NERS 576:
Project Presentation

Numerical Emission Spectra of Trapped Electrons in Relativistic Ion Cavities

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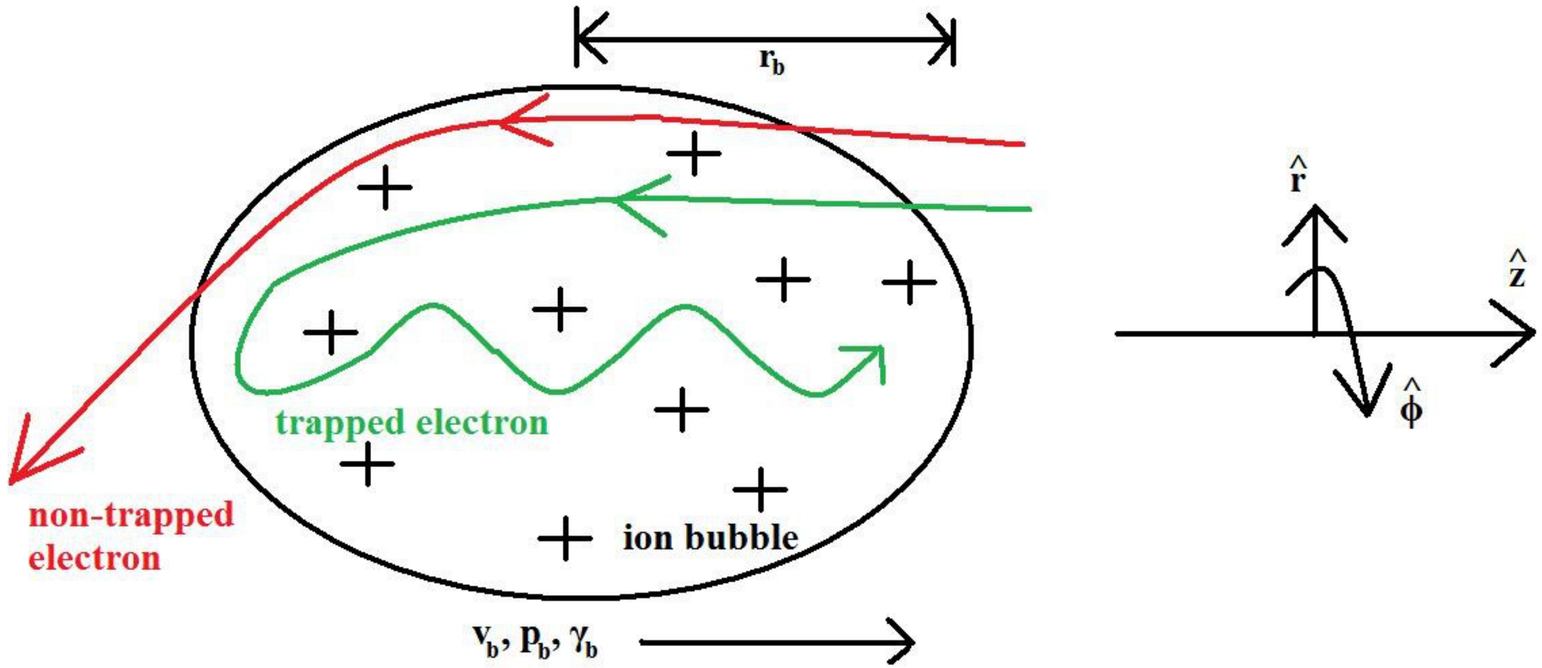
- PROBLEM: We want a high-energy, high-brightness X-ray source. At the moment, we can only do this well with a synchrotron accelerator, which is big and expensive.
- We could leverage the orders-of-magnitude shrinking that laser-driven plasma accelerators offer!
- How? Blast all the electrons out of a bubble of plasma to make an ion cavity. Send the ion cavity through a plasma.
- Some electrons that collide with the ion cavity will become trapped inside it and will rattle around.
- Periodic electron motion means radiation.
- We will consider the emission spectrum of an electron trapped in such a cavity in this talk.

Our model



- We will assume electrons the bubble traps are at rest in the lab frame.
- We will assume the ion bubble is strongly relativistic and spherical in the lab frame.

Electromagnetic fields from the bubble:

$$\mathbf{f}(z,r,t) = \mathbf{F}(z,r,t) + \mathbf{v}(t) \times \mathbf{B}(z,r,t)$$

$$= \begin{cases} -\frac{m_e\,\omega_p^2}{2}\left(\hat{z}\left(z(t) - v_b\,t - \beta_r(t)\,r(t)\right) + \hat{r}\left(1 + \beta_z(t)\right)\frac{r(t)}{2}\right), & \text{inside bubble} \\ 0, & \text{outside bubble} \end{cases}$$

$$\Big\downarrow \; \text{boost into bubble frame}$$

$$= \begin{cases} -\frac{m_e\,\omega_p^2}{2}\left(\hat{z}\;\frac{1}{\gamma_b}\left(z'(t) - \gamma_b^2\,\beta'_r(t)\,r'(t)\right) + \hat{r}\left(1 + \beta'_z(t)\right)r'(t)\right), & \text{inside} \\ 0, & \text{outside} \end{cases}$$

$$\Big\downarrow \; \text{normalize scales to } m_e,\ \omega_p\text{, and } c$$

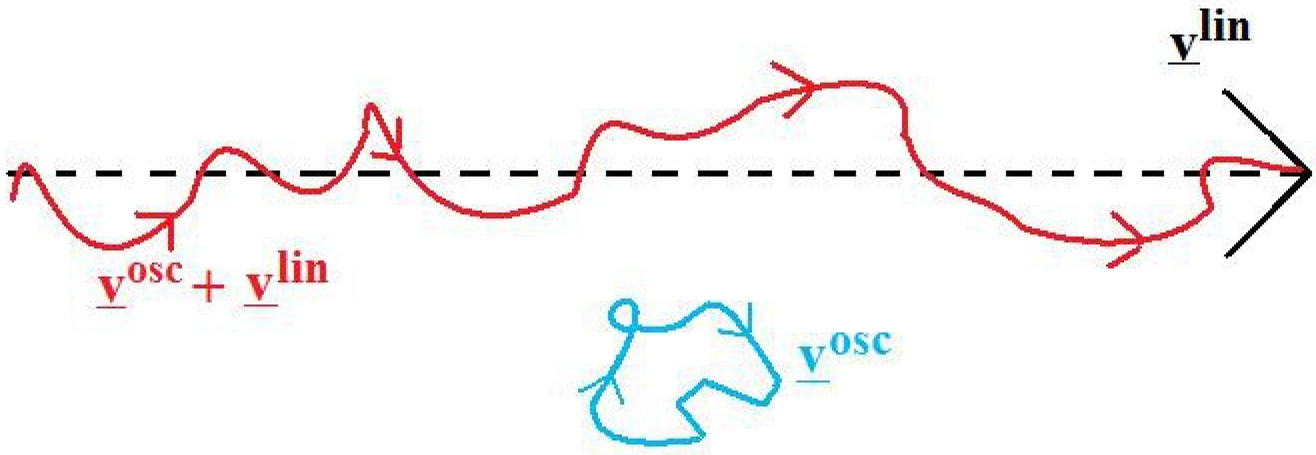
$$\tilde{\mathbf{f}}(\tilde{z},\tilde{r},\tilde{t}) = \begin{cases} -\frac{1}{2}\left(\hat{z}\;\frac{1}{\gamma_b}\left(\tilde{z}(\tilde{t}) - \gamma_b^2\,\tilde{v}_r(\tilde{t})\,\tilde{r}(\tilde{t})\right) + \hat{r}\left(1 + \tilde{v}_z(\tilde{t})\right)\tilde{r}(\tilde{t})\right), & \text{inside} \\ 0, & \text{outside} \end{cases}$$

$$\tilde{\mathbf{f}}(\tilde{z},\tilde{r},\tilde{t}) = \begin{cases} -\frac{1}{2}\left(\hat{z}\,\frac{1}{\gamma_b}\left(\tilde{z}(\tilde{t}) - \gamma_b^2\,\tilde{v}_r(\tilde{t})\,\tilde{r}(\tilde{t})\right) + \hat{r}\left(1+\tilde{v}_z(\tilde{t})\right)\tilde{r}(\tilde{t})\right), & \text{inside} \\ 0, & \text{outside} \end{cases}$$

For an electron inside the bubble (supressing tildes),

$$\frac{\mathrm{d}p_z}{\mathrm{d}t}(t) = \frac{1}{2\,\gamma_b}\left(\underline{\gamma_b^2\,v_r(t)\,r(t)} - \underline{z(t)}\right),$$
$$\frac{\mathrm{d}p_r}{\mathrm{d}t}(t) = -\frac{1}{2}\left(1+v_z(t)\right)r(t).$$

- γ_b is large, so the first term in the first equation dominates the second.
- This strongly couples p_z and p_r on short time scales.
- We can decompose the solution…



$$\mathbf{v}(t) \approx \mathbf{v}^{osc}(t) + \mathbf{v}^{lin}(t)$$

- Oscillating component varies quickly but averages to nearly zero over a cycle.
- Linear component varies slowly but does not cancel with itself.
- γ varies only with the linear component.

First the oscillating component:

$$\frac{\mathrm{d}\mathbf{p}^{\mathrm{osc}}}{\mathrm{d}t}(t) \approx \gamma^{\mathrm{lin}}\,\frac{\mathrm{d}\mathbf{v}^{\mathrm{osc}}}{\mathrm{d}t}(t) = \hat{z}\;\frac{1}{2}\Big(\gamma_b\, v_r(t)\, r(t)\Big) - \hat{r}\;\frac{1}{2}\Big(1 + v_z(t)\Big)\, r(t)$$

$$\Longrightarrow \mathbf{x}^{\mathrm{osc}} \approx \hat{z}\;\left(C_0\, t + C_1 + C_2\,\mathfrak{E}\Big(\mathfrak{F}^{-1}\Big(C_3\, t + C_4\,\Big|\,C_5\Big)\Big|C_5\Big)\right)$$

$$+\,\hat{r}\;\left(C_6\,\mathfrak{Sn}\Big(C_3\, t + C_4\,\Big|\,C_5\Big)\right)$$

- C_0, C_1, etc. are various (complicated!) constants.
- The fancy letters are for Jacobi elliptic functions/integrals.
- This is a pretty ugly function of t that typically has no obvious period.

Now the linear component:

$$\frac{\mathrm{d}p_z^{\mathrm{lin}}}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_z^{\mathrm{lin}}}{\sqrt{1-\left(v_z^{\mathrm{lin}}\right)^2}}\right) \approx -\,\hat{z}\,\frac{1}{2\,\gamma_b}\,z^{\mathrm{lin}}(t)$$

$$\Longrightarrow t \approx C + \sqrt{4\,\gamma_b\,(\gamma_0-1)+z_0^2}\;\mathfrak{E}\left(\sin^{-1}\left(\frac{z^{\mathrm{lin}}(t)}{\sqrt{4\,\gamma_b(\gamma_0+1)+z_0^2}}\right)\,\middle|\,\frac{4\,\gamma_b(\gamma_0+1)+z_0^2}{4\,\gamma_b(\gamma_0-1)+z_0^2}\right)$$
$$-\frac{4\,\gamma_b}{\sqrt{4\,\gamma_b\,(\gamma_0-1)+z_0^2}}\;\mathfrak{F}\left(\sin^{-1}\left(\frac{z^{\mathrm{lin}}(t)}{\sqrt{4\,\gamma_b(\gamma_0+1)+z_0^2}}\right)\,\middle|\,\frac{4\,\gamma_b(\gamma_0+1)+z_0^2}{4\,\gamma_b(\gamma_0-1)+z_0^2}\right)$$

- C is a constant describing the initial condition.
- Fancy letters are for Jacobi elliptic integrals again.
- Not a closed solution for z^{lin}, but this tells us z^{lin} should be (mostly) periodic.
- Exact period cannot be determined in general analytically.

When analytical methods fail, turn to numerics!

$$
\begin{aligned}
z'(t) &= v_z(t) = \frac{p_z(t)}{\gamma(t)} = \frac{p_z(t)}{\sqrt{1+p_z(t)^2+p_r(t)^2}} \\
r'(t) &= \frac{p_r(t)}{\sqrt{1+p_z(t)^2+p_r(t)^2}} \\
p_z'(t) &= \frac{1}{2}\left(\gamma_b\, r(t)\, v_r(t) - \frac{z(t)}{\gamma_b}\right) \\
&= \frac{1}{2}\left(\frac{\gamma_b\, r(t)\, p_r(t)}{\sqrt{1+p_z(t)^2+p_r(t)^2}} - \frac{z(t)}{\gamma_b}\right) \\
p_r'(t) &= -\frac{1}{2}\, r(t)\left(1+v_z(t)\right) \\
&= -\frac{1}{2}\, r(t)\left(1+\frac{p_r(t)}{\sqrt{1+p_z(t)^2+p_r(t)^2}}\right)
\end{aligned}
$$

- This is a coupled set of first-order ODEs, so we'll use a fourth-order explicit Runge-Kutta integrator to find solutions.

- We'll pick a single representative test case for now.

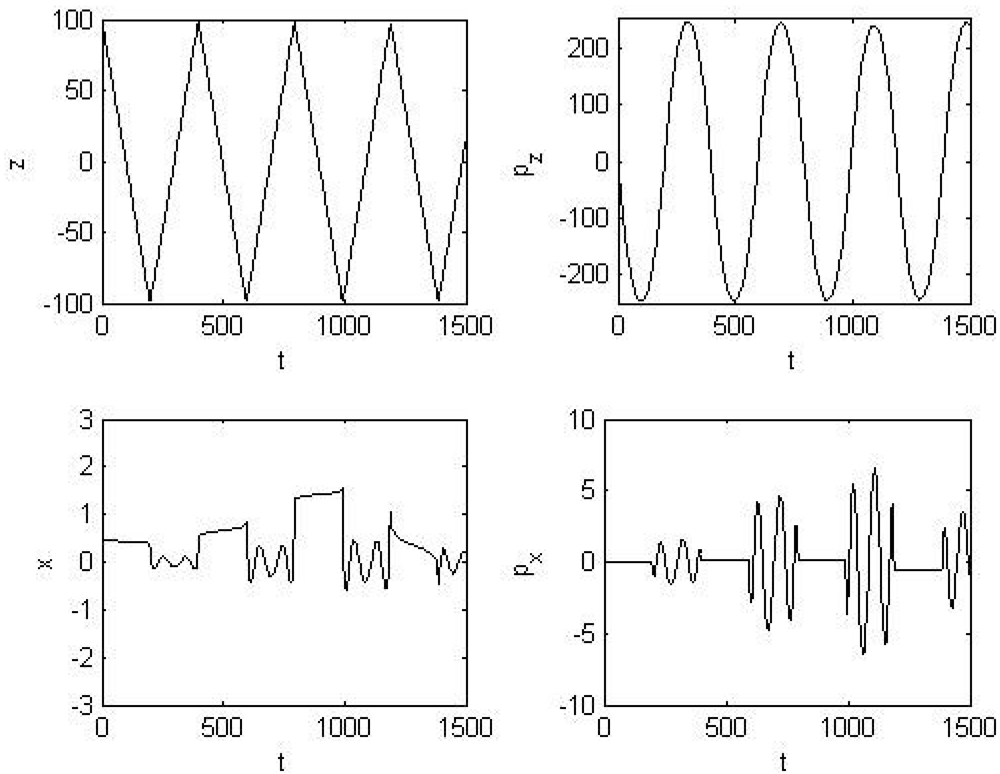
We’ll choose the case:

1. Bubble radius of 10,
2. γ_b of 10, and
3. Electron starts at rest at a position 1% of the bubble radius from the front of the bubble and 5% of a bubble radius off-axis.

- Afterward, we’ll Fourier-transform the z and r components in time to find expressed frequency modes.

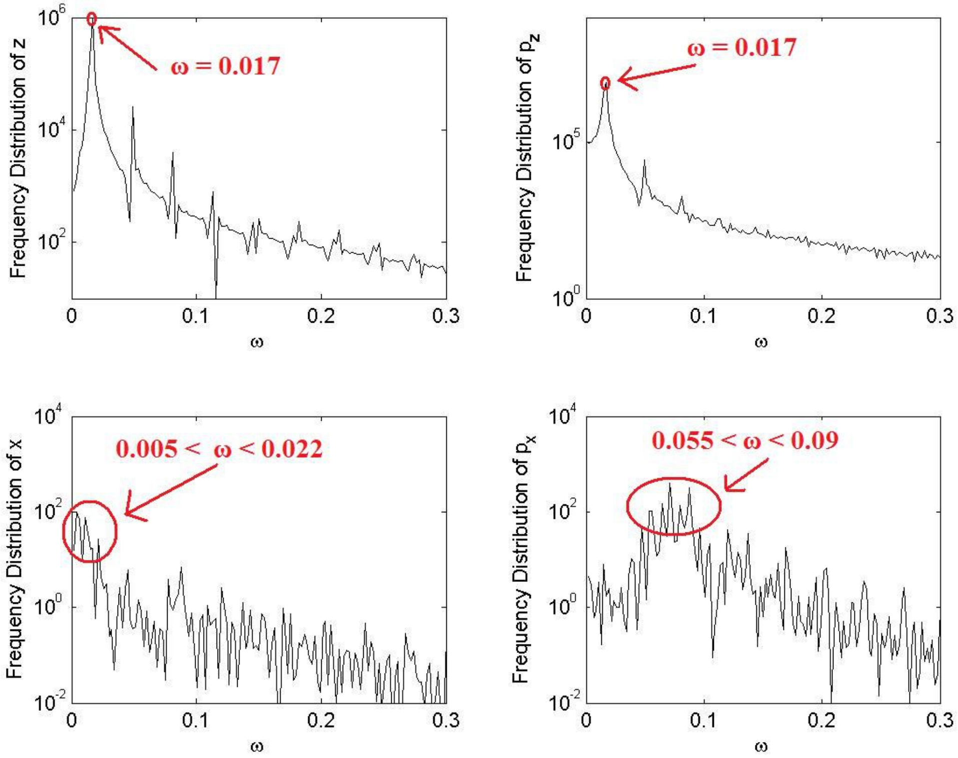
- Since the electron undergoes periodic motion at these modes, it will radiate at these frequencies (in the bubble frame).

Runge-Kutta Output: Spatial Data



- The variable r was computed as x to allow it to go negative (no changes to the physics result).
- As expected, z is dominated by very regular periodic motion.
- As expected, r is a mess. Interestingly, it only oscillates when the particle is moving forward in z.

Runge-Kutta Output: Frequency Data



- Both z and p_z have a very highly peaked distribution --- approaching monochromatic --- with rapidly decaying harmonics.
- Both r and v_r have very noisy graphs without a single dominant frequency.
- Their largest components don’t even match!

Conclusions

- The scenario described would lead to emission at

$$\omega_{\rm lab} \sim \gamma_b \left(1 + \frac{v_b}{c}\right) \omega_{\rm bubble} \approx 2\,\gamma_b\,\omega_{\rm b}\,\tilde{\omega}.$$

- For a plasma at a density of $n \sim 10^{24}$ m^{-3}, this yields a frequency of $\sim 2 \times 10^{13}$ Hz! (Using the z-frequency.)
- The electron's motion in z will be primarily responsible for radiation.
- This radiation will be strongly-peaked in frequency.
- The electron's motion in r will generate broadband noise at a much lower intensity.
- Future work could include investigation of how an electron's frequencies of oscillation vary with bubble parameters.

Questions?

References:

- Physics of Plasmas **17**, 056708 (2010). A.G.R. Thomas
- Nature Physics **6**, 980-983 (2010). S. Kneip, et al.
- Numerical Recipes – The Art of Scientific Computing (Third Edition, C++). Cambridge Press (2007).