3.(2) a_n收敛,证明如下:

不妨设 $p\ge 1$ $\quad |a_{n+1}-a_n|\le \frac{1}{n^2}$

则有 $|a_{n+p}-a_n| = |a_{n+p}-a_{n+p-1}+a_{n+p-1}-a_{n+p-2}+\cdots+a_{n+1}-a_n|$

$\le |a_{n+p}-a_{n+p-1}|+|a_{n+p-1}-a_{n+p-2}|+\cdots+|a_{n+p}-a_n|$

$\le \frac{1}{(n+p-1)^2}+\frac{1}{(n+p-2)^2}+\cdots+\frac{1}{n^2}$

$\le \frac{1}{(n+p-1)(n+p-2)}+\frac{1}{(n+p-2)(n+p-3)}+\cdots+\frac{1}{n(n-1)}$

$= \frac{1}{n+p-2}-\frac{1}{n+p-1}+\frac{1}{n+p-3}-\frac{1}{n+p-2}+\cdots+\frac{1}{n-1}-\frac{1}{n}$

$= \frac{1}{n-1}-\frac{1}{n+p-1}\le\frac{1}{n-1}<\varepsilon$.

$n>\frac{1}{\varepsilon}+1$

$\forall\varepsilon>0,\ \exists N=[\frac{1}{\varepsilon}]+1$, 使得 $n>N$, s.t. ~~$|a_n|$~~ $|a_{n+p}-a_n|<\varepsilon$

即 a_n为基本列, ⇒ a_n收敛.

4(1) $|a_{n+p}-a_n| = |(-1)^{n+2}\frac{1}{n+1}+\cdots+(-1)^{n+p+1}\frac{1}{n+p}|$

$\le |\frac{1}{n+1}+\cdots+\frac{1}{n+p}|$ $\quad \le |\frac{1}{n+1}-(\frac{1}{n+2}-\frac{1}{n+3})-\cdots|$

$\le\frac{p}{n+1}\le\varepsilon$. 放缩大有些了 ⇒ $n\ge\frac{p}{\varepsilon}-1$ $\quad \le\frac{1}{n+1}$.

$\forall\varepsilon>0,\ N=\frac{p}{\varepsilon}-1\ \ \forall n>N$ s.t. $|a_{n+p}-a_n|\le\varepsilon$.

即 a_n为基本列. ∴ a_n收敛

(2). $|x_{n+p}-x_n| = \left|\frac{\cos(n+p)!}{(p+n)(n+p)}+\frac{\cos(n+p-1)!}{(n+p)(n+p-1)}+\cdots+\frac{\cos(n+1)!}{(n+1)(n+2)}\right|$

$\le \frac{1}{n+p}-\frac{1}{n+p+1}+\frac{1}{n+p-1}-\frac{1}{n+p}+\cdots+\frac{1}{n+1}-\frac{1}{n+2}$

$=\frac{1}{n+1}-\frac{1}{n+p+1}\le\frac{1}{n+1}\le\varepsilon$.

$n\ge\frac{1}{\varepsilon}-1$

易得 x_n为基本列 ⇔ x_n收敛

5.(1) 证: ∵ $|y_n|$为基本列.

$|y_{n+p}-y_n| = |a_{n+1}|+|a_{n+2}|+\cdots+|a_{n+p}|$

$\le\varepsilon$.

$|x_{n+p}-x_n| = |a_{n+1}+a_{n+2}+\cdots+a_{n+p}|$

$\le |a_{n+1}|+|a_{n+2}|+\cdots+|a_{n+p}|$

$\le\varepsilon$

∴ x_n为基本列.

(2) 证: x_n收敛 ⇒ y_n为基本列 ⇒ x_n为基本列.

⇒ x_n收敛

(3). ∵ $|a_n|\ge 0$

$y_n-y_{n-1}=|a_n|\ge 0$. ∴ y_n ↗.

且 y_n有界 则 y_n收敛.

由(2) x_n收敛

7. ∵ $x_n\in[a,b]$

则一定可找到 x_{a_n}单调递增.

x_{b_n}单调递减. 且 x_n有界 ⇒收敛子列 x_{n_k}

∴ x_{a_n}与x_{b_n}收敛.

x_{a_n}收敛于b. x_{b_n}收敛于a. 设 $\lim_{k\to\infty}x_{n_k}=A$

$\boxed{或}$ ∵ x_n发散. ∵ x_n不收敛于A.

则存在两个子列收敛于不同极限.

$\exists\varepsilon_0>0\ \ \forall N_K\in N^*$

s.t. $n_k>N_k$

$|a_{n_k}-A|\ge\varepsilon_0$

$a_{n_{k_0}}\ge A+\varepsilon_0$ $\quad a_{n_{k_0}}\le A-\varepsilon$

则存在 $\underbrace{a_{n_{k_1}}}_{}$ 记作 y_n 子列不收敛于A

不妨设 $y_n\in[A+\varepsilon_0, b]$

则极限 $B\ge A+\varepsilon_0>A$.

1.(1) $\overline{\lim_{n\to\infty}} a_n = \sup\{\sqrt{5}, \sqrt[4]{7}, \cdots\} = \sqrt{2}$.

$\underline{\lim_{n\to\infty}} a_n = \inf\{\frac{3}{2}, \cdots, \frac{\sqrt{33}}{2}, \cdots\} =$ 分奇偶 是极限的

(2) $\overline{\lim_{n\to\infty}} a_n = \sup\{1, \frac{9}{5}, \frac{17}{9}, \cdots\} =$

$\underline{\lim_{n\to\infty}} a_n = \inf\{\frac{1}{2}, \frac{1}{7}, \frac{1}{11}, \cdots\} = 0$.

2. 证 ∵ $\{a_n\}$, $\{b_n\}$ 均为正 ∴ $0 < \underline{\lim_{n\to\infty}} b_n \le \overline{\lim_{n\to\infty}} b_n$

∴ $\overline{\lim_{n\to\infty}} a_n \cdot \underline{\lim_{n\to\infty}} b_n \le \overline{\lim_{n\to\infty}} (a_n \cdot b_n)$ ∵ a_n有界

~~右边易~~

$\overline{\lim_{n\to\infty}} (a_n \cdot b_n) \le \overline{\lim_{n\to\infty}} a_n \cdot \overline{\lim_{n\to\infty}} b_n$

同理即可证下.

$\overline{\lim_{n\to\infty}} a_n$ $\forall \varepsilon > 0$ $\exists N$, $n > N$, 有 $a_n > A_1 - \varepsilon$.

$a_n \cdot b_n > (A_1 - \varepsilon) b_n$ 从而

$\overline{\lim_{n\to\infty}} a_n b_n > \overline{\lim_{n\to\infty}} (A_1 - \varepsilon) \cdot b_n$

$\ge \overline{\lim_{n\to\infty}} (A_1 - \varepsilon) \cdot \inf b_n$

此处少写 a_n 为下, b_n 为上

3. ∵ $a_n > 0$ $\overline{\lim_{n\to\infty}} a_n = \frac{1}{\underline{\lim_{n\to\infty}} \frac{1}{a_n}} = \underline{\lim_{n\to\infty}} a_n$

∴ $\lim_{n\to\infty} a_n$ 存在.

∵ $a_n > 0$ ∴ $\overline{\lim_{n\to\infty}} a_n$ 存在. 设 $\overline{\lim_{n\to\infty}} a_n = A > 0$

1. $\underline{\lim_{n\to\infty}} a_n = 0$ ∃子列 $\{a_{n_k}\}$ $a_{n_k} < \varepsilon$

$-\varepsilon < 0 < a_{n_k} < \varepsilon$. $\lim_{n\to\infty} \frac{1}{a_{n_k}} = +\infty$ 与发散不存在

$\overline{\lim_{n\to\infty}} \frac{1}{a_n} = \frac{1}{\underline{\lim_{n\to\infty}} a_n} = \frac{1}{A}$

$\underline{\lim_{n\to\infty}} a_n = A$. $\boxed{\text{相等}}$ 极限存在

8. $\forall \varepsilon > 0$ $\exists \delta$ $|x - x_0| < \delta$ $|f(x) - A| < \varepsilon$.

~~$|[f(x)]^{\frac{1}{n}} - A^{\frac{1}{n}}|$~~

若 $f(x)$ 有极限, 则 $[f(x)]^{\frac{1}{n}}$ 有极限, 设其为 x,

则 $\lim_{x\to x_0} [f(x)]^{\frac{1}{n}} \cdot \lim_{x\to x_0} [f(x)]^{\frac{1}{n}} \cdots$ (n个)

$= x^n = A$. $x = \sqrt[n]{A}$

∴ $\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{A}$.

若 $A = 0$ $\lim_{x\to x_0} f(x) = 0 \Leftrightarrow \forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $0 < |x - x_0| < \delta$ $|f(x) - A| = |f(x)| < \varepsilon^n$.

则 $|\sqrt[n]{f(x)}| < \varepsilon$. 即 $\lim_{x\to x_0} \sqrt[n]{f(x)} = 0$

若 $A > 0$. $\lim_{x\to x_0} f(x) = A \Leftrightarrow \forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $0 < |x - x_0| < \delta$ $|f(x) - A| < A^{\frac{n-1}{n}} \varepsilon$.

$|\sqrt[n]{f(x)} - \sqrt[n]{A}| = \frac{|f(x) - A|}{f^{\frac{n-1}{n}} + f^{\frac{n-2}{n}} A^{\frac{1}{n}} + \cdots + A^{\frac{n-1}{n}}} \le \frac{|f(x) - A|}{A^{\frac{n-1}{n}}} < \varepsilon$.

证毕.

3.(2) $\varepsilon = \frac{B - A}{2}$

∵ $\lim_{x\to x_0^+} f(x) = A$, $\lim_{x\to x_0^+} g(x) = B$

∴ ~~∃N∈R~~ 使

$\exists \delta_1 > 0$ 使 $0 < x - x_0 < \delta$

$|f(x) - A| < \frac{B - A}{2}$

$f(x) < \frac{A + B}{2}$

$\exists \delta_2 > 0$, 使 $0 < x - x_0 < \delta$

$|g(x) - B| < \frac{B - A}{2}$

$\frac{A + B}{2} < g(x)$

∴ $0 < x - x_0 < \min\{\delta_1, \delta_2\}$

$f(x) < \frac{A + B}{2} < g(x)$

3. $\exists \varepsilon > 0$, 当 $|x - x_0| < \delta$ $|f(x) - A| < \varepsilon$

$||f(x)| - |A|| \le |f(x) - A| < \varepsilon$

∴ 当 $|x - x_0| < \delta$ 时 $\lim_{x\to x_0} |f(x)| = |A|$

5. $\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| = \left|\frac{f(x) \cdot B - A \cdot g(x)}{g(x) \cdot B}\right|$

$= \left|\frac{B[f(x) - A] - A[g(x) - B]}{g(x) \cdot B}\right|$

∵ $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$.

$\forall \varepsilon > 0$ $\exists \delta_1$ $|x - x_0| < \delta_1$ s.t. $|f(x) - A| \le \varepsilon$

$\forall \varepsilon > 0$ $\exists \delta_2$ $|x - x_0| < \delta_2$ s.t. $|f(x) - B| \le \varepsilon$.

$|x - x_0| < \min\{\delta_1, \delta_2\}$

$\le \frac{|B[f(x) - A]| + |A[g(x) - B]|}{|g(x) \cdot B|}$

$\le \frac{4(A + B)\varepsilon}{B^2}$, $g(x) \cdot B \to B^2 > \frac{B^2}{2}$

∴ 有 $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$.

1.(2) ~~$\lim_{x\to1}\frac{x-1}{x^2-1}=\lim_{x\to1}\frac{1}{x+1}$~~

~~$\forall\varepsilon>0\ \exists\delta$ 当 $|x-1|<\delta$~~

~~$\lim_{x\to1}\frac{x-1}{x^2-1}$~~

$\lim_{x\to1}\frac{x-1}{x^2-1}=\lim_{x\to1}\frac{1}{x+1}$

$\left|\frac{1}{x+1}-\frac{1}{2}\right|<\varepsilon$.　$\left|\frac{x-1}{2(x+1)}\right|$

对 $|x+1|$ 估计，在小范围变动.

$|x-1|<1 \Rightarrow x\in(0,2)$

$\Rightarrow 1<|x+1|<3$.

$<\frac{|x-1|}{2}<\varepsilon$.

$\Rightarrow \frac{-4\varepsilon}{1+2\varepsilon}<x-1<\frac{4\varepsilon}{1-2\varepsilon}$

~~$|x-1|<\min\{$~~ $\frac{4\varepsilon}{1+2\varepsilon},\frac{4\varepsilon}{1-2\varepsilon}\}$　$\delta=\min\{1,2\varepsilon\}$

不妨设 $\varepsilon\in(0,\frac{1}{2})$

$|x-1|<\frac{4\varepsilon}{1+2\varepsilon}$

有 $\lim_{x\to1}\frac{x-1}{x^2-1}=\frac{1}{2}$.

(3) $|\cos x-\cos x_0|<\varepsilon$.

和差化积

$\Rightarrow 2\left|\sin\frac{x+x_0}{2}\sin\frac{x-x_0}{2}\right|\le 2\left|\sin\frac{x-x_0}{2}\right|\le|x-x_0|<\varepsilon$

$|x-x_0|<\min\{|\arccos(\cos x_0-\varepsilon)-x_0|,$

$|\arccos(\cos x_0+\varepsilon)-x_0|\}$

$\forall\varepsilon>0$. $|x-x_0|<$

有 $\lim_{x\to x_0}\cos x=\cos x_0$.