Wiki: https://en.wikipedia.org/wiki/QR_decomposition

https://en.wikipedia.org/wiki/Givens_rotation

Lemma 1: QR decompostion $\boldsymbol{A} = \boldsymbol{QR}$ is unique whent the matrix $\boldsymbol{A}$ is inveritble with the requirement of positive diagonal entry for upper-triangular matrix $\boldsymbol{R}$.

Lemma 2: thin or reduced QR decompostion $\boldsymbol{A} = \boldsymbol{QR} = [\boldsymbol{Q_1}, \boldsymbol{Q_2}]\begin{bmatrix}\boldsymbol{R_1}\\ \boldsymbol{0}\end{bmatrix} = \boldsymbol{Q_1 R_1}$ is unique when A is full rank n.

where R_1 is an n×n upper triangular matrix, 0 is an $(m - n)$×n zero matrix, Q_1 is m×n, Q_2 is m×$(m - n)$, and Q_1 and Q_2 both have orthogonal columns.

Lemma 3: R_1 is equal to the upper triangular factor of the Cholesky decomposition of $A^H A$ (=$A^T A$ if A is real) if we require that the diagonal elements of R_1 are positive.

1 General least square problem

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{1-1}$$

Where matrix A is $m \times n$, x is one $n \times 1$ vector, and b is $m \times 1$ vector

1.1 Overdetermined

If the linear equations are overdetermined ($m \geq n$) (exactly, overdetermine means $m > n$), the least square solution is

$$\begin{aligned}\widehat{\boldsymbol{x}} &= \boldsymbol{A}^{\dagger}\boldsymbol{b}\\ &= (\boldsymbol{A^H A})^{-1}\boldsymbol{A^H b}\\ &= (\boldsymbol{A^H A})^{-1}\boldsymbol{A^H Ax}\\ &= \boldsymbol{x}\end{aligned} \tag{1-2}$$

So the problem becomes to caculate the inverse of matrix $\boldsymbol{A^H A}$.

By using QR decomposition, the least sqaure equation becomes to

$$\begin{aligned}&\boldsymbol{QRx} = \boldsymbol{b}\\ \lll=\ggg\ &\boldsymbol{Rx} = \boldsymbol{Q^H b}\\ \lll=\ggg\ &\boldsymbol{R_1 x} = \boldsymbol{Q_1^H b}\end{aligned} \tag{1-3}$$

The equation (1-3) can be solved by backward-substitution (for upper-triangular system). (same result can be got by using equation (1-2)).

1.2 Underdetermined (rare)

If the linear equations are underdetermined ($m < n$), the least square solution is

$$\begin{aligned}\widehat{\boldsymbol{x}} &= \boldsymbol{A}^{\dagger}\boldsymbol{b}\\ &= \boldsymbol{A^H}(\boldsymbol{AA^H})^{-1}\boldsymbol{b}\\ &= \boldsymbol{A^H}(\boldsymbol{AA^H})^{-1}\boldsymbol{Ax}\\ &\approx \boldsymbol{x}\end{aligned} \tag{1-4}$$

By using EQ decompostion for $\boldsymbol{A^H}$, the equation (1-1) becomes to

$$\begin{aligned}&(\boldsymbol{QR})^{\boldsymbol{H}}\boldsymbol{x} = \boldsymbol{b}\\ \lll=\ggg\ &\boldsymbol{R^H Q^H x} = \boldsymbol{b}\\ \lll=\ggg\ &\boldsymbol{Q^H x} = (\boldsymbol{R^H})^{-1}\boldsymbol{b}\end{aligned} \tag{1-5}$$

The $(\boldsymbol{R}^{\boldsymbol{H}})^{-1}\boldsymbol{b}$ can be calulated by forward-substitution (for lower-triangular system).

Another alternative slolution is to directly calculate the LQ decompostion of A.

$$\begin{aligned} &\boldsymbol{LQx} = \boldsymbol{b} \\ \lll=\ggg\ &\boldsymbol{Qx} = \boldsymbol{L}^{-1}\boldsymbol{b} \\ \lll=\ggg\ &\boldsymbol{x} = \boldsymbol{Q}^{\boldsymbol{H}}\boldsymbol{L}^{-1}\boldsymbol{b} \end{aligned} \tag{1-6}$$

The $\boldsymbol{L}^{-1}\boldsymbol{b}$ can be calcualted by forward-subsitution.

Another alternative slolution is to directly calculate the RQ decompostion of A.

$$\begin{aligned} &\boldsymbol{RQx} = \boldsymbol{b} \\ \lll=\ggg\ &\boldsymbol{Qx} = \boldsymbol{R}^{-1}\boldsymbol{b} \\ \lll=\ggg\ &\boldsymbol{x} = \boldsymbol{Q}^{\boldsymbol{H}}\boldsymbol{R}^{-1}\boldsymbol{b} \end{aligned} \tag{1-7}$$

The $\boldsymbol{R}^{-1}\boldsymbol{b}$ can be calcualted by backward-subsitution.

2 QR deocomposition (QRD)

In linear algebra, a **QR decomposition** (also called a **QR factorization**) of a matrix is a decomposition of a matrix *A* into a product *A* = *QR* of an orthogonal matrix *Q* and an upper triangular matrix *R*. QR decomposition is often used to solve the linear least squaresproblem, and is the basis for a particular eigenvalue algorithm, the QR algorithm.

3 QRD for complex rectangular matrix (overdetermined)

Generally, we can factor a complex *m*×*n* matrix A, with *m* ≥ *n*, as the product of an *m*×*m* unitary matrix *Q* and an *m*×*n* upper triangular matrix *R*. As the bottom (*m*−*n*) rows of an *m*×*n* upper triangular matrix consist entirely of zeroes, it is often useful to partition*R*, or both *R* and *Q*:

$$A = QR = [Q_1, Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1 \tag{3-1}$$

where R_1 is an *n*×*n* upper triangular matrix, *0* is an (*m* − *n*)×*n* zero matrix, Q_1 is *m*×*n*, Q_2 is *m*×(*m* − *n*), and Q_1 and Q_2 both have orthogonal columns.

Golub & Van Loan (1996, §5.2) call Q_1R_1 the *thin QR factorization* of *A*; Trefethen and Bau call this the *reduced QR factorization*.[1] If *A* is of full rank *n* and we require that the diagonal elements of R_1 are positive, then R_1 and Q_1 are unique, but in general Q_2 is not. R_1 is then equal to the upper triangular factor of the Cholesky decomposition of A^HA (=A^TA if *A* is real).

3.1 Triangular matrix: (Pseudo-) inversion

Moore-Penrose pseudo-inverse

$$\mathbf{A}^{+} := \left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}} = \mathbf{R}^{-1}\mathbf{Q}^{\mathrm{H}} \in \mathbb{C}^{N \times M}$$

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4 QRD for complex rectangular matrix (underdetermined)

If the matrix A is underdetermined (m<n), RQ or LQ decomposition can be used to slove the equation.

5 Givens Rotation

5.1 For real matrix

Vector to be rotated

$$\boldsymbol{v} = \begin{bmatrix} A \\ B \end{bmatrix} \\ = \sqrt{A^2+B^2}\begin{bmatrix} \frac{A}{\sqrt{A^2+B^2}} \\ \frac{B}{\sqrt{A^2+B^2}} \end{bmatrix} \\ = \sqrt{A^2+B^2}\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{bmatrix} \tag{5-1}$$

Where $\theta_0 = \mathrm{atan}\left(\frac{B}{A}\right)$.

Then one rotation is

$$\boldsymbol{Qv} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix}\sqrt{A^2+B^2}\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{bmatrix} \\ = \sqrt{A^2+B^2}\begin{bmatrix} \cos(\theta_1)\cos(\theta_0)+\sin(\theta_1)\sin(\theta_0) \\ -\sin(\theta_1)\cos(\theta_0)+\cos(\theta_1)\sin(\theta_0) \end{bmatrix} \\ = \sqrt{A^2+B^2}\begin{bmatrix} \cos(\theta_0-\theta_1) \\ \sin(\theta_0-\theta_1) \end{bmatrix} \tag{5-2}$$

Let's use a complex multiplication to rewrite the procedure in (5-2)

$$e^{-j\theta_1}\sqrt{A^2+B^2}\,e^{j\theta_0} = \sqrt{A^2+B^2}\,e^{j(\theta_0-\theta_1)} \tag{5-3}$$

This is the reason why the operation is called rotation.

Let's the $\theta_1 = \theta_0 = \theta$, then the vector $\boldsymbol{v}$ will be rotated to one $\boldsymbol{y}$ axis (image will be 0, and similarly, the real part can be rotated to 0 too). Then the rotation matrix

$$\boldsymbol{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{5-4}$$

Can be calculated by

$$\cos(\theta) = \frac{A}{\sqrt{A^2+B^2}} \\ \sin(\theta) = \frac{B}{\sqrt{A^2+B^2}} \tag{5-5}$$

5.2 For complex matrix

Vector to be rotated

$$
\begin{aligned}
\boldsymbol{v} &= \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} Ae^{j\theta_a} \\ Be^{j\theta_b} \end{bmatrix} \\
&= \sqrt{A^2+B^2}\begin{bmatrix} \dfrac{Ae^{j\theta_a}}{\sqrt{A^2+B^2}} \\ \dfrac{Be^{j\theta_b}}{\sqrt{A^2+B^2}} \end{bmatrix} \\
&= \frac{e^{j\theta_b}}{\sqrt{A^2+B^2}}\begin{bmatrix} \dfrac{Ae^{j(\theta_a-\theta_b)}}{\sqrt{A^2+B^2}} \\ \dfrac{B}{\sqrt{A^2+B^2}} \end{bmatrix} \\
&= \frac{e^{j\theta_b}}{\sqrt{A^2+B^2}}\begin{bmatrix} \cos(\theta_0)e^{j(\theta_a-\theta_b)} \\ \sin(\theta_0) \end{bmatrix}
\end{aligned} \tag{5-6}
$$

Where $\theta_0 = \text{atan}\left(\frac{B}{A}\right)$.

Then one complex rotation is

$$
\begin{aligned}
\boldsymbol{Qv} &= \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1)e^{j(\theta_a-\theta_b)} \\ -\sin(\theta_1)e^{-j(\theta_a-\theta_b)} & \cos(\theta_1) \end{bmatrix} \frac{e^{j\theta_b}}{\sqrt{A^2+B^2}}\begin{bmatrix} \cos(\theta_0)e^{j(\theta_a-\theta_b)} \\ \sin(\theta_0) \end{bmatrix} \\
&= \frac{e^{j\theta_b}}{\sqrt{A^2+B^2}}\begin{bmatrix} \cos(\theta_1)\cos(\theta_0)e^{j(\theta_a-\theta_b)} + \sin(\theta_1)e^{j(\theta_a-\theta_b)}\sin(\theta_0) \\ -\sin(\theta_1)e^{-j(\theta_a-\theta_b)}\cos(\theta_0)e^{j(\theta_a-\theta_b)} + \cos(\theta_1)\sin(\theta_0) \end{bmatrix} \\
&= \frac{e^{j\theta_b}}{\sqrt{A^2+B^2}}\begin{bmatrix} e^{j(\theta_a-\theta_b)}[\cos(\theta_1)\cos(\theta_0) + \sin(\theta_1)\sin(\theta_0)] \\ -\sin(\theta_1)\cos(\theta_0) + \cos(\theta_1)\sin(\theta_0) \end{bmatrix}
\end{aligned} \tag{5-7}
$$

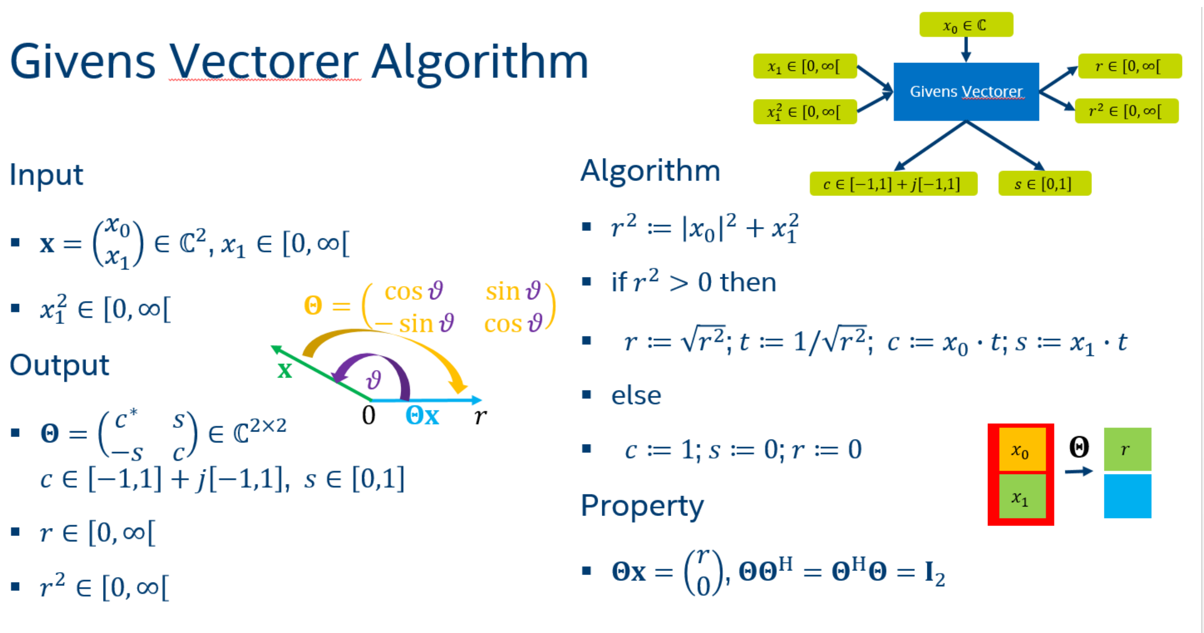
Let's the $\theta_1 = \theta_0 = \theta$, then the vector $\boldsymbol{v}$ will be rotated to one $\boldsymbol{y}$ axis (image will be 0, and similarly, the real part can be rotated to 0 too). Then the rotation matrix

$$
\boldsymbol{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta)e^{j(\theta_a-\theta_b)} \\ -\sin(\theta)e^{-j(\theta_a-\theta_b)} & \cos(\theta) \end{bmatrix} \tag{5-8}
$$

Can be calculated by

$$
\begin{aligned}
\cos(\theta) &= \frac{A}{\sqrt{A^2+B^2}} = \frac{|v_0|}{\sqrt{v_0^2+v_1^2}} \\
\sin(\theta)\, e^{j(\theta_a-\theta_b)} &= \frac{v_0 * v_1'}{|v_0|\sqrt{v_0^2+v_1^2}} \\
-\sin(\theta)\, e^{-j(\theta_a-\theta_b)} &= -conj\left(\sin(\theta)\, e^{j(\theta_a-\theta_b)}\right)
\end{aligned} \tag{5-9}
$$

Efficent implemenation



6 Relaization for QRD

$\boldsymbol{A}$ is one matrix ($m \geq n$).

$$A = \begin{bmatrix} \\ \\ \end{bmatrix}$$