Recursion & States

Recursion is an important technique/terminology in computer science. Expanding from that, its also an important foundation and a really useful technique in combinatorics

Now let's begin with the famous fibonacci Sequence.

Example 1.1:
The Fibonacci Sequence is defined as follows:

$$\begin{cases} f(0) = 1 \\ f(1) = 1 \\ \text{For all positive integers n such that } n \geq 2, f(n) = f(n-1) + f(n-2) \end{cases}$$

This is essentially a recursive sequence because we are trying to find a term with a specific index by getting the value of the previous terms! This can essentially be seen as a large tree:

Start off with $f(n)$, to evaluate $f(n)$ we need to find the value of $f(n-1)$ and $f(n-2)$, while we need $f(n-2)$ and $f(n-3)$ to find the value of $f(n-1)$. Furthermore, to find the value of $f(n-2)$ we need $f(n-3)$ and $f(n-4)$, and so on.

The Fibonacci Sequence $F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$ and from the definition above, we can continue to evaluat the terms in this sequence with only the first and the second term. Sounds interesting?

Now, this recursive terminology can be also incorporated into combinatorics by tracking the state of the system. To demonstrate this, I will present a solution to a self made problem:

Example 1.2: Recursively prove that the number of ways to choose any two balls among n balls is equal to $\binom{n}{2} = \frac{1}{2}n(n-1)$

In this problem, this identity is proven by another proof technique, induction. The process of an inductive proof is almost a resemble version of recursion. Inductive proof requires a ***base case*** and a ***Inductive Process*** that proves a hypothesis of positive integers like dominos(The base case is achieved, and for each n that satisfies, $n+1$ also satisfies).

Now, the base case suffices when n is equal to 2, since there is exactly one way to choose any two balls.

Inductive Step: Consider the recursion between n balls and $n-1$ balls. Let $f(n)$ be the number of ways to choose two balls from n balls. Now, consider the situation when there is another ball besides.

In this scenario, $f(n+1)$, the number of ways to choose two balls from $n+1$ balls, is equal to the sum of $f(n)$ and the cases when one of the ball is the **New** ball and the other ball is among the first n balls, giving us $f(n+1) = f(n) + n \cdot 1$. This suffices to prove that $f(n)$ is the sum of all positive integers from 1 to $n-1$. Finding the arithmetic series finishes.

Sounds interesting? Let's proceed to a problem.

Problem 1.3: How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

When encounter these types of problems, first, try to play around with it and find *patterns.*
Let's start with sequence of length 1. There is obviously 1 such sequence. Thus, $f(1) = 1$

For length 2, we cannot have a sequence that starts with 0 and ends with 0 since this will cause the two 0 s to be consecutive, which is a contradiction. Thus, $f(2) = 0$.

Now, $f(3) = 1$ because the only sequence possible is 010. For $f(4)$, there is exactly one possible sequence: 0110. With $f(5)$, we need to be careful! The only sequence possible is 01010 since we CANNOT have three consecutive 1s! $f(5) = 1$. Now, we actually do not even need the first two zeroes! We just want to shift all our index 2 left, and the sequences all starts with 1, ends with 1, and contains no two consecutive 1s!. This way, we just need to find $f(17)$.

In this scenario, $f(0) = 0, f(1) = 1, f(2) = 1, f(3) = 1$.

For each string that starts with 1 and ends with 1 while also following the conditions, it must end in either 101 or 011. If a string of length n ends in 101, then the string of length $n-3$ that comes before 101 can be any string with length $n-3$ that follows the conditions, since two 1s are allowed to be consecutive. Else if a string of length n ends in 011, then any string of length $n-2$ that follows the conditions can work. Thus, the recursion is

$$f(n) = f(n-2) + f(n-3) \quad n \geq 3$$

To proceed, we use a table to keep track of the values of $f(n)$, and we want to find $f(17)$.

n	$f(n)$
0	0
1	1
2	1
3	1
4	2
5	2
6	3
7	4
8	5
9	7
10	9
11	12
12	16
13	21
14	28
15	37
16	49
17	65

Thus, our answer is $f(17) = \boxed{065}$.

Let's proceed to another problem:

Problem 1.4: A particle moves in the Cartesian Plane according to the following rules:
1. From any lattice point (a, b), the particle may only move to $(a+1, b)$, $(a, b+1)$, or $(a+1, b+1)$.
2. There are no right angle turns in the particle's path.
How many different paths can the particle take from $(0, 0)$ to $(5, 5)$?

For this problem, its obvious not possible to casework on every possible combinations. Thus, we need to recursively find the terms. However, the condition that no right angle turns in the particle's path seems to be cringy. Let $A(x, y)$ be the number of ways to approach (x, y) from $(x-1, y)$, $B(x, y)$ be the number of ways to approach (x, y) from $(x, y-1)$, and $C(x, y)$ be the number of ways to approach (x, y) from $(x-1, y-1)$. Then, we have the following recursive formulas:

$$\begin{cases} A(x,y) = C(x-1,y) + A(x-1,y) \\ B(x,y) = C(x,y-1) + B(x,y-1) \\ C(x,y) = A(x-1,y-1) + B(x-1,y-1) + C(x-1,y-1) \end{cases}$$

Now, its the time to table bash. This will leave as an exercise to the readers
(Hint: *Block Walking*). You can find the full solution on youtube. This should give an answer of $\boxed{083}$ after completing the graph.

Practice problems:

1. Prove that
$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are spacy?

3. (Challenge) How many $2n$ step paths are there from (0, 0) to (n, n) such that for each (x, y) on that path, $x \geq y$. Let C_n be the such number of pathes, prove that
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Now, let's get into a larger topic: **DP** in Math! Recursion is an extremely powerful tool in Combinatorics. First, we are going to introduce a concept called *state*.

Definition 1.5:
State method is used when the problem involves relationship with state.

To demonstrate this, let's look at a problem:

Problem 1.6:
In a square $D_1D_2D_3D_4$, Michael is jumping around the square by strictly obeying to the rule: When he's on a vertex, in the next second, he will jump to one of the possible vertex that is not with equal probability. If Michael is on D_1 in the beginning, in 5 seconds what is the probability that he will land on D_3?

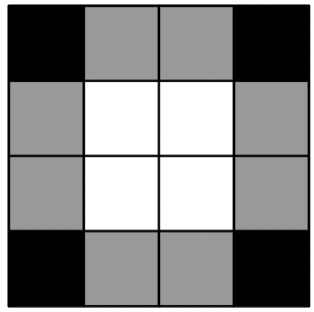
This can be solved by keeping track of the probability state in the nth second. Notice that the only way for Michael to get on D_3 in the nth second is if Michael is **NOT** on D_3 in the $n-1$ second, or else its impossible. Thus, the recursive formula is $P(n) = (1 - P(n-1))\frac{1}{3}$ Since whenever when you are at a vertex that is not D_3, there is a $\frac{1}{3}$ probability for Michael to jump onto D_3.

$$\begin{aligned}
P(1) &= \frac{1}{3} \\
P(2) &= \left(1 - \frac{1}{3}\right)\frac{1}{3} = \frac{2}{9} \\
P(3) &= \left(1 - \frac{2}{9}\right)\frac{1}{3} = \frac{7}{27} \\
P(4) &= \left(1 - \frac{7}{27}\right)\frac{1}{3} = \frac{20}{81} \\
P(5) &= \left(1 - \frac{20}{81}\right)\frac{1}{3} = \boxed{\frac{61}{243}}
\end{aligned}$$

A brief summary of this problem: We essentially just kept track of the state in the system of D_1, D_2, D_3 and D_4, and the important observation that enabled us to create the recursive formula $P(n) = \frac{1}{3}(1 - P(n-1)))$ is to notice that Michael cannot jump to the square on the nth second if he is on the same square on the $n-1$th second. The last step is to divide these into smaller problems: $P(1), P(2), P(3)$ and $P(4)$!

Problem 1.7:
Hamel is having fun jumping in a 4 x 4 grid. He starts off at the left bottom corner square, and he randomly jumps to a vertical or horizontal adjacent square. He cannot jump outside of the large square. After 5 jumps, the probability that he is at one of the squares in the center 2 x 2 square can be expressed as $\frac{m}{n}$ such that both m and n are positive integers and $\gcd(m, n) = 1$. Find $m + n$.

Solution:



We use probability states. Let $P_{n,1}$ be the probability that Hamel jumps onto a black square, $P_{n,2}$ be the probability that Hamel jumps onto a gray square, and $P_{n,3}$ be the probability that Hamel jumps onto the white square.

Now, instead of bashing, we can instead focus on the probability state in which we can reach each types of blocks. However, $P_{n,3}$ is tricky because if we reached a center square, then the probability to reach the diagonally opposite square will be tricky. In each second, Hamel must be on one of the square; therefore, $P_{n,3} = 1 - P_{n,2} - P_{n,1}$. This implies that $P_{5,3} = 1 - P_{5,2} - P_{5,1}$

In order to reach the black squares, the previous jump can ONLY land on one of the gray squares, and there is a probability of $\frac{1}{3}$ to jump from a gray square to a black square.

Furthermore, in order to reach a gray square, we can come from all of the three colors. From a black square, we have a probability of 1 since the only color of square adjacent to a black square is gray. Furthermore, we have a $\frac{1}{2}$ probability to enter from a white square because a white square has 2 adjacent white squares and 2 adjacent gray squares. Lastly, we have a $\frac{1}{3}$ probability to enter from another gray square.

Now, we can produce a system of three equations:

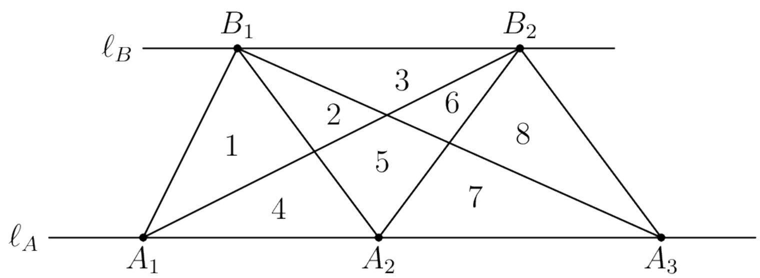
$$\begin{aligned} P_{n,1} &= \frac{1}{3}P_{n-1,2} \\ P_{n,2} = P_{n-1,1} + \frac{1}{3}P_{n-1,2} &+ \frac{1}{2}P_{n-1,3} \\ P_{n,3} &= 1 - P_{n,1} - P_{n,2} \end{aligned}$$

We can list a table to keep track of the probabilities:

n	$P_{n,1}$	$P_{n,2}$	$P_{n,3}$
0	1	0	0
1	0	1	0
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{9}$	$\frac{11}{18}$	$\frac{5}{18}$
4	$\frac{11}{54}$	$\frac{49}{108}$	$\frac{37}{108}$
5	$\frac{49}{324}$	$\frac{341}{648}$	$\frac{209}{648}$

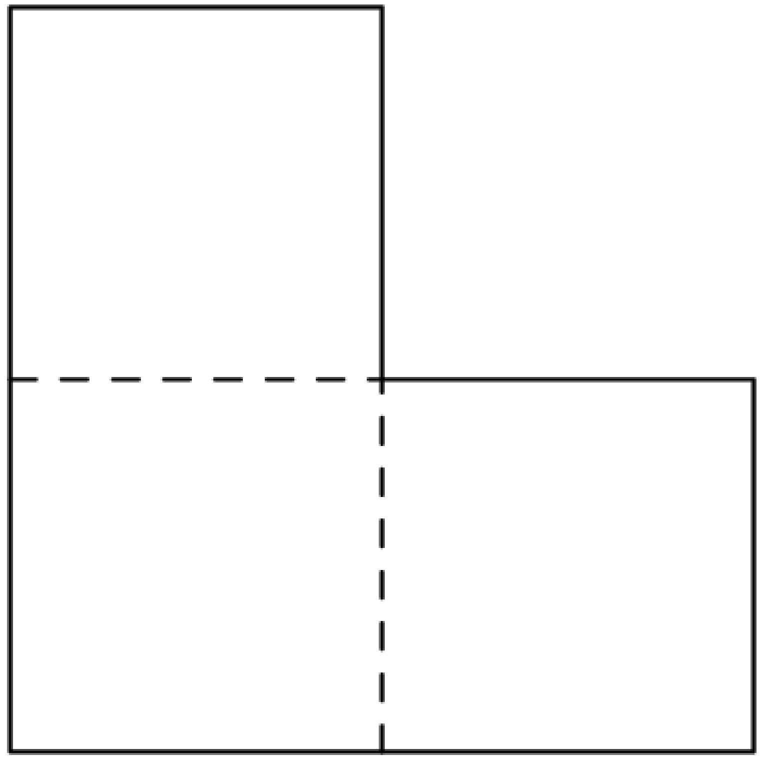
Which according to the table, $P_{5,3} = \frac{209}{648}$ which the answer is $\boxed{857}$.

Problem 1.8:
Let ℓ_A and ℓ_B be two distinct parallel lines. For positive integers m and n, distinct points $A_1, A_2, A_3, \ldots, A_m$ lie on ℓ_A, and distinct points $B_1, B_2, B_3, \ldots, B_n$ lie on ℓ_B. Additionally, when segments $\overline{A_iB_j}$ are drawn for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$, no point strictly between ℓ_A and ℓ_B lies on more than 1 of the segments. Find the number of bounded regions into which this figure divides the plane



In this problem, let $f(m, n)$ denote the number of bounded region in the scenario of m and n points at the top and bottom, respectively. Now, notice that for a line to divide a region A into one more region, the line must both enter and exit that region. This gives a hint on how to do this problem recursively. Furthermore, an important property in this problem is $f(m, n) = f(n, m)$, which is a trivial property for the problem statement.

Now, imagine that we have m points on l_A and n points on l_B.



This is an example of a point added from $f(2,2)$. Notice that there are only two more segments added into the diagram: one line segment each connected with the $n+1$th point and the m points. Now, let points on l_A be defined as $m_1, m_2, ..., m_m$ from left to right and L_{m_a,n_b} be the line segment that connects points m_a and n_b when points on l_b is defined analogously. Then, there is one more region enclosed by $L_{m_m,n_{n+1}}, L_{m_m,n_n}, L_{n_n,n_{n+1}}$. Then, its easy to see that each line expressed in the form L_{n_{n+1},m_k} such that $1 \leq k \leq m-1$ will enter the region from point n_{n+1} and exit at L_{m_m,n_n}, and thus, create m more regions.

Now, we will proceed with the following claim:

Claim: In the scenario with m points at the top and n points at the bottom, for L_{m_p,n_q} to intersect L_{m_x,n_y}, one of the following conditions must be satisfied.

$$\begin{cases} p < x \text{ and } q > y \\ p > x \text{ and } q < y \end{cases}$$

This claim is essentially an application of the Intermediate value theorem. let A and B be two different lines. Then, if the starting point of A is to the left of B, and the ending point of A is to the right of B, then there must be a crossing point that A transforms from the left of B to the right of B, and vice versa. This completes the proof.

Now, let's generate an equation.

Claim: $f(m, n+1) = f(m, n) + m + n\binom{m}{2}$

Proof: Use the same diagram, we will have m more regions as described, and for each line $L_{m_a, n_{n+1}}$, we will have $(m-a)(n)$ lines that will intersect with the line. The proof that there will have exactly $(m-a)(n)$ more regions created by the intersections will left as an exercise to the readers.

Now, we just have to sum up for all of a across 1 to $m-1$. This essentially gives

$$f(m, n+1) = f(m, n) + m + n\,(m - 1 + ... + 2 + 1)$$

$$f(m, n+1) = f(m, n) + m + n\binom{m}{2}$$

□

Now, we will finish by proving this claim. However, a simpler way would be recursively find $f(7, 5)$ based on $f(2, 2) = 4$.

$$\textbf{Claim: } f(m, n) = \binom{m}{2}\binom{n}{2} + mn - 1$$

Proof: We prove by induction.

The base case is $f(2, 2) = 4 = \binom{2}{2}\binom{2}{2} + 2 \cdot 2 - 1 = \boxed{4}$.

Inductive Step:

$$f(m, n+1) = f(m, n) + m + (n-1)\binom{m}{2}$$

$$\binom{m}{2}\binom{n+1}{2} + m(n+1) - 1 = \binom{m}{2}\binom{n}{2} + mn - 1 + m + n\binom{m}{2}$$

$$\text{This simplifies to } \binom{m}{2}\binom{n+1}{2} = \binom{m}{2}\binom{n}{2} + (n-1)\binom{m}{2}$$

$$\binom{m}{2}\binom{n+1}{2} = \binom{m}{2}\left(\binom{n}{2} + n\right)$$

Which is true.

q.e.d.

$$f(7,5) = \binom{7}{2}\binom{5}{2} + 7 \cdot 5 - 1 = \boxed{244}$$

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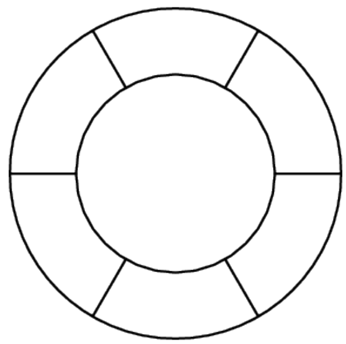
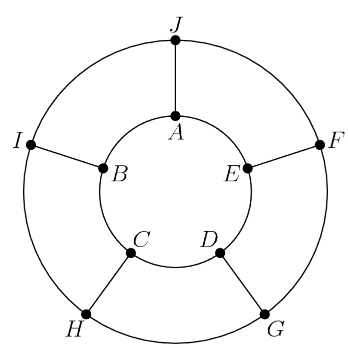
Practice Problems:

1. The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and you will paint each of the six sections a solid color. Find the number of ways you can choose to paint the sections if no two adjacent sections can be painted with the same color.



2. The wheel shown below consists of two circles and five spokes, with a label at each point where a spoke meets a circle. A bug walks along the wheel, starting at point A. At every step of the process, the bug walks from one labeled point to an adjacent labeled point. Along the inner circle the bug only walks in a counterclockwise direction, and along the outer circle the bug only walks in a clockwise direction. For example, the bug could travel along the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with 15 steps that begin and end at point A. Find the remainder when n is divided by 1000.



Now let's proceed to **Linear Recurrence!**

$$Q(x,y) = f(x,y) + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2}{\partial^2 x} & \frac{\partial^2}{\partial y \partial x} \\ \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$\frac{dG}{dt} = -K_{xgl} G(t) I(t) + \frac{T_{gh}}{V_G}$$