1.(4) $\sqrt{1+\tan x}-\sqrt{1-\sin x}$

$= \dfrac{\tan x+\sin x}{\sqrt{1+\tan x}+\sqrt{1-\sin x}} \triangleq f(x)$

$g(x)=x$

$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \cancel{\lim\limits_{x\to 0}\dfrac{\frac{\tan x}{x}+\frac{\sin x}{x}}{\sqrt{1+\tan x}+\sqrt{1-\sin x}}}$ 可为除 x^k.

$=\dfrac{1+1}{2}=1$ $\quad \lim\limits_{x\to 0}\dfrac{\tan x+\sin x}{2x^{k}} = b$

为1阶无穷小 $\Leftrightarrow k>1$ b为非零实数 当且仅当

(5) $f(x)=\dfrac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{1+\sqrt{x}}}+\sqrt{2}}$

$=\dfrac{\sqrt{x}}{(\sqrt{1+\sqrt{1+\sqrt{x}}}+\sqrt{2})(\sqrt{1+\sqrt{x}}+1)}$ → $x^{\frac{1}{2}-k}$ 剩下同理.

$g(x)=\sqrt{x}$

$\lim\limits_{x\to 0^+}\dfrac{f(x)}{g(x)} = \cancel{\dfrac{1}{(\sqrt{1+\sqrt{1+\sqrt{x}}}+\sqrt{2})(\sqrt{1+\sqrt{x}}+1)}}$

$=\dfrac{\sqrt{2}}{8}$ $\quad \frac{1}{2}$阶无穷小

(6) $g(x)=\sqrt[8]{x}$ $\sqrt{\sqrt{\sqrt{x}}} = x^{\frac{1}{8}}$

~~$\lim\limits_{x\to 0}$ ...~~ $\quad \sqrt{x^{1-2k}+\sqrt{x^{1-4k}+\sqrt{x^{1-8k}}}}$

$\lim\limits_{x\to 0^+}\dfrac{f(x)}{g(x)} = \cancel{\sqrt{x^{\frac{3}{4}}+\sqrt{x^{\frac{1}{2}}+1}}} = 1$

$\frac{1}{8}$阶无穷小

(9) $\lim\limits_{x\to\infty}\dfrac{\frac{x-2}{x^4+1}}{x^{-3}} = \cancel{\lim\limits_{x\to\infty}\dfrac{x^4-2x^3}{x^4+1}} = 1$ $\quad \dfrac{\frac{x-2}{x^4+1}}{x^{-k}} = \dfrac{x-2}{x^{4-k}+x^{-k}}$

3阶无穷小

(10) 原式 $=\dfrac{(1-x)(1+x)(1+x^2)\cdots(1+x^n)}{1-x}$

$=\dfrac{1-x^{2n}}{1-x}$. 不能简化? why?

~~原式~~ $\lim\limits_{x\to\infty}\dfrac{\frac{1-x^{2n}}{1-x}}{x^{2n-1}} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x^{2n-1}}-x}{1-x}$

$\because x\to\infty$ $n>1$ $\cancel{x}$

$\therefore \dfrac{1}{x^{2n-1}}\to 0$

$\lim\limits_{x\to\infty}\dfrac{(1+x)(1+x^2)\cdots(1+x^n)}{x^k}$

$\therefore$ 原式 $=1$

$=\lim\limits_{x\to\infty}\dfrac{x^{1+2+\cdots+n}+P_m(x)}{x^k} = b$

$\therefore 2n-1$阶无穷大. 根据为 $\frac{\infty}{\infty}$

当且仅当分子、分母最高次幂相同时为非零实数 $k=\dfrac{n(n+1)}{2}$

2.(3) $\forall f\in o(x^2)$

$\lim\limits_{x\to 0} f(x)\cdot x = \cancel{\lim\; 0\cdot 0 = 0}$

$\therefore f(x)\cdot x\in o(x^3)$

$\therefore x\, o(x^2)=o(x^3)$

(5) $\forall f\in o(x^2)$ $\forall g\in o(x)$

$\cancel{\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{\frac{f(x)}{x^2}}{\frac{g(x)}{x}}\cdot x}$

$\because$ ~~$\lim\limits_{x\to 0}\dfrac{g(x)}{x}=0$ $g(x)\sim o(x)$~~

~~又对原式変形, $\dfrac{o(x^2)}{o(x)\cdot x} = \dfrac{o(x)}{x}$~~

~~由(4) $\dfrac{o(x^2)}{x} = o(x)$~~

~~$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)\cdot x}$~~

$\lim\limits_{x\to 0}\dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$

$\therefore x^2$是x的高阶无穷小.

$f(x) = \cancel{\dfrac{o(x^2)}{o(x)}}$ $\quad \dfrac{f(x)}{g(x)}$

$\lim\limits_{x\to 0}\dfrac{F(x)}{x} = \dfrac{f(x)}{x^2}\cdot\dfrac{x}{g(x)}$ 为 $0\cdot\infty$型.

不一定为无穷小量, 原结论错误.

3.(2)

原式 $=\lim\limits_{x\to-\infty}\dfrac{x^2-x+1-(ax+b)^2}{\sqrt{x^2-x+1}+ax+b}$

$=\lim\limits_{x\to-\infty}\dfrac{(1-a^2)x^2-(1+2ab)x+1-b^2}{\sqrt{x^2-x+1}+ax+b}$

$=\lim\limits_{x\to-\infty}\dfrac{(1-a^2)\cdot x-(1+2ab)+\frac{1-b^2}{x}}{\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}+a+\frac{b}{x}}$

除时负数 直接拆进根号 数缺符号

$=\lim\limits_{x\to-\infty}\dfrac{(1-a^2)x-(1+2ab)}{-1+a}$

$\therefore \dfrac{(1-a^2)x-(1+2ab)}{1+a}$ 趋近于0.

$\therefore a^2=1$ $a\neq 1\neq 0$ $1+2ab=0$

$\therefore$ ~~$a=1$ $b=-\frac{1}{2}$~~.

$a=-1$ $b=\frac{1}{2}$

4.(3). 原式 $=\lim\limits_{x\to 0}\frac{e^{x\ln(1+x)}-1}{x^2}$

$=\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}$

$=1$

(5). 原式 $=\lim\limits_{x\to 0}\frac{\sqrt{1+x^2}-1}{x^2}$.

$=\lim\limits_{x\to 0}\frac{\frac{x^2}{2}}{x^2}$

$=\frac{1}{2}$.

(6) 原式 $=\lim\limits_{x\to 0}\frac{\frac{1}{n}x^2-\frac{2}{n}x^3}{-\frac{x^2}{2}}$

$=-\frac{2}{n}$

5.(1) 原式 $=\lim\limits_{x\to\infty} e^{x\ln(\sin\frac{1}{x}+\cos\frac{1}{x})}$

$=\lim\limits_{x\to\infty} e^{x\ln(\sin\frac{1}{x}+\cos\frac{1}{x}-1+1)}$

$=\lim\limits_{x\to\infty} e^{x(\sin\frac{1}{x}+\cos\frac{1}{x}-1)}$

$=e^{\lim\limits_{x\to\infty}x(\sin\frac{1}{x}+\cos\frac{1}{x}-1)}$

$=e^{\lim\limits_{x\to\infty}x(\frac{1}{x}-2\cdot\frac{1}{4x^2})}$

$=e^{\lim\limits_{x\to\infty}(1-\frac{1}{2x})}$

$=e$.

(2) $\lim\limits_{x\to 0} e^{\frac{1}{x(1-\cos x)}\ln(\frac{1+\sin x}{1+\tan x})}$

$=e^{\lim\limits_{x\to 0}\frac{\tan x(\cos x-1)}{x(1-\cos x)(1+\tan x)}}$

$=e^{\lim\limits_{x\to 0}\frac{-x}{x(1+\tan x)}}$

$=\frac{1}{e}$.

(3) 原式 $=\lim\limits_{x\to\frac{\pi}{4}} e^{\tan 2x\ln(\tan x-1+1)}$

$=\lim\limits_{x\to\frac{\pi}{4}} e^{\tan 2x(\tan x-1)}$

$=\lim\limits_{x\to\frac{\pi}{4}} e^{\frac{\sin 2x}{\cos 2x\cdot\cos x}(\sin x-\cos x)}$

$=e^{\lim\limits_{x\to\frac{\pi}{4}}\frac{-\sin 2x}{(\cos x+\sin x)\cos x}}$

$=\frac{1}{e}$

(4) $=\lim\limits_{x\to 1} e^{\frac{2x^2}{x-1}\ln(3e^{\frac{x-1}{x+2}}-2)}$

$=\lim\limits_{x\to 1} e^{\frac{2x^2}{x-1}\cdot 3(e^{\frac{x-1}{x+2}}-1)}$

$=\lim\limits_{x\to 1} e^{\frac{6x^2}{x-1}\cdot\frac{x-1}{x+2}}$

$=\lim\limits_{x\to 1} e^{\frac{6x^2}{x+2}}$

$=\lim\limits_{x\to 1} e^{\frac{6}{3}}=e^2$.

6. 原式 $=\lim\limits_{x\to 0} e^{\frac{1}{x}\ln(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n})}$

$=\lim\limits_{x\to 0} e^{\frac{1}{x}(\frac{a_1^x-1+a_2^x-1+\cdots+a_n^x-1}{n})}$

$=e^{\lim\limits_{x\to 0}\frac{1}{n}(\ln a_1\cdot a_2\cdots a_n)}$.

$=(a_1\cdot a_2\cdots a_n)^{\frac{1}{n}}$.

$=\sqrt[n]{a_1\cdot a_2\cdots a_n}$

3.5.2

$S_n=\sqrt{n+1}\quad t_n=\sqrt{n}$

$|S_n-t_n|=\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{2\sqrt{n+1}}<\delta$.

$|\sqrt{x_1}-\sqrt{x_2}|\le\sqrt{|x_1-x_2|}$ why?!

$\sqrt{x_1}-\sqrt{x_2}=\frac{x_1-x_2}{\sqrt{x_1}+\sqrt{x_2}}$

设 $x_1>x_2>0$

$\sqrt{x_1}+\sqrt{x_2}\ge\sqrt{x_2}\ge\sqrt{x_1-x_2}$

$|x_1-x_2|\le\delta$ 令 $\delta=\varepsilon^2$

则 $|\sqrt{x_1}-\sqrt{x_2}|\le\varepsilon$.

$\sqrt{x}$在区间$[0,+\infty)$上一致连续

3(1) $S_n=\sqrt{n+1}\quad t_n=\sqrt{n}$.

$|S_n-t_n|=\left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right|\le\frac{1}{2\sqrt{n+1}}<\varepsilon$ 令 ε

~~$\sin S_n^2-\sin t_n^2=\sin(n+1)-\sin n$~~ $\frac{1}{4\varepsilon}<n$.

$=$ ~~$2\cos\frac{2n+1}{2}\sin\frac{1}{2}<\cos\frac{2n+1}{2}$~~ $\quad n>\frac{1}{4\varepsilon^2}-1$

~~$n\to\infty\ \cos\frac{2n+1}{2}\in[-1,1]$~~ $\quad\therefore\lim\limits_{n\to\infty}(S_n-t_n)=0$

$\sin S_n^2-\sin t_n^2$

$=2\cos\frac{2n+1}{2}\cdot\sin\frac{1}{2}$

$\cos\frac{2n+1}{2}$在$[-1,1]$振荡 即 $\lim\limits_{n\to\infty}(f(S_n)-f(t_n))\ne 0$ $\quad\therefore f(x)$在R上不一致连续

3.(1)(b).

$S_n=\sqrt{n+1}$ $t_n=\sqrt{n}$

$\lim\limits_{n\to\infty}(S_n-t_n)=\lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0$

$\lim\limits_{n\to\infty}(f(S_n)-f(t_n))=\lim\limits_{n\to\infty}\sin(n+1)-\sin n$

$=\lim 2\cos\frac{2n+1}{2}\cdot\sin\frac{1}{2}$

$\left|\sin x_n^2-\sin y_n^2\right|=\left|2\cos\frac{x_n^2+y_n^2}{2}\sin\frac{x_n^2-y_n^2}{2}\right|$

$\le\left|2\sin\frac{x_n^2-y_n^2}{2}\right|\le\left|x_n^2-y_n^2\right|<\varepsilon$

取 $\delta=\varepsilon$ $=|(x_n+y_n)(x_n-y_n)|$

$\le 2A|(x_n-y_n)|<\varepsilon$

取 $\delta=\frac{\varepsilon}{2A}\Rightarrow f(x)$在$[0,A]$一致连续

(3)(a) $S_n=\frac{1}{2n\pi}$ $t_n=\frac{1}{2n\pi+\frac{\pi}{2}}$

$\lim\limits_{x\to\infty}(S_n-t_n)=\frac{\frac{\pi}{2}}{2n\pi(2n\pi+\frac{\pi}{2})}=0$

$\lim\limits_{x\to\infty}[f(S_n)-f(t_n)]=\left|2\cos(2n\pi+\frac{\pi}{4})\sin(-\frac{\pi}{4})\right|$

$=1$

∴不一致连续

(b)令 $t=\frac{1}{x}\in(0,\frac{1}{\delta})$

$|\sin t_1-\sin t_2|=\left|2\cos\frac{t_1+t_2}{2}\sin\frac{t_1-t_2}{2}\right|$

$\le|t_1-t_2|=\left|\frac{1}{x_1}-\frac{1}{x_2}\right|$

$=\left|\frac{x_2-x_1}{x_1\cdot x_2}\right|\le\frac{|x_2-x_1|}{\delta^2}<\varepsilon$

取 $\delta=\delta^2\cdot\varepsilon$

则 $f(x)$在$(\delta,+\infty)$上一致连续

4. ∵f在I上满足Lipschitz条件

∴$|f(x)-f(y)|\le L|x-y|<\varepsilon$

取 $\delta=\frac{\varepsilon}{L}$ 则f在I上一致连续

3.6 1. $\cancel{|f(x_{n+1})|\le q|f(x_n)|}$

$|f(x_2)|\le q|f(x_1)|$

∵$x_2\in[a,b]$

∴$|f(x_3)|\le q|f(x_2)|$

$|f(x_n)|\le q|f(x_{n-1})|$

$\Rightarrow|f(x_n)|\le q^{n-1}|f(x_1)|$

$q^{n-1}\to 0$

且 $x_1\in[a,b]$ $f(x)$连续

∴$f(x)$有界 ∴$|f(x)|\le M$

$x_0=\frac{a+b}{2}$ 取正数$M>f(x_0)$

∵$\lim\limits_{x\to a^+}f(x)=\lim\limits_{x\to b^-}f(x)=+\infty$

$\exists\delta>0$ 设 $\delta<\frac{b-a}{2}$ 时

$\forall x\in(a,a+\delta)\cup(b-\delta,b)$ $f(x)>M$.

$x_0\in[a+\delta,b-\delta]$ $\forall x\in(a,a+\delta)\cup(b-\delta,b)$

$f(\xi)\le f(x_0)<M<f(x)$

又$f(x)$连续

有界数列$\{x_n\}$

$\lim\limits_{k\to\infty}x_{n_k}=\xi\in[a,b]$

$\lim\limits_{k\to\infty}|f(x_{n_k})|$

$=f(\lim\limits_{k\to\infty}x_{n_k})$

$=f(\xi)=0$

$\lim\limits_{k\to\infty}|f(x_{n_k})|\le\lim\limits_{k\to\infty}q^{n_k-1}|f(x_1)|$

$\lim\limits_{n\to\infty}q^{n-1}|f(x_1)|\le\lim\limits_{n\to\infty}M\cdot q^{n-1}=0$

$|f(x_n)|\le 0$ 即 $f(x_n)=0$

∴至少存在一点$\xi\in[a,b]$,使$f(\xi)=0$

2. $\forall\varepsilon>0$, $\exists X>0$, $x_1,x_2>X$,有$|f(x_1)-f(x_2)|<\varepsilon$

由Cantor $f(x)$在$[0,X+1]$上一致连续 $\forall x\in[0,X+1]$

∴$[0,X+1]$上,$\forall\varepsilon>0$,$\exists\delta_1>0$ $m<f(x)<M$

$|x_1-x_2|<\delta_1$,有$|f(x_1)-f(x_2)|<\varepsilon$

∴在$[0,+\infty)$,$\forall\varepsilon>0$ $\delta=\min\{\delta_1,1\}$

$|x_1-x_2|<\delta_1$ $\lim\limits_{x\to+\infty}f(x)=A$ $\varepsilon_0>0$

$\Rightarrow|f(x_1)-f(x_2)|<\varepsilon$, $\exists X>0$, s.t. $x>X$时,

$f(x)$在$[0,+\infty)$上一致连续 $A-\varepsilon_0<f(x)<A+\varepsilon_0$

$\Rightarrow f(x)$在$[0,+\infty)$有界 $\min\{m,A-\varepsilon_0\}<f(x)<\max\{M,A+\varepsilon_0\}$

3. ∵$\lim\limits_{x\to a^+}f(x)=+\infty$ $\forall M>0$ $\exists\delta_1>0$ ∴$f(x)$有界

∴$0<x-a<\delta_1$ 时 $f(x)>M$.

同理 δ_2 $0<b-x<\delta_2$ $f(x)>M$.

∵$f(x)$在$[a+\delta_1,b-\delta_2]$里连续

∵$f(x)$在(a,b)连续

∴$\exists\xi\in[a+\delta_1,b-\delta_2]$使得对一切$x\in[a+\delta_1,b-\delta_2]$

$f(\xi)\le f(x)$ 令 $x=a+\delta_1$

$f(\xi)\le f(a+\delta_1)=M$

而 $0<x-a<\delta_1$, $0<b-x<\delta_2$

$f(x)>M$. ∴$x\in(a,b)$

$f(\xi)\le M\ f(x)$

$\Rightarrow f(x)$在(a,b)可取到最小值.

5. 证:∵$\lim\limits_{x\to b^-}f(x)=B$. $\forall\varepsilon>0$, $\exists\delta_1$

$0<b-x<\delta_1$ s.t. $|f(x)-B|<\varepsilon$.

① $x_1\in[a,b-\delta_1]$ 使 $f(x_1)>B$.

∵$f(x)$在$[a,b)$连续 ∴$f(x)$在$[a,b-\delta_1]$连续

∴$\exists\xi\in[a,b-\delta_1]$ $x\in[a,b-\delta_1]$

使 $f(\xi)\ge f(x_1)>B$

又ε的任意性 ∴$f(\xi)>f(x)$ $(x\in[a,b))$

② $x_1\in[b-\delta_1,b)$

再取δ_2 使 $0<b-x<\delta_2$ $|f(x)-B|\le\varepsilon$.

直到使得x_1归于第一种情况为止.

15. 设 $f(x)$ 周期为 T　x_0 为一常数

$f(x_0)=f(x_0+nT)$

~~$\lim\limits_{x_0\to x_0} f(x)=f(x+nT)$~~

$\lim\limits_{n\to\infty} f(x_0+nT)=\cancel{\lim\limits_{x\to\infty} f(x_0)}$

$\lim\limits_{n\to\infty} f(x_0)=\lim\limits_{n\to\infty} f(x_0+nT)$

$=\lim\limits_{x\to\infty} f(x)$

即 $x\to\infty$ 极限存在.

又 $f(x)$ 在 R 上连续

$\Rightarrow f(x)$ 在 R 上一致连续.

$f(mT+x)=f(x)$　则 $f(x)$ 在 $[0,2T]$ 上一致连续.

$\forall\varepsilon>0$ $\exists\delta_1>0$，$x_1,x_2\in[0,2T]$ 且 $|x_1-x_2|<\delta_1$.

$|f(x_1)-f(x_2)|<\varepsilon$　放进了不会超过

$\delta=\min\{T,\delta_1\}$　$\forall x_1,x_2\in R$　$|x_1-x_2|<\delta$　$\exists m\in Z^*$

使 $x_1,x_2\in[mT,mT+2T]$　$x_1-mT,\ x_2-mT\in[0,2T]$.

$|(x_1-mT)-(x_2-mT)|<\delta$.

$\Rightarrow|f(x_1)-f(x_2)|=|f(x_1-mT)-f(x_2-mT)|<\varepsilon$.

$\therefore f(x)$ 一致连续.

5. $f(x_1)>B=\lim\limits_{x\to b^-} f(x)$

让 $x_1\notin(b-\delta,b)$

$\exists\, 0<\delta<b-x_1$　$\forall\, 0<b-x<\delta$

$f(x)<f(x_1)$.

$f(x)$ 在 $[a,b-\frac{\delta}{2}]$ 连续.

$\therefore\exists\xi\in[a,b-\frac{\delta}{2}]$　$f(\xi)=M$ 为最大值

$\therefore f(x)\leq f(x_1)<M$　$\therefore M$ 也是 $f(x)$ 在 $[a,b)$ 上最大值

仅对 $0<b-x<\delta$