR ALGEBRA, fourth edition.

1.2 Controllability

Consider the system with m control inputs

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\ \mathbf{x}_0 \in \mathbb{R}^n, \tag{3}$$

or in short $(\mathbf{A}, \mathbf{B})$.

Definition 1.2. The LTI system in (3) is called "controllable" if for any initial state $\mathbf{x}_0$ and any desired state $\mathbf{x}_{\text{desired}}$, the input signal $\{\mathbf{u}_0, \ldots, \mathbf{u}_{k_1-1}\}$ can be designed such that the system reaches $\mathbf{x}_{k_1} = \mathbf{x}_{\text{desired}}$ in some finite time $k_1 \geq 1$.

The system $(\mathbf{A}, \mathbf{B})$ is controllable equals to:

1) **(Controllability matrix)** The controllability matrix $\mathcal{M}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{n \times nm}$ has full row rank, i.e., rank $(\mathcal{M}_c) = n$. This comes from the fact that

$$\begin{aligned} \mathbf{x}_{k_1} &= \mathbf{A}^{k_1}\mathbf{x}_0 + \mathbf{A}^{k_1-1}\mathbf{B}\mathbf{u}_0 + \cdots + \mathbf{A}\mathbf{B}\mathbf{u}_{k_1-2} + \mathbf{B}\mathbf{u}_{k_1-1} \\ \Leftrightarrow \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{k_1-1}\mathbf{B} \end{bmatrix} &\begin{bmatrix} \mathbf{u}_{k_1-1} \\ \mathbf{u}_{k_1-2} \\ \vdots \\ \mathbf{u}_0 \end{bmatrix} = \mathbf{x}_{k_1} - \mathbf{A}^{k_1}\mathbf{x}_0. \end{aligned}$$

   To make sure it has a solution for all $\mathbf{x}_{\text{desired}}$, a full row rank $\mathcal{M}_C$ is enough no matter whether $k_1 \geq n$ or not. [2]
2) **(Controllability Gramian)** The k-th step controllability Gramian, a symmetric matrix

$$\mathcal{W}_c(k) = \sum_{i=0}^{k-1} \mathbf{A}^i \mathbf{B}\mathbf{B}^\intercal \left(\mathbf{A}^i\right)^\intercal$$

   is nonsingular for all k, i.e., rank $(\mathcal{W}_c) = n$.
3) **(PBH test)** $\forall \lambda,\ \text{rank} \begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix} = n.$

1.2.1 Relationship between controllability & stability

Consider the system with control inputs and linear state feedback

$$\begin{aligned} \mathbf{x}_{k+1} =& \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\ \mathbf{x}_0 \in \mathbb{R}^n \\ =& (\mathbf{A} + \mathbf{B}\mathbf{F})\,\mathbf{x}_k. \end{aligned} \tag{4}$$

Given that (4) is controllable, there $\exists \mathbf{F}$ such that

$$(\mathbf{A}, \mathbf{B}) \text{ is controllable } \implies \rho(\mathbf{A} + \mathbf{B}\mathbf{F}) < 1.$$

1.3 Observability

Consider the system with p observations

$$\begin{aligned} \mathbf{x}_{k+1} =& \mathbf{A}\mathbf{x}_k,\ \mathbf{x}_0 \in \mathbb{R}^n \\ \mathbf{y}_k =& \mathbf{C}\mathbf{x}_k \end{aligned} \tag{5}$$

2. See UCR-lecture for details of controllability.

or in short $(\mathbf{A}, \mathrm{C})$. **Observability is a dual concept of controllability.**

Definition 1.3. The LTI system in (5) is observable if $\forall \mathbf{x}_0$, $\exists k_1 \geq 0$ such that $\mathbf{x}_0$ can be computed from $\{\mathbf{y}_0, \cdots, \mathbf{y}_{k_1}\}$.

The system $(\mathbf{A}, \mathbf{C})$ is observable equals to:

1) **(Observability matrix)** The observability matrix

$$\mathcal{M}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$$

has full column rank, i.e., $\operatorname{rank}(\mathcal{M}_o) = n$.

2) **(Observability Gramian)** The k-th step observability Gramian, a symmetric matrix

$$\mathcal{W}_o(k) = \sum_{i=0}^{k-1} \left(\mathbf{A}^i\right)^\intercal \mathbf{C}^\intercal \mathbf{C} \mathbf{A}^i$$

is nonsingular for all k, i.e., $\operatorname{rank}(\mathcal{W}_o) = n$.

3) **(PBH test)** $\forall \lambda$, $\operatorname{rank} \begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} \\ \mathbf{C} \end{bmatrix} = n$.

1.3.1 *Duality between observability & controllability*

$$(\mathbf{A}, \mathbf{C}) \text{ is observable} \iff (\mathbf{A}^\intercal, \mathbf{C}^\intercal) \text{ is controllable}$$

2 LECTURE 3-4

Consider $\mathbf{X}$ is the an unknown random variable we want to estimate, and $\mathbf{x} \in \mathbb{R}^n$ is corresponding realization. $\mathbf{Y}$ is another random variable, $\mathbf{y} \in \mathbb{R}^m$ is the corresponding measurement or observation. According to Bayes' theorem, we have

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{x})\, p(\mathbf{y}|\mathbf{x})}{\sum_{\mathbf{x}} p(\mathbf{x})\, p(\mathbf{y}|\mathbf{x})}$$

where $p(\mathbf{x})$ is the prior distribution of $\mathbf{x}$, $p(\mathbf{y}|\mathbf{x})$ is the likelihood function, $p(\mathbf{x}|\mathbf{y})$ is the posterior distribution of $\mathbf{x}$.[3]

2.1 Non-Bayesian estimation

1) **MLE**:

$$\hat{\mathbf{x}}_{\text{MLE}}(\mathbf{y}) = \arg\max_{\mathbf{x}} \; p(\mathbf{y}; \mathbf{x}).$$

Here $\mathbf{x}$ is a deterministic, but unknown constant.[4]

3. Here we use condition on $\mathbf{x}$ because $\mathbf{x}$ is a random variable.
4. Here we use given $\mathbf{x}$ because $\mathbf{x}$ is fixed.

2.2 Bayesian estimation

Two concepts:

- Bayesian estimator: $\hat{\mathbf{X}} = g(\mathbf{Y})$ is a random variable. An estimator is a function of samples, i.e., a rule that tells you how to calculate an estimate of a parameter from a sample.
- Bayesian point estimate: $\hat{\mathbf{x}} = g(\mathbf{y})$ isn't a random variable. An estimate is a value of an estimator calculated from a sample.

Some differences: Unlike non-Bayesian approach where parameters of interest are assumed to be deterministic but unknown constants, the Bayesian estimator seeks to estimate a parameter that is itself a random variable. The likelihood is not a probability distribution over $\mathbf{x}$, and its integral with respect to $\mathbf{x}$ does not necessarily equal one.

Within the Bayesian framework, there are two kinds of estimators:

1) **MAP**:
$$\hat{\mathbf{x}}_{\text{MAP}}(\mathbf{y}) = \arg\max_{\mathbf{x}} p(\mathbf{x}|\mathbf{y}),$$
the maximum value of $p(\mathbf{x}|\mathbf{y})$.
2) **MMSE**: The estimation error vector is given by $e = \hat{\mathbf{x}} - \mathbf{x}$, and its mean squared error (quadratic cost function) is given by
$$\text{MSE} = \mathbb{E}\left[(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}(\hat{\mathbf{x}} - \mathbf{x})|\mathbf{y}\right] = \text{Tr}\left(\mathbb{E}\left[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}|\mathbf{y}\right]\right),$$
where the expectation is taken over both $\mathbf{x}$ and $\mathbf{y}$. The MMSE estimator is then
$$\begin{aligned}\hat{\mathbf{x}}_{\text{MMSE}}(\mathbf{y}) =& \arg\min_{\hat{\mathbf{x}}} \text{MSE} \\ =& \mathbb{E}[\mathbf{x}|\mathbf{y}],\end{aligned}$$
the expected value of $p(\mathbf{x}|\mathbf{y})$.

2.3 LMMSE

Direct numerical evaluation of $\mathbb{E}[\mathbf{x}|\mathbf{y}]$ is usually computationally expensive with the methods like Monte Carlo experiments. One alternative is to consider the linear MMSE estimators, a subclass of MMSE estimators, given by
$$\min_{\hat{\mathbf{x}}} \text{MSE} \quad \text{s.t. } \hat{\mathbf{x}} = \mathbf{K}\mathbf{y}. \tag{6}$$
From the definition of MSE, we just need to minimize $\text{Tr}(\mathbf{P}(\mathbf{K}) = \mathbb{E}\left[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}|\mathbf{y}\right])$. Here we want a stronger condition
$$\mathbf{P}(\mathbf{K}_0) \preceq \mathbf{P}(\mathbf{K}) \ \forall \mathbf{K},$$
and so the problem (6) changes into
$$\min_{\hat{\mathbf{x}}} \mathbf{P}(\mathbf{K}) \quad \text{s.t. } \hat{\mathbf{x}} = \mathbf{K}\mathbf{y}. \tag{7}$$

Notation: Define the covariance matrices $\mathbf{R}_y = \mathbb{E}[\mathbf{y}\mathbf{y}^{\mathsf{T}}]$ and $\mathbf{R}_x = \mathbb{E}[\mathbf{x}\mathbf{x}^{\mathsf{T}}]$. Define the cross-covariance matrix $\mathbf{R}_{xy} = \mathbb{E}[\mathbf{x}\mathbf{y}^{\mathsf{T}}] \in \mathbb{R}^{n\times m}$, whose $\mathbf{R}_{xy_{(i,j)}} = \text{cov}(x_i, y_j)$ and x_i, y_j are random elements. Same for $\mathbf{R}_{yx} = \mathbb{E}[\mathbf{y}\mathbf{x}^{\mathsf{T}}] \in \mathbb{R}^{m\times n}$.

Derivation

The lecture note provides the solution of (7). According to the completing the square, we have

$$\begin{aligned}\mathbf{P}(\mathbf{K}) =& \mathbb{E}\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^{\intercal}\right] \\ =& \mathbb{E}\left[(\mathbf{K}\mathbf{y}-\mathbf{x})(\mathbf{K}\mathbf{y}-\mathbf{x})^{\intercal}\right] \\ =& \left[\mathbf{K}\mathbf{R}_y\mathbf{K}^{\intercal}-\mathbf{K}\mathbf{R}_{yx}-\mathbf{R}_{xy}\mathbf{K}^{\intercal}\right]+\mathbf{R}_x \\ =& \left[\left(\mathbf{K}-\mathbf{R}_{xy}\mathbf{R}_y^{-1}\right)\mathbf{R}_y\left(\mathbf{K}-\mathbf{R}_{xy}\mathbf{R}_y^{-1}\right)^{\intercal}-\mathbf{R}_{xy}\mathbf{R}_y^{-1}\mathbf{R}_{yx}\right]+\mathbf{R}_x.\end{aligned}$$

The optimal $\mathbf{K}=\mathbf{K}_0$ and linear MMSE estimator are given by

$$\begin{aligned}\mathbf{K}_0 &= \mathbf{R}_{xy}\mathbf{R}_y^{-1} \\ \hat{\mathbf{x}}_{\text{LMMSE}} &= \mathbf{K}_0\mathbf{Y}.\end{aligned} \tag{8}$$

The auto-covariance of $\hat{\mathbf{x}}$ is then

$$\mathbf{R}_{\hat{x}} = \mathbb{E}\left[\mathbf{K}_0\mathbf{Y}\mathbf{Y}^{\intercal}\mathbf{K}_0^{\intercal}\right] = \mathbf{R}_{xy}\mathbf{R}_y^{-1}\mathbf{R}_{yx},$$

and the minimal error covariance is

$$\begin{aligned}\mathbf{P}(\mathbf{K}_0) =& \mathbf{R}_x-\mathbf{R}_{xy}\mathbf{R}_y^{-1}\mathbf{R}_{yx} \\ \Leftrightarrow& \mathbf{R}_x-\mathbf{R}_{\hat{x}} \\ \Leftrightarrow& \mathbf{R}_x-\mathbf{K}_0\mathbf{R}_{yx} \\ \Leftrightarrow& \mathbf{R}_x-\mathbf{R}_{xy}\mathbf{K}_0^{\intercal}.\end{aligned} \tag{9}$$

Properties

- Orthogonal Principle (originated from MMSE)

  Theorem 2.1. *The $\hat{\mathbf{x}}_{LMMSE}(\mathbf{y})$ is the orthogonal projection of $\mathbf{x}$ onto the linear span of $\mathbf{y}$ denoted as $L\{\mathbf{y}\}$.*

  Corollary 2.2. *In other words, estimation error is perpendicular to all measurements: given $\mathbf{y}=\{\mathbf{y}_1,\ldots,\mathbf{y}_M\}$,then $\mathbf{e}=\mathbf{x}-\hat{\mathbf{x}}\perp\mathbf{y}_i,\ \forall i=1,\ldots,M$.*

- If $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian, then the MMSE estimator is linear. As a consequence, to find the MMSE estimator, it is sufficient to find the linear MMSE estimator.
- It is not necessary to explicitly calculate $p(\mathbf{x}|\mathbf{y})$, and only the first two moments of $\mathbf{x}$ and $\mathbf{y}$ are required.

3 LECTURE 4-5

3.1 LMMSE for linear observation process

Consider the linear process:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v},$$

where

- $\mathbf{H}$ is a known matrix;
- $\mathbf{v} \sim \mathcal{N}(0, \mathbf{R}_v)$ is a random noise vector with cross-covariance $\mathbf{R}_{xv} = \mathbf{0} \Leftrightarrow \mathbf{x} \perp \mathbf{v}$.

Now we can replace the y-relevant matrices in (8) (9). Consider

$$\begin{aligned}\mathbf{R}_y =& \mathbf{H}\mathbf{R}_x\mathbf{H}^\intercal + \mathbf{R}_v \\ \mathbf{R}_{xy} =& \mathbb{E}\left[\mathbf{x}\left(\mathbf{H}\mathbf{x} + \mathbf{v}\right)^\intercal\right] = \mathbf{R}_x\mathbf{H}^\intercal\end{aligned}$$

we have

$$\begin{aligned}\hat{\mathbf{x}}_{\text{LMMSE}} =& \mathbf{R}_{xy}\mathbf{R}_y^{-1}\mathbf{y} \\ \Rightarrow& \mathbf{R}_x\mathbf{H}^\intercal\left(\mathbf{H}\mathbf{R}_x\mathbf{H}^\intercal + \mathbf{R}_v\right)^{-1}\mathbf{y}\end{aligned}$$

$$\begin{aligned}\mathbf{P}\left(\mathbf{K}_0\right) =& \mathbf{R}_x - \mathbf{R}_{xy}\mathbf{R}_y^{-1}\mathbf{R}_{yx} \\ \Rightarrow& \mathbf{R}_x - \mathbf{R}_x\mathbf{H}^\intercal\left(\mathbf{H}\mathbf{R}_x\mathbf{H}^\intercal + \mathbf{R}_v\right)^{-1}\mathbf{H}\mathbf{R}_x \\ =& \left(\mathbf{R}_x^{-1} + \mathbf{H}\mathbf{R}_v^{-1}\mathbf{H}^\intercal\right)^{-1}.\end{aligned}$$

3.2 Hilbert space of random variables

This subsection covers the knowledge in why we can derive the Kalman filter from the projection. Here we take the important messages from EE126 at UCB, and there are violations of notations (and we correct the reverse meaning of X, Y).[5]

3.2.1 Hilbert space

The definition of a Hilbert space is a set

$$\mathcal{H} \coloneqq \left\{X : X \text{ is a real-valued random variable with } \mathbb{E}\left[X^2\right] < \infty\right\}.$$

It is a inner product space that allows us to analyze the random variables geometrically.

Definition 3.1. For a real vector space V, a map $\langle\cdot,\cdot\rangle : V \times V \to [0, \infty)$ satisfying, for all $u, v, w \in V$ and $c \in \mathbb{R}$,

- (symmetry) $\langle u, v\rangle = \langle v, u\rangle$,
- (linearity) $\langle u + cv, w\rangle = \langle v, w\rangle + c\langle v, w\rangle$,
- (positive definiteness) $\langle u, u\rangle > 0$ if $u \neq 0$,

is called a real inner product on V, and V along with the map $\langle\cdot,\cdot\rangle$ is a real inner product space. We can say that u and v are orthogonal if $\langle u, v\rangle = 0$. The inner product of real-valued random variables X, Y is defined as $\langle X, Y\rangle = \mathbb{E}[XY]$.[6]

A real Hilbert space is a real inner product space which satisfies an additional analytic property called completeness. In general $\mathcal{H}$ is infinite-dimensional, though the analysis is from a finite-dimensional perspective.

5. THE HILBERT SPACE OF RANDOM VARIABLES: link
6. VECTOR SPACES OF RANDOM VARIABLES: link

3.2.2 Projection with Gram-Schmidt process

Now our problem is to estimate a random variable $X \in \mathcal{H}$ based on a linear function of the observable random variable Y denoted as $a + bY$, i.e, span $\{1, Y\}$. Again, we emphasize that Y is not a constant.

(Projection) The problem can be formulated as the projection of x onto the subspace U:

$$\text{Given } x \in V \text{ and a subspace } U \subseteq V, \text{ find the closet point } \hat{x} \in U \text{ to } x.$$

The orthogonal projection onto a subspace U is the map $P : V \to U$ such that $Px \coloneqq \arg\min_{\hat{x} \in U} \|x - \hat{x}\|$. According to the properties of least square estimation, we have

$$Px \in U \quad \text{and} \quad x - Px \in U^{\perp},$$

where $U^{\perp}$ is the orthogonal complement of U. Suppose that U is n-dimensional with a finite basis $\{v_i\}_{i=1}^n$, we can find an orthonormal set of vectors $\{u_i\}_{i=1}^n$, where span $\{u_i\}_{i=1}^j =$ span $\{v_i\}_{i=1}^j$, $j = 1, \ldots, n$. We have $Px = \sum_{i=1}^n \langle x, u_i \rangle u_i$, and P can be represented as $P = \sum_{i=1}^n u_i u_i^{\intercal}$.

To find such an orthonormal basis, we follow the Gram-Schmidt process:

1) Let $u_1 \coloneqq v_1 / \|v_1\|$.
2) For $j = 1, \ldots, n-1$:
   a) Set $w_{j+1} \coloneqq v_{j+1} - \sum_{i=1}^j \langle v_{j+1}, u_i \rangle u_i$.
   b) Set $u_{j+1} \coloneqq w_{j+1} / \|w_{j+1}\|$.

3.2.3 Linear least squares estimation

(Formal Formulation) Combined with the linear process, we have a formal formulation:

$$\text{Given } X, Y \in \mathcal{H}, \text{ minimize } \mathbb{E}\left[(X - a - bY)^2\right] \text{ over } a, b \in \mathbb{R}.$$

The solution to this problem is called the linear least squares estimator (LLSE).

4 LECTURE 5-7

4.1 Sequential LMMSE and Kalman Filter

In practice, observational data doesn't often come in a batch but more like a time series. If we consider the dataset with a new measurement as a fresh dataset, we can recompute everything without previous information. It is naive because we will face an increasing demand of computational power and storage with an increasing dimension. Instead, we prefer to update an old estimate with the new measurement.

Key message: When we are talking about estimation/filtering/prediction, we are always talking about estimating the same random variable, but not the instances of the random variable. The random variable is an infinite-dimensional function.

Consider the system model:

$$\begin{aligned}\mathbf{x}_{k+1} =& \mathbf{A}\mathbf{x}_k + \mathbf{w}_k, \qquad \mathbf{w}_k \sim \mathcal{N}(0, \mathbf{Q}),\ \mathbf{x}_0 \sim \mathcal{N}(0, \boldsymbol{\pi})\\ \mathbf{y}_k =& \mathbf{C}\mathbf{x}_k + \mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}).\end{aligned}$$

The Kalman filter:

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1} =& \mathbf{A}\hat{\mathbf{x}}_{k-1|k-1},\\ \mathbf{P}_{k|k-1} =& \mathbf{A}\mathbf{P}_{k-1|k-1}\mathbf{A}^\intercal + \mathbf{Q}.\end{aligned}$$

$$\begin{aligned}\mathbf{K}_k =& \mathbf{P}_{k|k-1}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{P}_{k|k-1}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1}\\ \hat{\mathbf{x}}_{k|k} =& \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \left[\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}\right]\\ \mathbf{P}_{k|k} =& \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{P}_{k|k-1}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1}\mathbf{C}\mathbf{P}_{k|k-1}\end{aligned}$$

Key ideas:

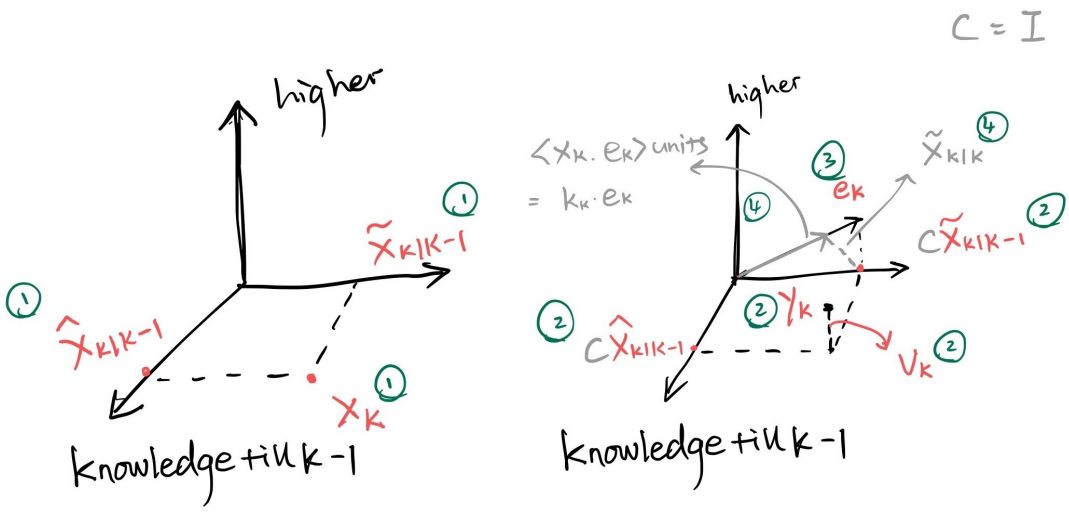
1) The instances of $\mathbf{x}$ and $\mathbf{y}$ have a mapping relationship through $\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k$. $\mathbf{y}$ and $\mathbf{x}$ are in the same space, and the observed subspace is simply spanned by $\mathbf{y}$. In calculation, we tend to replace $\mathbf{y}_k$ with $\mathbf{C}\mathbf{x}_k + \mathbf{v}_k$, because $\mathbf{x}_k$ is our interest.
2) An LMMSE estimator of $\mathbf{x}$ requires to project $\mathbf{x}$ onto the $\mathbf{L}_k = \text{span}\{\mathbf{y}\}$. More $\mathbf{y}_k$ we receive, the higher dimensional $\mathbf{L}_k$ we can construct, and thus the higher accuracy of modeling the infinite-dimensional $\mathbf{x}$ we can achieve.
3) Every time when a new $\mathbf{y}_k$ comes, we enlarge the $\mathbf{L}_{k-1}$ by one dimension $\mathbf{e}_k$ in a Gram-Schmidt process, called the innovation, perpendicular to $\mathbf{L}_{k-1}$.
4) In the past, $\mathbf{x}_k$ can only be projected on $\mathbf{L}_{k-1}$ left with $\tilde{\mathbf{x}}_{k|k-1}$. Now we can consider the additional projection of $\tilde{\mathbf{x}}_{k|k-1}$ on this new dimension $\mathbf{e}_k$, and find the new estimate as $\hat{\mathbf{x}}_{k|k}$. Because of the existence of $\mathbf{v}_k$, we cannot expect that $\tilde{\mathbf{x}}_{k|k-1}$ can be fully described by $\mathbf{e}_k = \mathbf{C}\tilde{\mathbf{x}}_{k|k-1} + \mathbf{v}_k$, i.e., there is $\tilde{\mathbf{x}}_{k|k}$ left.
5) Since the random variables are infinite-dimensional, each observation can continue providing one additional perpendicular dimension.

The important relationships are summarized as

$$\begin{aligned}\mathbf{x}_k =& \hat{\mathbf{x}}_{k|k-1} + \tilde{\mathbf{x}}_{k|k-1} = \hat{\mathbf{x}}_{k|k} + \tilde{\mathbf{x}}_{k|k}\\ \mathbf{y}_k =& \mathbf{C}\mathbf{x}_k + \mathbf{v}_k\\ =& \mathbf{C}\hat{\mathbf{x}}_{k|k-1} + \mathbf{C}\tilde{\mathbf{x}}_{k|k-1} + \mathbf{v}_k\\ \text{proj}\{\mathbf{x}_k|\mathbf{L}_{k-1}\} =& \hat{\mathbf{x}}_{k|k-1}\\ \text{proj}\{\mathbf{y}_k|\mathbf{L}_{k-1}\} =& \mathbf{C}\hat{\mathbf{x}}_{k|k-1}.\end{aligned}$$

Illustration:

1) Given $\mathbf{x}_k$, we has the portion $\hat{\mathbf{x}}_{k|k-1}$ based on the knowledge till $k-1$, and a perpendicular portion $\tilde{\mathbf{x}}_{k|k-1}$. In the plot, we condense all information provided till $k-1$ into one axis, the prediction error as another axis, and other higher dimensions as the other axis.
2) Because of the $\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k$, we can easily draw $\mathbf{y}_k$ as the combination of three components, $\mathbf{y}_k = \mathbf{C}\hat{\mathbf{x}}_{k|k-1} + \mathbf{C}\tilde{\mathbf{x}}_{k|k-1} + \mathbf{v}_k$ ($\mathbf{v}_k$ is uncorrelated with $\hat{\mathbf{x}}_{k|k-1}$ and $\tilde{\mathbf{x}}_{k|k-1}$). We draw another coordinate system spanned by $\{\mathbf{y}\}$ and lets $\mathbf{C} = \mathbf{I}$ for simplicity.
3) The knowledge of k is introduced by the innovation $\mathbf{e}_k = \mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}$, i.e., the portion perpendicular to $\text{proj}\{\mathbf{y}_k|\mathbf{L}_{k-1}\} = \mathbf{C}\hat{\mathbf{x}}_{k|k-1}$.
4) As we mentioned before, the next step is to project $\tilde{\mathbf{x}}_{k|k-1}$ onto $\mathbf{e}_k$ denoted as $\langle \mathbf{x}_k, \mathbf{e}_k \rangle$. The corrected estimate is then $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \langle \mathbf{x}_k, \mathbf{e}_k \rangle$. There is $\tilde{\mathbf{x}}_{k|k}$ left.



4.1.1 Measurement update

Measurement update of Kalman filter

$$\begin{aligned}\hat{\mathbf{x}}_{k|k} =&\hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\mathbf{e}_k,\\ \mathbf{P}_{k|k} =&\mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{R}_{e,k}\mathbf{K}_k^\intercal,\\ =&\mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{C}^\intercal\mathbf{R}_{e,k}^{-1}\mathbf{C}\mathbf{P}_{k|k-1},\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}_k =&\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}\\ \mathbf{R}_{e,k} =& \langle \mathbf{e}_k, \mathbf{e}_k\rangle = \mathbb{E}\left[\mathbf{e}_k\mathbf{e}_k^\intercal\right] = \mathbf{C}\mathbf{P}_{k|k-1}\mathbf{C}^\intercal + \mathbf{R}\\ \mathbf{K}_k =&\mathbf{P}_{k|k-1}\mathbf{C}^\intercal\mathbf{R}_{e,k}^{-1}.\end{aligned}$$

Notations:

- $\hat{\mathbf{x}}_{k|k}$: LMMSE estimate of $\mathbf{x}_k$ given $\{\mathbf{y}_0, \dots, \mathbf{y}_k\}$; $\hat{\mathbf{x}}_{k|k-1}$: LMMSE estimate of $\mathbf{x}_k$ given $\{\mathbf{y}_0, \dots, \mathbf{y}_{k-1}\}$
- $\tilde{\mathbf{x}}_{k|k} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k}$; same for $\tilde{\mathbf{x}}_{k|k-1}$
- $\mathbf{P}_{k|k} = \left\langle \tilde{\mathbf{x}}_{k|k}, \tilde{\mathbf{x}}_{k|k}\right\rangle = \mathbb{E}\left[\tilde{\mathbf{x}}_{k|k}\tilde{\mathbf{x}}_{k|k}^\intercal\right]$; same for $\mathbf{P}_{k|k-1}$

Derivation: Consider the $\mathbf{L}_{k-1} = \text{span}\{\mathbf{y}_0, \dots, \mathbf{y}_{k-1}\} = \text{span}\{\mathbf{e}_0, \dots, \mathbf{e}_{k-1}\}$.

1) (Innovation in span $\{\mathbf{y}\}$) The new dimension of span $\{\mathbf{y}\}$ is the direction of the innovation

$$\begin{aligned}\mathbf{e}_k =&\mathbf{y}_k - \text{proj}\left\{\mathbf{y}_k|\mathbf{L}_{k-1}\right\}\\ =&\mathbf{y}_k - \sum_{j=0}^{k-1}\langle \mathbf{y}_k, \mathbf{e}_j\rangle \|\mathbf{e}_j\|^{-2}\mathbf{e}_j\\ =&\mathbf{y}_k - \sum_{j=0}^{k-1}\langle \mathbf{C}\mathbf{x}_k + \mathbf{v}_k, \mathbf{e}_j\rangle \|\mathbf{e}_j\|^{-2}\mathbf{e}_j\\ =&\mathbf{y}_k - \mathbf{C}\underbrace{\sum_{j=0}^{k-1}\langle \mathbf{x}_k, \mathbf{e}_j\rangle \|\mathbf{e}_j\|^{-2}\mathbf{e}_j}_{\text{proj}\{\mathbf{x}_k|\mathbf{L}_{k-1}\}} - \underbrace{\sum_{j=0}^{k-1}\langle \mathbf{v}_k, \mathbf{e}_j\rangle \|\mathbf{e}_j\|^{-2}\mathbf{e}_j}_{=\mathbf{0}}\\ =&\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}.\end{aligned} \tag{10}$$

2) (Updated LMMSE estimation) The projection of $\mathbf{x}$ on this new measurement dimension is given by

$$\begin{aligned}\langle \mathbf{x}_k, \mathbf{e}_k\rangle &= \left\langle \mathbf{x}_k, \mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}\right\rangle\\ &= \left\langle \mathbf{x}_k, \mathbf{C}\tilde{\mathbf{x}}_{k|k-1}\right\rangle + \underbrace{\langle \mathbf{x}_k, \mathbf{v}_k\rangle}_{=0}\\ &= \left\langle \hat{\mathbf{x}}_{k|k-1} + \tilde{\mathbf{x}}_{k|k-1}, \mathbf{C}\tilde{\mathbf{x}}_{k|k-1}\right\rangle\\ &=\mathbf{P}_{k|k-1}\mathbf{C}^\intercal.\end{aligned} \tag{11}$$

An understanding is that if $\mathbf{P}_{k|k-1}$ is large, the $\mathbf{x}_k$ will have a large projection on the

innovation, i.e., the $\mathbf{x}_k$ cannot be well predicted. We have the complete projection as

$$\begin{aligned}\hat{\mathbf{x}}_{k|k} =& \sum_{j=0}^{k} \langle \mathbf{x}_k, \mathbf{e}_j \rangle \|\mathbf{e}_j\|^{-2} \mathbf{e}_j \\ =& \hat{\mathbf{x}}_{k|k-1} + \langle \mathbf{x}_k, \mathbf{e}_k \rangle \|\mathbf{e}_k\|^{-2} \mathbf{e}_k \\ =& \hat{\mathbf{x}}_{k|k-1} + \underbrace{\mathbf{P}_{k|k-1}\mathbf{C}^\intercal \mathbf{R}_{e,k}^{-1}}_{\mathbf{K}_k} \mathbf{e}_k,\end{aligned} \tag{12}$$

where the Kalman gain $\mathbf{K}_k$ can be interpreted as how many $\mathbf{e}_k$'s contribute to $\hat{\mathbf{x}}_{k|k}$. (recall that another interpretation is how much we should trust in the observation)

3) (Error covariance) In (12) we require $\mathbf{R}_{e,k}$, the error covariance matrix of the proj $\{\mathbf{y}_k|\mathbf{L}_{k-1}\}$ is

$$\begin{aligned}\mathbf{R}_{e,k} =& \langle \mathbf{e}_k, \mathbf{e}_k \rangle \\ =& \left\langle \mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1}, \mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_{k|k-1} \right\rangle \\ =& \left\langle \mathbf{C}\tilde{\mathbf{x}}_{k|k-1} + \mathbf{v}_k, \mathbf{C}\tilde{\mathbf{x}}_{k|k-1} + \mathbf{v}_k \right\rangle \\ =& \mathbf{C}\mathbf{P}_{k|k-1}\mathbf{C}^\intercal + \mathbf{R}.\end{aligned} \tag{13}$$

The error covariance of $\hat{\mathbf{x}}_{k|k}$ is

$$\begin{aligned}\mathbf{P}_{k|k} =& \left\langle \tilde{\mathbf{x}}_{k|k}, \tilde{\mathbf{x}}_{k|k} \right\rangle \\ =& \left\langle \tilde{\mathbf{x}}_{k|k-1} - \mathbf{K}_k\mathbf{e}_k, \tilde{\mathbf{x}}_{k|k-1} - \mathbf{K}_k\mathbf{e}_k \right\rangle \quad \text{(mentioned before)} \\ =& \mathbf{P}_{k|k-1} \underbrace{-\mathbf{K}_k \left\langle \mathbf{e}_k, \tilde{\mathbf{x}}_{k|k-1} \right\rangle - \left\langle \tilde{\mathbf{x}}_{k|k-1}, \mathbf{e}_k \right\rangle \mathbf{K}_k^\intercal + \mathbf{K}_k\mathbf{R}_{e,k}\mathbf{K}_k^\intercal}_{\text{same when plugging in}} \\ =& \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{P}_{k|k-1}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1} \mathbf{C}\mathbf{P}_{k|k-1}.\end{aligned} \tag{14}$$

4.1.2 Time update

Time update of Kalman filter

$$\begin{aligned}\hat{\mathbf{x}}_{k+1|k} =& \mathbf{A}\hat{\mathbf{x}}_{k|k}, \\ \mathbf{P}_{k+1|k} =& \mathbf{A}\mathbf{P}_{k|k}\mathbf{A}^\intercal + \mathbf{Q}.\end{aligned}$$

There is no new information, and thus we can only follows the state transition $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{w}_k$. Firstly, the LMMSE estimate $\hat{\mathbf{x}}_{k+1|k}$ is given by

$$\begin{aligned}\hat{\mathbf{x}}_{k+1|k} =& \text{proj}\left\{\mathbf{x}_{k+1}|\mathbf{L}_k\right\} \\ =& \text{proj}\left\{\mathbf{A}\mathbf{x}_k + \mathbf{w}_k|\mathbf{L}_k\right\} \\ =& \mathbf{A}\hat{\mathbf{x}}_{k|k}.\end{aligned} \tag{15}$$

The error covariance matrix of the estimate is

$$\begin{aligned}\mathbf{P}_{k+1|k} =& \left\langle \mathbf{A}\mathbf{x}_k + \mathbf{w}_k - \mathbf{A}\hat{\mathbf{x}}_{k|k}, \mathbf{A}\mathbf{x}_k + \mathbf{w}_k - \mathbf{A}\hat{\mathbf{x}}_{k|k} \right\rangle \\ =& \left\langle \mathbf{A}\tilde{\mathbf{x}}_{k|k} + \mathbf{w}_k, \mathbf{A}\tilde{\mathbf{x}}_{k|k} + \mathbf{w}_k \right\rangle \\ =& \mathbf{A}\mathbf{P}_{k|k}\mathbf{A}^\intercal + \mathbf{Q}.\end{aligned} \tag{16}$$

4.2 Properties of Kalman filter

4.2.1 Convergence

Here we introduce two operators to describe the transition of error covariance matrices:

$$\begin{aligned} h(\mathbf{X}) =& \mathbf{A}\mathbf{X}\mathbf{A}^\intercal + \mathbf{Q} && \text{(Lyapunov operator)} \\ \tilde{g}(\mathbf{X}) =& \mathbf{X} - \mathbf{X}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{X}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1} \mathbf{C}\mathbf{X} && \text{(Riccatti operator)} \\ =& \left(\mathbf{X}^{-1} + \mathbf{C}^\intercal \mathbf{R}^{-1}\mathbf{C}\right)^{-1} \end{aligned}$$

and then their composition[7]

$$\begin{aligned} g(\mathbf{X}) =& h \circ \tilde{g}(\mathbf{X}) \triangleq h\left(\tilde{g}(\mathbf{X})\right) && \text{(DARE)} \\ =& \mathbf{A}\mathbf{X}\mathbf{A}^\intercal + \mathbf{Q} - \mathbf{A}\mathbf{X}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{X}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1} \mathbf{C}\mathbf{X}\mathbf{A}^\intercal. \end{aligned}$$

With these operators, we can simplify the Kalman filter equations as

$$\mathbf{P}_{k-1|k-1} \xrightarrow{h} \mathbf{P}_{k|k-1} \xrightarrow{\tilde{g}} \mathbf{P}_{k|k} \xrightarrow{h} \mathbf{P}_{k+1|k} \xrightarrow{\tilde{g}} \cdots$$

$$\begin{aligned} \mathbf{P}_{k|k} =& \tilde{g} \circ h\left(\mathbf{P}_{k-1|k-1}\right) \\ \mathbf{P}_{k+1|k} =& h \circ \tilde{g}\left(\mathbf{P}_{k|k-1}\right) = g\left(\mathbf{P}_{k|k-1}\right). \end{aligned}$$

One convergence sign is

$$g(\mathbf{X}) = \mathbf{X},$$

which is called the discrete time algebraic Riccati equation (DARE). If $\mathbf{P}_{k|k} \to \bar{\mathbf{P}}$ as $k \to \infty$, then $\mathbf{P}_{k+1|k} \to \mathbf{P}^\star = h\left(\bar{\mathbf{P}}\right)$.

4.2.2 Packet drop

Packet drop case is the situation when some measurements $\mathbf{y}_k$'s are missing. It has the modified Kalman filter

$$\hat{\mathbf{x}}_{k|k} = \begin{cases} \mathbf{A}\hat{\mathbf{x}}_{k-1|k-1} & \text{if } \mathbf{y}_k \text{ is lost} \\ \mathrm{KF}\left(\hat{\mathbf{x}}_{k-1|k-1}\right) & \text{if } \mathbf{y}_k \text{ is received} \end{cases}$$

$$\mathbf{P}_{k|k} = \begin{cases} h\left(\mathbf{P}_{k-1|k-1}\right) & \text{if } \mathbf{y}_k \text{ is lost} \\ \tilde{g} \circ h\left(\mathbf{P}_{k-1|k-1}\right) & \text{if } \mathbf{y}_k \text{ is received.} \end{cases}$$

An operator

$$g_\lambda(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^\intercal + \mathbf{Q} - \lambda \cdot \mathbf{A}\mathbf{X}\mathbf{C}^\intercal \left[\mathbf{C}\mathbf{X}\mathbf{C}^\intercal + \mathbf{R}\right]^{-1} \mathbf{C}\mathbf{X}\mathbf{A}^\intercal$$

lies between h and g. It is roughly corresponding to $\mathbb{E}\left[\mathbf{P}_{k|k}\right]$ given the packet drop rate $(1-\lambda)$. $g_\lambda(\mathbf{X})$ has the following properties:

- (Non-decreasing) If $\mathbf{X} \geq \mathbf{Y} \geq \mathbf{0}$, then $g_\lambda(\mathbf{X}) \geq g_\lambda(\mathbf{Y})$.[8]
- (Concave) If $\lambda \in [0,1]$, then $g_\lambda(\alpha\mathbf{X} + (1-\alpha)\mathbf{Y}) \geq \alpha g_\lambda(\mathbf{X}) + (1-\alpha) g_\lambda(\mathbf{Y})$.
- (Better with higher receive rate) If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $g_{\lambda_1}(\mathbf{X}) \geq g_{\lambda_2}(\mathbf{X})$.

4.2.3 Multiple sensors

- (The more sensors, the better) For any $\mathbf{X} \geq \mathbf{0}$, $\tilde{g}_i(\mathbf{X}) \geq \tilde{g}(\mathbf{X}), i = 1, 2, \ldots$.

7. We are not interested in $\tilde{g}(h(\mathbf{X}))$ because of its complicated form.
8. The proof of the $0 < \lambda < 1$ case requires another operator $\phi(\mathbf{K}_x, \mathbf{X})$. See LECTURE 7 for details.

5 LECTURE 8

5.1 Optimal control problems

[9]There are two components of an optimal control problem. One is a system model, and the other one is a cost function (performance index).

1) **A general system model**:

$$\mathbf{x}_{k+1} = f\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right).$$

2) **Cost function**:

$$\begin{aligned} J =& g_N\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} g_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right) \\ =& \text{terminal state cost + per state cost.} \end{aligned}$$

The optimal control problem is to obtain the optimal control sequence

$$\left\{\mathbf{u}_0^\star, \mathbf{u}_1^\star, \ldots, \mathbf{u}_{N-1}^\star\right\},$$

so that the cost can be minimized.

5.1.1 Dynamic programming for optimal control

We define $\mathbf{U}_k \triangleq \{\mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{N-1}\}$ as the control sequences that start at time k. The optimal cost-to-go at time k (minimum cost starting at time k) is

$$\begin{aligned} J_k^\star\left(\mathbf{x}_k\right) \triangleq & \min_{\mathbf{U}_k}\left\{g_N\left(\mathbf{x}_N\right) + \sum_{i=k}^{N-1} g_i\left(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i\right)\right\} \\ =& \min_{\mathbf{u}_k} \min_{\mathbf{U}_{k+1}}\left\{g_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right) + \left[g_N\left(\mathbf{x}_N\right) + \sum_{i=k+1}^{N-1} g_i\left(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i\right)\right]\right\} \\ =& \min_{\mathbf{u}_k}\left\{g_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right) + \min_{\mathbf{U}_{k+1}}\left[g_N\left(\mathbf{x}_N\right) + \sum_{i=k+1}^{N-1} g_i\left(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i\right)\right]\right\} \\ =& \min_{\mathbf{u}_k}\left\{g_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right) + J_{k+1}^\star\left(\mathbf{x}_{k+1}\right)\right\}. \end{aligned}$$

The meaning is the optimal cost at $t = k$ equals to the optimal cost at $t = k + 1$ plus a state cost connecting $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$. This recursive relationship aligns to dynamic programming: the problem can be solved by solving a sequence of problems $J_{N-1}^\star, J_{N-2}^\star, \ldots, J_1^\star, J_0^\star$.

5.2 Linear quadratic regulator (LQR)

Here we consider a simple case with a linear system and a quadratic cost function[10]

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$

9. DYNAMIC PROGRAMMING, ME233, UCB: video, lecture note
10. LQR also allows time-variant parameters $\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}$.

$$\begin{aligned} J =& g_N\left(\mathbf{x}_N\right)+\sum_{k=0}^{N-1} g_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right) \\ =& \mathbf{x}_N^{\intercal} \mathbf{Q} \mathbf{x}_N+\sum_{k=0}^{N-1}\left(\mathbf{x}_k^{\intercal} \mathbf{Q} \mathbf{x}_k+\mathbf{u}_k^{\intercal} \mathbf{R} \mathbf{u}_k\right). \end{aligned}$$

5.2.1 From $J_N^\star$ to $J_{N-1}^\star$

At $t=N$, the optimal cost is

$$J_N^\star\left(\mathbf{x}_N\right)=\mathbf{x}_N^{\intercal} \mathbf{Q} \mathbf{x}_N=\mathbf{x}_N^{\intercal} \mathbf{P}_N \mathbf{x}_N,$$

where we introduce a new notation $\mathbf{P}_N=\mathbf{Q}$ and will explain it later.

At $t=N-1$, the optimal cost is[11]

$$\begin{aligned} J_{N-1}^\star\left(\mathbf{x}_{N-1}\right) =& \min_{\mathbf{u}_{N-1}}\left\{\mathbf{x}_N^{\intercal} \mathbf{Q} \mathbf{x}_N+\left[\mathbf{x}_{N-1}^{\intercal} \mathbf{Q} \mathbf{x}_{N-1}+\mathbf{u}_{N-1}^{\intercal} \mathbf{R} \mathbf{u}_{N-1}\right]\right\} \\ & \downarrow \text{(plug in } \mathbf{x}_N=\mathbf{A} \mathbf{x}_{N-1}+\mathbf{B} \mathbf{u}_{N-1}\text{)} \\ =& \min_{\mathbf{u}_{N-1}}\left\{\left[(\underline{\ldots})^{\intercal} \mathbf{Q}\left(\underline{\mathbf{A} \mathbf{x}_{N-1}+\mathbf{B} \mathbf{u}_{N-1}}\right)\right]+\left[\mathbf{x}_{N-1}^{\intercal} \mathbf{Q} \mathbf{x}_{N-1}+\mathbf{u}_{N-1}^{\intercal} \mathbf{R} \mathbf{u}_{N-1}\right]\right\} \\ & \downarrow \text{(optimal value of QP)} \\ =& \mathbf{x}_{N-1}^{\intercal}\left\{\mathbf{A}^{\intercal} \mathbf{Q} \mathbf{A}+\mathbf{Q}-(\underline{\ldots})^{\intercal}\left[\mathbf{B}^{\intercal} \mathbf{Q} \mathbf{B}+\mathbf{R}\right]^{-1} \underline{\mathbf{B}^{\intercal} \mathbf{Q} \mathbf{A}}\right\} \mathbf{x}_{N-1} \\ =& \mathbf{x}_{N-1}^{\intercal}\left\{\mathbf{A}^{\intercal} \mathbf{P}_N \mathbf{A}+\mathbf{Q}-(\underline{\ldots})^{\intercal}\left[\mathbf{B}^{\intercal} \mathbf{P}_N \mathbf{B}+\mathbf{R}\right]^{-1} \underline{\mathbf{B}^{\intercal} \mathbf{P}_N \mathbf{A}}\right\} \mathbf{x}_{N-1} \\ & \downarrow \text{(use } P_{N-1} \text{ to denote } \{\cdot\}\text{)} \\ =& \mathbf{x}_{N-1}^{\intercal} \mathbf{P}_{N-1} \mathbf{x}_{N-1}, \end{aligned}$$

and the optimal control (state-feedback law) is

$$\mathbf{u}_{N-1}^\star=-\mathbf{L}_{N-1} \mathbf{x}_{N-1}$$

$$\mathbf{L}_{N-1}=\left[\mathbf{B}^{\intercal} \mathbf{P}_N \mathbf{B}+\mathbf{R}\right]^{-1} \mathbf{B}^{\intercal} \mathbf{P}_N \mathbf{A}.$$

By introducing $\mathbf{P}$, we identify the following messages of LQR:

- Optimal costs are quadratic terms of $\mathbf{P}$.
- The optimal control at $\mathbf{u}_{N-1}^\star$ is just a state feedback control, parameterized by $\mathbf{L}_{N-1}\left(\mathbf{P}_N\right)$. Same for other moments.
- As $\mathbf{P}$ is irrelevant to $\mathbf{x}$, it can be prepared offline.

5.2.2 From $k+1$ to k

We can generalize the previous content in the following way.

Assume at $t=k+1$:

$$J_{k+1}^\star\left(\mathbf{x}_{k+1}\right)=\mathbf{x}_{k+1}^{\intercal} \mathbf{P}_{k+1} \mathbf{x}_{k+1},$$

and we can get

$$J_k^\star\left(\mathbf{x}_k\right)=\mathbf{x}_k^{\intercal} \mathbf{P}_k \mathbf{x}_k,$$

with Riccati equation

$$\mathbf{P}_k=\mathbf{A}^{\intercal} \mathbf{P}_{k+1} \mathbf{A}+\mathbf{Q}-(\underline{\ldots})^{\intercal}\left[\mathbf{B}^{\intercal} \mathbf{P}_{k+1} \mathbf{B}+\mathbf{R}\right]^{-1} \underline{\mathbf{B}^{\intercal} \mathbf{P}_{k+1} \mathbf{A}},$$

11. See Appendix for facts about quadratic functions.

and state-feedback law

$$\mathbf{u}_k^\star = -\left[\mathbf{B}^\intercal \mathbf{P}_{k+1}\mathbf{B} + \mathbf{R}\right]^{-1} \mathbf{B}^\intercal \mathbf{P}_{k+1}\mathbf{A}\mathbf{x}_k.$$

APPENDIX

Facts about quadratic functions

Consider the problem

$$f\left(\mathbf{u}\right) = \frac{1}{2}\mathbf{u}^\intercal \mathbf{M}\mathbf{u} + \mathbf{p}^\intercal \mathbf{u} + \mathbf{q},$$

its optimality condition is

$$\frac{\partial f}{\partial \mathbf{u}} = \mathbf{M}\mathbf{u}^\star + \mathbf{p} = \mathbf{0} \Rightarrow \mathbf{u}^\star = -\mathbf{M}^{-1}\mathbf{p},$$

and the optimal value is

$$f\left(\mathbf{u}^\star\right) = -\frac{1}{2}\mathbf{p}^\intercal \mathbf{M}^{-1}\mathbf{p} + \mathbf{q}.$$