

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).
3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.
4. Some guidance on teaching (Section 8).
5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle,

sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line³ is short for straight line. Take a fixed **plane**⁴, once and for all, and consider just lines that lie in it. The plane and the lines are **sets**⁵ of **points**⁶. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A , B , C , etc., and lines by lower-case roman letters l , m , n , etc.

Axioms are statements we will accept as true⁷.

Axiom 1 (Two Points Axiom). *There is exactly one line through any two given points. (We denote the line through A and B by AB .)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all

³Line is undefined.

⁴Undefined term

⁵Undefined term

⁶Undefined term

⁷ An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at A and passes through B by $[AB]$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A , B and C be points that are not collinear. The **triangle** ΔABC is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**⁸ by \mathbb{R} .

Definition 4. We denote the **distance**⁹ between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a , b , and c . The usual thing for a triangle ΔABC is to take $a = |BC|$, i.e. the length of the side opposite the vertex A , and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom¹⁰). *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. $|AB| = |BA|$;
3. *if C lies on AB , between A and B , then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A , and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k .*

⁸Undefined term

⁹Undefined term

¹⁰ Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with ¹¹

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A , called its **vertex**;
2. two rays $[AB]$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;
3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle. Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

¹¹ Students may notice that the first equality implies the second.

Definition 12. Suppose that A , B , and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB]$ and $[AC]$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A , B , C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α, β, γ , etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle \alpha|$, as the case may be.

Axiom 3 (Protractor Axiom). *The number of degrees in an angle (also known as its degree-measure) is always a number between 0° and 360° . The number of degrees of an ordinary angle is less than 180° . It has these properties:*

1. *A straight angle has 180° .*
2. *Given a ray $[AB]$, and a number d between 0 and 180 , there is exactly one ray from A on each side of the line AB that makes an (ordinary) angle having d degrees with the ray $[AB]$.*
3. *If D is a point inside an angle $\angle BAC$, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned 0° , full angles 360° , and reflex angles have more than 180° . To be more exact, if A , B , and C are noncollinear points, then the reflex angle “outside” the angle $\angle BAC$ measures $360^\circ - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD]$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is ‘an angle of’ (for instance) 45° , if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly 90° .

Definition 15. An angle is **acute** if it has less than 90° , and **obtuse** if it has more than 90° .

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC , then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180° .

Definition 17. When two lines AB and AC cross at a point A , they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC , and also between D and E on the line DE . Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

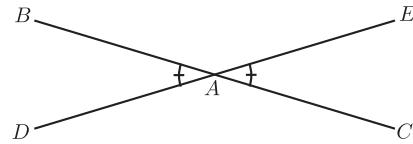


Figure 1.

Theorem 1 (Vertically-opposite Angles).

Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180° . In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180^\circ, \\ |\angle CAE| + |\angle BAE| &= 180^\circ, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0^\circ, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

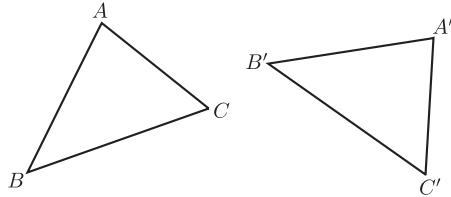


Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS¹²).

If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$,

or

(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, and $|\angle C| = |\angle C'|$,

or

(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $|CA| = |C'A'|$

then the triangles ΔABC and $\Delta A'B'C'$ are congruent.

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to 90° , by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal¹³. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles).

(1) In an isosceles triangle the angles opposite the equal sides are equal.

(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS] $\therefore \angle B = \angle C$.

¹²It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.

¹³ The simple “equal” is preferred to “of equal length”

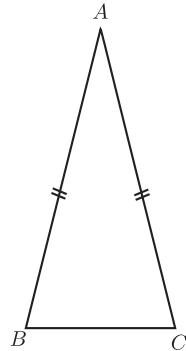


Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
 ΔABC is congruent to ΔACB [ASA]
 $\therefore |AB| = |AC|$, ΔABC is isosceles. \square

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SAS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). \square

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l \parallel m$ for “ l is parallel to m ”.

Axiom 5 (Axiom of Parallels). *Given any line l and a point P , there is exactly one line through P that is parallel to l .*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

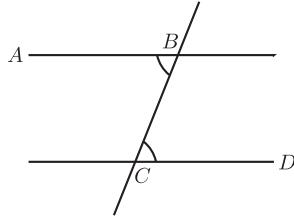


Figure 4.

Theorem 3 (Alternate Angles). *Suppose that A and D are on opposite sides of the line BC.*

- (1) *If $|\angle ABC| = |\angle BCD|$, then $AB \parallel CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.*
- (2) *Conversely, if $AB \parallel CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.*

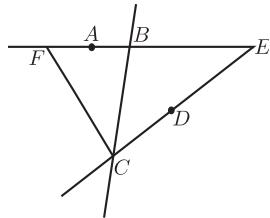


Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E . Let us assume that E is on the same side of BC as D .¹⁴ Take F on EB , on the same side of BC as A , with $|BF| = |CE|$ [Ruler Axiom]

¹⁴Fuller detail: There are three cases:

1°: E lies on BC . Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
 2°: E lies on the same side of BC as D . In that case, take F on EB , on the same side of BC as A , with $|BF| = |CE|$. [Ruler Axiom]
 Then ΔBCE is congruent to ΔCBF . [SAS]

Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF . [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that F lies on DC . [Ruler Axiom]
Thus AB and CD both pass through E and F , and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

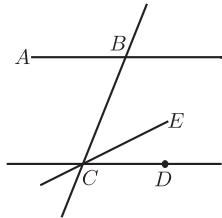


Figure 6.

(2) To prove the converse, suppose $AB \parallel CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB . By Axiom 5, there is only one line through C parallel to AB , so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. \square

Theorem 4 (Angle Sum 180). *The angles in any triangle add to 180° .*

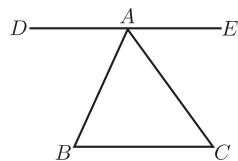


Figure 7.

so that F lies on DC . [Protractor Axiom]
Thus AB and CD both pass through E and F , and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A . Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A , parallel to BC , with D on the opposite side of AB from C , and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]

Then AB is a transversal of DE and BC , so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC , so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ &= |\angle DAB| + |\angle CAE| + |\angle BAC| \\ &= |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. \square

Definition 25. Given two lines AB and CD , and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding angles**¹⁵.

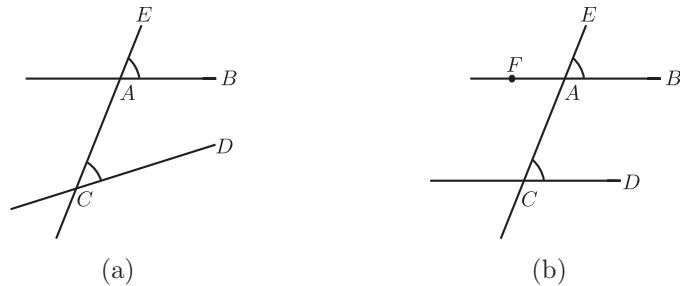


Figure 8.

Theorem 5 (Corresponding Angles). *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

¹⁵with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE . Then we have

$$|\angle EAB| = |\angle FAC| \quad [\text{Vertically opposite angles}]$$

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$$|\angle EAB| = |\angle FAC| \quad [\text{Vertically opposite angles}]$$

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. \square

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.¹⁶

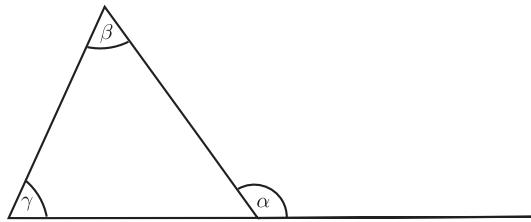


Figure 9.

Theorem 6 (Exterior Angle). *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A . Then

$$|\alpha| + |\angle A| = 180^\circ \quad [\text{Supplementary angles}]$$

and

$$|\angle B| + |\angle C| + |\angle A| = 180^\circ. \quad [\text{Angle sum } 180^\circ]$$

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. \square

¹⁶The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

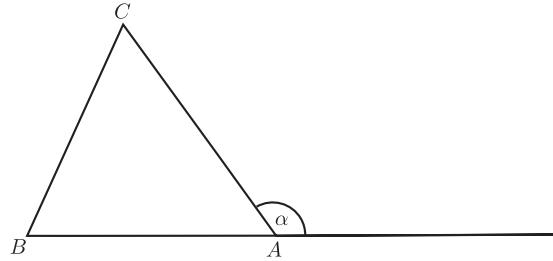


Figure 10.

Theorem 7.

- (1) In $\triangle ABC$, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
- (2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.

- (1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
 $|AD| = |AB|$. [Ruler Axiom]

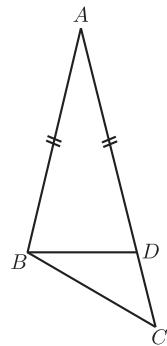


Figure 11.

See Figure 11. Then $\triangle ABD$ is isosceles, so

$$\begin{aligned} |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\ &= |\angle ABD| && \text{[Isosceles Triangle]} \\ &< |\angle ABC|. \end{aligned}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)

Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

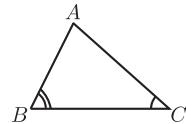


Figure 12.

If it could happen that $|AC| \leq |AB|$, then

either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. \square

Theorem 8 (Triangle Inequality).

Two sides of a triangle are together greater than the third.

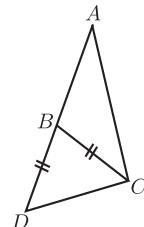


Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$ ¹⁷ we have

$$|\angle BCD| < |\angle ACD|.$$

¹⁷ B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180^\circ$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to $\triangle ADC$ we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1.¹⁸ *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing through a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB .

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

¹⁸In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

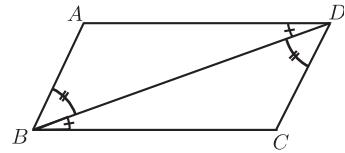


Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB \parallel CD$ and $AD \parallel BC$. Then

$$|\angle ABD| = |\angle BDC| \quad [\text{Alternate Angle Theorem}]$$

$$|\angle ADB| = |\angle CBD| \quad [\text{Alternate Angle Theorem}]$$

ΔDAB is congruent to ΔBCD . [ASA]

$$\therefore |AB| = |CD|, |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|. \quad \square$$

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360° . It follows that adjacent angles add to 180° . Theorem 3 then yields the result. \square

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. \square

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

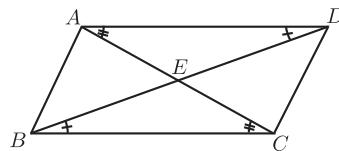


Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of $\triangle ADE$ and $\triangle CBE$.

In detail: Let AC cut BD in E . Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && [\text{Alternate Angle Theorem}] \\ |AD| &= |BC|. && [\text{Theorem 9}] \end{aligned}$$

$\therefore \triangle ADE$ is congruent to $\triangle CBE$. [ASA] \square

Proposition 6 (Converse). *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of $\triangle ABE$ and $\triangle CDE$. Then use Alternate Angles. \square

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180° , so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

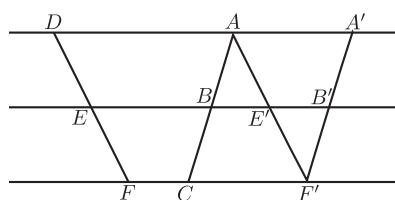


Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD \parallel BE \parallel CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE' \parallel DE$, cutting EB at E' and CF at F' .
Draw $F'B' \parallel AB$, cutting EB at B' . See Figure 16.

Then

$$\begin{aligned} |B'F'| &= |BC| && [\text{Theorem 9}] \\ &= |AB|. && [\text{by Assumption}] \\ |\angle BAE'| &= |\angle E'F'B'|. && [\text{Alternate Angle Theorem}] \\ |\angle AE'B'| &= |\angle F'E'B'|. && [\text{Vertically Opposite Angles}] \\ \therefore \Delta ABE' &\text{ is congruent to } \Delta F'B'E'. && [\text{ASA}] \\ \therefore |AE'| &= |F'E'|. \end{aligned}$$

But

$$\begin{aligned} |AE'| &= |DE| \text{ and } |F'E'| = |FE|. && [\text{Theorem 9}] \\ \therefore |DE| &= |EF|. \end{aligned}$$
□

Definition 36. Let s and t be positive real numbers. We say that a point C divides the segment $[AB]$ in the ratio $s : t$ if C lies on the line AB , and is between A and B , and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l cuts $[AB]$ in the ratio $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B , and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n . Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \dots, D_{m-1}, D_m = D, D_{m+1}, \dots, D_{m+n-1}, D_{m+n} = B,$$

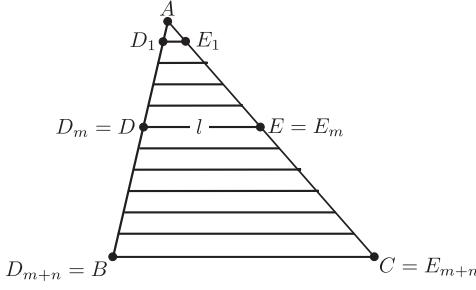


Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \dots, [D_iD_{i+1}], \dots, [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines D_1E_1, D_2E_2, \dots parallel to BC with E_1, E_2, \dots on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \dots, [E_{m+n-1}C]$$

have the same length,

[Theorem 11]

and $E_m = E$ is the point where l cuts $[AC]$.

[Axiom of Parallels]

Hence E divides $[AC]$ in the ratio $m : n$. \square

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB]$ and C'' on $[AC]$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D . Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so ΔABC is similar to $\Delta A'B'C'$.

[Definition of similar]

□

Remark 5. The Converse to Theorem 12 is true:

Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC .

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

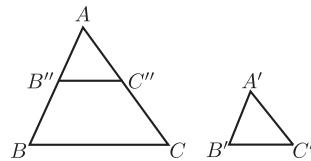


Figure 18.

Proof. We may suppose $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$$\begin{aligned}
 \Delta AB''C'' &\text{ is congruent to } \Delta A'B'C' & [\text{SAS}] \\
 \therefore |\angle AB''C''| &= |\angle ABC| \\
 \therefore B''C'' &\parallel BC & [\text{Corresponding Angles}] \\
 \therefore \frac{|A'B'|}{|A'C'|} &= \frac{|AB''|}{|AC''|} & [\text{Choice of } B'', C''] \\
 &= \frac{|AB|}{|AC|} & [\text{Theorem 12}] \\
 \frac{|AC|}{|A'C'|} &= \frac{|AB|}{|A'B'|} & [\text{Re-arrange}]
 \end{aligned}$$

Similarly, $\frac{|BC|}{|B'C'|} = \frac{|AB|}{|A'B'|}$

□

Proposition 8 (Converse). *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC , so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC . \square

6.9 Pythagoras

Theorem 14 (Pythagoras). *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

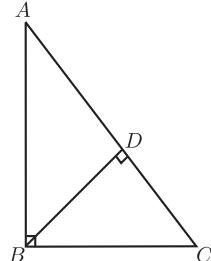


Figure 19.

Proof. Let ΔABC have a right angle at B . Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A .
 $\therefore \Delta ABC$ is similar to ΔADB .

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC .

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC|(|AD| + |DC|) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras). *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

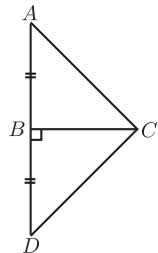


Figure 20.

Proof. (Idea: Construct a second triangle on the other side of $[BC]$, and use Pythagoras and SSS to show it congruent to the original.)

In detail: We wish to show that $|\angle ABC| = 90^\circ$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned}
 |DC| &= \sqrt{|DC|^2} \\
 &= \sqrt{|BD|^2 + |BC|^2} && [\text{Pythagoras}] \\
 &= \sqrt{|AB|^2 + |BC|^2} && [|AB| = |BD|] \\
 &= \sqrt{|AC|^2} && [\text{Hypothesis}] \\
 &= |AC|.
 \end{aligned}$$

$\therefore \Delta ABC$ is congruent to ΔDBC . [SSS]
 $\therefore |\angle ABC| = |\angle DBC| = 90^\circ$. \square

Proposition 9 (RHS). *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B' , and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. \square

Proposition 10. *Each point on the perpendicular bisector of a segment $[AB]$ is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C , hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. \square

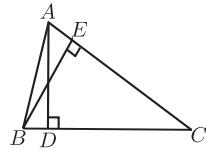


Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by “area of ΔABC ”¹⁹.

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

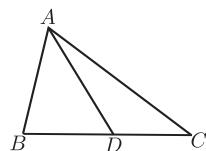


Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h , so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

¹⁹ $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles²⁰.

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas²¹.

Proposition 14. *The area of a rectangle having sides of length a and b is ab .*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC .

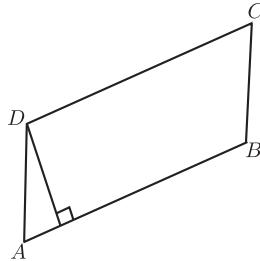


Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD , and as the length of the perpendicular segment from D onto AB .*

²⁰ If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

²¹ Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, ΔABD and ΔCDB . These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. \square

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called a radius. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the diameter** of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B , and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π .

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

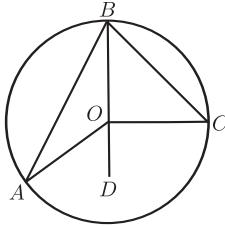


Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D . Then

$$\begin{aligned} |OA| &= |OB|. && [\text{Definition of circle}] \\ \therefore |\angle BAO| &= |\angle ABO|. && [\text{Isosceles triangle}] \\ \therefore |\angle AOD| &= |\angle BAO| + |\angle ABO| && [\text{Exterior Angle}] \\ &= 2 \cdot |\angle ABO|. \end{aligned}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{aligned} |\angle AOC| &= |\angle AOD| + |\angle COD| \\ &= 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\ &= 2 \cdot |\angle ABC|. \end{aligned}$$

□

Corollary 2. All angles at points of the circle, standing on the same arc, are equal. In symbols, if A , A' , B and C lie on a circle, and both A and A' are on the same side of the line BC , then $\angle BAC = \angle BA'C$.

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC :

Converse to Corollary 2: If points A and A' lie on the same side of the line BC , and if $|\angle BAC| = |\angle BA'C|$, then the four points A , A' , B and C lie on a circle.

Proof. Consider the circle s through A , B and C . If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s . We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. \square

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90^\circ$.*

Proof. The angle at the centre is a straight angle, measuring 180° , and half of that is 90° . \square

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180° , so is straight, and so the line BC passes through the centre. \square

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to 180° .*

Proof. The two angles at the centre standing on the same arcs add to 360° , so the two halves add to 180° . \square

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to 180° , then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. \square

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2 , where r is its radius. For this reason, we say that the area of the disc is πr^2 .

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. \square

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.

- (1) *Each tangent is perpendicular to the radius that goes to the point of contact.*
- (2) *If P lies on the circle s , and a line l through P is perpendicular to the radius to P , then l is tangent to s .*

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP .

Let the perpendicular to the tangent from the centre O meet it at Q . Pick R on PQ , on the other side of Q from P , with $|QR| = |PQ|$ (as in Figure 25).

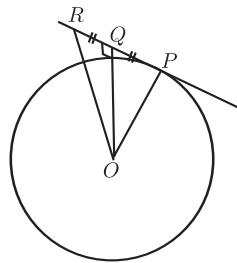


Figure 25.

Then ΔOQR is congruent to ΔOQP .

[SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP , as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P , and hence is not on the circle.)

In detail: Let Q be any point on l , other than P . See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && [\text{Pythagoras}] \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

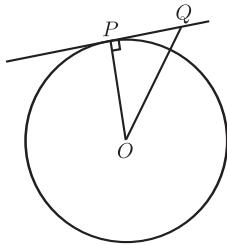


Figure 26.

- $\therefore Q$ is not on the circle. [Definition of circle]
- $\therefore P$ is the only point of l on the circle.
- $\therefore l$ is a tangent. [Definition of tangent] \square

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. \square

The circles described in Corollary 6 are shown in Figure 27.

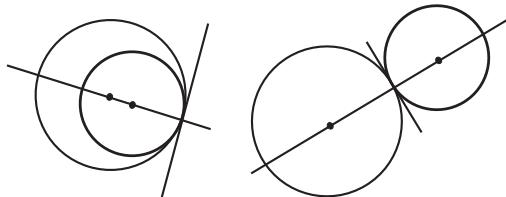


Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *taking* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.

- (1) *The perpendicular from the centre to a chord bisects the chord.*
- (2) *The perpendicular bisector of a chord passes through the centre.*

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.)
See Figure 28.

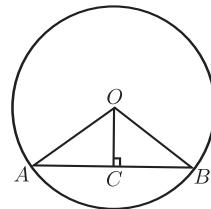


Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| \\ |OC| &= |OC| \end{aligned} \quad [\text{Definition of circle}]$$

$$\begin{aligned} |AC| &= \sqrt{|OA|^2 - |OC|^2} \\ &= \sqrt{|OB|^2 - |OC|^2} \\ &= |CB|. \end{aligned} \quad [\text{Pythagoras}]$$

$\therefore \Delta OAC$ is congruent to ΔOBC . [SSS]
 $\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.

Let C be the foot of the perpendicular from O on AB .

By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.

Thus CO is the perpendicular bisector of AB .

Hence the perpendicular bisector of AB passes through O . \square

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A , B , and C , then its centre lies on the perpendicular bisector of each side of the triangle ΔABC .*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

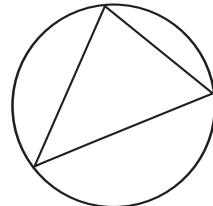


Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

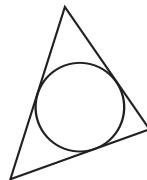


Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

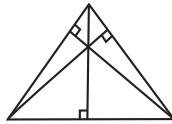


Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A , such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB]$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of 30° , 60° , and 45° . It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l , passing through a given point not on l .

4. Line perpendicular to a given line l , passing through a given point on l .
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of 60° , without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.