Something about Trigonometry

"Have you ever consider about the square root of i?"
"Have you ever wonder about the relationship between e, π and i?"

Hopefully, after this handout, you will be able to answer the above questions. Now, lets begin!

Consider the equation $i^2 = -1$. From this, we can conclude that i is a solution to the equation $x^2 + 1 = 0$. In fact, $-i$ is indeed the other solution. But what if we want to find $\sqrt{i}$, which is equivalent to finding the solution $x^2 - i = 0$?

This can seem really counterintuitive: Do we have to invent another imaginary unit? $s?w?z?$. However, this question can actually be tackled, and we do not need any of the letters above.

$$\sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

The first way can be straightforward. We start from $x^2 - i = 0$, and now, if we let $x = a + bi$, then we will get $(a+bi)^2 = i \to a^2 - b^2 + 2abi = i \to 2ab = 1$ and $a^2 - b^2 = 0$.

Now, we need to find the principle root, which $a > 0$ and $b > 0$. This generates a system of equations:

$$a^2 - b^2 = 0$$
$$2abi = i$$

Which $ab = \frac{1}{2}$ and $a^2 = b^2$. Since both are positive integers, $a = b = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and thus, proves the result. Now, we can also see that $-\sqrt{i}$ is also another solution to $x^2 - i = 0$. Now, consider this equation:

$$x^4 + 1 = 0$$

Can we conclude that all the solutions of $f(x) = x^2 - i$ are the ones of $g(x) = x^4 + 1$? The answer is no. By the Fundamental Theorem of Algebra, if a polynomial can be expressed as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

Then it must have n **COMPLEX** solutions in the form of $a_k + b_k i$. Therefore, we can essentially conclude that the solutions of $f(x)$ are definitely the one for $g(x)$, but it cannot be reversed. This works the same from $g(x) = x^4 + 1$ to $p(x) = x^8 - 1$.

Now, draw the solution set of $f(x) = x^2 - i, g(x) = x^4 + 1, p(x) = x^8 - 1$ and $q(x) = x^{16} - 1$ to see what can you observe. From the sketch, we can probably notice that there exists symmetry between the solutions, and the distance between two adjacent solutions starts to get smaller and smaller. Construct a straight line between each two adjacent solutions. Before proceeding to the next step, make a hypothesis of what the diagram of the solution set of $x^{2^k} - 1$ would look like if $k \to \infty$.

Theorem 1.1: ***The Unit Circle:***
The diagram of the solution set of $x^{2k} - 1$ as k approach to ∞ will be a circle with radius 1. A solution of $f(x) = x^{2k} - 1$ will be in the form of $z = a_k + b_k i$, and $\sqrt{a_k^2 + b_k^2}$, in other words, $|z| = 1$.

This can be proven even visually. Now, let O be the origin, and WLOG let A_1, A_2 be random two adjacent solutions. Now, $\angle A_n O A_{n-1} = \frac{360}{2^k}$ when $1 \leq n \leq 2^k$. Let the endpoint of A_1 be (x_1, y_1) and the endpoint of A_2 be (x_2, y_2). We have

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1x_2 + y_1y_2)}$$

$$= \sqrt{2 - 2(x_1x_2 + y_1y_2)} = \sqrt{2 - 2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}} = \sqrt{2 - 2|A_1||A_2| \cos\left(\frac{360}{2^k}\right)^{\circ}}$$

Which since $\lim_{x \to 0} \cos x^{\circ} = 1$, this distance approaches to 0. End proof!

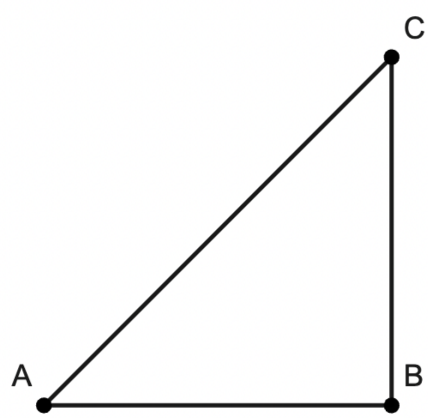
Exercise:

1. Show that all solutions in the form $x^m - 1$ lies on the unit circle on a complex plane.

Now, since for all such complex numbers, the modulos is 1. This implies that it can be expressed as $a+bi$ such that $a^2+b^2=1$. Recall the trig identity of $\sin^2\theta+\cos^2\theta=1$, we can let $a=\cos\theta$ and $b=\sin\theta$, which is essentially true that there are, in fact, an infinite number of such angle θ in radians that satisfies this equation for a fixed radian of y, as each rotation of 2π around the origin will yield the same angle with respect to the unit circle.

Fact 1.1:
All solutions of x^m-1 can be expressed in the form $\cos\theta+i\sin\theta$ for some radian θ.

This should be pretty intuitive when we plot this onto the complex plane. Now, if we let $m=8$ from the original question, and we see that $\pm\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i$ are indeed solutions to the equation. Is there anything special with $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$?



$AB=BC=\frac{\sqrt{2}}{2}$ and $AC=1$ by the pythagorean theorem. Then, we can see that this is actually an isoceles right triangle, which is a $45-45-90$ triangle.

Now, $45=\frac{360}{8}$, is this a coincidence? Furthermore, $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ has an angle of $315°$ with respect to the origin. Rewrite in the radian form, we see that $\frac{\pi}{4}$, $\frac{7\pi}{4}$, $\frac{2\pi}{4}$, $\frac{6\pi}{4}$, $\frac{4\pi}{4}$ and $\frac{0\pi}{4}$ all works when we plug $\cos\theta+i\sin\theta$ into the original equation $x^8-1=0$. This generates a hypothesis: Would all solutions in the form of

$$\cos\frac{2k\pi}{8} + i\sin\frac{2k\pi}{8}$$

work? Can we find a generalized form of the solutions of $x^k - 1 = 0$ for which k is a positive integer? A way to test the hypothesis would be to plug in $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$, which both works. In fact, $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ and $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ are the two solutions of $x^2 + i = 0$.

Theorem 1.2: Roots of Unity
All the solutions of the equation $x^n - 1$, $k \in \mathbb{Z}^+$ are in the form of

$$\cos\frac{2\pi k}{n} + \sin\frac{2\pi k}{n} i \quad k, n \in \mathbb{Z}^+$$

We begin by proving that if $\cos\theta + i\sin\theta$ is a solution, then $\cos(\theta + \frac{2\pi}{n}) + i\sin(\theta + \frac{2\pi}{n})$ is also a solution. By symmetry, the solutions are equally spaced out on a unit circle. when the angle is 0, this is always true, and while the solutions are equally distributed and by Theorem 1, the angle between two solutions are always the same, $\frac{2\pi}{n}$ in radians.

Now, it suffices to show that all such solutions will have an angle of $\frac{2\pi k}{n}$ with respect to origin, when $k \in \mathbb{Z}/n\mathbb{Z}$.

q.e.d.

However, doesn't the equation $x^n - 1$ still seems kind of weird for the solutions to be in that form? What if we do not have $x^n - 1$, but $x^n - c$ for any constant? If we can express the solutions to be a^k for some values of a, k for which $a^{kn} = 1$, then this will seem more natural. We will start to explore the exponential form of $\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$.

First, consider the functions $\cos x$ and $\sin x$ start with $\cos x$, we begin by generating the function

$$\cos x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

Notice that $\frac{d^2y}{dx^2}\cos x = -\cos x$ and $\frac{d^2y}{dx^2}\sin x = -\sin x$, we seek to find a relationship. Notice that $\cos 0 = 1$, thus, $a_0 = 1$. Now, if we take the second derivative, a_0 and a_1 vanishes, leaving $= 2a_2 - 6a_3 x - 24a_4 x^3 + ...$ Set the coefficient of each term equal, we have $a_0 = -2a_2$, $a_1 = -12a_3$, ...

However, a_{2k+1}, the odd coefficients, vanishes because $\frac{d^{2k+1}y}{dx^{2k+1}}\cos x = \pm\sin x$, which $\sin 0 = 0$.

Theorem 1.3: Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Now lets prove $\cos x$, and the rest will be left as exercises to the reader.

Claim: $a_{2k} = (-1)^k \frac{1}{(2k)!}$

Proof: We prove by induction.
Base Case: $a_0 = (-1)^0 \cdot \frac{1}{0!} = 1.$ $\square$

Inductive Step: If $a_{2k} = (-1)^k \frac{1}{(2k)!}$, then $a_{2k} = -1(2k+2)(2k+1)a_{2(k+1)} \rightarrow -\frac{a_{2k}}{(2k+2)(2k+1)} = a_{2(k+1)}$. This gives

$$a_{2(k+1)} = (-1)^{k+1} \frac{1}{(2k+2)(2k+1)(2k)!} = (-1)^{k+1} \frac{1}{(2k+2)!} \square$$

0.1 *Exercise:*

1. Prove the Taylor expansion of e^x.
2. Prove the Taylor expansion of $\sin x$
3. Find the Taylor expansion of $\sqrt{1-x}$

Now, doesn't the sine and cosine expansion look suspicious? The expansion of e^x looks like the sum of the expansions of $\sin x$ and $\cos x$, except for some negative ones. We can see that for every cycles of 4 in the exponent of x, the odd ones are parts of $\cos x$ and the even ones are parts of $\sin x$.

Now, its the time for the identity that we will use throughout this handout:

Identity 1.1: Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

proof: Use Taylor expansion.

$$(1)e^{\imath\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

$$(2)\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...$$

$$(3)i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!}$$

Now, its easy to see that $e^{i\theta} = \cos\theta + i\sin\theta$ by adding equation 2 and equation 3 and set the sum equal to equation 1. $\square$

Instead of writing the solutions of $x^k - 1 = 0$ as $\cos\frac{2\pi n}{k} + i\sin\frac{2\pi n}{k}i$, we can just write it as $e^{\frac{2\pi n}{k}i}$. When $k \mid n$, $e^{\frac{2\pi n}{k}i} = 1$. Now, consider mutliply $e^{\frac{2\pi n}{k}i}$ by $e^{2\pi i}$. By the addition of exponents, we have

$$e^{\frac{2\pi n}{k}i} \cdot e^{2\pi i} = e^{\frac{2\pi n}{k}i + 2\pi i} = e^{\frac{2\pi i(n+k)}{k}}$$

Which since $e^{2\pi i}$ is equal to 1, this does not change the result. Now, describe it on the complex plane, its essentially a rotation of 360° around the origin, which the angle with respect to the x axis remains invariant.

0.2 *Exercises*

1. How many different solutions are there to the equation $x^k - 1$? (General a formula for all such positive integers k)

2. (Medium) Find the product of the distinct solutions of $x^k - 1$. (Try not to use the Vieta formulas)
3. (Hard) Find the sum of distinct solutions of $x^7 - 1$.
4. (Challenging) Find $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$ *Hint: 12, 13*

Theorem 1.4:
The roots of unity are the complex numbers that can be expressed in the form $e^{i\theta}$, which when raised to some integer power, the expression will result to 1.
The kth roots of unity are basically the solutions to $x^k - 1 = 0$.

Recall the problem $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$. Then, these are all 7th roots of unity.

If you tried the practice problem, then you should notice that $e^{ix} + e^{-ix} = 2\cos x$. This is because $e^{ix}+e^{-ix} = \cos x + i\sin x + \cos x - i\sin x = 2\cos x$, as desired. Therefore,

$$\cos\frac{2\pi k}{7} = \frac{e^{\frac{2\pi ki}{7}} + e^{\frac{-2\pi ki}{7}}}{2}$$

Which the sum desired is just

$$\sum_{k=1}^{3}\frac{e^{\frac{2\pi ki}{7}} + e^{\frac{-2\pi ki}{7}}}{2} = \sum_{k=1}^{6}\frac{1}{2}\left(e^{\frac{2\pi k}{7}}\right)$$

Since the sum of kth roots of unity is simply 0 for all k, we have $\frac{0-1}{2} = \boxed{-\frac{1}{2}}$

0.3 *Exercises*

1. Evaluate $\sin\frac{2\pi}{9}+\sin\frac{4\pi}{9}+\sin\frac{6\pi}{9}+\sin\frac{8\pi}{9}$

2. Find the value of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$

3. *(Source: AMC 12)* Find the number of ordered pairs (a,b) such that $(a+bi)^{2003}=a-bi$

4. *(Source: AMC 12)* Let $f(x)=1+\cos x+i\sin x-\cos 2x-i\sin 2x+\cos 3x+i\sin 3x$. Show that no root of $f(x)$ has a magnitude of 1.

5. (Challenging) Let
$$\sum_{n=0}^{m} x^m = 1+x+x^2+x^3+...+x^m$$
Prove that for all positive m, all the m roots are $m+1$th roots of unity.

6. (Extremely Challenging) Let $f(x)=x^7+x^2+1$. Find ALL roots of $f(x)$. *hint: 6*

1 *Hints*

1. Take the derivative each time, and plug in $x = 0$ to find the constants.

2. Consider prove by contradiction(Assume that there is a root of $f(x)$ with a modulo of 1).

3. Induct on m.

4. Consider rewriting $\cos x$ in terms of Euler's identity.

5. What will happen if u add e^{ix} by e^{-ix}?

6. Try to think outside of the box. Think about turning $f(x)$ into $a(x^{kn} - 1) + x^{n-1} + ...x + 1$. Then since $x^n - 1 \mid x^{kn} - 1$ by factoring and all solutions of $S = x^{n-1} + x^{n-2} + ... + 1$ are the nth roots of unity, $S \mid x^{kn} - 1$