MATH3013 Final Project

Student Name: Choi Wan Ioi

Student ID: DC011407

1 Introduction

Various numerical schemes for solving 1D heat equation, 2D heat equation and 1D wave equation with homogeneous Dirichlet boundary condition is implemented using C++. The error and stability of the schemes is analyzed using Taylor's Theorem and Von Neumann Analysis. The numerical solutions of the problems are visualized using gnuplot.

Code can be found on https://github.com/Michael1119/finalForMATH3013

2 Problem Statement

1. A rod has a uniform initial temperature of 100 Celsius. The length of the rod is 1 meter. The temperature of the 2 ends of the rod is suddenly decreased and maintained at 20 Celsius. How will the temperature change inside the rod?
2. A square plate has a uniform initial temperature of 100 Celsius. The length of each edge is 1 meter. The temperature of the 4 edges of the plate is suddenly decreased and maintained at 20 Celsius. How will the temperature change inside the plate?

5. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1,\ 0 < t;$$
$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$
$$u(x,0) = \sin 2\pi x, \quad 0 \le x \le 1,$$
$$\frac{\partial u}{\partial t}(x,0) = 2\pi \sin 2\pi x, \quad 0 \le x \le 1,$$

using Algorithm 12.4 with $h = 0.1$ and $k = 0.1$. Compare your results at $t = 0.3$ to the actual solution $u(x,t) = \sin 2\pi x(\cos 2\pi t + \sin 2\pi t)$.

This problem is from [1] p.764.

3 Numerical schemes for 1D heat equation

A 1D heat equation with homogeneous Dirichlet boundary condition is defined as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x) & 0 \le x \le L \end{cases}$$

To approximate $u(x,t)$ at $t = T$, choose integers n_x, n_t and define $\Delta x = \frac{L}{n_x}$, $\Delta T = \frac{T}{n_t}$, $x_j = j\Delta x$ for $0 \le j \le n_x$, $t_k = k\Delta t$ for $0 \le k \le n_t$ and $u_j^k = u(x_j, t_k)$.

3.1 Forward Euler (Forward Time Centered Space) scheme

Replace $\frac{du}{dt}(x_j, t_k)$ by $\frac{u_j^{k+1} - u_j^k}{\Delta t}$ and $\frac{d^2u}{dx^2}(x_j, t_k)$ by $\frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$$

$$u_j^{k+1} = \lambda u_{j-1}^k + (1 - 2\lambda) u_j^k + \lambda u_{j+1}^k \quad \text{where } \lambda = \frac{c^2 \Delta t}{\Delta x^2} > 0$$

3.1.1 Error analysis

$$u_j^{k+1} = u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2)$$

$$u_{j-1}^k = u_j^k - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

$$u_{j+1}^k = u_j^k + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

Substitute into $\frac{u_j^{k+1} - u_j^k}{\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2)$$

Therefore, the local truncation error of the Forward Euler scheme is $O(\Delta t + \Delta x^2)$.

3.1.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $u_j^{k+1} = \lambda u_{j-1}^k + (1 - 2\lambda)u_j^k + \lambda u_{j+1}^k$.

$$\alpha^{k+1} e^{i\beta j} = \lambda \alpha^k e^{i\beta(j-1)} + (1 - 2\lambda)\alpha^k e^{i\beta j} + \lambda \alpha^k e^{i\beta(j+1)}$$

$$\alpha = \lambda e^{-i\beta} + (1 - 2\lambda) + \lambda e^{i\beta} = (1 - 2\lambda) + 2\lambda \cos\beta = 1 - 2\lambda(1 - \cos\beta) = 1 - 4\lambda \sin^2\frac{\beta}{2}$$

The Forward Euler scheme is stable if $|\alpha| \leq 1$ for all $\beta \in \mathbb{R}$.

$|\alpha| \leq 1 \Longrightarrow 1 - 4\lambda \sin^2\frac{\beta}{2} \geq -1 \Longrightarrow \lambda \leq \frac{1}{2\sin^2\frac{\beta}{2}}$ for all $\beta \in \mathbb{R}$

Therefore, the Forward Euler scheme is stable if $\lambda \leq \frac{1}{2}$.

3.2 Backward Euler (Backward Time Centered Space) scheme

Replace $\frac{du}{dt}(x_j, t_k)$ by $\frac{u_j^k - u_j^{k-1}}{\Delta t}$ and $\frac{d^2u}{dx^2}(x_j, t_k)$ by $\frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{u_j^k - u_j^{k-1}}{\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$$

$$-\lambda u_{j-1}^k + (1 + 2\lambda)u_j^k - \lambda u_{j+1}^k = u_j^{k-1}$$

Backward Euler scheme is an implicit scheme and can be written as

$$\begin{bmatrix} 1+2\lambda & -\lambda & & & \\ -\lambda & 1+2\lambda & -\lambda & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & 1+2\lambda & -\lambda \\ & & & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{nx-2}^k \\ u_{nx-1}^k \end{bmatrix} = \begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{nx-2}^{k-1} \\ u_{nx-1}^{k-1} \end{bmatrix}$$

This tridiagonal system can be solved by tridiagonal matrix algorithm.

3.2.1 Error analysis

$$u_j^{k-1} = u_j^k - \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2)$$

$$u_{j-1}^k = u_j^k - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

$$u_{j+1}^k = u_j^k + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

Substitute into $\frac{u_j^k - u_j^{k-1}}{\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2)$$

Therefore, the local truncation error of the Backward Euler scheme is $O(\Delta t + \Delta x^2)$.

3.2.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $-\lambda u_{j-1}^k + (1+2\lambda)u_j^k - \lambda u_{j+1}^k = u_j^{k-1}$.

$$-\lambda \alpha^k e^{i\beta(j-1)} + (1+2\lambda)\alpha^k e^{i\beta j} - \lambda \alpha^k e^{i\beta(j+1)} = \alpha^{k-1} e^{i\beta j}$$

$$\alpha = \frac{1}{-\lambda e^{-i\beta} + (1+2\lambda) - \lambda e^{i\beta}} = \frac{1}{1 + 4\lambda \sin^2\frac{\beta}{2}} \leq 1 \text{ for all } \lambda > 0 \text{ and } \beta \in \mathbb{R}.$$

Therefore, the Backward Euler scheme is unconditionally stable.

3.3 Richardson scheme

Replace $\frac{du}{dt}(x_j, t_k)$ by $\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t}$ and $\frac{d^2u}{dx^2}(x_j, t_k)$ by $\frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$$

$$u_j^{k+1} = u_j^{k-1} + 2\lambda\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$$

This is an explicit 3-level scheme. u^k and u^{k-1} are required to compute u^{k+1}.

u^1 is computed from u^0 (initial condition) using Forward or Backward Euler scheme.

3.3.1 Error analysis

$$u_j^{k+1} = u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + O(\Delta t^3)$$

$$u_j^{k-1} = u_j^k - \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t}(x_j, t_k) + O(\Delta t^3)$$

$$u_{j-1}^k = u_j^k - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

$$u_{j+1}^k = u_j^k + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

Substitute into $\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2)$$

Therefore, the local truncation error of Richardson scheme is $O(\Delta t^2 + \Delta x^2)$.

3.3.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $u_j^{k+1} = u_j^{k-1} + 2\lambda\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$.

$$\alpha^{k+1} e^{i\beta j} = \alpha^{k-1} e^{i\beta j} + 2\lambda\left(\alpha^k e^{i\beta(j-1)} - 2\alpha^k e^{i\beta j} + \alpha^k e^{i\beta(j+1)}\right)$$

$$\alpha^2 = 1 + 2\lambda\left(\alpha e^{-i\beta} - 2\alpha + \alpha e^{i\beta}\right)$$

$$\alpha^2 - 2\lambda(2\cos\beta - 2)\alpha - 1 = 0$$

$$\alpha^2 + \left(8\lambda \sin^2\frac{\beta}{2}\right)\alpha - 1 = 0$$

$$\alpha_{1,2} = \frac{-8\lambda \sin^2\frac{\beta}{2} \pm \sqrt{64\lambda^2 \sin^4\frac{\beta}{2} + 4}}{2} = -4\lambda \sin^2\frac{\beta}{2} \pm \sqrt{16\lambda^2 \sin^4\frac{\beta}{2} + 1}$$

One can verify that $|\alpha_2| = \left|-4\lambda \sin^2\frac{\beta}{2} - \sqrt{16\lambda^2 \sin^4\frac{\beta}{2} + 1}\right| > 1$ for all $\lambda > 0$.

Therefore, the Richardson scheme is unconditionally unstable.

3.4 Dufort-Frankel scheme

The Richardson scheme can be made stable by replacing u_j^k by $\frac{1}{2}\left(u_j^{k+1} + u_j^{k-1}\right)$.

$$\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t} = c^2 \frac{u_{j-1}^k + u_{j+1}^k - u_j^{k+1} - u_j^{k-1}}{\Delta x^2}$$

$$u_j^{k+1} = \frac{1}{1+2\lambda}\left[2\lambda\left(u_{j-1}^k + u_{j+1}^k\right) + (1-2\lambda)u_j^{k-1}\right]$$

This is an explicit 3-level scheme. u^k and u^{k-1} are required to compute u^{k+1}.

u^1 is computed from u^0 (initial condition) using Forward or Backward Euler scheme.

3.4.1 Error analysis

$$u_j^{k+1} = u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + O(\Delta t^3)$$

$$u_j^{k-1} = u_j^k - \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + O(\Delta t^3)$$

$$u_{j-1}^k = u_j^k - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

$$u_{j+1}^k = u_j^k + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

Substitute into $\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t} = c^2 \frac{u_{j-1}^k + u_{j+1}^k - u_j^{k+1} - u_j^{k-1}}{\Delta x^2}$.

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2) = c^2 \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^4) - \Delta t^2 \frac{\partial u}{\partial t}(x_j, t_k) - O(\Delta t^3)}{\Delta x^2}$$

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2) - \frac{\Delta t^2}{\Delta x^2}\frac{\partial u}{\partial t}(x_j, t_k) - O\left(\frac{\Delta t^3}{\Delta x^2}\right)$$

$$\frac{\partial u}{\partial t}(x_j, t_k) + O(\Delta t^2) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2) + O\left(\frac{\Delta t^2}{\Delta x^2}\right)$$

Therefore, the local truncation error of the Dufort-Frankel scheme is $O\left(\Delta t^2 + \Delta x^2 + \frac{\Delta t^2}{\Delta x^2}\right)$.

Denote $r = \frac{\Delta t}{\Delta x}$, then the local truncation error becomes $O\left(\Delta t^2 + \Delta x^2 + \frac{\Delta t^2}{\Delta x^2}\right) =$

$O(r^2\Delta x^2 + \Delta x^2 + r^2) = O(\Delta x^2 + r^2)$.

3.4.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $u_j^{k+1} = \frac{1}{1+2\lambda}\left[2\lambda\left(u_{j-1}^k + u_{j+1}^k\right) + (1-2\lambda)u_j^{k-1}\right]$.

$$\alpha^{k+1}e^{i\beta j} = \frac{1}{1+2\lambda}\left[2\lambda\left(\alpha^k e^{i\beta(j-1)} + \alpha^k e^{i\beta(j+1)}\right) + (1-2\lambda)\alpha^{k-1}e^{i\beta j}\right]$$

$$\alpha^2 = \frac{1}{1+2\lambda}\left[2\lambda\left(\alpha e^{-i\beta} + \alpha e^{i\beta}\right) + (1-2\lambda)\right] = \frac{1}{1+2\lambda}\left[(4\lambda\cos\beta)\alpha + (1-2\lambda)\right]$$

$$(1+2\lambda)\alpha^2 - (4\lambda\cos\beta)\alpha - (1-2\lambda) = 0$$

$$\alpha_{1,2} = \frac{4\lambda\cos\beta \pm \sqrt{(4\lambda\cos\beta)^2 + 4(1+2\lambda)(1-2\lambda)}}{2(1+2\lambda)} = \frac{2\lambda\cos\beta \pm \sqrt{4\lambda^2\cos^2\beta + 1 - 4\lambda^2}}{1+2\lambda} = \frac{2\lambda\cos\beta \pm \sqrt{1-4\lambda^2\sin^2\beta}}{1+2\lambda}$$

Case 1: $1 - 4\lambda\sin^2\beta \geq 0$

$$\left|\alpha_{1,2}\right| \leq \frac{|2\lambda\cos\beta| + \left|\sqrt{1-4\lambda^2\sin^2\beta}\right|}{1+2\lambda} \leq \frac{2\lambda+1}{1+2\lambda} = 1 \text{ for all } \lambda > 0 \text{ and } \beta \in \mathbb{R}$$

Case 2: $1 - 4\lambda\sin^2\beta < 0$, so $\sqrt{1-4\lambda^2\sin^2\beta} = i\sqrt{4\lambda^2\sin^2\beta - 1}$

$$\left|\alpha_{1,2}\right|^2 = \left|\frac{2\lambda\cos\beta \pm i\sqrt{4\lambda^2\sin^2\beta - 1}}{1+2\lambda}\right|^2 = \left(\frac{2\lambda\cos\beta}{1+2\lambda}\right)^2 + \left(\frac{\sqrt{4\lambda^2\sin^2\beta-1}}{1+2\lambda}\right)^2 = \frac{4\lambda^2\cos^2\beta}{1+4\lambda+4\lambda^2} + \frac{4\lambda^2\sin^2\beta - 1}{1+4\lambda+4\lambda^2} =$$

$$\frac{4\lambda^2-1}{1+4\lambda+4\lambda^2} < 1 \text{ for all } \lambda > 0 \text{ and } \beta \in \mathbb{R}$$

Therefore, the Dufort-Frankel scheme is unconditionally stable.

3.5 Crank-Nicolson scheme

Crank-Nicolson scheme is a combination of the Forward and Backward Euler scheme.

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{c^2}{2}\left(\frac{u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}}{\Delta x^2} + \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}\right)$$

$$-\frac{\lambda}{2}u_{j-1}^{k+1} + (1+\lambda)u_j^{k+1} - \frac{\lambda}{2}u_{j+1}^{k+1} = \frac{\lambda}{2}u_{j-1}^k + (1-\lambda)u_j^k + \frac{\lambda}{2}u_{j+1}^k$$

Crank-Nicolson scheme is an implicit scheme and can be written as a tridiagonal system which can be solved by tridiagonal matrix algorithm.

3.5.1 Error analysis

The local truncation error of Crank-Nicolson scheme is $O(\Delta t^2 + \Delta x^2)$ by tedious calculation.

3.5.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $-\frac{\lambda}{2}u_{j-1}^{k+1} + (1+\lambda)u_j^{k+1} - \frac{\lambda}{2}u_{j+1}^{k+1} = \frac{\lambda}{2}u_{j-1}^k + (1-\lambda)u_j^k + \frac{\lambda}{2}u_{j+1}^k$

$$-\frac{\lambda}{2}\alpha^{k+1}e^{i\beta(j-1)} + (1+\lambda)\alpha^{k+1}e^{i\beta j} - \frac{\lambda}{2}\alpha^{k+1}e^{i\beta(j+1)} = \frac{\lambda}{2}\alpha^k e^{i\beta(j-1)} + (1-\lambda)\alpha^k e^{i\beta j} + \frac{\lambda}{2}\alpha^k e^{i\beta(j+1)}$$

$$\left(-\frac{\lambda}{2}e^{-i\beta} + (1+\lambda) - \frac{\lambda}{2}e^{i\beta}\right)\alpha = \frac{\lambda}{2}e^{-i\beta} + (1-\lambda) + \frac{\lambda}{2}e^{i\beta}$$

$$\alpha = \frac{\frac{\lambda}{2}e^{-i\beta} + (1-\lambda) + \frac{\lambda}{2}e^{i\beta}}{-\frac{\lambda}{2}e^{-i\beta} + (1+\lambda) - \frac{\lambda}{2}e^{i\beta}} = \frac{(1-\lambda) + \lambda\cos\beta}{(1+\lambda) - \lambda\cos\beta} = \frac{1 - 2\lambda\sin^2\frac{\beta}{2}}{1 + 2\lambda\sin^2\frac{\beta}{2}} \leq 1 \text{ for } \lambda > 0 \text{ and } \beta \in \mathbb{R}$$

Therefore, the Crank-Nicolson scheme is unconditionally stable.

4 Numerical schemes for 2D heat equation

A 2D heat equation with square and homogeneous Dirichlet boundary condition is defined as follows:

$$\begin{cases} \frac{\partial u}{\partial t}(x,y,t) = c^2\left[\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t)\right] & (x,y)\in\Omega \quad t>0 \\ u(x,y,t)=0 & (x,y)\in\partial\Omega \\ u(x,y,0)=f(x,y) & (x,y)\in\Omega \end{cases}$$

Where Ω is the square $(0,L)\times(0,L)$ and $\partial\Omega$ is the boundary of Ω.

To approximate $u(x,y,t)$ at $t=T$, choose integers n_x, n_y, n_t and define $\Delta x = \frac{L}{n_x}$, $\Delta y = \frac{L}{n_y}$, $\Delta T = \frac{T}{n_t}$, $x_p = p\Delta x$ for $0\le p\le n_x$, $y_q = q\Delta y$ for $0\le q\le n_y$, $t_k = k\Delta t$ for $0\le k\le n_t$ and $u_{p,q}^k = u(p,q,k)$.

4.1 Forward Euler (Forward Time Centered Space) scheme

Replace $\frac{\partial u}{\partial t}(x_p,y_q,t_k)$ by $\frac{u_{p,q}^{k+1}-u_{p,q}^k}{\Delta t}$, $\frac{\partial^2 u}{\partial x^2}(x_p,y_q,t_k)$ by $\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2}$ and $\frac{\partial^2 u}{\partial y^2}(x_p,y_q,t_k)$ by $\frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}$.

$$\frac{u_{p,q}^{k+1}-u_{p,q}^k}{\Delta t} = c^2\left(\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2} + \frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}\right)$$

$$u_{p,q}^{k+1} = (1-2\lambda_x-2\lambda_y)u_{p,q}^k + \lambda_x\left(u_{p-1,q}^k+u_{p+1,q}^k\right) + \lambda_y\left(u_{p,q-1}^k+u_{p,q+1}^k\right)$$

Where $\lambda_x = \frac{c^2\Delta t}{\Delta x^2}$ and $\lambda_y = \frac{c^2\Delta t}{\Delta y^2}$

4.1.1 Error analysis

$$u_{p,q}^{k+1} = u_{p,q}^k + \Delta t\frac{\partial u}{\partial t}(x_p,y_q,t_k) + O(\Delta t^2)$$

$$u_{p-1,q}^k = u_{p,q}^k - \Delta x\frac{\partial u}{\partial x}(x_p,y_q,t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_p,y_q,t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_p,y_q,t_k) + O(\Delta x^4)$$

$$u_{p+1,q}^k = u_{p,q}^k + \Delta x\frac{\partial u}{\partial x}(x_p,y_q,t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_p,y_q,t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_p,y_q,t_k) + O(\Delta x^4)$$

$$u_{p,q-1}^k = u_{p,q}^k - \Delta y\frac{\partial u}{\partial y}(x_p,y_q,t_k) + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2}(x_p,y_q,t_k) - \frac{\Delta y^3}{6}\frac{\partial^3 u}{\partial y^3}(x_p,y_q,t_k) + O(\Delta y^4)$$

$$u_{p,q+1}^k = u_{p,q}^k + \Delta y\frac{\partial u}{\partial y}(x_p,y_q,t_k) + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2}(x_p,y_q,t_k) + \frac{\Delta y^3}{6}\frac{\partial^3 u}{\partial y^3}(x_p,y_q,t_k) + O(\Delta y^4)$$

Substitute into $\frac{u_{p,q}^{k+1}-u_{p,q}^k}{\Delta t} = c^2\left(\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2} + \frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}\right)$

$$\frac{\partial u}{\partial t}(x_p,y_q,t_k) + O(\Delta t) = c^2\left(\frac{\partial^2 u}{\partial x^2}(x_p,y_q,t_k) + O(\Delta x^2) + \frac{\partial^2 u}{\partial y^2}(x_p,y_q,t_k) + O(\Delta y^2)\right)$$

Therefore, the local truncation error of the Forward Euler scheme is $O(\Delta t + \Delta x^2 + \Delta y^2)$.

4.1.2 Stability analysis

Substitute $u(x_p,y_q,t_k) = \alpha^k e^{i(\beta p+\gamma q)}$ into $u_{p,q}^{k+1} = (1-2\lambda_x-2\lambda_y)u_{p,q}^k + \lambda_x\left(u_{p-1,q}^k+u_{p+1,q}^k\right) + \lambda_y\left(u_{p,q-1}^k+u_{p,q+1}^k\right)$

$\alpha^{k+1}e^{i(\beta p+\gamma q)} = (1-2\lambda_x-2\lambda_y)\alpha^k e^{i(\beta p+\gamma q)} + \lambda_x\left(\alpha^k e^{i(\beta p-\beta+\gamma q)} + \alpha^k e^{i(\beta p+\beta+\gamma q)}\right) +$

$\lambda_y\left(\alpha^k e^{i(\beta p+\gamma q-\gamma)}+\alpha^k e^{i(\beta p+\gamma q+\gamma)}\right)$

$\alpha=\left(1-2\lambda_x-2\lambda_y\right)+\lambda_x\left(e^{-i\beta}+e^{i\beta}\right)+\lambda_y\left(e^{-i\gamma}+e^{i\gamma}\right)=\left(1-2\lambda_x-2\lambda_y\right)+$

$2\lambda_x\cos\beta+2\lambda_y\cos\gamma=1-4\lambda_x\sin^2\frac{\beta}{2}-4\lambda_y\sin^2\frac{\gamma}{2}$

It can be shown that $|\alpha|\leq 1$ for all $\beta,\gamma\in\mathbb{R}$ when $\lambda_x+\lambda_y\leq\frac{1}{2}$.

Therefore, the Forward Euler scheme is stable when $\lambda_x+\lambda_y\leq\frac{1}{2}$.

4.2 Backward Euler (Forward Time Centered Space) scheme

Replace $\frac{\partial u}{\partial t}\left(x_p,y_q,t_k\right)$ by $\frac{u_{p,q}^k-u_{p,q}^{k-1}}{\Delta t}$, $\frac{\partial^2 u}{\partial x^2}\left(x_p,y_q,t_k\right)$ by $\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2}$ and $\frac{\partial^2 u}{\partial y^2}\left(x_p,y_q,t_k\right)$ by $\frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}$.

$$\frac{u_{p,q}^k-u_{p,q}^{k-1}}{\Delta t}=c^2\left(\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2}+\frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}\right)$$

$$\left(1+2\lambda_x+2\lambda_y\right)u_{p,q}^k-\lambda_x\left(u_{p-1,q}^k+u_{p+1,q}^k\right)-\lambda_y\left(u_{p,q-1}^k+u_{p,q+1}^k\right)=u_{p,q}^{k-1}$$

Backward Euler scheme is an implicit scheme and can be written as a block tridiagonal matrix. The block tridiagonal system can be solved using conjugate gradient.

4.2.1 Error analysis

Using similar technique for the error analysis of the Forward Euler scheme, it can be shown that the local truncation error of the Backward Euler scheme is $O(\Delta t+\Delta x^2+\Delta y^2)$.

4.2.2 Stability analysis

Using similar technique for the error analysis of the Forward Euler scheme, it can be shown that $\alpha=\frac{1}{1+4\lambda_x\sin^2\frac{\beta}{2}+4\lambda_y\sin^2\frac{\gamma}{2}}\leq 1$ for all $\beta,\gamma\in\mathbb{R}$.

Therefore, the Backward Euler scheme is unconditionally stable.

4.3 Crank-Nicolson scheme

Crank-Nicolson scheme is a combination of the Forward and Backward Euler scheme.

$$\frac{u_{p,q}^{k+1}-u_{p,q}^k}{\Delta t}=\frac{c^2}{2}\left[\left(\frac{u_{p-1,q}^{k+1}-2u_{p,q}^{k+1}+u_{p+1,q}^{k+1}}{\Delta x^2}+\frac{u_{p,q-1}^{k+1}-2u_{p,q}^{k+1}+u_{p,q+1}^{k+1}}{\Delta y^2}\right)+\left(\frac{u_{p-1,q}^k-2u_{p,q}^k+u_{p+1,q}^k}{\Delta x^2}+\frac{u_{p,q-1}^k-2u_{p,q}^k+u_{p,q+1}^k}{\Delta y^2}\right)\right]$$

$$\left(1+\lambda_x+\lambda_y\right)u_{p,q}^{k+1}-\frac{\lambda_x}{2}\left(u_{p-1,q}^{k+1}+u_{p+1,q}^{k+1}\right)-\frac{\lambda_y}{2}\left(u_{p,q-1}^{k+1}+u_{p,q+1}^{k+1}\right)=\left(1-\lambda_x-\lambda_y\right)u_{p,q}^k+\frac{\lambda_x}{2}\left(u_{p-1,q}^k+u_{p+1,q}^k\right)+\frac{\lambda_y}{2}\left(u_{p,q-1}^k+u_{p,q+1}^k\right)$$

Crank-Nicolson scheme is an implicit scheme and can be written as a block tridiagonal matrix. The block tridiagonal system can be solved using conjugate gradient method.

4.3.1 Error analysis

Using similar technique for the error analysis of the Forward Euler scheme, it can be shown that the local truncation error of the Crank-Nicolson scheme is $O(\Delta t^2+\Delta x^2+\Delta y^2)$.

4.3.2 Stability analysis

Using similar technique for the error analysis of the Forward Euler scheme, it can be shown that $\alpha = \frac{1-2\lambda_x \sin^2\frac{\beta}{2} - 2\lambda_y \sin^2\frac{\gamma}{2}}{1+2\lambda_x \sin^2\frac{\beta}{2} + 2\lambda_y \sin^2\frac{\gamma}{2}} \leq 1$ for all $\beta, \gamma \in \mathbb{R}$.

Therefore, the Crank-Nicolson scheme is unconditionally stable.

4.4 Alternating-direction Implicit (ADI) scheme

A disadvantage of the Crank-Nicolson scheme is that the direct solution of the block tridiagonal system is costly. ADI scheme split the finite difference equations into two, one with the x-derivative taken implicitly and the next with the y-derivative taken implicitly.

$$\frac{u_{p,q}^{k+1/2} - u_{p,q}^{k}}{\Delta t} = \frac{c^2}{2}\left(\frac{u_{p-1,q}^{k+1/2} - 2u_{p,q}^{k+1/2} + u_{p+1,q}^{k+1/2}}{\Delta x^2} + \frac{u_{p,q-1}^{k} - 2u_{p,q}^{k} + u_{p,q+1}^{k}}{\Delta y^2}\right)$$

$$\frac{u_{p,q}^{k+1} - u_{p,q}^{k+1/2}}{\Delta t} = \frac{c^2}{2}\left(\frac{u_{p-1,q}^{k+1/2} - 2u_{p,q}^{k+1/2} + u_{p+1,q}^{k+1/2}}{\Delta x^2} + \frac{u_{p,q-1}^{k+1} - 2u_{p,q}^{k+1} + u_{p,q+1}^{k+1}}{\Delta y^2}\right)$$

Notice that the 2 equations have almost the same form as the 1D Crank-Nicolson scheme and therefore can be solved by tridiagonal matrix algorithm instead of conjugate gradient method.

4.4.1 Error analysis

It is shown in [2] that the local truncation error of the ADI scheme is $O(\Delta t^2 + \Delta x^2 + \Delta y^2)$.

4.4.2 Stability analysis

It is shown in [2] that the ADI scheme is unconditionally stable.

5 Numerical schemes for 1D wave equation

A 1D wave equation with homogeneous Dirichlet boundary condition is defined as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x) & 0 \le x \le L \\ \frac{\partial u}{\partial t}(x,0) = g(x) & 0 \le x \le L \end{cases}$$

To approximate $u(x,t)$ at $t = T$, choose integers n_x, n_t and define $\Delta x = \frac{L}{n_x}$, $\Delta T = \frac{T}{n_t}$, $x_j = j\Delta x$ for $0 \le j \le n_x$, $t_k = k\Delta t$ for $0 \le k \le n_t$ and $u_j^k = u(x_j, t_k)$.

5.1 Explicit Finite Difference method

Replace $\frac{\partial^2 u}{\partial t^2}(x_j, t_k)$ by $\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{\Delta t^2}$ and $\frac{\partial^2 u}{\partial x^2}(x_j, t_k)$ by $\frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$.

$$\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{\Delta t^2} = c^2 \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2}$$

$u_j^{k+1} = 2(1-\lambda^2)u_j^k + \lambda^2(u_{j-1}^k + u_{k+1}^k) - u_j^{k-1}$ where $\lambda = \frac{c\Delta t}{\Delta x}$

Notice that u^k and u^{k-1} is required to compute u^{k+1}.

$$f(x_{j-1}) = f(x_j) - \Delta x f'(x_j) + \frac{\Delta x^2}{2} f''(x_j) - \frac{\Delta x^3}{6} f'''(x_j) + O(\Delta x^4)$$

$$f(x_{j+1}) = f(x_j) + \Delta x f'(x_j) + \frac{\Delta x^2}{2} f''(x_j) + \frac{\Delta x^3}{6} f'''(x_j) + O(\Delta x^4)$$

$u_j^1 = u_j^0 + \Delta t \frac{\partial u}{\partial t}(x_j, 0) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, 0) + O(\Delta t^3) = f(x_j) + \Delta t g(x_j) + \frac{\Delta t^2}{2} c^2 \frac{\partial^2 u}{\partial x^2}(x, 0) + O(\Delta t^3) = f(x_j) + \Delta t g(x_j) + \frac{\Delta t^2}{2} c^2 f''(x_j) + O(\Delta t^3) = f(x_j) + \Delta t g(x_j) + \frac{c^2 \Delta t^2}{2} \left(\frac{f(x_{j-1}) - 2f(x_j) + f(x_{j+1})}{\Delta x^2} + O(\Delta x^2) \right) + O(\Delta t^3) = f(x_j) + \Delta t g(x_j) + \frac{c^2 \Delta t^2}{2\Delta x^2} [f(x_{j-1}) - 2f(x_j) + f(x_{j+1})] + O(\Delta t^2 \Delta x^2) + O(\Delta t^3) = f(x_j) + \Delta t g(x_j) + \frac{\lambda^2}{2} [f(x_{j-1}) - 2f(x_j) + f(x_{j+1})] + O(\Delta t^2 \Delta x^2 + \Delta t^3)$

Therefore, u^1 can be computed by $f(x_j) + \Delta t g(x_j) + \frac{\lambda^2}{2} [f(x_{j-1}) - 2f(x_j) + f(x_{j+1})]$.

5.1.1 Error analysis

$$u_j^{k+1} = u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_k) + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}(x_j, t_k) + O(\Delta t^4)$$

$$u_j^{k-1} = u_j^k - \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_k) - \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}(x_j, t_k) + O(\Delta t^4)$$

$$u_{j-1}^k = u_j^k - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2}(x_j, t_k) - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

$$u_{j+1}^k = u_j^k + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3}(x_j, t_k) + O(\Delta x^4)$$

Substitute into $\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{\Delta t^2} = c^2 \frac{u_{j-1}^k-2u_j^k+u_{j+1}^k}{\Delta x^2}$.

$$\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + O(\Delta t^2) = c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + O(\Delta x^2)$$

Therefore, the local truncation error of the Explicit Finite Difference method is $O(\Delta t^2 + \Delta x^2)$.

5.1.2 Stability analysis

Substitute $u_j^k = \alpha^k e^{i\beta j}$ into $u_j^{k+1} = 2(1-\lambda^2)u_j^k + \lambda^2(u_{j-1}^k + u_{k+1}^k) - u_j^{k-1}$.

$$\alpha^{k+1}e^{i\beta j} = 2(1-\lambda^2)\alpha^k e^{i\beta j} + \lambda^2\left(\alpha^k e^{i\beta(j-1)} + \alpha^k e^{i\beta(j+1)}\right) - \alpha^{k-1}e^{i\beta j}$$

$$\alpha^2 = 2(1-\lambda^2)\alpha + \lambda^2\left(e^{-i\beta} + e^{i\beta}\right)\alpha - 1 = 2(1-\lambda^2)\alpha + (2\lambda^2\cos\beta)\alpha - 1$$

$$\alpha^2 + (2\lambda^2 - 2\lambda^2\cos\beta - 2)\alpha + 1 = 0$$

$$\alpha^2 + \left(4\lambda^2\sin^2\frac{\beta}{2} - 2\right)\alpha + 1 = 0$$

$$\alpha_{1,2} = \frac{\left(4\lambda^2\sin^2\frac{\beta}{2}-2\right) \pm \sqrt{\left(4\lambda^2\sin^2\frac{\beta}{2}-2\right)^2 - 4}}{2} = \left(2\lambda^2\sin^2\frac{\beta}{2} - 1\right) \pm \sqrt{\left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2 - 1}$$

If $2\lambda^2\sin^2\frac{\beta}{2} - 1 > 1$ then $|\alpha_1| > 1$ and if $2\lambda^2\sin^2\frac{\beta}{2} - 1 < -1$ then $|\alpha_2| > 1$.

If $\left|2\lambda^2\sin^2\frac{\beta}{2} - 1\right| \le 1$ then $\sqrt{\left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2 - 1} = i\sqrt{1 - \left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2}$

$$\left|\alpha_{1,2}\right|^2 = \left|\left(2\lambda^2\sin^2\frac{\beta}{2} - 1\right) \pm i\sqrt{1 - \left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2}\right|^2 = \left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2 + 1 - \left(2\lambda^2\sin^2\frac{\beta}{2}-1\right)^2 = 1$$

$\left|2\lambda^2\sin^2\frac{\beta}{2} - 1\right| \le 1$ for all $\beta \in \mathbb{R} \Longrightarrow \lambda^2 \le 1$

Therefore, the Explicit Finite Difference method is stable when $\lambda \le 1$.

5.2 Implicit Finite Difference

This implicit scheme is introduced in [3] p.199.

$$\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{\Delta t^2} = c^2\left[\frac{1}{4}\left(\frac{u_{j-1}^{k+1}-2u_j^{k+1}+u_{j+1}^{k+1}}{\Delta x^2}\right) + \frac{1}{2}\left(\frac{u_{j-1}^{k}-2u_j^{k}+u_{j+1}^{k}}{\Delta x^2}\right) + \frac{1}{4}\left(\frac{u_{j-1}^{k-1}-2u_j^{k-1}+u_{j+1}^{k-1}}{\Delta x^2}\right)\right]$$

$$-\frac{\lambda^2}{4}u_{j-1}^{k+1} + \left(1+\frac{\lambda^2}{2}\right)u_j^{k+1} - \frac{\lambda^2}{4}u_{j+1}^{k+1} = \frac{\lambda^2}{2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right) + \frac{\lambda^2}{4}\left(u_{j-1}^{k-1} - 2u_j^{k-1} + u_{j+1}^{k-1}\right) + 2u_j^k - u_j^{k-1}$$

u^1 can be computed by $f(x_j) + \Delta t g(x_j) + \frac{\lambda^2}{2}\left[f(x_{j-1}) - 2f(x_j) + f(x_{j+1})\right]$.

This difference equation can be written as a tridiagonal system and can be solved by tridiagonal matrix algorithm.

Error analysis is omitted as the calculation is very tedious.

It can be shown that this scheme is unconditionally stable.

6 Summary

Equation	Scheme	Local Truncation Error	Stability	Algorithm
1D Heat	FTCS	$O(\Delta t + \Delta x^2)$	$c^2\Delta t/\Delta x^2 \leq 0.5$	
	BTCS	$O(\Delta t + \Delta x^2)$	Stable	Tridiag
	Richardson	$O(\Delta t^2 + \Delta x^2)$	Unstable	
	Dufort-Frankel	$O(\Delta t^2 + \Delta x^2 + \Delta t^2/\Delta x^2)$	Stable	
	Crank-Nicolson	$O(\Delta t^2 + \Delta x^2)$	Stable	Tridiag
2D Heat	FTCS	$O(\Delta t + \Delta x^2 + \Delta y^2)$	$c^2\Delta t/\Delta x^2 + c^2\Delta t/\Delta y^2 \leq 0.5$	
	BTCS	$O(\Delta t + \Delta x^2 + \Delta y^2)$	Stable	Conjgrad
	Crank-Nicolson	$O(\Delta t^2 + \Delta x^2 + \Delta y^2)$	Stable	Conjgrad
	ADI	$O(\Delta t^2 + \Delta x^2 + \Delta y^2)$	Stable	Tridiag
1D Wave	Explicit FD	$O(\Delta t^2 + \Delta x^2)$	$c\Delta t/\Delta x \leq 1$	
	Implicit FD		Stable	Tridiag

7 Reference

[1] Burden, R. L., Faires, J. D., & Burden, A. M. (2015). *Numerical analysis*. Cengage learning.

[2] https://web.archive.org/web/20061127231320/http://www.cs.rpi.edu/~flaherje/pdf/pde5.pdf

[3] Mitchell, A. R., & Griffiths, D. F. (1980). The finite difference method in partial differential equations. *A Wiley-Interscience Publication*.