1.4 2.(2) 下确界=0 上确界1

∵$n\in N^*$ ∴$\frac{1}{\sqrt{n}}\in(1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{2},\cdots)$

$n\uparrow$ $\frac{1}{\sqrt{n}}\downarrow$ ∴$\frac{1}{\sqrt{n}}<1$

$\forall\varepsilon>0$. $1>1-\varepsilon$.

∴1为上确界

∵$\sqrt{n}>0$, ∴$\frac{1}{\sqrt{n}}>0$.

$\forall\varepsilon>0$ $\exists x\in\frac{1}{\sqrt{n}}$, 使得 ~~即0+ε~~ $\frac{1}{\sqrt{n}}<\varepsilon$.

$n>\frac{1}{\varepsilon^2}$时成立.

∴0为下确界.

(3) 下确界−3 上确界−1

$\{x\mid x^2+4x+3<0\}=x\in(-3,-1)$

$\forall x\in(-3,-1)$, $x>-3$.

$\forall\varepsilon>0$, $\exists x\in(-3,-1)$, s.t. $x<-3+\varepsilon$.

故−3为下确界. 上确界证明同理.

3. ∵S有上界, 即设其中一个上界为a

则 $\forall x\in S$. $x<a$.

$-x>-a$. 又$-x\in T$:

即所有$x\in T$都有$x>-a$ 即T有下界.

~~sups~~ $\forall x\in S$. $x<\sup S$.

且$\forall\varepsilon>0$, $\exists x\in S$ s.t. $x>\sup S-\varepsilon$

2) $\begin{cases}-x>-\sup S\\ -x<\varepsilon-\sup S\end{cases}$ 令$-x=y\in T$

$\begin{cases}y>-\sup S\\ \forall\varepsilon>0,\ \exists y\in T,\ \text{s.t. } y<-\sup S+\varepsilon\end{cases}$

∴$-\sup S=\inf T$

即 $\sup S=-\inf T$.

5. 证: $\forall x\in A$ $x<\sup A$

$\forall y\in B$ $y\le\sup B$.

∵又$x,y>0$

∴$x\cdot y<\sup A\cdot\sup B$.

$\forall\varepsilon>0$, $\exists x\in A$. 使 $x>\sup A-\varepsilon$. 即 $x\cdot y>(\sup A-\varepsilon)\cdot y$.

且y最大为$\sup B$. $y>\sup B-\varepsilon$

则. $x\cdot y>\sup A\cdot\sup B-\varepsilon(\varepsilon+\sup A+\sup B$
$-\varepsilon$.

且ε可任意取

∴$x\cdot y>\sup A\cdot\sup B-\varepsilon$.

即 $\sup AB=\sup A\cdot\sup B$.

2.3 7. 证. $n\to\infty$成立时 $2n\to\infty$同样成立.

$\lim\limits_{2n\to\infty}\frac{a_1+a_2+\cdots a_{2n}}{2n}$

$=\lim\limits_{2n\to\infty}\frac{a_1+a_3+a_5+\cdots+a_{2n-1}}{2n}+\lim\limits_{2n\to\infty}\frac{a_2+a_4+\cdots a_{2n}}{2n}$

$=\lim\limits_{2n\to\infty}\frac{a_{2n-1}}{2}+\lim\limits_{2n\to\infty}\frac{a_{2n}}{2}=\frac{a+b}{2}$ 也可分类拆解

n取2n+1时 $=\left(\frac{a_1+\cdots+a_{2n}}{2n}+\underbrace{\frac{a_{2n+1}}{2n}}_{\to 0}\right)\left(\underbrace{\frac{2n}{2n+1}}_{\to 1}\right)$

$\lim\limits_{2n+1\to\infty}\frac{a_1+\cdots+a_{2n+1}}{2n+1}=\lim\limits_{2n+1\to\infty}\frac{a_1+\cdots+a_{2n}}{2n+1}+\lim\limits_{2n+1\to\infty}\frac{a_{2n+1}}{2n+1}$

$=\frac{a+b}{2}+0=\frac{a+b}{2}$.

8.(1) 证 A. $a=0$时 对$|a_n|\le\varepsilon$.

$\forall\varepsilon>0$, ∃$n>N$时 ~~∃N∈N* $|a_n|\le\varepsilon$~~

$0\le\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}$ ~~$=\lim\limits_{n\to\infty}\sqrt[n]{a_1\cdots a_N}\cdot\lim\limits_{n\to\infty}\sqrt[n]{a_{N+1}\cdots a_n}$~~

~~$\le\lim\limits_{n\to\infty}\sqrt[n]{a_1\cdots a_N}\cdot\varepsilon^{\frac{n-N}{n}}$~~ 夹逼

∵N固定 ~~$\ge\lim\limits_{n\to\infty}\sqrt[n]{a_1\cdots a_N}\cdot\varepsilon$~~

ε可无穷小 ∴$\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}=0$.

B. $a\ne0$时. $\forall\varepsilon>0$, $\exists N\in N^*$, $n>N$ $|a_n-a|\le\varepsilon$.

~~$\lim\sqrt[n]{a_1\cdots a_n}=\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_N}\cdot\lim\limits_{n\to\infty}\sqrt[n]{a_{N+1}\cdots a_n}$~~

~~$\le(a+\varepsilon)^{\frac{n-N}{n}}$~~

~~$\lim\limits_{n\to\infty}\sqrt[n]{a_1\cdots a_N}(a-\varepsilon)^{\frac{n-N}{n}}$~~ $\le$ ~~$\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}$~~

$\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}\le\lim\limits_{n\to\infty}\frac{a_1+\cdots+a_n}{n}$

$\underbrace{\lim\limits_{n\to\infty}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}}_{=a}\le\lim\sqrt[n]{a_1a_2\cdots a_n}$

(2) 证:

$\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{a_2}{a_1}\cdots\frac{a_n}{a_{n-1}}\cdot a_1}$

$\to\lim\limits_{n\to\infty}\frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}}$

又 $\lim\limits_{n\to\infty}a_n=a$, $\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}=a$.

有限次思路 $=\frac{1}{\frac{1}{a}}=a$

∴$\lim\limits_{n\to\infty}\sqrt[n]{x_1x_2\cdots a_n}=a$.

∴$\lim\limits_{n\to\infty}\sqrt[n]{a_n}=l$.

0. 令 $S_n = a_1 + \cdots + a_n$

$$\lim_{n\to\infty} \frac{a_1+2a_2+\cdots+na_n}{n} = \lim_{n\to\infty} \frac{S_1+2(S_2-S_1)+3(S_3-S_2)+\cdots+n(S_n-S_{n-1})}{n}$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} \frac{S_1+S_2+\cdots+S_{n-1}}{n}$$

$$= S - \lim_{n\to\infty} \frac{S_{n-1}}{n}$$

$$= S - S = 0$$

11.(1) $\lim_{n\to\infty} \frac{1!+2!+3!+\cdots+n!}{n!}$

$$= \lim_{n\to\infty} \frac{n!}{n!-(n-1)!}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} = 1.$$

(3) $\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n}$

$$= \lim_{n\to\infty} \sqrt[n]{n}$$

$= 1$

12.(2) 左边 $= \lim_{n\to\infty} \left(\frac{(1^p+2^p+\cdots+n^p)(1+p) - n^{p+1}}{n^p(1+p)} \right)$

$$= \lim_{n\to\infty} \left(\frac{n^p(1+p)+(n-1)^{p+1}-n^{p+1}}{(1+p)[n^p-(n-1)^p]} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^p(1+p)+n^{p+1}+C_{p+1}^1\cdot n^p(-1)^1 + C_{p+1}^2\cdot n^{p-1}\cdot(-1)^2 - n^{p+1}+\cdots}{(1+p)[n^p-(n^p+C_p^1\cdot n^{p-1}\cdot(-1)^1+\cdots)]} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{p(p+1)}{2}\cdot n^{p-1}+\cdots}{(1+p)(pn^{p-1}+\cdots)} \right)$$

$= \frac{1}{2}$.

9. a 分情况 ① $a=0$. $n>N$ $|a_n|<\varepsilon$.

$\lim_{n\to\infty}|a_n|=0 \Rightarrow \lim_{n\to\infty} \frac{|a_1|+|a_2|+\cdots+|a_n|}{n} = 0$.

$\because \lim_{n\to\infty} b_n = b$ $\therefore b_n$ 有界. $|b_n|<M$. $n>N_1$ 时 $\Rightarrow \frac{|a_1|+\cdots+|a_n|}{n} < \varepsilon$.

$$\left| \frac{a_1 b_n + a_2 b_{n-1}+\cdots+a_n b_1}{n} \right| \le \frac{|a_1|+\cdots+|a_n|}{n} M < \varepsilon M$$

$\therefore \lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1}+\cdots+a_n b_1}{n} = 0$.

② $\lim_{n\to\infty} a_n = a \neq 0$ $\alpha_n = a_n - a$ 由①可知 $\lim \frac{\alpha_1 b_n+\cdots+\alpha_n b_1}{n} = 0$.

$$\lim_{n\to\infty} \frac{a_1 b_n+\cdots+a_n b_1}{n} = \lim_{n\to\infty} \frac{(\alpha_1+a)b_n+(\alpha_2+a)b_{n-1}+\cdots+(\alpha_n+a)b_1}{n}$$

$$= \lim_{n\to\infty} \frac{\alpha_1 b_n+\cdots+\alpha_n b_1}{n} \quad ab$$

0. $a=+\infty$ $\lim_{n\to\infty} a_n = +\infty$ $\forall M>0$, $n>N$, $a_n>M$

$$\sqrt[n]{a_1\cdots a_n} \le \frac{a_1+\cdots+a_n}{n} \le \frac{S_N}{n} + \frac{a_{N+1}+\cdots+a_n}{n-N}\cdot\frac{n-N}{n}$$

$$\sqrt[n]{a_1\cdots a_n} = \sqrt[n]{a_1\cdot a_2\cdots a_N(a_{N+1}\cdots a_n)}$$

$$= \sqrt[n]{S_N}\,\sqrt[n]{a_{N+1}\cdots a_n}$$

$$> \sqrt[n]{S_N}\cdot\sqrt[n]{M^{n-N}} = M\sqrt[n]{S_N}\cdot\sqrt[n]{M^{-N}}$$

N 为固定常量.

$$\lim_{n\to\infty} M\sqrt[n]{S_N}\sqrt[n]{M^{-N}} = M$$

$\sqrt[n]{a_1\cdots a_n} \ge M$. $\therefore$ 证毕.