Maximal Subset without chains of k multiples in $\mathbb{N}_n$ and *divisor surfaces*

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1 Introduction

Problem 1.1. Define $f(n, 2)$ to be the maximal size of a subset $S \in \mathbb{N}_n$, which $\mathbb{N}_n = \{a | a \in \mathbb{N} \ \& \ a \leq n\}$ such that it follows the axiom that if $b \in S$, $2b$ is not in S. What is $f(n, 2)$?

Cases:

1. $f(1, 2) = 1$, only 1.
2. $f(2, 2) = 1$, being 1 or 2.
3. $f(3, 2) = 2$, either $1, 3$ or $2, 3$.
4. $f(4, 2) = 3$, being $1, 3, 4$.

Corollary 1.2. $f(n, 2) \geq \lceil \frac{n}{2} \rceil$

Proof: Trivial because all odd positive integers less than or equal to n would work. $\square$

Now, consider the power of 2 for each positive integers in the range, rewriting it in the form of $k = 2^n p_i$ for which $n = v_2(k)$, then when $n = 0$ would always work. Now, we define

$$G_k = \{2^k a \mid a \nmid 2 \ \& \ 2^k a < n\}$$

For n, the sets are defined as $G_0, G_1, ..., G_{\lfloor \log_2 n \rfloor}$, and define a set S_G that encompasses all of the above subsets.

Then clearly, we cannot take $2G_k$ if we take all the elements of G_k. Therefore, we either take $G_0, G_2, G_4, ..., G_{2s}$ or $G_1, G_3, ..., G_{2s+1}$. Now, it suffices to find the value of G_s.

Claim 1.3: $|G_s| = \left\lceil \frac{\lfloor \frac{n}{2^s} \rfloor}{2} \right\rceil$ which $\lfloor a \rfloor$ is defined as the largest integer less than or equal to a, and $\lceil a \rceil$ is defined as the smallest integer greater than or equal to a.

Proof: We divide this into two claims:

Subclaim 1.3.1: The number of positive integers k less than or equal to n such that $2^s \mid k$ is equal to $\lfloor \frac{n}{2^s} \rfloor$.

Proof: $2^s a \leq n$ for some positive integer a, and from here, $a \leq \frac{n}{2^s}$. If $\frac{n}{2^s}$ is not an integer, then the maximum pos.int a that is less than or equal to $\lfloor \frac{n}{2^s} \rfloor$ by definition. $\square$

Subclaim 1.3.2: The number of odd positive integers less than n is $\lceil \frac{n}{2} \rceil$

Proof: Followed by the prove of observation 1.1. $\square$

The claim follows by combining these two subclaims. $\square$

Corollary 1.4: $|G_s|$ can be written as $\lceil \frac{m}{2^{s+1}} \rceil$ for some positive integer m.

$$\text{Now we define } p(x,y) = \left\lceil \frac{\lfloor \frac{x}{2^y} \rfloor}{2} \right\rceil \text{, then } p(a_1 2^m + b_1, m) = \left\lceil \frac{a_1}{2} \right\rceil \text{ such that } 0 \leq b_1 < 2^m$$

From here, define n_s to be $n \pmod{2^s}$, then $f(n,s) = \lceil \frac{n-n_s}{2^{s+1}} \rceil$. Therefore, $\boxed{m = n - n_s}$. Then, we split into cases about the parity of a_1 :

1. If a_1 is even, then $a_1 2^m + b_1 \equiv b_1 \pmod{2^{m+1}}$
2. If a_1 is odd, then $a_1 2^m + b_1 = (a_1 - 1)2^m + 2^m + b_1 \equiv 2^m + b_1 \pmod{2^{m+1}}$

Assume that $n = 50$, then

1. $|G_0| = \left\lceil \frac{\lfloor \frac{50}{2^0} \rfloor}{2} \right\rceil = 25,\ 50 \equiv 0 \pmod 2$
2. $|G_1| = \left\lceil \frac{\lfloor \frac{50}{2^1} \rfloor}{2} \right\rceil = 13,\ 50 \equiv 2 \pmod 4$
3. $|G_2| = \left\lceil \frac{\lfloor \frac{50}{2^2} \rfloor}{2} \right\rceil = 6,\ 50 \equiv 2 \pmod 8$
4. $|G_3| = \left\lceil \frac{\lfloor \frac{50}{2^3} \rfloor}{2} \right\rceil = 3,\ 50 \equiv 2 \pmod{16}$
5. $|G_4| = \left\lceil \frac{\lfloor \frac{50}{2^4} \rfloor}{2} \right\rceil = 2,\ 50 \equiv 18 \pmod{32}$
6. $|G_5| = \left\lceil \frac{\lfloor \frac{50}{2^5} \rfloor}{2} \right\rceil = 1,\ 50 \equiv 50 \pmod{64}$

Preposition 1.5. let $g(m,s) = \lceil \frac{m}{2^{s+1}} \rceil$, then $p(m,s) = g(m - 2^s, s)$

Proof. Now let $m = a_1 2^s + b_1$, which by euclidean algorithm, this will always be possible. Then, $f(m, s) = \lceil \frac{a_1}{2} \rceil$.

$$g(m - 2^s, s) = \left\lceil \frac{a_1 2^s + b_1}{2^{s+1}} \right\rceil$$

Which because of the bound that $0 \leq b_1 < 2^s$, $\frac{b_1}{2^{s+1}} < \frac{2^s}{2^{s+1}} = \frac{1}{2}$. Therefore, b_1 will have no impact on the value of $g(m - 2^s, s)$. □

Therefore, $g(m - 2^s, s) = \lceil \frac{a_1}{2} \rceil$, and the prove is complete.

In conclusion, $|G_m| = \lceil \frac{n-2^m}{2^{m+1}} \rceil$, and this allows us to massively simplify our expression. Before that, we need to prove that $|G_0| + |G_2| + ... > |G_1| + |G_3| + ...$ Consider the differences $|G_0| - |G_1|, |G_2| - |G_3|, ..., |G_{2n}| - |G_{2n+1}|$. For the remaining, consider $u = \lfloor \log_2 n \rfloor \equiv 1 \pmod 2$. Now we separate the cases by parity:

1. if $u \equiv 1 \pmod 2$, then we will be able to partition $G_0, ..., G_u$ into $\frac{u+1}{2}$ subsets of $\{G_{2n}, G_{2n+1}\}$ such that $0 \leq n \leq \frac{u-1}{2}$, and clearly $|G_{2n}| - |G_{2n+1}| > 0$, thus, the inequality

$$\sum_{m=0}^{\frac{u-1}{2}} |G_{2m}| - |G_{2m+1}| > 0$$

holds. □

2. If $u \equiv 0 \pmod 2$, then consider the set $S \backslash \{G_u\}$. Then, $|S \backslash \{G_u\}| \equiv 1 \pmod 2$, which by subcase 1, the inequality holds. Now, this simply transforms to

$$\left(\sum_{m=0}^{\frac{u-2}{2}} |G_{2m}| - |G_{2m+1}| \right) + |G_u| > 0$$

Which is true since $|G_u|$ is always larger than 0 unless if $2^u > n$. □

In conclusion, let $a = \lfloor \log_2 n \rfloor$, then $\boxed{f(n, 2) = \sum_{m=0}^{\lfloor \frac{a}{2} \rfloor} \left\lceil \frac{n - 2^{2m}}{2^{2m+1}} \right\rceil}$

However, this will become much more complicated when k becomes large. Now, consider D_n to be the multiples of 2^n. Then, we can express all of these as $|D_{2n}| - |D_{2n-1}|$, and we can just combine as a single summation.

$$\boxed{f(n, 2) = \sum_{i=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^i} \right\rfloor}$$

And the idea is the same for all k other than 2 because we just want to take G_{2n} by subtracting all the multiples of k^{2n} by all the multiples of k^{2n+1}.

Preposition 1.6.

$$f(n,k) = \sum_{i=0}^{\lfloor \log_k n \rfloor} \left\lfloor \frac{n}{k^i} \right\rfloor$$

2 Extension of 2 variables

Problem 2.1. Define $f(n, k_1, k_2)$ to be the maximal size of a subset $S \in \mathbb{N}_n$, which $\mathbb{N}_n = \{a | a \in \mathbb{N} \,\&\, a \leq n\}$ such that it follows the axiom that if $b \in S$, kb is not in S for all k_1, k_2. What is $f(n, k_1, k_2)$?

Consider the multiples of k^0 with the previous strategy. Notice that for all $a \in \mathbb{N}_n$, $a \equiv b \pmod n$ such that $b \in \frac{\mathbb{Z}}{k\mathbb{Z}}$, and I will first solve the problem when $\gcd(k_1, k_2) = 1$ and $k_1, k_2 \mid n$.

Observation 2.1: Assume that $\gcd(k_1, k_2) = 1$, then $f(n,k) \geq (k_1 - 1)(k_2 - 1)\left\lfloor \frac{n}{k_1 k_2} \right\rfloor$
Proof: This is a result of the Chinese Remainder Theorem: We can take every positive integers a that aren't divisible by k_1 or k_2, which by modular arithmetics,

1. $a \equiv 1, ..., k_1 - 1 \pmod{k_1}$
2. $a \equiv 1, ..., k_2 - 1 \pmod{k_2}$

Which since $\gcd(k_1, k_2) = 1$, then by Chinese Remainder Theorem, every pair of $a_1 \pmod{k_1}$ and $a_2 \pmod{k_2}$ will produce a unique residue modulo $k_1 k_2$. Therefore, there are a total $(k_1 - 1)(k_2 - 1)$ of residues in a cycle of length $k_1 k_2$ that is divisible by neither k_1 nor k_2. From here, we simply want to find the number of such cycles, which there are $\left\lfloor \frac{n}{k_1 k_2} \right\rfloor$ of them. $\square$.

However, this is not the end. We will also have $k_1^{2m}(k_1 x_1 + 1), k_1^{2m}(k_1 x_1 + 2)...$ and $k_2^{2m}(k_2 x_2 + 1), k_2^{2m}(k_2 x_2 + 2), ...$ Therefore, let $G_{k_1,m}$ be the set of all positive integers a such that $k_1^m a < n$ and $a \in \frac{\mathbb{Z}}{k_1\mathbb{Z}} - \{0\}$.

Preposition 2.2: Define $G_{k_1,m}$ be the set of positive integers p less than or equal to n such that $v_{k_1}(p) = m$. $G_{k_2,m}$ is defined analogously. then, the sets

$$S_{k_1,m}\{a \mid a \in k_1^{2m} G_{k_1 k_2, 0} \text{ and } a \leq n\}$$

$$S_{k_2,m}\{b \mid b \in k_2^{2m} G_{k_1 k_2, 0} \text{ and } b \leq n\}$$

will work.

Proof: Follow directly from the Chinese Remainder Theorem as a result of *Observation 2.1* and the extended version of *Conjecture 1.4*. $\square$.

Now, we simply want to find the set of quotients when a positive integer $a \in \mathbb{N}_n$ is divided by powers of k_1 and k_2. There yields $p_1 = \left\lfloor \frac{n}{k_1^m} \right\rfloor$ such positive integers, and analogous defined for p_2, which the set will be in the form $\{1, ..., a_{p_1}\} \leftrightarrow \{k_1 m, ..., k_1^m a_{p_1}\}$ and the former set is simply K_{p_1}.

From here, use the Euclid's Algorithm by rewriting $p_1 = xk_1k_2 + r$, which there will simply be $x(k_1-1)(k_2-1)+r$ positive integers in the set $S_{k_1,m}$. From here, the answer is

$$x\left(\frac{p_1-r}{xk_1k_2}\right)\varphi(k_1k_2)+r = \frac{(p_1-r)\varphi(k_1k_2)}{k_1k_2}+r$$

To further simplify this, let's make clear of what $p_1 - r$ is. $p_1 - r$ is basically the largest multiple of k_1k_2 less than p_1, and p_1 is the largest quotient when a positive integer less than or equal to n divided by k_1^m, from our conjecture. $p_1 = \left\lfloor \frac{n}{k_1^m} \right\rfloor$
$p_1 - r$ is simply $k_1k_2 \left\lfloor \frac{\left\lfloor \frac{n}{k_1^m} \right\rfloor}{k_1k_2} \right\rfloor$. The conjecture so far is that this is equal to $\left\lfloor \frac{n}{k_1^{m+1}k_2} \right\rfloor$.

This conjecture is flawed because for $f(6,2,3)$, you can take $1,4,5,6$ which will yield a result of 4 instead of 2. Conjecture 2.2 is disproved by this counterexample. $\blacksquare$.

Salvaged Preposition 2.3: Assume that $k_1 < k_2$, then

$$f(n,k_1,k_2) = \sum_{\zeta=0}^{\infty}\left(\left\lfloor \frac{n}{(k_1k_2)^\zeta}\right\rfloor - \left\lfloor \frac{n}{k_1(k_1k_2)^\zeta}\right\rfloor\right)$$

Proof: It's always possible to partition $\mathbb{N}_n$ into various strings of $\{p, ..., pk_1^a k_2^b\}$ such that $\gcd(k_1,k_2,p) = 1$. First, pick out all positive integers $a \in (\frac{n}{k_1}, n]$. It's easy to prove that this construction works because the function $\frac{n}{x}$ is monotonically decreasing, and the maximum value of of such multiple chain will be extended from n, which is $\frac{n}{k_1}$ and this is excluded from the set. $\square$

Claim: that the next stage is between the set $\left(\left\lfloor \frac{n}{k_1^2k_2} \right\rfloor, \frac{n}{k_1k_2}\right]$
Proof: For the sake of contradiction, assume that there exists an integer between $\left[\frac{n}{k_1k_2}, \frac{n}{k_1}\right]$. This is equivalent to the statement that there exist a chain of multiples a_1, k_1a_1 or a_1, k_2a_1 for any integer $a_1 \in \left[\frac{n}{k_1k_2}+1, \frac{n}{k_1}-1\right]$.

Proceed with bounding: $\frac{n}{k_2} + k_1 \leq a_1k_1 \leq n - k_1 < n$. $\square$

Example: Consider $f(1296,2,3)$, first we take all positive integers between $[649, 1296]$, and then we take $[109, 216], [19, 36], [4, 6], [1]$. This yields $648 + 108 + 18 + 3 + 1 = \boxed{778}$

For $f(1296,3,5)$ instead, we first take $[433, 1296]$. Then, the maximum value of $x \in [1, 432]$ is 86. Then, we take from 86 to $\left\lfloor \frac{86}{3} \right\rfloor + 1$, which is $[29, 86]$. Continue with this, we take $[2, 5]$. This gives $864 + 58 + 4 = \boxed{926}$

However, we see a counterexample that if $n = 20, k_1 = 2, k_2 = 5$, then we can choose $[11, 20]$ and $[2]$ by our algorithm, but notice that 5 can also be taken because $5 \cdot 5 > n$. Therefore, we have to also take positive integers $a \in \left[\left\lfloor \frac{n}{k_1k_2} \right\rfloor, \left\lfloor \frac{n}{k_1} \right\rfloor\right]$, such that $\frac{a}{k_1} > \frac{n}{k_1^2k_2}$ and $k_2a > n$ and $k_1a < \frac{n}{k_1}$.

Salvaged Preposition 2.4.: Assume that $k_1 < k_2$, then

$$f(n, k_1, k_2) = \sum_{\zeta=0}^{\infty} \left(\left\lfloor \frac{n}{(k_1k_2)^\zeta} \right\rfloor - \left\lfloor \frac{n}{k_1(k_1k_2)^\zeta} \right\rfloor \right) + S_n$$

Such that S_n is a set that contains all positive integers a such that

$$\begin{cases} \frac{a}{k_1} > \frac{n}{k_1^2k_2} \\ k_2a > n \\ k_1a < \frac{n}{k_1} \end{cases}$$

Proof. We bound the possible values a: $\frac{n}{k_2} < a < \frac{n}{k_1^2}$. We can ignore the first bound of $a > \frac{n}{k_1k_2}$ because $\frac{n}{k_1k_2} < \frac{n}{k_2}$. From here, we deduce that $k_1^2 < k_2$ must be true for $|S_n| > 0$, and this will produce about $n\frac{k_2-k_1^2}{k_1^2k_2}$ values.

Now, we continue to approximate the maximal number of elements.

$$f(n, k_1, k_2) = \sum_{\zeta=0}^{\infty} \left(\left\lfloor \frac{n}{(k_1k_2)^\zeta} \right\rfloor - \left\lfloor \frac{n}{k_1(k_1k_2)^\zeta} \right\rfloor \right) + S_n$$

Which when $k_1^2 < k_2$, $|S_n| \approx \frac{(k_2-k_1^2)n}{k_1^2k_2}$. For the sake of approximation, assume that the floor values vanish. Then,

$$f(n, k_1, k_2) \approx \sum_{\zeta=0}^{\infty} \frac{n}{(k_1k_2)^\zeta} \left(\frac{k_1-1}{k_1} \right) = \frac{(k_1-1)n}{k_1} \left(\frac{k_1k_2}{k_1k_2-1} \right) = n \left(\frac{k_1k_2-k_2}{k_1k_2-1} \right)$$

Conjecture 2.5:

1. If $k_1^2 > k_2$, then
$$f(n, k_1, k_2) \approx n \left(\frac{k_1k_2-k_2}{k_1k_2-1} \right)$$

2. If $k_1^2 < k_2$, then
$$f(n, k_1, k_2) \approx n \left(\frac{k_1k_2-k_2}{k_1k_2-1} + \frac{k_2-k_1^2}{k_1^2k_2} \right)$$

Proof. consider the map $f : \mathbb{N} \to \mathbb{Z}^2$ defined as $f(k_1^{n_1} k_2^{n_2} r) = (n_1, n_2)$ for all r such that $\gcd(r, k_1 k_2) = 1$.
Then, for each value $a = k_1^{n_1} k_2^{n_2} r$ for some r such that $\gcd(r, k_1k_2) = 1$, then f^{-1} maps every divisors of $\frac{a}{r}$ into lattice points $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Therefore, the number of lattice points is evidently the number of divisors of $\frac{a}{r}$, which is $(n_1 + 1)(n_2 + 1)$.

Subclaim 2.5.1. The maximum number of lattice points that can be selected such that no two are directly connected by an edge is $\left\lceil \frac{(n_1+1)(n_2+1)}{2} \right\rceil$

Proof: FTSOC assume there exists a number c larger than the bound, then it's equivalent to saying there is at least one pair of the points that are directly connected by the Pigeonhole Principle. □

Furthermore, we define

$$f(p, q) : \mathbb{Z}^2 \to \mathbb{Z} \text{ defined by } f(p, q) = |p - n_1| + |q - n_2|$$

And we define layer a to be the set of lattice points (m_1, m_2) such that $f(m_1, m_2) = a$.
And if we call $f(p, q)$ to be the layer of point (p, q) in $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, then from the proof of subclaim 2.5.1, we can only take $1, 3, 5, ...2y + 1$th layers, or $2, 4, 6, .., 2y$ layers, which the layers cannot be adjacent, and we can take all the numbers in that layer to maximize the size of the subset.

Now, notice that if a exists such that $ak_1^{m_1} k_2^{m_2}, ak_1^{n_1} k_2^{n_2} \in \mathbb{N}_n$, then a would exist on the graph of both. However, this ensures our algorithm is correct when we select all the positive integers within the range of $(\frac{n}{k_1}, n]$, then every positive integers between will be the "peak" of the lattice points. Assume we take m as the peak, then we will need the 3rd layer, which is $\frac{m}{rk_1k_2}$, $\frac{m}{rk_2^2}$, $\frac{m}{rk_1^2}$ if they exist. This corresponds to $(p - 2, q), (p, q - 2)$ and $(p - 1, q - 1)$ if (p, q) is the peak.

This bounding gives

$$\frac{n}{k_1^2 k_2} < \frac{m}{k_2^2} < \frac{m}{k_1 k_2} \leq \frac{n}{k_1 k_2}$$

If $k_1^2 < k_2$, then $\frac{n}{k_2} < \frac{n}{k_1^2} < \frac{n}{k_1}$, which accounts for the second part of the conjecture and we are done. □

3 Generalized problem to $|P_n|$ variables, and generalize a closed formula

Problem 3.1. Let P_n be the set of prime divisors of n(for example, $P_{50} = \{2, 5\}$ and $P_{30} = \{2, 3, 5\}$). Define $\mathbb{N}_n = \{a \in \mathbb{N} | a \leq n\}$ What is the maximal size of a subset $S \in \mathbb{N}_n$ such that it follows the axiom that if $b \in S$, $kb \notin S$ for all $k \in P_n$.

Consider the mapping $f : \mathbb{N} \to \mathbb{Z}^{|P_n|}$ defined as

$$f(k) = \left(v_{p_1}(k), v_{p_2}(k), ..., v_{p_{|P_n|}}(k)\right)$$

Then, we can show that

$$f^{-1}\left(\prod_{m=1}^{|p_n|} p_m^{a_m}\right)$$

Will form an object with at most $|p_n|$ dimensional linear surface, which is trivial because the map connects a with $\frac{a}{p_i}$ for all $1 \leq i \leq |p_n|$. □

Assume that we choose the peak to be an arbitrary value $a = p_1^{a_1} p_2^{a_2} ... p_{|p_n|}^{a_{|p_n|}}$.

Claim 3.2. The most number of vertices we can choose such that no two are directly connected is

$$\left\lceil \frac{1}{2} \prod_{m=1}^{|p_n|} (a_m + 1) \right\rceil$$

Proof. Assume contradiction that we can take more than that, then by the Pigeonhole Principle, we must have two nodes chosen that are consecutive. □

Define the function

$$\zeta_a : \mathbb{Z}^{|P_n|} \to \mathbb{Z}_s \ni \zeta(b_1, ..., b_{P_n}) = \sum_{w=1}^{P_n} |a_w - b_w|$$

Which $s = \sum_{i=0}^{|P_n|} a_i$, and $a = p_1^{a_1} p_2^{a_2} ... p_{|p_n|}^{a_{|p_n|}}$.

Now, we denote the $|P_n|$ dimensional linear surface produced by performing f^{-1} on particular number m to be the *divisor surface* of m, or define as D_m. For the sake of simplicity, we define $g(m, P_n)$ to be the maximum number of vertices that can be selected with no direct adjacency on the divisor surface of m.

Preposition 3.2.:

$$g(m, P_n) = \left\lceil \frac{\tau(m)}{2} \right\rceil$$

For which $m \in \mathbb{N}_n$ and $P_m \subseteq P_n$, and the maximum occurs by taking all numbers p such that

$2 \mid \zeta(f^{-1}(p))$.

Proof. When $\tau(m) \equiv 0 \pmod 2$, then let

$$A_{m,1} = \left\{ a \mid a \in \prod_{m=1}^{|P_n|} \mathbb{Z}_m \text{ and } \zeta_m(f^{-1}(a)) \equiv 0 \pmod 2 \right\}$$
$$A_{m,2} = \left\{ a \mid a \in \prod_{m=1}^{|P_n|} \mathbb{Z}_m \text{ and } \zeta_m(f^{-1}(a)) \equiv 1 \pmod 2 \right\}$$

Then $|A_1| = |A_2|$ because of symmetry in the cube. $\square$

When $\tau(m) \equiv 1 \pmod 2$, then by parity, D_n will be a $|P_n|$ dimensional linear surface with dimensions being $2a_1 \times 2a_2 \times ... \times 2a_{|P_n|}$, that is, $m = p_1^{2a_1} p_2^{2a_2}....$ By symmetry, we can arbitrarily select one of the $2^{|P_n|}$ vertices. Now, consider the $|P_n|$ edges that are connected to the vertice, then the number of lattice points on the edge will be odd because the numbers of line segments between two adjacent lattice points on the edge will be even, and there will be 1 more points then the edge that connects them. Let the edge be e_1, and let its length be $2a_1$, the set of lattice points on it to be E

Now, we can partition the $2a_1 + 1$ points into two sets: $B_{m,1} = A_{m,1} \cap E$ and $B_{m,2} = A_{m,2} \cap E$, and

$$|A_1| + |A_2| - (|B_1| + |B_2|) = (2a_1) \prod_{w=2}^{|P_n|} (2a_w + 1)$$

Which is even. Furthermore,

$$D_m \backslash E \cong D_{\frac{m}{p_1}}$$

Which since $|D_{\frac{m}{p_1}}| \equiv 0 \pmod 2$,

$$A_{\frac{m}{p_1},1} = A_{\frac{m}{p_1},2}$$

Therefore, I proved that for once any points in B_1 and B_2 is reached, and it must proceed right, then this is equivalent to the problem of $D_{\frac{m}{P_1}}$. Now, this only remains to consider B_1 and B_2. For the sake of convenience, let $G_a(p) = \zeta_a(f^{-1}(p))$, which becomes trivial because all elements in B_1 has an odd distance away, and all elements in B_2 has an even distance away, and $|B_1| = |B_2| + 1$. $\square$

However, there are duplicates in these constructions: a positive integer a can appear on the divisor surface of multiple positive integers, and it can potentially be double counted. Now, we simply just need consider how many times each element in $\mathbb{N}_n$ will be selected, and subtract from here.
When $n = 60$ and we want to find the number of times 5 is contained. We have $20, 30, 45$, which corresponds to $(2,0,0), (1,1,0), (1,0,1), (0,2,0)$.

Lemma 3.3.. Let $s = |P_n|$, and let $q_1 = \left\lfloor \log_{p_1} \frac{n}{a} \right\rfloor, q_2 = \left\lfloor \log_{p_s} \frac{n}{a} \right\rfloor$ for some positive integer a, then we show that the number of times a occurs in the divisor surface of divisors of n is

$$\sum_{x=0}^{\left\lfloor \frac{q_2}{2} \right\rfloor} \binom{2x+s-1}{s-1}$$

Proof. Notice that $p_i^{q_2}$ is monotonically increasing, and this implies that $p_1^{a_1}...p_s^{a_s} \leq p_s^{2q_2}$ such that $\sum_{\alpha=1}^{s} a_\alpha = q_2$, and $\log_{p_i} n - \log_{p_i} a > 2q_2$ for all $p_i \in P_n$. Therefore, this is simply finding the number of ordered s-tuples $(a_1, a_2, ..., a_s)$ of nonnegative integers such that

$$\sum_{i=1}^{s} a_s = 2k \text{ for all } k \in \mathbb{Z}_p$$

Which, by the Pigeonhole Principle, is $\binom{2k+s-1}{s-1} + ... + \binom{s-1}{s-1}$. □

Issue 3.4.: Recognized there is a problem because there are duplicates, and there can be another configuration that has the same size of the cardiality of $A_{m,1}$. Disproved.

Preposition 3.5. Define $C_{a,P_b} \subseteq \mathbb{Z}^{|P_b|}$ as the set of lattice points for some positive integers a, b, and S_{a,P_b} as the set of positive integers $\{x | P_x \subseteq P_b \text{ and } x \leq a\}$ such that the bijective function $f : S_{a,P_b} \to C_{a,P_b}$ is defined as

$$f(x) = (v_{p_1}(x), v_{p_2}(x), ..., v_{p_{|P_m|}}(x))$$

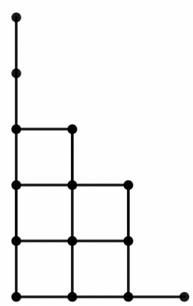

Figure 1: $C_{441,\{3,7\}}$

issue 3.4 is solved because for each point in C_{a,P_b}, there are at least 1 value that corresponds, and will produce different values by multiplying with a number that is relatively prime to all elements in P_b. Therefore, there won't be any duplicates that we need to consider.

Now, we just need to find the cardiality of C_{a,P_n}, and then sum up across all c such that $\gcd(c, n) = 1$ with $\left|C_{\lfloor \frac{n}{c} \rfloor, |P_n|}\right|$

Corollary 3.6.

$$|P_n| = 1 : \lceil \log_{p_1} a \rceil$$

$$|P_n| = 2 : \sum_{k=1}^{\lceil \log_{p_1 p_2} a \rceil} \lceil \log_{p_1} a + \log_{p_2} a - 2k - k(\log_{p_1} p_2 + \log_{p_2} p_1) \rceil$$

$$= \lceil \log_{p_1 p_2} a \rceil \lceil \log_a p_1 p_2 \log_a p_1 \log_a p_2 \rceil - \lceil \log_{p_1 p_2} a \rceil \left(2 + \log_{p_1} p_2 + \log_{p_2} p_1\right)$$

Proof. The case of $|P_n| = 1$ is trivial because $p_1^{\lfloor \log_{p_1} a \rfloor}$ is the maximum power of p_1 less than or equal to a, and the minimum power of p_1 is p_1^0, which yields $\lfloor \log_{p_1} a \rfloor + 1 = \lceil \log_{p_1} a \rceil$.

In the case of $|P_n| = 2$, rewrite all elements in S_{a,P_b} as $(p_1 p_2)^b \cdot k^c$ for $k \in P_b$, then for all $0 \leq b \leq \lfloor \log_{p_1 p_2} a \rfloor$, the number of c will be equal to $\left\lceil \frac{a}{(p_1 p_2)^b} \right\rceil$, and summing all of them yields the result.

□