The Spectral Hermite Transform as a Means of Simulating Brownian Motion

Mike Johnson

NERS 590: Stochastic Differential Equations

April 23rd, 2024

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$$f_n^m = \frac{1}{m!} \int_{-\infty}^{\infty} f(\bar{x})\, \mathcal{H}_\sigma^m\left(\frac{\bar{x} - n\,\mu}{\sigma}\right) d\bar{x}, \quad \text{and}$$

$$\mathcal{H}_C^m(x) = \left(-\frac{d}{dx}\right)^m \left[\frac{1}{C\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\right].$$

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- Looks complicated!! Let’s try building up slowly...

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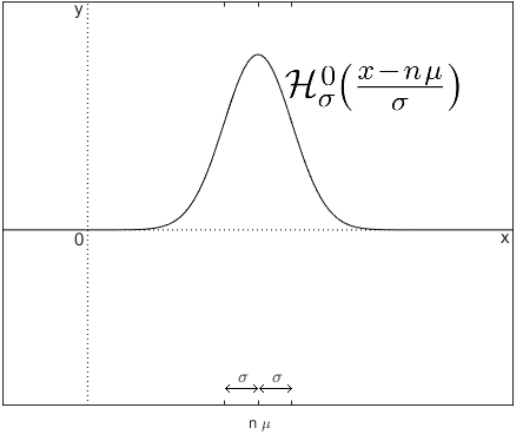

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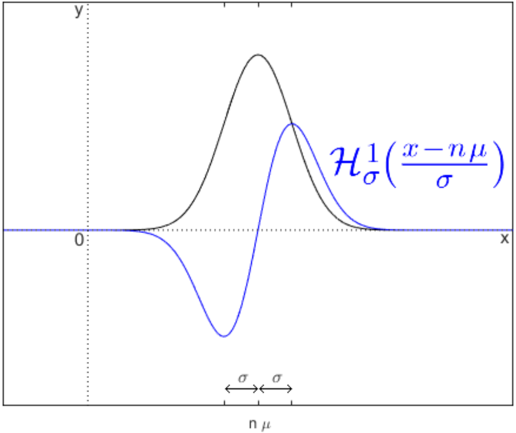

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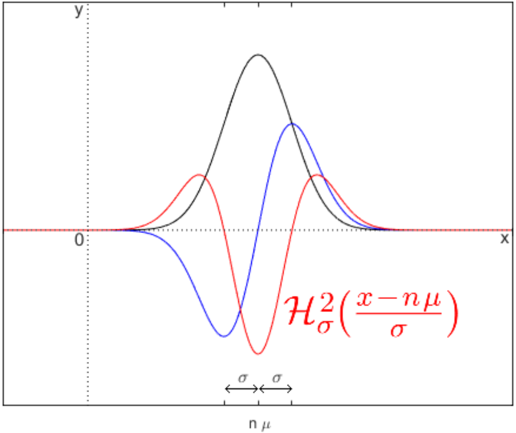

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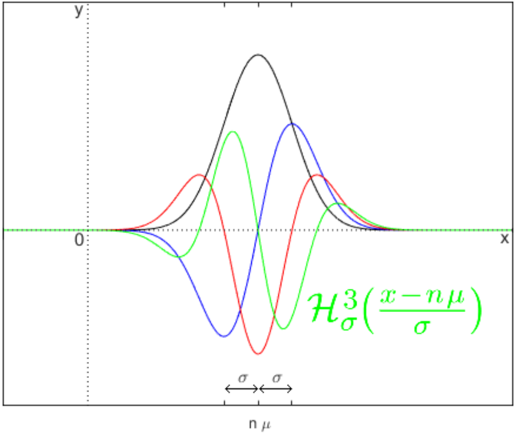

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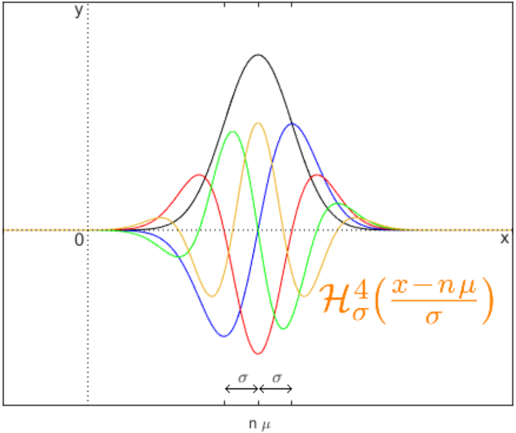

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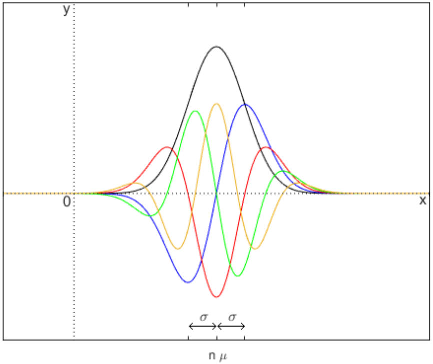

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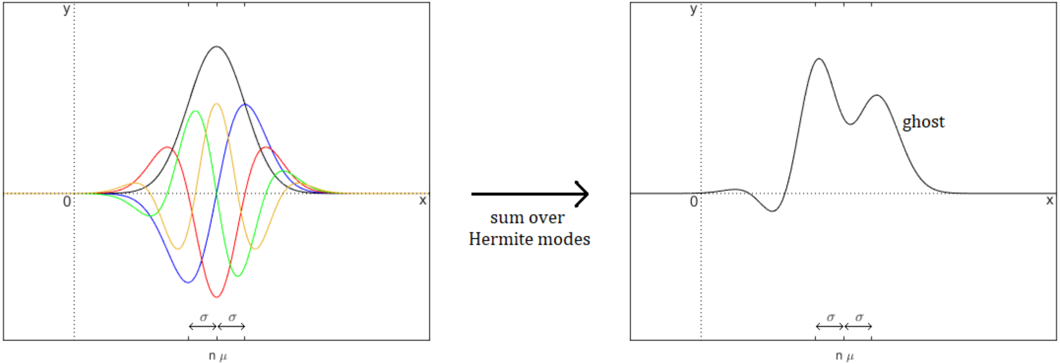

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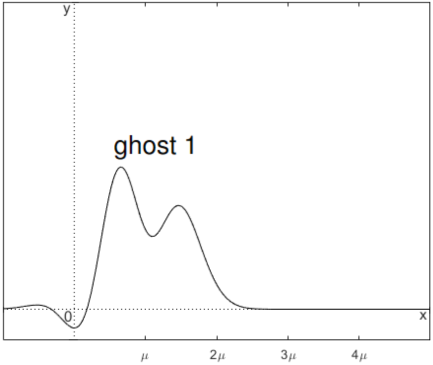

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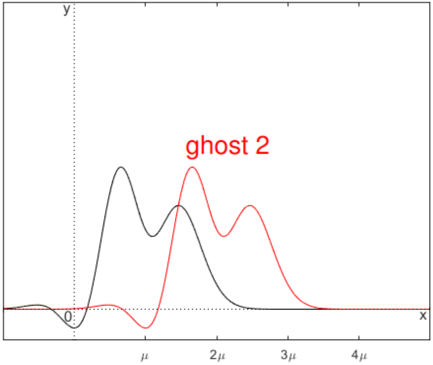

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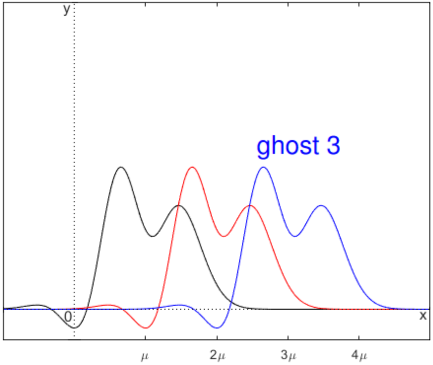

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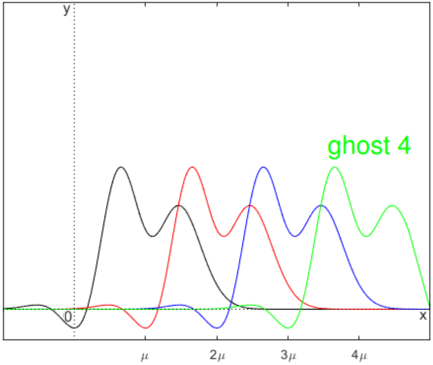

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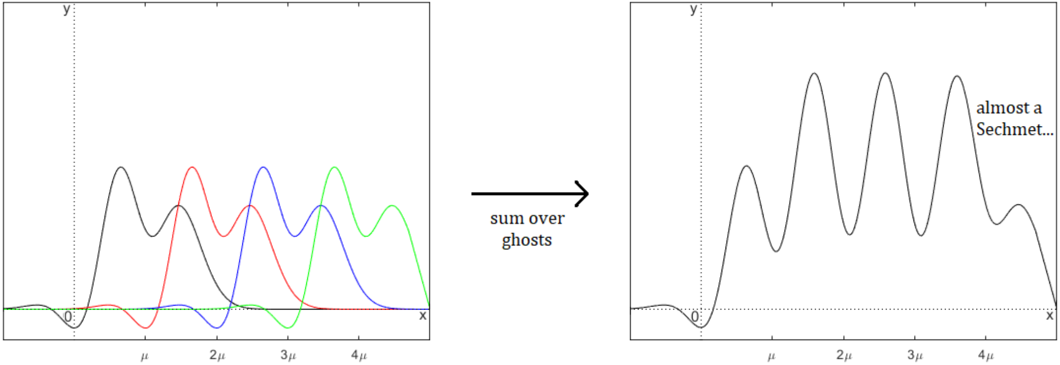

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- Hopefully this doesn't look like gibberish now:

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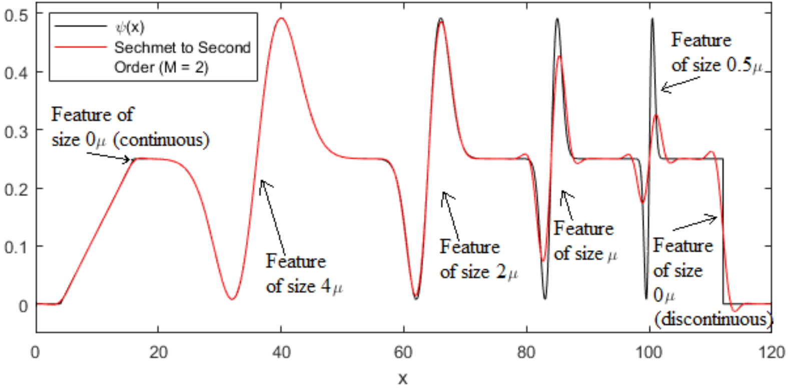
- A picture of the Sechmet of a complicated function, $\psi(x)$:



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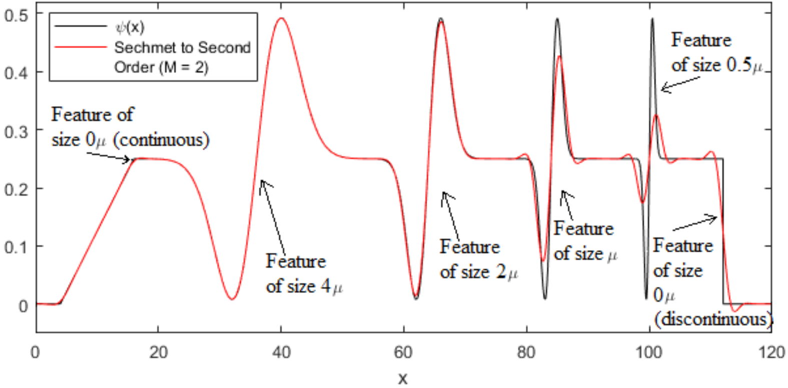

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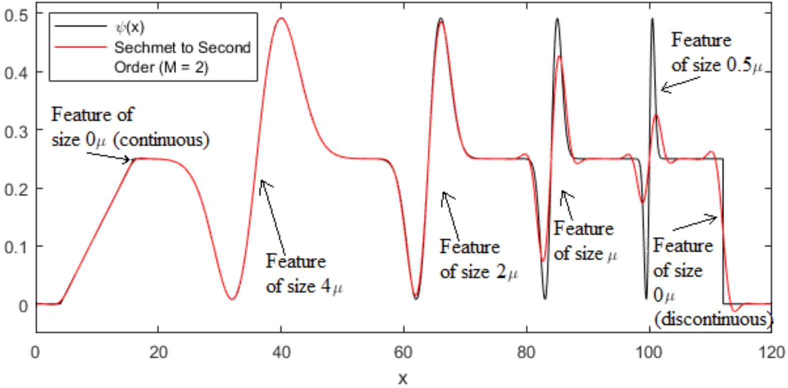

- Features larger than $\sim\mu$ are accurate.
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- Basis functions are localized (unlike e.g. sine or cosine series).

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- Finally $f(x) - f(y) \approx f(x-y) \approx \mathcal{N}(0; x-y)$ for large N. (Very non-trivially!)
- In other words, $f(x) \approx B_x$!

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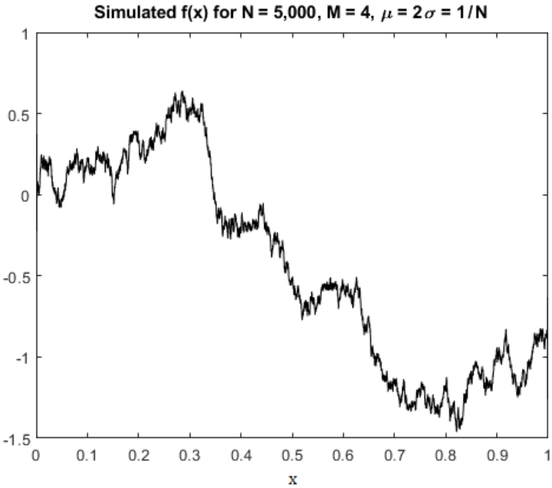
- A picture of $f(x)$:



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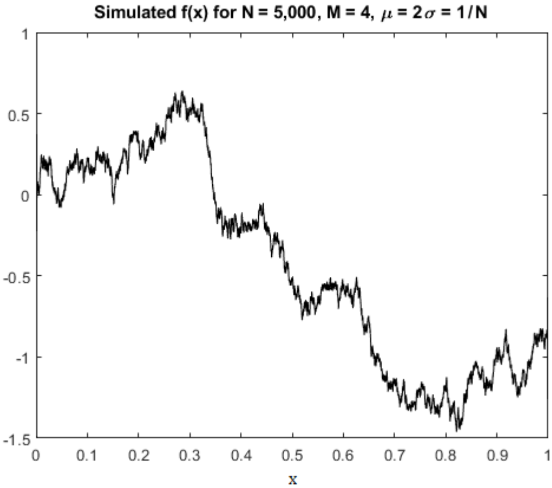

- $f(x)$ looks like Brownian motion, but let's test with some integrals.

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- If $f(x) \approx B_x$, then $f(x)$ approximately obeys

$$\int_I \equiv \int_0^x B_s \, dB_s = \frac{1}{2}\left(B_x^2 - x\right) \quad \text{(Itô's Integral), and}$$

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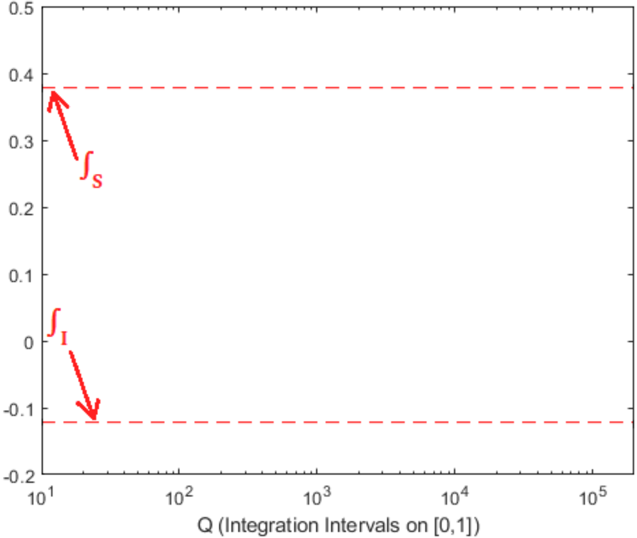
- Numerical results for integrals on $[0, 1]$ over the $f(x)$ shown previously:



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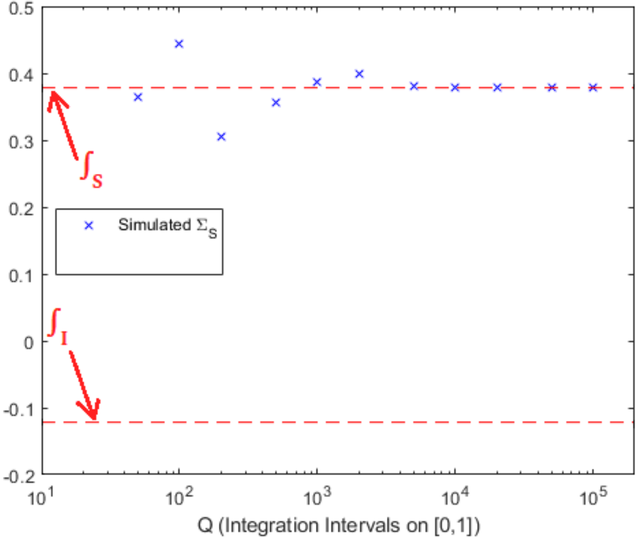

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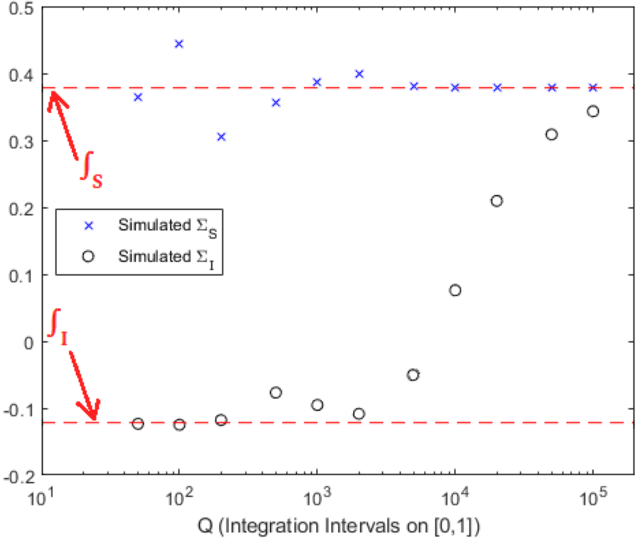

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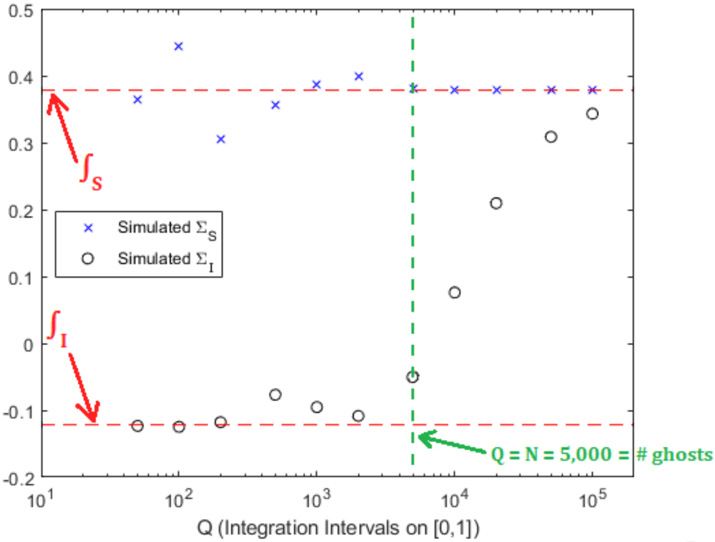

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- Allows us to cheaply modify particle advection in plasma simulations to include collisional effects via a Langevin-like term. (Future work!)

Thank you for your attention!

Any questions?