8.1) $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$.

2) $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = -\det(A)$

3) $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$.

4) 0

5) $\begin{vmatrix} a_{11} & b_{12} & a_{13} & a_{14} \\ a_{21} & b_{22} & a_{23} & a_{24} \\ a_{31} & b_{32} & a_{33} & a_{34} \\ a_{41} & b_{42} & a_{43} & a_{44} \end{vmatrix}$

(5) $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 1-n \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & \cdots & n-1 & n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 1-n \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 & 6 & \cdots & \frac{n(n-1)}{2} & \frac{n(n+1)}{2} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \end{vmatrix}$

$= (-1)^{\tau(n12\cdots n-1)}(n-1)!\,\frac{n(n+1)}{2}$

$= (-1)^{n-1}\frac{(n+1)!}{2}$.

13.1) $p(x) = \sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1\cdots j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n}$

其中,多项式次数由x的最高次数决定

且仅有a_{1j_1}包含x, $a_{1j_1}\in\{1,x,\cdots,x^{n-1}\}$ 最高次为$n-1$ $\therefore p(x)$为$n-1$次多项式

2) $p(x)$的根为$a_1, a_2, \cdots, a_{n-1}$

19.1) 左边等于 $a_1a_2\cdots a_n \begin{vmatrix} a_0 & 1 & \cdots & 1 \\ \frac{1}{a_1} & 1 & 0 \cdots & 0 \\ \vdots & 0 & 1 \cdots & 0 \\ \frac{1}{a_n} & 0 & 0 \cdots & 1 \end{vmatrix}$

$= a_1a_2\cdots a_n \begin{vmatrix} a_0-\sum\frac{1}{a_n} & 0 & 0 & 0 \\ \frac{1}{a_1} & 1 & 0 & 0 \\ \vdots & & 1 & \ddots \\ \frac{1}{a_n} & \cdots & & 1 \end{vmatrix}$

$= a_1a_2\cdots a_n\left(a_0 - \sum_{i=1}^{n}\frac{1}{a_i}\right)$

18.1) $\begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix} = x^n + (-1)^{\tau(234\cdots n1)}y^n = x^n + (-1)^{n-1}y^n$

2) 原式 $= (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)\begin{vmatrix} a_1-b_1 & a_1-b_2 & \cdots & a_1-b_n \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}$

$n=1$ a_1-b_1

$= 0$ $n=2$ 时 $(a_1-a_2)(b_1-b_2)$.

3) 原式 $= \begin{vmatrix} \sum x_i - m & x_2 & \cdots & x_n \\ \sum x_i - m & x_2-m & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ \sum x_i - m & x_2 & \cdots & x_n-m \end{vmatrix}$

$= \begin{vmatrix} \sum x_i - m & x_2 & x_3 & \cdots & x_n \\ 0 & -m & 0 & \cdots & \vdots \\ 0 & 0 & -m & \cdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & -m \end{vmatrix}$

$= (-m)^{n-1}\left(\sum x_i - m\right)$

2) 数学归纳法. $n=2$

原式 $= D_n$ $D_2 = \begin{vmatrix} x & a_0 \\ -1 & x+a_1 \end{vmatrix} = x^2+a_1x+a_0$ 满足

假设$n-1$阶行列式满足

$D_n = x\begin{vmatrix} x & 0 & \cdots & 0 & a_1 \\ & & \ddots & & \vdots \\ 0 & \cdots & & & x+a_{n-1} \end{vmatrix}_{n-1阶} + (-1)^{n+1}a_0\begin{vmatrix} -1 & x & \cdots \\ & -1 & \cdots \\ & & \ddots \end{vmatrix}_{n-1阶}$

$= x(x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_2x+a_1)+a_0$

$= x^n + a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$

4) 原式 $= \begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & -2 & -2 & \cdots & -2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{vmatrix} = (-2)(n-2)!$

3) $D_n=\begin{vmatrix} \alpha+\beta & \alpha\beta & \cdots & \cdots & 0 \\ 1 & \alpha+\beta & & & \vdots \\ & & & & \vdots \\ 0 & \cdots & 0 & 1 & \alpha+\beta \end{vmatrix}=\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$

$D_2=\begin{vmatrix} \alpha+\beta & \alpha\beta \\ 1 & \alpha+\beta \end{vmatrix}=\alpha^2+\alpha\beta+\beta^2$ 显然成立.

下验证 D_{n-1} 成立时，D_n 也成立 设 D_{n-1} 成立.

对第一行展开

$D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}$

$=\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$ 成立.

4) 设 $D_n=\begin{vmatrix} \cos\alpha & 1 & \cdots & 0 & 0 \\ 1 & 2\cos\alpha & \cdots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & 2\cos\alpha \end{vmatrix}$

$=\cos n\alpha.$

$D_2=\begin{vmatrix} \cos\alpha & 1 \\ 1 & 2\cos\alpha \end{vmatrix}=2\cos^2\alpha-1$ 显然成立.

下验证 D_{n-1} 成立时 D_n 同样成立. 设 D_{n-1} 成立.

$D_n=\cancel{D_{n-1}-D_{n-2}}$

$2\cos\alpha D_{n-1}-D_{n-2}$

$=2\cos\alpha\cdot\cos(n-1)\alpha-\cos(n-2)\alpha$

$=\cos n\alpha+\cos(2\alpha-n\alpha)-\cos[(n-2)\alpha]$

$=\cos n\alpha$ 证毕

(5) 原式 $=a_1a_2a_3\cdots a_n\begin{vmatrix} 1+\frac{1}{a_1} & \frac{1}{a_1} & \frac{1}{a_1} & \cdots & \frac{1}{a_1} \\ \frac{1}{a_2} & 1+\frac{1}{a_2} & & & \vdots \\ \frac{1}{a_3} & \frac{1}{a_3} & & & \vdots \\ \vdots & \vdots & & & \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & & 1+\frac{1}{a_n} \end{vmatrix}$

$=a_1a_2\cdots a_n\begin{vmatrix} 1+\sum_{i=1}^{n}\frac{1}{a_i} & 1+\sum_{i=1}^{n}\frac{1}{a_i} & \cdots & 1+\sum_{i=1}^{n}\frac{1}{a_i} \\ \frac{1}{a_2} & 1+\frac{1}{a_2} & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{a_n} & \frac{1}{a_n} & \cdots & 1+\frac{1}{a_n} \end{vmatrix}$

$=a_1\cdots a_n\begin{vmatrix} 1+\sum_{i=1}^{n}\frac{1}{a_i} & 0 & \cdots & 0 & 0 \\ \frac{1}{a_2} & 1 & & & \vdots \\ \frac{1}{a_3} & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{1}{a_n} & \cdots & & & 1 \end{vmatrix}$

$=a_1\cdots a_n\left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)$

20. 原式 $=\begin{vmatrix} x & y & z & 0 \\ 3 & 0 & 2 & -5 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 2 & -5 \end{vmatrix}=2x+y-3z$

目标式: $\begin{vmatrix} x & y & z & x+y+z \\ 6 & 0 & -1 & 0 \\ 2 & 2 & 0 & 2 \\ 5 & 2 & -1 & 1 \end{vmatrix}$

$=\begin{vmatrix} x & y & z & 0 \\ 6 & 0 & -1 & -5 \\ 2 & 2 & 0 & -2 \\ 5 & 2 & -1 & -5 \end{vmatrix}$

$=x\begin{vmatrix} 0 & -1 & -5 \\ 2 & 0 & -2 \\ 2 & -1 & -5 \end{vmatrix}-y\begin{vmatrix} 6 & -1 & -5 \\ 2 & 0 & -2 \\ 5 & -1 & -5 \end{vmatrix}+z\begin{vmatrix} 6 & 0 & -5 \\ 2 & 2 & -2 \\ 5 & 2 & -5 \end{vmatrix}$

$=4x+2y-6\cancel{x}z=$