Expected Value

In this handout, we are going to explore the idea of Expected Value. The applications of Expected Value are very broad, as it can give an insight of the expected outcomes of an event. For example, the expected winning amount of money in a lottery.

To begin, let's look at a simple example:

Example 1.1:
James rolls a 6 sided standard dice. What is the expected number he rolled?

Now, the expected value is nothing fancy, but the average outcome. In an even-distributed probability scenario, we can just take the average of all possible outcomes. However, this technique will not solve the problems when the probability aren't distributed equally.

For this problem, its simply $\frac{1+2+3+4+5+6}{6}$ since we have a total of 6 outcomes, and each number from $1-6$, inclusive, has a $\frac{1}{6}$ probability to be rolled. Evaluate it out gives 3.5 as our answer.

Now, we will dig into the official expression of Expected value:

Lemma 1.1–Expected Value
Let X be a random discrete variable, and suppose that X would attain the value X_i with probability P_i, then

$$E[X] = \sum_{n=1}^{\infty} X_n P_n$$

In this formula, we are essentially multiplying the probability that a value will be obtained by the value itself. This can be seen as a weighted average of all values based on the distribution of their probability.

Now, going back to example 1.1, since each value from 1 to 6, inclusive, can be obtained by the probability $\frac{1}{6}$, we can evaluate the Expected Value by

$$\frac{1}{6}\cdot 1+\frac{1}{6}\cdot 2+\frac{1}{6}\cdot 3+\frac{1}{6}\cdot 4+\frac{1}{6}\cdot 5+\frac{1}{6}\cdot 6=\frac{21}{6}\rightarrow \boxed{3.5}$$

Which yields the same result.

Now, James got a larger dice and surprisingly, each of the integers from 1 to 5 are obtained with probability $\frac{1}{12}$ while 6 can be rolled with probability $\frac{7}{12}$. Since the probability is distributed with $1:1:1:1:1:7$, we cannot use the way of add all possible outcomes and divide by the number of the outcomes. We need to evaluated the ***Weighted Average*** in order to get our desired result.

$$E[\text{Roll}] = \frac{1}{12}(1+2+3+4+5) + \frac{7}{12}(6) = \frac{5}{4} + \frac{7}{2} = \boxed{\frac{19}{4}}$$

0.1 *Exercise:*

1. Michael has an unfair 6 sided die which the probability to roll n is proportional to n. What is the expected number that he will roll?

2. Josh has a fair coin. He flips it 4 times. What is the expected number of heads he will get?

3. Josh has a fair coin. He flips it 10 times and records the orientation(h/f) for each flip. A consecutive string of orientation is called *Same* if its the same orientations(Example: TFFFTFTFTF, FFF is a *Same* string with length 3 while T is a *Same* orientation string with length 1). What is the expected number of *Same* strings in a random selected orientation record of length 10?

Next, we will explore some problems that will involve **Nested** Expected value:

Example 1.2: Roll another die
Hanna has an eight side die with each number being equally possible to be rolled. In each second, Hanna rolls the die until she rolls 8 the first time which she will stop. What is the expected number of seconds Hanna will roll the die?

For the sake of simplicity, let $E[8]$ denote the expected number of seconds she will roll until she reaches 8. Now, we can see that each roll is independent of the previous roll. Thus, $E[8]$ resets every-time after a roll. Therefore, in the next roll, we have a $\frac{7}{8}$, the probability of not rolling a 8, of $E[8]$ again, and the second increases by 1.

Mathematically, we get the following equation:

$$E[8] = \frac{7}{8}(E[8] + 1) + \frac{1}{8}(1)$$

Which we can just treat $E[8]$ as a variable and this is the variable that we are solving for. Isolate $E[8]$ gives $\frac{1}{8}E[8] = 1 \rightarrow E[8] = \boxed{8}$.

Now, think about problem 3 from the exercise. The first flip can be both head and tail, and each successive flip will have a $\frac{1}{2}$ probability of restarting a *Same* string of 1 or add 1 to the *Same* string, which does not change the number of *Same* String.

Therefore, we can write it in the equation form

$$E[i] = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2}$$

and it works with all successive flips after the first flip, giving the expected number of $1 + \frac{1}{2}(10-1) = \boxed{5.5 \textit{ Same} \text{ Strings}}$

Sounds convenient right? This way, you do not need to use recursion to find all possible outcomes of the number of same strings, which will take forever.

Now, let's walk through a really challenging problem.

Problem 1.1 (Source: 2022 AoPS Mock AMC 12/22)
Kara forms a list $a_1, a_2, \ldots, a_n$ as follows: She sets $a_1 = 1$. For each $k \geq 2$, she sets S_{k-1} to be the set of positive integers that are both multiples of a_{k-1} and factors of 30. She then sets a_k equal to a number picked at random from S_{k-1}. She stops forming the list once she sets $a_i = 30$ for some i. Find the expected value of n, the number of elements in the list.

This problem can arguably be a blend of NT and Combo problem. $30 = 2 \cdot 3 \cdot 5$. Therefore, we separate into stages.

$E[0]$ is the expected value of 1 occurring. $E[1]$ is the expected value of $2, 3, 5$ occurring, $E[2]$ is the expected value of $2 \cdot 3, 3 \cdot 5, 2 \cdot 5$ occurring.

$$E[0] = \frac{1}{8}E[0] + \frac{3}{8}E[1] + \frac{3}{8}E[2] + 1$$

.

This is because $E[3]$ will always be equal to 1 because once we get to 30, we add 1 to the length of the set and then stops. Since we can pick 1 again as 1 is a multiple of 1, we will have $\frac{1}{1+3+3+1}E[0] = \frac{1}{8}E[0]$. Now, we have $\frac{3}{8}$ probability of choosing $2, 3, 5$, and $\frac{3}{8}$ probability of choosing one of the pairwise multiples of $2, 3, 5$.

With the same idea,

$$E[1] = \frac{1}{4}E[1] + \frac{1}{2}E[2] + 1$$

$$E[2] = \frac{1}{2}E[2] + 1$$

Now, we can solve the equations backward. $E[2] = 2$, $E[1] = \frac{8}{3}$ and finally, $E[0] = \frac{22}{7}$. However, we need to add 1 since we already have 1 in the beginning. This yields the answer of $\frac{22}{7} + 1 = \boxed{\frac{29}{7}}$

0.2 ***Exercise***

1. *(Source: 2022 Fall MIMC)* If I roll a fair dice repeatedly until I roll a six, what is the expected value of the sum of the values that I rolled in the process?

2. (Challenging) Let $a = 123456$. Now, he swaps any two digits in a. What is the expect value of the swapped number?

Now, I will introduce a really powerful tool in solving Expected Value problems.

Lemma 1.2-The Linearity of Expectation
Let $X_1, X_2, ..., X_n$ be the n outcomes that can be attained by the random value X. Regardless of the dependency of the events,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Which

$$E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2] + ... + E[X_n]$$

This seems to be counterintuitive, especially thinking about the utility of this when dealing with a series of dependent events. Now, let's look at an example:

Problem 1.2: Tapping
2022 babies sit in a circle. In the next second, each baby taps the shoulder of the baby sitting to the left or to the right, with a probability of $\frac{1}{3}$ tapping the left and $\frac{2}{3}$ tapping the right. What is the expected number of baby who got tapped?

Using the Linearity of Expectation, we can analyze the probability that a baby is tapped. let the probability be P, since $X_1, ..., X_{2022}$ are equal, its just $2022P$.

Since there is a $\frac{2}{3}$ probability to be tapped from the left, and $\frac{1}{3}$ probability to be tapped from the right, $P = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$. Now, by LOE, $E = 2022 \cdot \frac{2}{9} = \boxed{\dfrac{1348}{3}}$

OCMC TST(Adjusted)
A ruthless farmer wants to kill a cow for dinner tonight, and a kind farmer wants to save the cow from the ruthless farmer. If the cow makes 3 "Moo" sound, then the farmer will slaughter the cow. Starting from the 0 In each 5 second, there is a $\frac{1}{5}$ chance for the cow to "Moo". What is the expected number of seconds the kind farmer can save the cow?

Now, we can break down onto the expected number of seconds before the first "Moo" sound and multiply by 3 by Linearity of Expectation since they are same random events.

Let it be $E[1]$. Then, we have

$$E[1] = \frac{4}{5}(E[1] + 5) + \frac{1}{5} \cdot 5 \rightarrow E[1] = 25$$

.

Now, since we can treat $E[3]$ as three identical random events $E[1]s$ according to LOE. $E[3] = 3E[1] = 3 \cdot 25 = \boxed{75}$

0.3 Problems

1. *(Source: AMC 10)* Five balls are arranged around a circle. Chris chooses two adjacent balls at random and interchanges them. Then Silva does the same, with her choice of adjacent balls to interchange being independent of Chris's. What is the expected number of balls that occupy their original positions after these two successive transpositions?

2. There is a square in the coordinate plane with the four vertices $(0,0),(0,2),(2,2),(2,0)$. The frog starts at $(1,1)$, and in each second, it will randomly jump in a direction that is available with equal probability. What is the expected number of seconds that the frog takes to return to $(1,1)$?

3. In the Cartesian coordinate plane, I stand at the origin. I begin at $(0,0)$, and in each second, if I currently occupy (x,y), then I have a $\frac{1}{4}$ probability to land on $(x+1,y)$ and $(x,y+1)$ and a $\frac{1}{2}$ probability to land on $(x+1,y+1)$. After 100 seconds, what is the expected value of the sum of the coordinates of the point I'm currently on?

4. Mam has four boxes, and she counts the score of each position. In each second, Mam switches two boxes randomly, and add n point if the box at position n is switched. After 24 seconds, what is the expected value of the sum of the 4 scores?

5. *(Source: OCMC TST)* I have 2 buttons, one which adds 1 to the current number and the other button doubles the current number. I press the two buttons 6 times each with random order. If the initial number is 1, what is the expected value of the number after these 12 operations?

6. *(Source: AMC 12)* What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1,2,3,\dots,30\}$? (For example the set $\{1,17,18,19,30\}$ has 2 pairs of consecutive integers.)

$$\sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} dist(i,j)$$

To find an explicit formula for the sequence $b_{n,c} = cb_{n-1,c} + c^{n-1}$ with $b_{0,c} = 0$, we can analyze the recursion.

Let's expand the recursion for a few terms:

$b_{n,c} = cb_{n-1,c} + c^{n-1} = c(cb_{n-2,c} + c^{n-2}) + c^{n-1} = c^2 b_{n-2,c} + cc^{n-2} + c^{n-1} = c^2(cb_{n-3,c} + c^{n-3}) + cc^{n-2} + c^{n-1} = c^3 b_{n-3,c} + c^2 c^{n-3} + cc^{n-2} + c^{n-1}$

Continuing this process, we can observe the pattern:

$b_{n,c} = c^n b_{0,c} + c^{n-1}c^0 + c^{n-2}c^1 + \ldots + c^1 c^{n-2} + c^0 c^{n-1} = c^n \cdot 0 + c^{n-1} + c^{n-1} + \ldots + c^{n-1} + c^{n-1} = n \cdot c^{n-1}$

Therefore, we have an explicit formula for the sequence $b_{n,c}$, where $c \in \mathbb{N}$ and $b_{0,c} = 0$:

$b_{n,c} = n \cdot c^{n-1}$

This formula holds for all values of n and satisfies the given recursion and initial condition.