2.(2). $x_n=\sqrt{2+x_{n-1}}$

$x_{n+1}-x_n=\sqrt{2+x_n}-\sqrt{2+x_{n-1}}$

$=\dfrac{x_n-x_{n-1}}{\sqrt{2+x_n}+\sqrt{2+x_{n-1}}}$

$=\dfrac{x_2-x_1}{\cdots\cdots}$

$\sqrt{2+\sqrt{2}}>\sqrt{2}$.

$2+\sqrt{2}>2$

即 $x_2>x_1$

设 $a_k<a_{k+1}$

$\sqrt{2+a_k}<\sqrt{2+a_{k+1}}$

$\therefore x_{n+1}-x_n>0$ $\therefore x_n$ 单调递增. $a_{k+1}<a_{k+2}$.

$x_n>x_{n-1}\Rightarrow\sqrt{2+x_{n-1}}>x_{n-1}$

$x_{n-1}^2-x_{n-1}-2<0$

$\boxed{\begin{array}{l} a_1=\sqrt{2}<2 \\ a_{k+1}=\sqrt{2+a_k}<\sqrt{2+2}=2,\end{array}}$

$(x_{n-1}-2)(x_{n-1}+1)<0$.

$-1<x_{n-1}<2\Rightarrow x_{n-1}\in(0,2)$.

$\therefore x_n$ 收敛. 设 $\lim\limits_{n\to\infty}x_n=x$.

$\lim\limits_{n\to\infty}x_n^2=\lim\limits_{n\to\infty}2+x_{n-1}$

$\therefore x^2-x-2=0\Rightarrow x=2$ 或 -1(舍)

3. 证: 由题: $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=t<1$ 反证

设 $\lim\limits_{n\to\infty}a_n=a\neq0$

则 $\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=\dfrac{a}{a}=1$

与 $\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=\dfrac{1}{t}$ 相矛盾

$\lim\limits_{n\to\infty}a_n=0$

$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}a_1\cdot\dfrac{a_2}{a_1}\cdot\dfrac{a_3}{a_2}\cdots\dfrac{a_n}{a_{n-1}}$

$=\lim\limits_{n\to\infty}\dfrac{1}{t^n}<\varepsilon\Rightarrow n>-\log_t\varepsilon$

$\forall\varepsilon>0,\exists N=[-\log_t\varepsilon]+1$

s.t. $n>N$ 时 $\lim\limits_{n\to\infty}a_n<\varepsilon$.

即 $\lim\limits_{n\to\infty}a_n=0$

6.(2) $\because x_{n+1}=\dfrac{1}{3}(2x_n+\dfrac{a}{x_n^2})$

且 $a>0$, $x_1>0$ $\therefore x_n>0$

也可以直接减

$x_{n+1}-x_n=\dfrac{1}{3}\left[2(x_n-x_{n-1})+\dfrac{a(x_{n-1}^2-x_n^2)}{x_n^2\cdot x_{n-1}^2}\right]$

$=\dfrac{1}{3}(x_n-x_{n-1})\left[2-\dfrac{a(x_n+x_{n-1})}{x_n^2\cdot x_{n-1}^2}\right]$.

$x_{n+1}=\dfrac{1}{3}(x_n+x_n+\dfrac{a}{x_n^2})$

$\geqslant\dfrac{1}{3}\cdot3\cdot\sqrt[3]{x_n\cdot x_n\cdot\dfrac{a}{x_n^2}}=\sqrt[3]{a}$.

$\dfrac{x_{n+1}}{x_n}=\dfrac{1}{3}(2+\dfrac{a}{x_n^3})\leqslant\dfrac{1}{3}(2+\dfrac{a}{a})=1$.

$\therefore x_n$ 单调递减.

$\therefore x_n$ 收敛.

$\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\dfrac{1}{3}(2x_n+\dfrac{a}{x_n^2})$.

$x=\dfrac{2}{3}x+\dfrac{a}{3x^2}$.

$x=\sqrt[3]{a}$

$\therefore\lim\limits_{n\to\infty}x_n=\sqrt[3]{a}$. 细心

7. $x_{n+1}\geqslant\dfrac{1}{4(1-x_n)}$

$(2x_n-1)^2\geqslant0$.

$1\geqslant4(x_n-x_n^2)$

$\dfrac{1}{4(1-x_n)}\geqslant x_n$.

$\therefore x_{n+1}\geqslant x_n$ 即 x_n 单调递增.

又 $x_n\in[0,1]$ $\therefore x_n$ 收敛.

$\lim\limits_{n\to\infty}x_{n+1}\geqslant\lim\limits_{n\to\infty}\dfrac{1}{4(1-x_n)}$ $\lim\limits_{n\to\infty}x_n=x$.

$x\geqslant\dfrac{1}{4(1-x)}$

$(2x-1)^2\leqslant0$

$\therefore x=\dfrac{1}{2}$.

$\lim\limits_{n\to\infty}x_n=\dfrac{1}{2}$.

8.(3). $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{\frac{n^2}{n-1}}\right)^{\frac{n^2}{n-1}\cdot\frac{n-1}{n}}$ 1^∞ $=e$

$1+\dfrac{1}{n}-\dfrac{1}{n^2}<1+\dfrac{1}{n}$

$1+\dfrac{1}{n}-\dfrac{1}{n^2}=1+\dfrac{n-1}{n^2}$

$>1+\dfrac{n-1}{n^2+n-2}$

$=1+\dfrac{1}{n+2}$

夹逼可得

$\lim\limits_{n\to\infty}\dfrac{n-1}{n}=1$.

(4). $(1-\dfrac{1}{n})^{\frac{1}{n}}<1^{\frac{1}{n}}=1$

$\lim\limits_{n\to\infty}(1-\frac{1}{n})^{\frac{1}{n}}<\lim\limits_{n\to\infty}1^{\frac{1}{n}}=1$

$\lim\limits_{n\to\infty}(1-\frac{1}{n})^{\frac{1}{n}}>\lim\limits_{n\to\infty}(1-\dfrac{1}{n})=1$

也可 $\dfrac{\sqrt[n]{n-1}}{\sqrt[n]{n}}$

$\therefore\lim\limits_{n\to\infty}(1-\frac{1}{n})^{\frac{1}{n}}=1$.

9.(1) $\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=\lim\limits_{n\to\infty}\ln\sqrt[n]{n}$.

$\because\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

也可使用Stolz定理 分母趋于正无穷.

$\therefore\lim\limits_{n\to\infty}\ln\sqrt[n]{n}=0$

(2) $\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}}{\ln\frac{n}{n-1}}$

$\boxed{\begin{array}{l}\dfrac{1}{1+n}<\ln(1+\dfrac{1}{n}) \\ \qquad<\dfrac{1}{n}\end{array}}$

$\lim\limits_{n\to\infty}n\ln\dfrac{n}{n-1}>\lim\limits_{n\to\infty}n\ln\dfrac{n}{n}=0$.

10. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

$\boxed{\geq \lim_{n\to\infty}\left(\frac{1}{2n}+\frac{1}{2n}\cdots+\frac{1}{2n}\right)=\frac{1}{2}.}$

$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\leq \ln\left(\frac{1+n}{n}\right)+\ln\left(\frac{n+2}{n+1}\right)+\cdots+\ln\frac{2n}{2n-1}$

$=\ln 2.$

$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$

$>\ln\left(1+\frac{1}{n+1}\right)+\ln\left(1+\frac{1}{n+2}\right)+\cdots+\ln\left(1+\frac{1}{2n}\right)$

$=\ln\frac{n+2}{n+1}+\ln\frac{n+3}{n+2}+\cdots+\ln\frac{2n+1}{2n}$

$=\ln\frac{2n+1}{n+1}$

$\lim_{n\to\infty}\frac{2n+1}{n+1}=2.$

$\therefore \lim_{n\to\infty}\ln\frac{2n+1}{n+1}\leq\lim\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)\leq\lim_{n\to\infty}\ln\frac{2n}{n}=\ln 2$

$=\ln 2$

$\therefore \lim\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)=\ln 2$

11. 证 1) $\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{2\sqrt{n}}$

$\frac{1}{\sqrt{n+1}+\sqrt{n}}>\frac{1}{2\sqrt{n+1}}$

证毕

2) $x_n>2(\sqrt{2}-1+\sqrt{3}-\sqrt{2}\text{ —— }+\sqrt{n+1}-\sqrt{n})-2\sqrt{n}$

$=2(\sqrt{n+1}-\sqrt{n}-1)$

$x_n-x_{n-1}=\frac{1}{\sqrt{n}}-2(\sqrt{n}-\sqrt{n-1})\leq\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}=0$

$\therefore x_n\downarrow.$

$\because \sqrt{n+1}>\sqrt{n}$

$\therefore x_n>-2.$ $\therefore x_n$收敛,极限存在.