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The code and figures included in this report can be found on:

https://github.com/Michael1119/midtermForMATH3005

1. Problem 1
   1. Derivation

Consider an initial value problem .

From page 58, an explicit Runge-Kutta formula with 2 stages has the following form:

This can be rewritten as

To derive a method of order 2, the truncation error is required to be :

Where .

Denote , and .

Then and .

By Taylor’s theorem,

Therefore

implies

If , then .

If , then .

The numerical scheme and Butcher tableu for the second-order Runge-Kutta method with is

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

The numerical scheme and Butcher tableu for the second-order Runge-Kutta method with is

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

* 1. Theoretical properties

From page 63, where .

Therefore , i.e, the order of convergence of the numerical scheme is 2.

1. Problem 2

The exact solution to the initial value problem is .

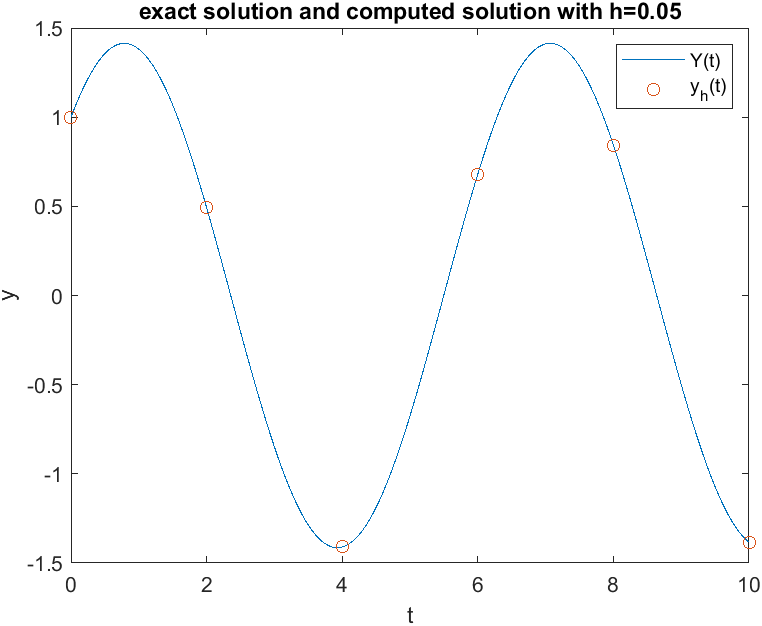
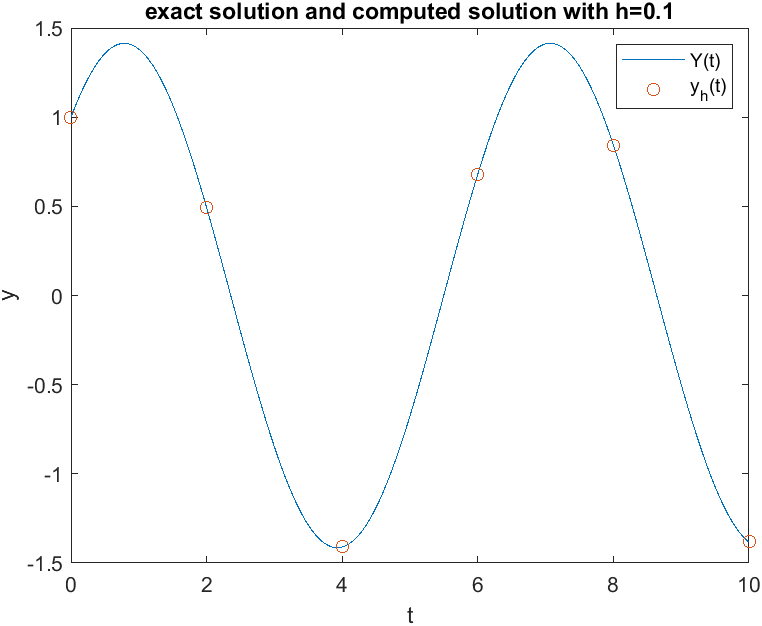
Let be the computed solution with step size .

Denote the error between the exact solution and the computed solution with step size as .

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| 0.1 | 2 | 0.49152 | 1.6266e-03 |
| 4 | -1.4092 | 1.2952e-03 |
| 6 | 0.6813 | 5.4062e-04 |
| 8 | 0.84211 | 1.7463e-03 |
| 10 | -1.3822 | 9.1265e-04 |
| 0.05 | 2 | 0.49276 | 3.9241e-04 |
| 4 | -1.4101 | 3.1968e-04 |
| 6 | 0.68088 | 1.2540e-04 |
| 8 | 0.84343 | 4.2418e-04 |
| 10 | -1.3829 | 2.2762e-04 |

The exact solution and computed solution are shown in the following figure.

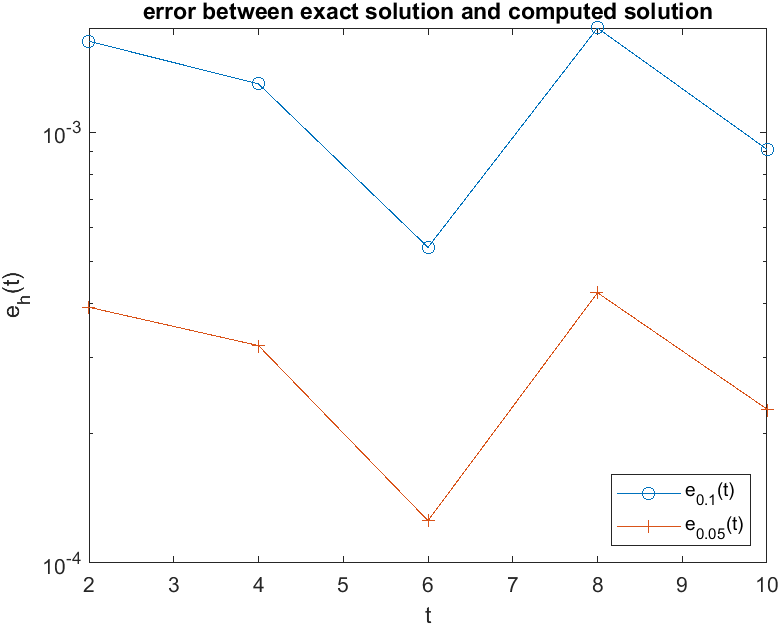
It can be seen that the computed solution is an accurate estimate of the exact solution.



The error between the exact solution and computed solution are shown in the following semi-log plot.

It can be seen that the error between the exact solution and computed solution decreases as step size decreases from 0.1 to 0.05.

Moreover, the error between the exact solution and computed solution does not increase over time, which suggests that the numerical scheme is stable for this initial value problem.



Section 1.2 implies , so the expected value of is .

The actual values of is shown in the following table.

It can be seen that the values are quite close to 4, which agrees with the theoretical results.

|  |  |
| --- | --- |
|  |  |
| 2 | 4.1452 |
| 4 | 4.0517 |
| 6 | 4.311 |
| 8 | 4.1168 |
| 10 | 4.0095 |

Alternatively, one can use log-log plot to find the order of convergence of the numerical scheme.

Suppose , i.e., the order of convergence of the numerical scheme is , then , so the function appears as a straight line with slope and intercept in the log-log plot.

To estimate the order of convergence of the numerical scheme, one can find the slope of the least square regression line .

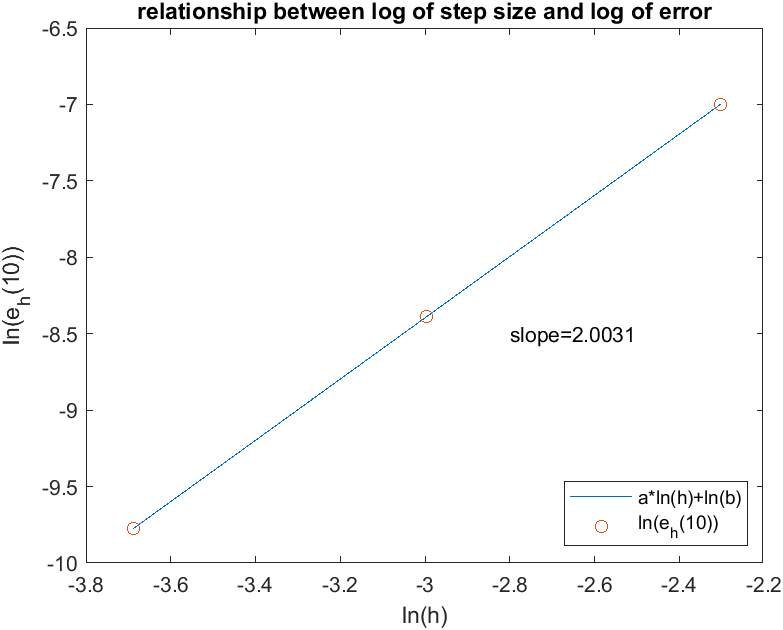
The slope and intercept of the least square regression line is found by solving the least square problem:

Where , and

It can be shown that satisfies the linear system .

By solving the linear system, one can obtain which agrees with the theoretical results.

The errors and the regression line are shown in the following figure.



We compare the computed solution to those on page 57.

The average absolute error of the computed solution when and are 1.2243e-03 and 2.9786e-04 respectively.

On page 57, the average absolute error when and are 1.8296e-03 and 4.4802e-04 respectively.

This shows that the computed solution when are more accurate than those on page 57.

1. Problem 3
   1. Question (a)

Let the exact solution and the computed solution be and respectively.

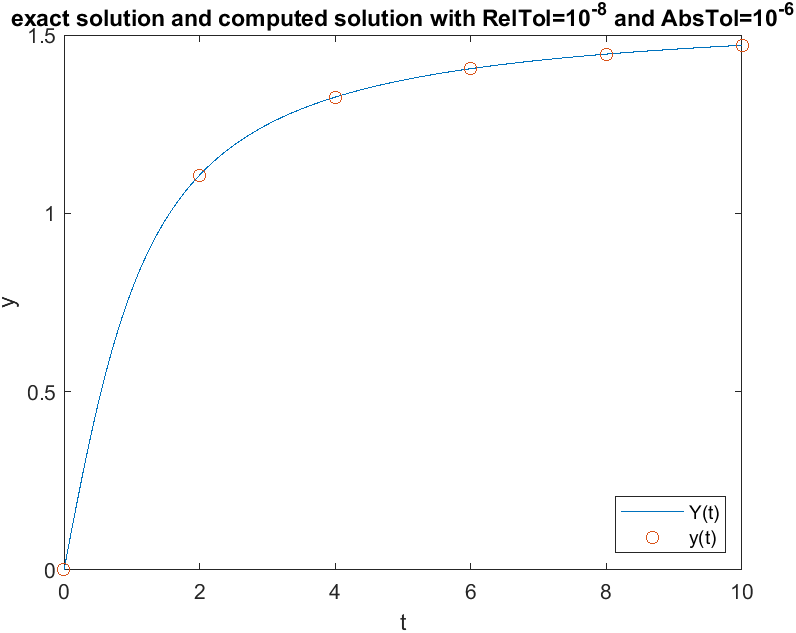
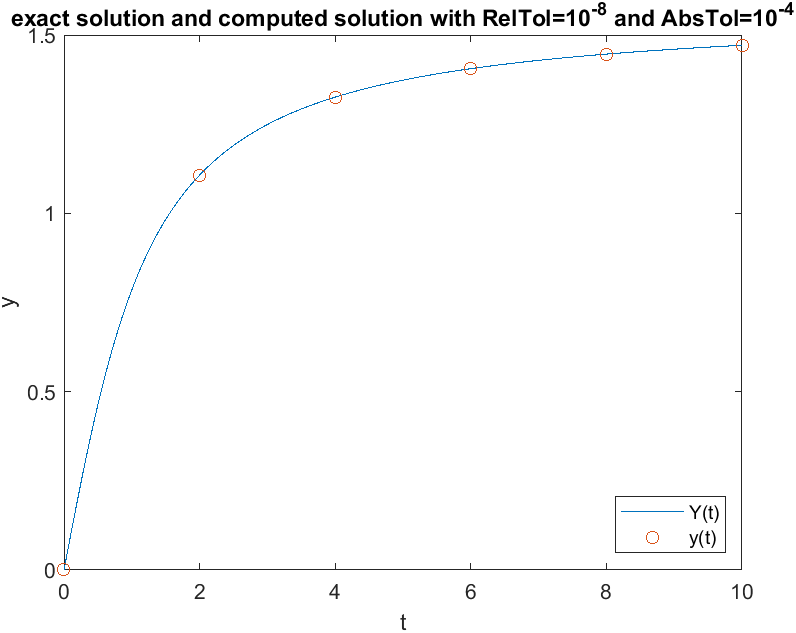
Denote the error between the exact solution and the computed solution as .

Then .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| RelTol | AbsTol |  |  |  |
|  |  | 2 | 1.1071 | 1.4910e-05 |
| 4 | 1.3258 | 2.7787e-05 |
| 6 | 1.4056 | 1.5064e-05 |
| 8 | 1.4464 | 8.6718e-06 |
| 10 | 1.4711 | 5.5854e-06 |
|  | 2 | 1.1071 | 1.3400e-07 |
| 4 | 1.3258 | 9.6140e-08 |
| 6 | 1.4056 | 1.8570e-08 |
| 8 | 1.4464 | 1.5654e-07 |
| 10 | 1.4711 | 1.4887e-07 |

The exact solution and computed solution are shown in the following figure.

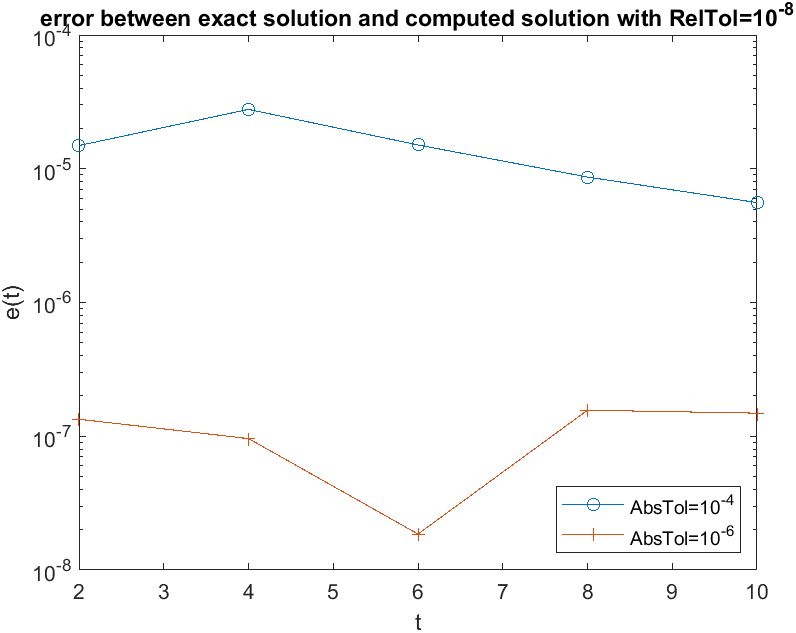
It can be seen that the computed solution is an accurate estimate of the exact solution.



The error between the exact solution and computed solution are shown in the following semi-log plot.

It can be seen that the error between the exact solution and computed solution decreases as the absolute tolerance decreases from to .

Moreover, the error between the exact solution and computed solution does not increase over time, which suggests that the ode45 method is stable for this initial value problem.

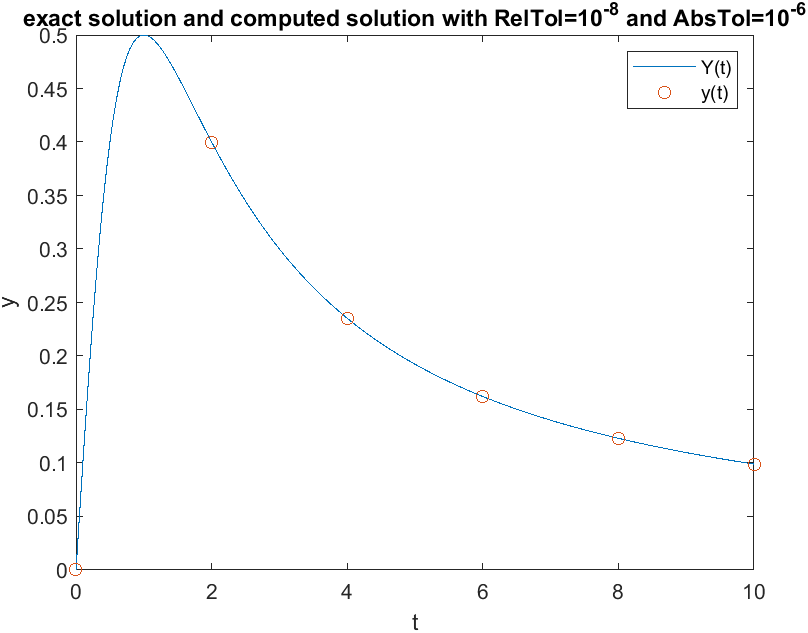
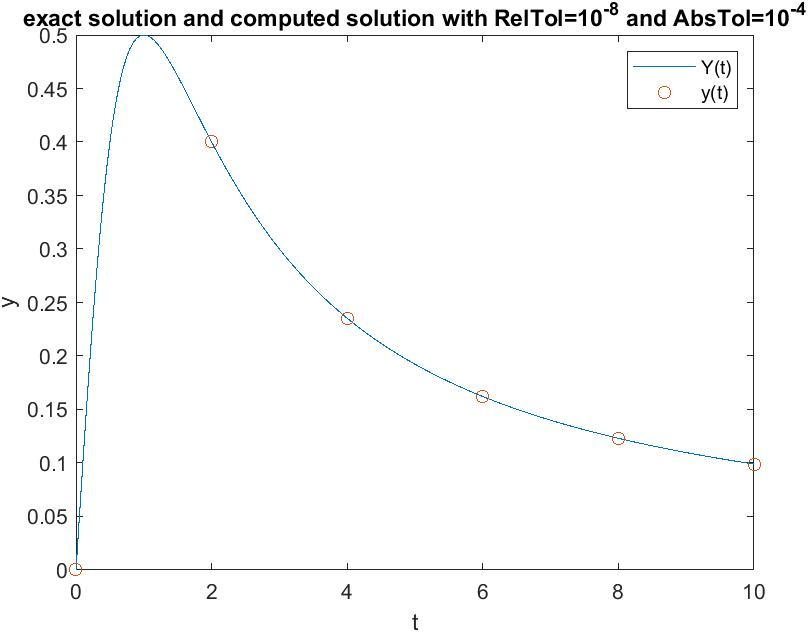


* 1. Question (b)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| RelTol | AbsTol |  |  |  |
|  |  | 2 | 0.40002 | 1.6045e-05 |
| 4 | 0.23533 | 3.8651e-05 |
| 6 | 0.16218 | 1.5356e-05 |
| 8 | 0.12308 | 6.0588e-06 |
| 10 | 0.099013 | 2.7536e-06 |
|  | 2 | 0.4 | 6.0309e-08 |
| 4 | 0.23529 | 1.1271e-07 |
| 6 | 0.16216 | 1.5294e-07 |
| 8 | 0.12308 | 2.6486e-08 |
| 10 | 0.09901 | 1.2950e-07 |

The exact solution and computed solution are shown in the following figure.

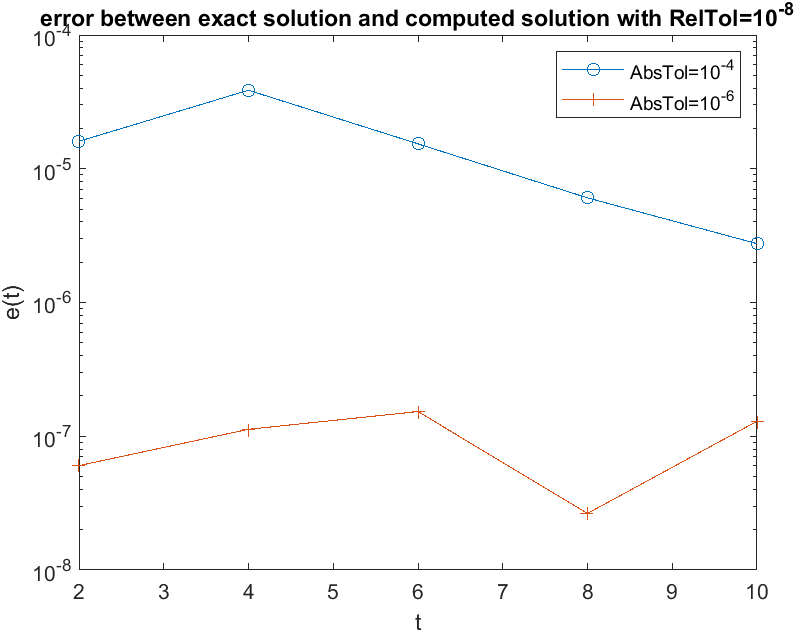
It can be seen that the computed solution is an accurate estimate of the exact solution.



The error between the exact solution and computed solution are shown in the following semi-log plot.

It can be seen that the error between the exact solution and computed solution decreases as the absolute tolerance decreases from to .

Moreover, the error between the exact solution and computed solution does not increase over time, which suggests that the ode45 method is stable for this initial value problem.

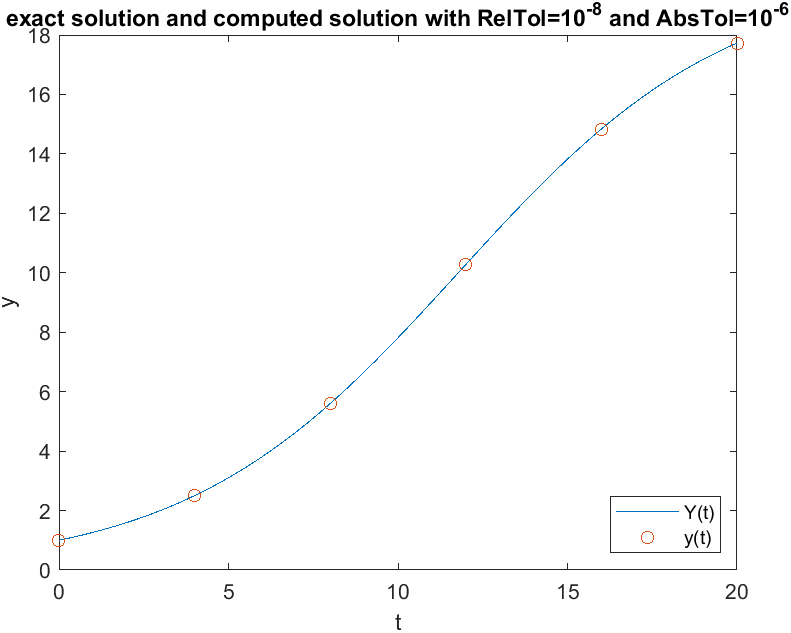
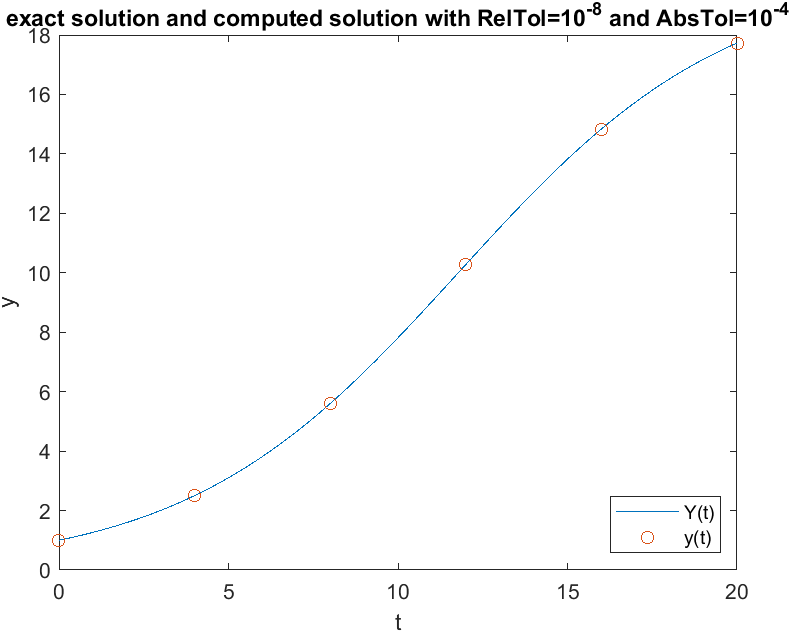


* 1. Question (c)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| RelTol | AbsTol |  |  |  |
|  |  | 2 | 2.5032 | 8.9301e-06 |
| 4 | 5.6001 | 2.8719e-05 |
| 6 | 10.278 | 3.8231e-05 |
| 8 | 14.837 | 2.9475e-05 |
| 10 | 17.73 | 1.6545e-05 |
|  | 2 | 2.5032 | 2.2273e-07 |
| 4 | 5.6001 | 6.9716e-07 |
| 6 | 10.278 | 1.5046e-06 |
| 8 | 14.837 | 5.5990e-07 |
| 10 | 17.73 | 2.9031e-07 |

The exact solution and computed solution are shown in the following figure.

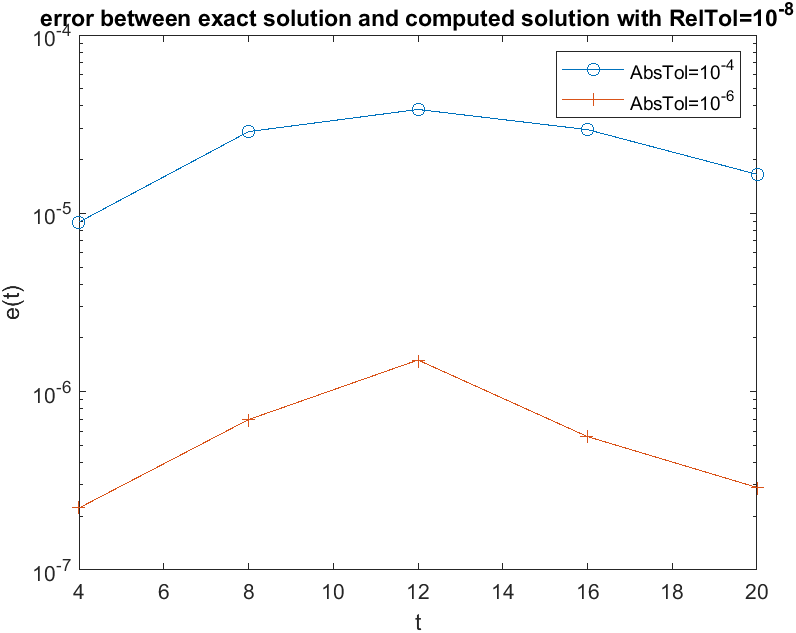
It can be seen that the computed solution is an accurate estimate of the exact solution.



The error between the exact solution and computed solution are shown in the following semi-log plot.

It can be seen that the error between the exact solution and computed solution decreases as the absolute tolerance decreases from to .

Moreover, the error between the exact solution and computed solution does not increase over time, which suggests that the ode45 method is stable for this initial value problem.



1. Appendix
   1. Second-order Runge-Kutta method with

function y = RK2(h)

y=zeros(5,1);

n=10/h;

yold=1;

k=1;

for i=1:n

ynew=yold+h\*(1/4\*f((i-1)\*h,yold)+3/4\*f((i-1)\*h+2/3\*h,yold+2/3\*h\*f((i-1)\*h,yold)));

if abs(mod(i\*h,2))<1e-8

y(k)=ynew;

k=k+1;

end

yold=ynew;

end

end

function dy = f(t,y)

dy=-y+2\*cos(t);

end

* 1. Main function of problem 2

clc

clear

tspan=2:2:10;

Y=@(t) sin(t)+cos(t);

exact=transpose(Y(tspan));

y1=RK2(0.1);

y2=RK2(0.05);

y3=RK2(0.025);

error1=abs(exact-y1);

error2=abs(exact-y2);

error3=abs(exact-y3);

ratio=error1./error2;

avg1=mean(error1);

avg2=mean(error2);

format shortG

y1

y2

ratio

format shortE

error1

error2

avg1

avg2

figure

plot(0:0.05:10,Y(0:0.05:10),'-',0:2:10,[1;y1],'o')

title('exact solution and computed solution with h=0.1')

xlabel('t')

ylabel('y')

legend('Y(t)','y\_h(t)')

figure

plot(0:0.05:10,Y(0:0.05:10),'-',0:2:10,[1;y2],'o')

title('exact solution and computed solution with h=0.05')

xlabel('t')

ylabel('y')

legend('Y(t)','y\_h(t)')

figure

semilogy(tspan,error1,'-o',tspan,error2,'-+')

title('error between exact solution and computed solution')

xlabel('t')

ylabel('e\_h(t)')

legend({'e\_{0.1}(t)','e\_{0.05}(t)'},'Location','southeast')

x=transpose(log([0.1,0.05,0.025]));

y=transpose(log([error1(5),error2(5),error3(5)]));

A=[x,ones(length(x),1)];

par=(A'\*A)\(A'\*y);

a=par(1);

b=par(2);

figure

plot(log([0.1,0.05,0.025]),a\*log([0.1,0.05,0.025])+b,'-',x,y,'o')

title('relationship between log of step size and log of error')

xlabel('ln(h)')

ylabel('ln(e\_h(10))')

legend({'a\*ln(h)+ln(b)','ln(e\_h(10))'},'Location','southeast')

text(-2.8,-8.5,strcat('slope=',string(a)))

* 1. Main function of problem 3a

clc

clear

t=0:2:10;

f=@(t,y) (cos(y)).^2;

Y=@(t) atan(t);

exact=transpose(Y(t));

opts1=odeset('RelTol',1e-8,'AbsTol',1e-4);

opts2=odeset('RelTol',1e-8,'AbsTol',1e-6);

[~,y1]=ode45(f,t,0,opts1);

[~,y2]=ode45(f,t,0,opts2);

error1=abs(exact-y1);

error2=abs(exact-y2);

format shortG

y1

y2

format shortE

error1

error2

figure

plot(0:0.05:10,Y(0:0.05:10),'-',t,y1,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-4}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'},'Location','southeast')

figure

plot(0:0.05:10,Y(0:0.05:10),'-',t,y2,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-6}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'},'Location','southeast')

figure

semilogy(t,error1,'-o',t,error2,'-+')

title('error between exact solution and computed solution with RelTol=10^{-8}')

xlabel('t')

ylabel('e(t)')

legend({'AbsTol=10^{-4}','AbsTol=10^{-6}'},'Location','southeast')

* 1. Main function of problem 3b

clc

clear

t=0:2:10;

f=@(t,y) 1./(1+t.^2)-2\*y.^2;

Y=@(t) t./(1+t.^2);

exact=transpose(Y(t));

opts1=odeset('RelTol',1e-8,'AbsTol',1e-4);

opts2=odeset('RelTol',1e-8,'AbsTol',1e-6);

[~,y1]=ode45(f,t,0,opts1);

[~,y2]=ode45(f,t,0,opts2);

error1=abs(exact-y1);

error2=abs(exact-y2);

format shortG

y1

y2

format shortE

error1

error2

figure

plot(0:0.05:10,Y(0:0.05:10),'-',t,y1,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-4}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'})

figure

plot(0:0.05:10,Y(0:0.05:10),'-',t,y2,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-6}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'})

figure

semilogy(t,error1,'-o',t,error2,'-+')

title('error between exact solution and computed solution with RelTol=10^{-8}')

xlabel('t')

ylabel('e(t)')

legend({'AbsTol=10^{-4}','AbsTol=10^{-6}'})

* 1. Main function of problem 3c

clc

clear

t=0:4:20;

f=@(t,y) y/4.\*(1-y/20);

Y=@(t) 20./(1+19\*exp(-t/4));

exact=transpose(Y(t));

opts1=odeset('RelTol',1e-8,'AbsTol',1e-4);

opts2=odeset('RelTol',1e-8,'AbsTol',1e-6);

[~,y1]=ode45(f,t,1,opts1);

[~,y2]=ode45(f,t,1,opts2);

error1=abs(exact-y1);

error2=abs(exact-y2);

format shortG

y1

y2

format shortE

error1

error2

figure

plot(0:0.05:20,Y(0:0.05:20),'-',t,y1,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-4}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'},'Location','southeast')

figure

plot(0:0.05:20,Y(0:0.05:20),'-',t,y2,'o')

title('exact solution and computed solution with RelTol=10^{-8} and AbsTol=10^{-6}')

xlabel('t')

ylabel('y')

legend({'Y(t)','y(t)'},'Location','southeast')

figure

semilogy(t,error1,'-o',t,error2,'-+')

title('error between exact solution and computed solution with RelTol=10^{-8}')

xlabel('t')

ylabel('e(t)')

legend({'AbsTol=10^{-4}','AbsTol=10^{-6}'})