Decomposing Risk in Dynamic Stochastic General Equilibrium

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Decomposing Risk in Dynamic Stochastic General Equilibrium[*]

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We analyze the theoretical moments of a nonlinear approximation to a model of business cycles and asset pricing with stochastic volatility and recursive preferences. We find that heteroskedastic volatility operationalizes a time-varying risk adjustment channel that induces variability in conditional asset pricing measures and assigns a substantial portion of the variance of macroeconomic variables to variations in precautionary behavior, both while leaving its ability to match key macroeconomic and asset pricing facts untouched. Our method decomposes moments into contributions from realized shocks and differing orders of approximation and from shifts in the distribution of future shocks, enabling us to identify the common channel through which stochastic volatility in isolation operates and through which conditional asset pricing measures vary.

JEL classification: C63, E32, G12

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1   Introduction

Assessing the statistical and structural implications of nonlinear DSGE models with recursive preferences and stochastic volatility for asset pricing and business cycle dynamics is an unfinished task in macroeconomics. We derive the theoretical moments of nonlinear moving average approximations to the model and decompose these moments into contributions from the individual orders of nonlinearity in realized shocks (amplification effects) and from the moments of future shocks (risk adjustment effects). With this decomposition, we find that stochastic volatility activates a time-varying risk adjustment channel in macroeconomic variables accounting for a substantial amount of total variation. We identify this conditional heteroskedastic mechanism as the sole driving force of the conditional asset pricing measures under study. This enables us to tell the story of a varying pattern of risk in the economy eliciting changes in households' precautionary responses as priced by measures such as the conditional market price of risk.

While there is growing interest in stochastic volatility and Epstein and Zin's (1989) recursive preferences[1] in recent literature, there is little work that studies the joint effect of these two elements for both asset pricing and business cycle dynamics.[2] Andreasen (2012), focusing on the different specifications of the conditional heteroskedasticity and the consequential difference in the quantitative performance of a New Keynesian model, takes a brief look at the implications of the model on both sides. Bidder and Smith (2012), taking a model uncertainty perspective à la Hansen and Sargent (2007), study fluctuations in the worst-case distribution as sources for business cycles in a model with stochastic volatility and recursive preferences. We differ from both their work in our

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[1]See also Kreps and Porteus (1978) and Weil (1990). Backus, Routledge, and Zin (2005) offers a recent review of these and related preferences.

[2]Bloom (2009) notes the impact of stochastic volatility on macroeconomic variables. Justiniano and Primiceri (2008) add stochastic volatility to DSGE models to study the documented reduction in volatility of U.S. economy since the early 1980's (See also Blanchard and Simon (2001), Stock and Watson (2003) and Sims and Zha (2006) for a review.). Tallarini (2000) among many others, note recursive preferences can contribute to resolving the longstanding asset pricing puzzles (equity premium and risk free rate) documented in Mehra and Prescott (1985) and Weil (1989) without compromising the model's ability of replicating macroeconomic dynamics; and Rudebusch and Swanson (2012) and van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012) use a model with recursive preferences to study the dynamics of the yield curve.

aim to analyze the propagation mechanism of stochastic volatility implemented as a volatility shock, and we examine the role of stochastic volatility in attaining the Hansen-Jagannathan bounds (See Hansen and Jagannathan (1991)) to complement the empirical evaluation of the model regarding replicating asset pricing regularities.

We solve the model using the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2012), following the documentation of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) regarding the accuracy of perturbation for a business cycle model with recursive preferences and stochastic volatility and Bidder and Smith's (2012) perturbation study using the same specific functional form for continuation utility (the exponential certainty equivalent of Tallarini (2000)). We approximate the policy function to third order since it is the minimum order needed to capture the time-varying shifts in risk premium as noted in Andreasen (2012, p. 300) and van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012, p. 638). The nonlinear moving average policy function takes the infinite sequence of realized shocks, past to present, as its state variable basis and adjusts the deterministic policy function for the effect of future shocks by scaling their distribution with the perturbation parameter. This policy function and its third order approximation can be decomposed straightforwardly into the order of the amplification effects (the impact of the realized shocks) and risk adjustment (the anticipation effect of future shocks). We find, in the analysis of the impulse responses of both macroeconomic and asset pricing variables, a volatility shock by itself propagates solely through the time-varying risk adjustment channel. For conditional asset pricing measures such as the expected risk premium, volatility shocks and productivity growth shocks propagate individually through the time-varying risk adjustment channel only. Moreover, the effect of stochastic volatility shocks on the expected risk premium is several orders of magnitude larger than that of productivity growth shocks, highlighting again the importance of this time variation in the dispersion of probability measures used to form expectations for conditional asset pricing.

Using our third order nonlinear moving approximation, we are able to derive theoretical moments that are in general not available in the nonlinear DSGE models. We further derive a decomposition of the theoretical variance that neatly dissects the individual contributions of amplification and risk adjustment effects to the total variance of the model. With this variance decomposition, we find that adding stochastic volatility changes the composition of the variance of the macroeconomic variables. In the presence of stochastic volatility, more variation is generated in the time-varying risk adjustment channel. As for macroeconomic variables, movements in the risk adjustment channel can be explained by the household's precautionary motive. This finding implies households aware of shifts in the distributions of future shocks will adjust their precautionary behavior commensurately.

The nonlinear moving average approximation, as its policy function directly maps exogenous shocks into the endogenous variables, only needs the moments of the exogenous shocks when computing the theoretical moments. We implement our approach numerically by providing an add-on for the popular Dynare package.[3] A state space perturbation policy function, by contrast, maps the endogenous variables into themselves and resulting in an infinite regression in theoretical moments requiring higher moments than moments being computed. In a similar vein to our nonlinear moving average, Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2012) compute theoretical moments using a pruned state space perturbation,[4] since after pruning, the unknown higher moments are nonlinear functions of the known moments of lower order approximations.

The paper is organized as follows. The competitive real business cycle model with recursive preferences and stochastic volatility is derived in section 2. In section 3, we present the nonlinear moving average perturbation solution to the model. The calibrations are introduced in section 4. We then derive the theoretical moments in section 5 and apply our method to analyze the model in section 6. Section 7 concludes.

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[3]See Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011) for Dynare.
[4]See Lan and Meyer-Gohde (2013) for an overview of pruning and its relation to our nonlinear moving average.

2 The Model

In this section, we lay out the stochastic neoclassical growth model with the recursive preferences and stochastic volatility. We parameterize the model close to the production model described in Tallarini (2000). The economy is populated by an infinitely lived household seeking to maximize its expected discounted lifetime utility given by the recursive preferences

$$(1) \qquad U_t = \ln C_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} U_{t+1} \right) \right] \right)$$

where C_t is consumption, N_t labor, $\beta \in (0, 1)$ the discount factor and

$$(2) \qquad \gamma \equiv 2 \frac{(1 - \beta)(1 - \chi)}{1 + \psi}$$

indexes the deviation with respect to the expected utility. χ denotes the coefficient of relative risk aversion (CRRA) and $\psi > 0$ controls labor supply. With χ equal to the elasticity of intertemporal substitution (EIS) which is equal to one here, (1) collapses to the expected utility. The household optimizes over consumption and labor supply subject to

$$(3) \qquad C_t + K_t = W_t N_t + r_t^K K_{t-1} + (1 - \delta) K_{t-1}$$

where K_t is capital stock accumulated today for productive purpose tomorrow, W_t real wage, r_t^K the capital rental rate and $\delta \in [0, 1]$ the depreciation rate. Investment is the difference between the current capital stock and the capital stock in the previous period after depreciation

$$(4) \qquad I_t = K_t - (1 - \delta) K_{t-1}$$

We assume a perfectly competitive production side of the economy, where output is produced using the labor augmented Cobb-Douglas technology $Y_t = K_{t-1}^{\alpha} \left(e^{Z_t} N_t \right)^{1-\alpha}$. Z_t is a stochastic productivity process and $\alpha \in [0, 1]$ the capital share. Productivity is assumed to be a random walk with drift, incorporating long-run risk into the model[5]

$$(5) \qquad a_t \equiv Z_t - Z_{t-1} = \overline{a} + \overline{\sigma_z} e^{\sigma_{z,t}} \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, 1)$$

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[5]As noted by Bansal and Yaron (2004, p. 1502), in an endowment economy with recursive preferences and stochastic volatility, better long-run growth prospects leads to a rise in the wealth-consumption and the price-dividend ratios. Rudebusch and Swanson (2012, p. 108) incorporate both real and nominal long-run risk in a production economy with recursive preference, and find long-run nominal risk improves the model's ability to fit the data.

with $\varepsilon_{z,t}$ the innovation to Z_t. $\overline{\sigma}_z e^{\sigma_{z,t}}$ can be interpreted as the standard deviation of the productivity growth with $\overline{\sigma}_z$ the homoskedastic component. Following, e.g., Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011) and Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), we specify the heteroskedastic component, $\sigma_{z,t}$, as

(6)
$$\sigma_{z,t} = \rho_\sigma \sigma_{z,t-1} + \tau \varepsilon_{\sigma_{z,t}}, \ \ \varepsilon_{\sigma_{z,t}} \sim \mathcal{N}(0,1)$$

where $|\rho_\sigma| < 1$ and τ is the standard deviation of $\varepsilon_{\sigma_{z,t}}$. The model is closed by the market clearing condition

(7)
$$Y_t = C_t + I_t$$

that prevents consumption and investment from exceeding output in each period.

The solution is characterized by the intratemporal labor supply/productivity condition equalizing the utility cost of marginally increasing labor supply to the utility value of the additional consumption

(8)
$$\frac{\psi}{1-N_t} = \frac{1}{C_t}(1-\alpha)K_{t-1}^\alpha e^{Z_t(1-\alpha)}N_t^{-\alpha}$$

The stochastic discount factor, or pricing kernel, from the household's intertemporal maximization of utility is given by

(9)
$$m_{t+1} \equiv \frac{\partial V_t/\partial C_{t+1}}{\partial V_t/\partial C_t} = \beta\frac{C_t}{C_{t+1}}\frac{\exp\left(\frac{\gamma}{2}V_{t+1}\right)}{E_t\left[\exp\left(\frac{\gamma}{2}V_{t+1}\right)\right]}$$

where V_t is the maximum attainable utility, i.e., the value function of the household

(10)
$$V_t = \ln C_t + \psi\ln(1-N_t) + \beta\frac{2}{\gamma}\ln\left(E_t\left[\exp\left(\frac{\gamma}{2}V_{t+1}\right)\right]\right)$$

Combining firms' profit and households' utility maximization yields the real risky rate r_t

(11)
$$1+r_t = \alpha K_{t-1}^{\alpha-1}\left(e^{z_t}N_t\right)^{1-\alpha}+1-\delta = r_t^K+1-\delta$$

The fundamental asset pricing equation takes the form

(12)
$$E_t\left[m_{t+1}\left(1+r_{t+1}\right)\right] = 1$$

As the economy is nonstationary, growing at the rate a_t, we detrend output, consumption, investment, capital stock and value function to stationarize the model. This is achieved by dividing all

nonstationary variables but the value function, which must detrended differently, by the contemporaneous level of productivity e^{Z_t}.[6] Labor supply N_t and leisure $1 - N_t$ as well as the returns r_t and r_t^K are stationary and therefore do not need to be transformed. Stationary variables will be denoted by lower case letters.

Reexpressing the pricing kernel in terms of stationary variables, the effect of incorporating long-run risk can be seen directly in the pricing kernel

$$(13) \qquad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\overline{a} + \overline{\sigma_z} e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})} \frac{\exp\left(\frac{\gamma}{2}\left[v_{t+1} + \frac{1}{1-\beta}(\overline{a} + \overline{\sigma_z} e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})\right]\right)}{E_t\left[\exp\left(\frac{\gamma}{2}\left[v_{t+1} + \frac{1}{1-\beta}(\overline{a} + \overline{\sigma_z} e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})\right]\right)\right]}$$

with the stochastic trend, $\overline{\sigma_z} e^{\sigma_{z,t+1}}$, entering the kernel directly.

To analyze asset prices, we append the model with the following variables: the real risk-free rate

$$(14) \qquad 1 + r_t^f \equiv E_t(m_{t+1})^{-1}$$

the conditional market price of risk—the ratio of the conditional standard deviation of the pricing kernel to its conditional mean

$$(15) \qquad cmpr_t \equiv \frac{\left(E_t\left[(m_{t+1} - E_t m_{t+1})^2\right]\right)^{\frac{1}{2}}}{E_t m_{t+1}}$$

that measures the excess return the household demands for bearing an additional unit of risk, the expected (ex ante) risk premium

$$(16) \qquad erp_t \equiv E_t\left(r_{t+1} - r_t^f\right)$$

and the (ex post) risk premium

$$(17) \qquad rp_t = r_t - r_{t-1}^f$$

as the difference between the risky and risk-free rate.

3   Perturbation Solution and Risk Adjustment Channel

As stated by Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), local approximations via perturbation methods can solve models such as ours quickly with a degree of accuracy com-

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[6]See the appendix for details.

parable to global methods. Moreover, as at least a third order approximation is necessary for the analysis of time-varying shifts in risk premia and related measures at the heart of our analysis, we solve the model to third order. We choose the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2012) as it delivers stable impulse responses and simulations at all orders, hence including our third order approximation of interest, and, as we shall show, enables the direct calculation and decomposition of moments.

For the implementation of the nonlinear moving average perturbation, we collect the (stationarized) equilibrium conditions into a vector of functions

(18) $$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)]$$

where $y_t = \begin{bmatrix} k_t & c_t & N_t & a_t - \overline{a} & \sigma_{z,t} & v_t & m_t & r_t & r_t^f & cmpr_t & erp_t & rp_t \end{bmatrix}'$ is the vector of the endogenous variables, and $\varepsilon_t = \begin{bmatrix} \varepsilon_{z,t} & \varepsilon_{\sigma_{z,t}} \end{bmatrix}'$ the vector of the exogenous shocks, assuming the function f in (18) is sufficiently smooth and all the moments of ε_t exist and finite[7].

The solution to (18) is a time-invariant function y, taking as its state variable basis the infinite sequence of realized shocks, past and present, and indexed by the perturbation parameter $\sigma \in [0, 1]$ scaling the distribution of future shocks

(19) $$y_t = y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \ldots)$$

Assuming normality of all the shocks and setting $\sigma = 1$ as we are interested in the stochastic model, the third order approximation—a Volterra expansion, see Lan and Meyer-Gohde (2012)—of (19), takes the form

(20) $$y_t^{(3)} = \overline{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{2} \sum_{i=0}^{\infty} \left(y_i + y_{\sigma^2, i} \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i})$$
$$+ \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

where $\overline{y}$ denotes the deterministic steady state of the model, at which all the partial derivatives

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[7]See for example, Judd (1998, ch. 13) and Jin and Judd (2002) for a complete characterization of these assumptions. While the normal distribution for shocks we choose is at odds with Jin and Judd's (2002) assumption of bounded support, Kim, Kim, Schaumburg, and Sims (2008) dispute the essentiality of this assumption, lending support to our distribution choice

$y_{\sigma^2}, y_{\sigma^2,i}, y_i, y_{j,i}$ and $y_{k,j,i}$ are evaluated. (20) is naturally decomposed into order of nonlinearity and risk adjustment—y_i, $y_{j,i}$ and $y_{k,j,i}$ capture the amplification effects of the realized shocks $(\varepsilon_t, \varepsilon_{t-1}, \dots)$ in the policy function (19) at first, second and third order respectively. The two partial derivatives with respect to σ, y_{σ^2} and $y_{\sigma^2,i}$ adjust the approximation for future risk.[8] While y_{σ^2} is a constant adjustment for risk and a linear function of the variance of future shocks[9], $y_{\sigma^2,i}$ varies over time, interacting the linear response to realized shocks with the variance of future shocks essentially adjusting the model for time variation in the conditional volatility of future risk.

4 Calibration

We select three calibrations for the numerical analysis of the model. For the baseline calibration, most of the parameter values are taken from Tallarini (2000) and are listed below. For the parameters of the volatility shock, the literature varies in the range of the persistence—ρ_σ, from 0.9, Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) and Bidder and Smith (2012), to 0.95, Fernández-Villaverde and Rubio-Ramírez (2010a), and to 0.99 or 1, Andreasen (2012) and Justiniano and Primiceri (2008)—and in the range of its instantaneous standard deviation—τ, from 0.01, Andreasen (2012) and Justiniano and Primiceri (2008), to 0.1, Fernández-Villaverde and Rubio-Ramírez (2010b), and to 0.15, Bidder and Smith (2012). We follow the parameterization of Bidder and Smith (2012), implying a cumulative variance comparable to the value in Fernández-Villaverde and Rubio-Ramírez (2010a, p. 20), described as "generat[ing] changes in volatility similar to the ones observed in the [post-war] U.S." Following Tallarini (2000), we adjust the homoskedastic component of the standard deviation of productivity growth to match the standard deviation of (log) consumption growth.

[Table 1 about here.]

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[8]More generally, a constant term, y_{σ^3}, at third order adjusts (20) for the skewness of the shocks. See Andreasen (2012). As we assume all the shocks are normally distributed, y_{σ^3} is zero and not included in (20) and the rest of our analysis.

[9]See, Lan and Meyer-Gohde (2012, p. 13) for the derivation of this term.

The discount factor $\beta = 0.9926$ generates an annual interest rate of about 3 percent. The capital share $\alpha = 0.331$ matches the ratio of labor share to national income. The depreciation rate $\delta = 0.021$ matches the ratio of investment to output. The risk aversion parameter χ and labor supply parameter ψ are chosen such that labor in the deterministic steady state, $\overline{N}$, is 0.2305 to align with the mean level of hours in data and such that $\gamma = -0.3676$ in line with Tallarini (2000).

While still allowing preferences to be recursive, the constant volatility calibration shuts down stochastic volatility by setting $\tau = 0$, this enables direct comparison with Tallarini's (2000) results. In addition, by comparing with the results from the baseline calibration, this exercise helps identify the contribution of the stochastic volatility, by itself and/or in interaction with recursive preferences, to the model.

[Table 2 about here.]

The expected utility calibration shuts stochastic volatility down and is implemented by setting $\chi = 1$ (equivalently, $\gamma = 0$). We will be using all the three calibrations to analyze the contributions of recursive preferences and stochastic volatility to the model's performance evaluated by the Hansen-Jagnannathan bounds.

5 Theoretical Moments

In this section, we derive the theoretical moments of the third order approximation (20). The nonlinear moving average policy function (19) and its third order approximation (20) both map exogenous shocks directly into endogenous variables. The moments of endogenous variables can therefore be computed directly as they are functions of the known moments of exogenous shocks. We further decompose the theoretical variance, disentangling the individual contributions of the risk adjustment and amplification channels to the total variance. Note that throughout the derivation of theoretical moments, we assume normality of the exogenous shocks[10] and all processes involved are, as proved

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[10]While removing normality does not disable the calculation of theoretical moments, the derivation will be more complicated as additional terms involving skewness and higher (up to fifth) moments of the shocks emerge.

in Lan and Meyer-Gohde (2012), covariance stationary.

By contrast, the state space perturbation policy function and its nonlinear approximations map the endogenous variables into themselves. Computing the *m*-th theoretical moment of such a non-linear approximations of *n*-th order, for example, requires the knowledge of higher (than *m*-th) moments of endogenous variables that are in general nonlinear functions of the approximations up to and including *n*-th order. To this end, the calculation results in an infinite regression in the moments of endogenous variables. While theoretical moments of nonlinear state space perturbation approximations are in general not available, there are attempts in recent literature. Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2012) calculate theoretical moments by pruning the nonlinear approximations, such that the higher (than *m*-th) moments are functions of approximations lower than the current order of approximation, and therefore computable given the results from all lower orders.

5.1   Mean

The mean (first moment) of the third order approximation (20) is straightforward to calculate. Applying the expectations operator to (20) yields

(21)
$$E\left[y_t^{(3)}\right] = \bar{y} + \frac{1}{2}y_{\sigma^2} + \frac{1}{2}\sum_{j=0}^{\infty} y_{j,j}E\left[\varepsilon_t \otimes \varepsilon_t\right]$$

The last term in (20) vanishes as the triple Kronecker product in expectation is the columnwise vectorization of the third moment of the exogenous shocks, equal to zero under normality. Likewise, the Kronecker product in expectation is the columnwise vectorization of the second moment of the exogenous shocks. Only the contemporaneous variance appears because the shock vector is assumed serially uncorrelated. The other two terms containing ε_{t-i} in (20) also disappear as the shock is mean zero. From a different perspective, the deterministic steady state is the mean of the zeroth order approximation where all shocks, past, present and future are zero. It remains the mean in a first order approximation, as the exogenous shocks are mean zero (first moment is zero). At second order, the second moments of the shocks are included—both past and present (in the

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term $\sum_{j=0}^{\infty} y_{j,j} E[\varepsilon_t \otimes \varepsilon_t]$) as well as future shocks (in the term y_{σ^2})—which are assumed nonzero, generating an adjustment from the deterministic steady state. When the approximation moves to the third order, the calculation of the mean of (20) would be accordingly adjusted for the first three moments of all the realized and future shocks, but the mean zero and normality assumptions render the first and third moments of the shocks zero, thus leaving the first moment at third order identical to its value from a second order approximation.

5.2   Variance and Autocovariances

While we could conceivably compute the second moments (variance and autocovariances) of (20) using the Volterra expansion directly, it would be a rather complicated operation on the products of multi-layered infinite summation of coefficients. As an alternative, we use the recursive expression of (20) derived in Lan and Meyer-Gohde (2013) to compute the second moments.

Computing the second moments using the recursive expression of (20), we need to proceed sequentially through the orders of approximation and exploit the linearly recursive (in order) structure of the solution.[11] That is, the second moments of the approximation at any order can always be expressed as the sum of the second moments of the approximation of the previous order and the second moments of all the previous order increments (the difference between two approximations of adjacent order, subtracting the constant risk adjustment of the higher order). In other words, the embedded decomposition into order of approximation in the nonlinear approximations of the policy function (19) is preserved its second moments.

The first order approximation of (19) takes the form of a linear moving average, $y_t^{(1)} = \overline{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i}$, and can be expressed recursively as[12]

$$(22) \qquad\qquad y_t^{(1)} - \overline{y} = \alpha \left(y_t^{(1)state} - \overline{y}^{state} \right) + \beta_0 \varepsilon_t$$

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[11]The terminology if Lombardo's (2010). In Lan and Meyer-Gohde (2013), we compare Lombardo's (2010), others, and our recursive representation.

[12]See Lan and Meyer-Gohde (2013). This is, of course, an standard result for linear models. Compare, e.g., the state space representations of Uhlig (1999) with the infinite moving average representations of Taylor (1986).

where the difference $y_t^{(1)} - \bar{y}$ is the deviation of the first order approximation with respect to the deterministic steady state, and identical to the first order increment

$$(23) \qquad dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$$

which captures the addition to the approximation contributed by the time varying terms of the current, here first, order of approximation, as $\bar{y}$ is the zeroth order approximation[13] and the constant risk adjustment of first order, y_σ, is zero. In addition

$$(24) \qquad E\left[dy_{t-1}^{(1)} \varepsilon_t'\right] = 0$$

as the current shock is not correlated with the endogenous variables in the past. Under the orthogonality condition (24), the sequence of autocovariances of endogenous variables or, at this order equivalently, of the first order increment $\Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E\left[dy_t^{(1)} dy_{t-j}^{(1)'}\right]$, solves the following Lyapunov equation

$$(25) \qquad \Gamma_j^{y^{(1)}} = \alpha \Gamma_j^{y^{(1)}} \alpha' + \beta_0 E[\varepsilon_t \varepsilon_{t-j}]\beta_0'$$

The second order approximation of the policy function (19) captures the amplification effects of the realized shocks up to second order, and the constant risk adjustment for future shocks

$$(26) \qquad y_t^{(2)} = \bar{y} + \frac{1}{2}y_{\sigma^2} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_{j,i}(\varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

Defining the second order increment

$$(27) \qquad dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2}y_{\sigma^2}$$

which more clearly illustrates the notion of increment we use here; the addition the approximation contributed by time varying components of current order (or the difference between the current and previous order of approximation, here $y_t^{(2)} - y_t^{(1)}$, less the additional constant contributed by the current order, here $\frac{1}{2}y_{\sigma^2}$). The second moments of the second order approximation (26) can be expressed as the sum of the second moments of the first order approximation and those of the order increment. We summarize the results for a second order approximation in the following proposition

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[13]This is the terminology in Anderson, Levin, and Swanson (2006, p. 17) and Borovicka and Hansen (2012, p. 22).

Proposition 5.1. *Assuming the exogenous shocks are normally distributed, the j'th autocovariance of the second order approximation (26) is of the form*

$$\Gamma_j^{y^{(2)}} = \Gamma_j^{y^{(1)}} + \Gamma_j^{(2)}$$
(28)

where

(29)
$$\Gamma_j^{y^{(2)}} = E\left[\left(y_t^{(2)} - Ey_t^{(2)}\right)\left(y_{t-j}^{(2)} - Ey_t^{(2)}\right)'\right]$$

(30)
$$\Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E\left(dy_t^{(1)}dy_{t-j}^{(1)'}\right)$$

(31)
$$\Gamma_j^{(2)} = E\left[\left(dy_t^{(2)} - Edy_t^{(2)}\right)\left(dy_{t-j}^{(2)} - Edy_t^{(2)}\right)'\right]$$

Proof. See the appendices. □

The second order increment $dy_t^{(2)}$ can likewise be expressed recursively.[14] With that recursive expression in hand, the unknown $\Gamma_j^{(2)}$ in (28) can be computed by formulating an appropriate Lyapunov equation. We regelate all details to the appendices.

Likewise, to compute the second moments of endogenous variables using the third order approximation (20), we define the third order increment

$$dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$$
(32)

which is merely the difference between the third and second order approximations, as the third order approximation adds no additional constant terms under normality. We summarize the resulting second moment calculations at third order in the following proposition

Proposition 5.2. *Assuming the exogenous shocks are normally distributed, the j'th autocovariance of the third order approximation (20) takes the form*

(33)
$$\Gamma_j^{y^{(3)}} = \Gamma_j^{y^{(2)}} + \Gamma_j^{(3)} + \Gamma_j^{(1),(3)} + \left(\Gamma_j^{(1),(3)}\right)'$$

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[14]See, again, Lan and Meyer-Gohde (2013).

where

$$(34) \quad \Gamma_j^{y^{(3)}} = E\left[\left(y_t^{(3)} - Ey_t^{(3)}\right)\left(y_{t-j}^{(3)} - Ey_t^{(3)}\right)'\right]$$

$$(35) \quad \Gamma_j^{(3)} = E\left(dy_t^{(3)}dy_{t-j}^{(3)'}\right)$$

$$(36) \quad \Gamma_j^{(1),(3)} = E\left(dy_t^{(1)}dy_{t-j}^{(3)'}\right)$$

and $\Gamma_j^{y^{(2)}}$ *is as defined in Proposition 5.1.*

Proof. See the appendices. □

$\Gamma_j^{y^{(3)}}$ is the j'th autocovariance of endogenous variables computed using the third order approximation (20), $\Gamma_j^{(3)}$ the j'th autocovariance of the third order increment $dy_t^{(3)}$, and $\Gamma_j^{(1),(3)}$ the j'th autocovariance between the first and the third order increments $dy_t^{(1)}$ and $dy_t^{(3)}$. Analogous to (28) in Proposition 5.1, (33) decomposes the second moments into order of approximation: When the approximation moves to the third order, the second moments of endogenous variables are those computed using second order approximation (26), adjusted by the second moments of $dy_t^{(3)}$ itself and the interaction with the first order increment $dy_t^{(1)}$.

With the recursive form of the third order increment $dy_t^{(3)}$,[15] the two unknown quantities, $\Gamma_j^{(3)}$ and $\Gamma_j^{(1),(3)}$, in (33) for calculating the covariance matrices of the third order approximation can be computed by formulating appropriate Lyapunov equations. The details are in the appendices.

5.3   A Variance Decomposition

The third order approximation, (20), decomposes naturally into orders of nonlinearity and risk adjustment. This dissects the individual contributions of the sequence of realized shocks and future shocks and a variance decomposition can be accordingly derived to analyze the composition of the volatility of endogenous variables.

Let $y_t^{(3)risk} \equiv \frac{1}{2}y_{\sigma^2} + \frac{1}{2}\sum_{i=0}^{\infty}y_{\sigma^2,i}\varepsilon_{t-i}$ denote risk adjustment channel, with a constant risk adjustment at second order ($\frac{1}{2}y_{\sigma^2}$) and a time-varying risk adjustment channel at third order ($\frac{1}{2}\sum_{i=0}^{\infty}y_{\sigma^2,i}\varepsilon_{t-i}$)

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[15]See, again, Lan and Meyer-Gohde (2013).

and $y_t^{(3)amp}$ collect all the other terms in the third order approximation (20) capturing the amplification effects, we can rewrite (20) as

(37) $$y_t^{(3)} \equiv y_t^{(3)risk} + y_t^{(3)amp}$$

Centering the previous equation around its mean,[16] multiplying the resulting expression with its transposition and applying the expectations operator yields the following variance decomposition

(38) $$\Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(3)risk}} + \Gamma_0^{y^{(3)risk,amp}} + \Gamma_0^{y^{(3)amp}}$$

where $\Gamma_0^{y^{(3)}}$ is the variance of the endogenous variables. $\Gamma_0^{y^{(3)risk}} = E\left[\left(y_t^{(3)risk} - Ey_t^{(3)risk}\right)\left(y_t^{(3)risk} - Ey_t^{(3)risk}\right)'\right]$ stores the variations in the endogenous variables come from the time-varying risk adjustment channel alone. $\Gamma_0^{y^{(3)amp}} = E\left[\left(y_t^{(3)amp} - Ey_t^{(3)amp}\right)\left(y_t^{(3)amp} - Ey_t^{(3)amp}\right)'\right]$ stores the variations come from the amplification channels of all three orders. $\Gamma_0^{y^{(3)risk,amp}}$ is the sum of $E\left[\left(y_t^{(3)amp} - Ey_t^{(3)amp}\right)y_t^{(3)risk'}\right]$ and its transposition, storing the variations come from the interaction between the two types of channels.

Both $y_t^{(3)risk}$ and $y_t^{(3)amp}$ can be expressed recursively. With those recursive expressions, $\Gamma_0^{y^{(3)risk}}$ and $\Gamma_0^{y^{(3)amp}}$ can be computed by formulating appropriate Lyapunov equations (See the appendices for details). As $\Gamma_0^{y^{(3)}}$ is already known from Proposition 5.2, $\Gamma_0^{y^{(3)risk,amp}}$ can be computed by subtracting $\Gamma_0^{y^{(3)risk}}$ and $\Gamma_0^{y^{(3)amp}}$ from $\Gamma_0^{y^{(3)}}$.

5.4   Simulated Moments

Apart from the theoretical moments, we can simulate the third order approximation (20) and compute the moments of the simulated series to analyze the statistical implications of the model. Lan and Meyer-Gohde (2012) show that nonlinear approximation of the policy function (19) preserve the stability of the linear approximation or first order approximation and, hence, does not generate explosive time paths in simulations.

Simulation methods for moment calculations are, however, not always feasible for state space

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[16]Note $Ey_t^{(3)risk} = \frac{1}{2}y_{\sigma^2}$ and $Ey_t^{(3)amp} = \bar{y} + \frac{1}{2}\sum_{j=0}^{\infty} y_{j,j}E\left[\varepsilon_t \otimes \varepsilon_t\right]$.

perturbations. Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006), Fernández-Villaverde and Rubio-Ramírez (2006) and Kim, Kim, Schaumburg, and Sims (2008) note that higher order Taylor approximations to state space perturbation policy function can be potentially explosive in simulations. Truncation of the distribution from which exogenous shocks are drawn or the application of pruning schemes, like proposed by Kim, Kim, Schaumburg, and Sims (2008) for a second order approximation,[17] can prevent such behavior. While this imposes stability on simulations of higher order approximations, pruning is an ad hoc procedure as noted by Lombardo (2010) and potentially distortive even when the simulation is not on an explosive path (See, Den Haan and De Wind (2012)). Though this might give rise to reasonable doubts regarding the accuracy and validity of moments calculated using perturbations, we will show that this is not the case with our nonlinear moving average.

As (20) generates stable time paths, moments computed by simulating (20) should asymptotically converge to their theoretical counterparts.

[Figure 1 about here.]

Figure 1 is an example of this check. It depicts the evolution path of the density of the simulated variance of the pricing kernel in the model described in Section 2 under the benchmark calibration. Densities of the simulated variance of the pricing kernel are calculated using a kernel density estimation and 100 simulations at the indicated length. The theoretical variance, denoted by the red dashed line, is 0.0666 and all densities are in general centered around this value. The distributions of simulated variance are more dispersed in short-run simulations, tightening up to the theoretical value as the length increases consistent with asymptotic convergence of the simulated moments to their theoretical couterparts we calculated above.

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[17]See Lan and Meyer-Gohde (2013) for an overview and comparison of pruning algorithms at second and third order and their relation to our nonlinear moving average.

6   Analysis

In this section, we report the performance of the model under the different calibrations. We present impulse responses of shocks in productivity growth and its volatility for both macroeconomic and asset pricing variables, to analyze the role of the amplification and risk adjustment channel in shock propagation. We then proceed to the moments and the results of the variance decomposition introduced in Section 5.3 to identify and quantify the individual contribution of the time-varying risk adjustment channel to the total variation. In addition, we analyze effect of adding stochastic volatility on model's ability of attaining the Hansen-Jagnanthan bounds.

6.1   Impulse Responses and Simulations

We analyze the impulse responses to shocks in productivity growth and shock in its volatility for macroeconomic and asset pricing variables. We also simulate the conditional market price of risk under stochastic volatility and with growth shocks of constant variance to observe the change in the variations of this variable under conditional heteroskedasticity.

[Figure 2 about here.]

Figure 2 depicts the impulse response and its contributing components for capital to a positive, one standard deviation shock in $\varepsilon_{\sigma^a,t}$. The upper panel displays the impulse responses at first, second and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel). In the the middle left panel and the middle column of panels in the lower half of the figure, the contributions to the total impulse responses from the first, second and third order amplification channels, that is, y_i, $y_{i,i}$ and $y_{i,i,i}$ in the third order approximation (20), are displayed. Notice that there is no response in these amplification channels. All responses to this volatility shock come from the lower left panel of the figure where the time-varying risk adjustment channel $y_{\sigma^2,i}$ is displayed. In other words, for capital, a volatility shock by itself propagates solely through the time-varying risk adjustment channel.

Capital responds positively to a positive volatility shock. This captures the household's precautionary reaction to the widening of the distribution of future shocks. Our risk-averse household accumulates a buffer stock in capital to ensure itself against the increased future risk of productivity growth shocks from a more dispersed distribution.

[Figure 3 about here.]

Figure 3 depicts the systematic responses at the third order of macroeconomic variables as deviations from their nonstochastic steady states to a positive, one standard deviation volatility shock. The household accumulates a buffer stock of capital by increasing current investment on impact of the shock. As the allocation has not changed, the household finances this investment through a decrease in current consumption, resulting in an increase in the marginal utility of consumption. The intratemporal labor supply equation (8) implies this increased marginal utility of consumption leads to an increase in the marginal utility of leisure, and therefore a decrease in time spend on leisure. The increased labor effort, with the capital stock being fixed on impact as it is a state variable and with the productivity having not changed,[18] translates into an increase in current output partially offsetting the costs borne by consumption of the increased investment for the buffer stock of capital. Thus, this model predicts a boom in economic activity following an increase in risk, as firms produce and households work to accumulate the necessary buffer stock. A richer model of investment that, for example, includes variable capacity utilization can overturn this result, see Bidder and Smith (2012). While the impulse responses for the macroeconomic variables are not pictured with their contributing components, responses of these variables to a volatility shock come solely from the time-varying risk adjustment channel. The volatility shock is persistent but not permanent. As the shock dies out and productivity shocks fail to materialize from their widened distribution, the household winds down its buffer stock of capital by increasing consumption and leisure, leading to a fall in output and investment.

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[18]Remember, it is the distribution governing future productivity shocks that is being shocked here, not the level of productivity itself.

[Figure 4 about here.]

Figure 4 depicts the impulse responses and their contributing components for the expected risk premium to positive, one standard deviation shocks in $\varepsilon_{\sigma^a,t}$ and $\varepsilon_{z,t}$ (Figure 4a and 4b respectively). Note that both the volatility shock and productivity growth shock propagate solely through $y_{\sigma^2,i}$ the time-varying risk adjustment channel for this variable and there are no responses in the amplification channels of any of the three orders. Moreover, the responses to the volatility shock are almost two orders magnitude larger than the responses to the growth shock. Hence, figure 4 implies that almost all the variations in this variable are driven solely by volatility shocks with the contribution of the growth shock to the total variation negligible.

[Figure 5 about here.]

Figure 5 depicts the simulated time paths of the squared conditional market price of risk[19] under the second and the baseline calibration of the model (Figure 5a and 5b respectively). When there is no volatility shock, the conditional market price exhibits minimal fluctuations along the simulation path. Adding stochastic volatility, however, induces a substantial amount of variations in this variable. This is consistent with the interpretation that volatility shocks are a source of conditional heteroskedasticity. The displayed time variation in the conditional market price of risk is roughly consistent with the empirical variations in the (lower bound of) market price of risk as measured over different periods of time the past 130 odd years (See, Cogley and Sargent (2008, p. 466)).

6.2 Moments Comparison

We compare the mean and standard deviations of the third order approximation (20) to those reported in Tallarini (2000) for his model and post-war U.S. data. The results of the variance decomposition in Section 5.3 are reported, allowing us to pin down the contribution from the time-varying risk adjustment channel to the total variance of the endogenous variables.

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[19]We square this variable to eliminate the kink at the deterministic steady state, so that perturbation methods can be applied.

[Table 3 about here.]

The third column of Table 3 reports the theoretical means under the baseline calibration of the model. The fourth column reports means of Tallarini's (2000) model under the same calibration but without the volatility shock. For both macroeconomic and asset pricing variables, our theoretical means are in line with those of Tallarini (2000)[20]. As (21) implies, the theoretical means will generally differ from the deterministic steady states reported in the second column of the table since the mean at second and third order is adjusted for the infinite sum of variance over realized shocks $\frac{1}{2}\sum_{j=0}^{\infty} y_{j,j} E\left(\varepsilon_t \otimes \varepsilon_t\right)$ and the variance of future shocks $\frac{1}{2} y_{\sigma^2}$.

[Table 4 about here.]

The second column of Table 4 reports the theoretical standard deviations of the third order approximation (20) under the baseline calibration of the model. Comparing to the standard deviations reported in the third and fourth column, the theoretical standard deviations are in line with those reported in Tallarini (2000), both model based and empirical.

[Table 5 about here.]

Table 5 reports the results of the variance decomposition under the baseline and the second calibration without stochastic volatility. For each calibration, the table reports the percentage contributions of the first order amplification channel $y_t^{(1)}$ and the time-varying risk adjustment channel $y_t^{(3)risk}$ to the total variance of the endogenous variables as the overall majority of variations come from these two channels. The second and third column report the decomposition results in absence of volatility shock and the last two columns in presence of volatility shock. For the conditional market price of risk and the expected risk premium, all variation comes from the time-varying risk

[20]The fact that Tallarini chooses an iterative implementation of a modified LQ approximation method proposed by Hansen and Sargent (1995) to solve his model may account for the remaining difference.

adjustment channel regardless of whether there is volatility shock. This is consistent with the impulse responses for the expected risk premium (Figure 4), where we observed that both the growth and volatility shock propagate solely through the time-varying risk adjustment channel.

For the risk premium and macroeconomic variables, adding the volatility shock alters the composition of variance. In the absence of the volatility shock, the contribution of the time-varying risk adjustment channel is negligible and almost all variation comes from the first order amplification channel. Adding stochastic volatility, however, operationalizes the time-varying risk channel, as a large portion of variance now comes through this channel. Since, for macroeconomic variables, actions in the time-varying risk adjustment channel can be explained by the risk-averse household's precautionary motives, this variance decomposition result implies that such motives account for a larger portion of variance in the presence of stochastic volatility than in the absence thereof.

From a methodological point of view, in the absence of stochastic volatility shock, a first order linear approximation would thus appear sufficient for computing the theoretical variance of macroeconomic variables. However, theoretical variances need to be computed using a third order approximation in the presence of stochastic volatility and for conditional asset pricing measures, as otherwise a large portion or all of the variance will be missed through the neglect of time varying risk adjustment and higher order amplification effects.

6.3 Stochastic Volatility and Hansen-Jagannathan Bounds

We evaluate the model's ability of attaining the Hansen-Jagannathan bounds under the three different calibrations, as they are an important empirical measure for a model's ability to replicate asset pricing facts that depend on the first two moments of the pricing kernel.

[Figure 6 about here.]

Figure 6 depicts the unconditional mean standard deviation pairs of the pricing kernel generated by the model under the three different calibrations. Under the baseline (stochastic volatility) and

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second (constant variance) calibrations, the preferences are in recursive form, and therefore when the volatility of the kernel increases with risk aversion (here from one to five, ten, twenty, thirty, forty, fifty, and one hundred), the unconditional mean of the kernel is left (essentially) unchanged and the model approaches the Hansen-Jagannathan Bounds from below. The expected utility calibration generates a volatile pricing kernel at the cost of reducing its unconditional mean, as the EIS and risk aversion are inversely correlated in the expected utility, generating Weil's (1989) risk free rate puzzle. Figure 6a shows that given the same value of risk aversion, the calibration with stochastic volatility (baseline calibration) generates a more volatile pricing kernel than the constant volatility calibration. In other words, to generate certain amount of volatility in the pricing kernel, the model with volatility shock appears to need less risk aversion than the model without volatility shock. This is achieved, however, at the cost of increasing the variance of the log consumption growth. As figure 6b shows, if we hold that variance constant at its empirical counterpart by reducing the homoskedastic component of the productivity growth shock, the effect of volatility shock in terms of further increasing the volatility in the pricing kernel vanishes, reiterating the conditional heteroskedastic interpretation of volatility shocks.

7   Conclusion

We have solved a business cycle model with recursive preferences and stochastic volatility with a third order perturbation approximation to the nonlinear moving average policy function. We use the impulse responses generated by this third order approximation to analyze the propagation mechanism of a volatility shock, and find that for macroeconomic variables, a volatility shock by itself propagates solely through a time-varying risk adjustment channel. For conditional asset pricing variables, this time-varying risk adjustment channel is the only working channel for the transmission of shocks, both to productivity growth and its volatility.

We have derived a closed-form calculation of the theoretical moments of the endogenous vari-

ables using a third order approximation. Our calculation of moments lends itself to a decomposition that disentangles the individual contributions of time-varying risk adjustment and amplification channels to the total variance. In our model, we find that adding stochastic volatility alters the composition of variance, making a time-varying risk channel a substantial contributor of variance. For macroeconomic variables, variations that come from the time-varying risk adjustment channel can be explained by the household's precautionary savings desires and, in the presence of stochastic volatility, we find a large portion of variations in macroeconomic variables is driven by precautionary behavior.

In linear approximations, variance decompositions can be applied to study the individual contribution of each shock to the total variance. The channels of risk adjustment and amplification we have derived here are perhaps a step towards extending this shock-specific decomposition to nonlinear perturbation approximations.

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A Appendices

A.1   Detrending the Model

Stationary consumption, investment, capital stock and output, denoted by the lower case letters, are defined as follows

$$(39) \qquad c_t \equiv \frac{C_t}{e^{Z_t}}, \ \ I_t \equiv \frac{I_t}{e^{Z_t}}, \ \ k_t \equiv \frac{K_t}{e^{Z_t}}, \ \ y_t \equiv \frac{Y_t}{e^{Z_t}},$$

For notational ease in detrending the model, we define a combined shock $\varepsilon_{a,t}$, containing both the homoskedastic and heteroskedastic components of the productivity growth shock

$$(40) \qquad \varepsilon_{a,t} \equiv \overline{\sigma_z} e^{\sigma_{z,t}} \varepsilon_{z,t}$$

The productivity growth process can therefore be written as

$$(41) \qquad a_t \equiv Z_t - Z_{t-1} = \overline{a} + \varepsilon_{a,t}$$

While detrending, the exponential form of the foregoing will be frequently used

$$(42) \qquad e^{a_t} = \frac{e^{Z_t}}{e^{Z_{t-1}}} = e^{\overline{a} + \varepsilon_{a,t}}$$

The goal is essentially to substitute C_t, I_t, K_t and Y_t for their stationary counterparts in the relevant model equations. We start with the production function

$$(43) \qquad \left(y_t e^{Z_t} \right) = \left(k_{t-1} e^{Z_{t-1}} \right)^{\alpha} \left(e^{Z_t} N_t \right)^{1-\alpha}$$

$$(44) \qquad \Rightarrow y_t = \left(\frac{e^{Z_t}}{e^{Z_{t-1}}} \right)^{-\alpha} k_{t-1}^{\alpha} N_t^{1-\alpha}$$

$$(45) \qquad \Rightarrow y_t = e^{-\alpha(\overline{a} + \varepsilon_{a,t})} k_{t-1}^{\alpha} N_t^{1-\alpha}$$

Detrending the capital accumulation law

$$(46) \qquad \left(k_t e^{Z_t} \right) = (1 - \delta) \left(k_{t-1} e^{Z_{t-1}} \right) + \left(i_t e^{Z_t} \right)$$

$$(47) \qquad \Rightarrow k_t = (1 - \delta) \frac{e^{Z_{t-1}}}{e^{Z_t}} k_{t-1} + i_t$$

$$(48) \qquad \Rightarrow k_t = (1 - \delta) e^{-\overline{a} - \varepsilon_{a,t}} k_{t-1} + i_t$$

Detrending the market clearing condition is straightforward as it is a contemporaneous relation-

ship

(49)
$$\left(y_t e^{Z_t}\right) = \left(c_t e^{Z_t}\right) + \left(i_t e^{Z_t}\right)$$

(50)
$$\Rightarrow y_t = c_t + i_t$$

Combing (45), (48) and (50) yields the detrended resource constraint

(51)
$$c_t + k_t = e^{-\alpha(\bar{a}+\varepsilon_{a,t})} k_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) e^{-\bar{a}-\varepsilon_{a,t}} k_{t-1}$$

Detrending the labor supply equation

(52)
$$\frac{\psi}{1-N_t} = \frac{1}{c_t e^{Z_t}} (1-\alpha) \left(k_{t-1} e^{Z_{t-1}}\right)^{\alpha} e^{Z_t(1-\alpha)} N_t^{-\alpha}$$

(53)
$$\Rightarrow \frac{\psi}{1-N_t} = (1-\alpha) e^{-\alpha(\bar{a}+\varepsilon_{a,t})} \frac{1}{c_t} k_{t-1}^{\alpha} N_t^{-\alpha}$$

The risky rate r_t is stationary and we reexpress it in terms of the stationary variables

(54)
$$1 + r_t = (1-\delta) + \alpha \left(k_{t-1} e^{Z_{t-1}}\right)^{\alpha-1} \left(e^{Z_t} N_t\right)^{1-\alpha}$$

(55)
$$\Rightarrow 1 + r_t = (1-\delta) + \alpha k_{t-1}^{\alpha-1} e^{(\bar{a}+\varepsilon_{a,t})(1-\alpha)} N_t^{1-\alpha}$$

We now move to the value function. As the felicity function is logarithmic in nonstationary consumption, removing the trend in consumption will leave a term linear in the level of productivity that when subtracted from V_t gives the stationary value function v_t

(56)
$$v_t = V_t - b \ln e^{Z_t} = V_t - b Z_t$$

Substituting the relevant variables for their stationary counterparts yields

(57)
$$v_t + b Z_t = \ln \left(c_t e^{Z_t}\right) + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} [v_{t+1} + b Z_{t+1}]\right)\right]\right)$$

(58)
$$\Rightarrow v_t = \ln c_t + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + b \left(Z_{t+1} - \frac{b-1}{b\beta} Z_t\right)\right]\right)\right]\right)$$

It follows that the remaining nonstationarities can be offset if

(59)
$$\frac{b-1}{b\beta} = 1$$

which pins down b as

(60)
$$b = \frac{1}{1-\beta}$$

Inserting (60) in (58) yields the stationary value function

$$(61) \qquad v_t = \ln c_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right] \right)$$

While stationary, the foregoing value function does not fit in the problem statement (18) in the text, thus can not be implemented directly in perturbation software packages like Dynare. This problem is caused by nonlinear twisting of the expected continuation value, and can be fixed by redefining this conditional expectation as a new variable known in period t. Besides, the twisted expected continuation value is numerically unstable, due to the logarithmic transformation, when γ approaches zero or becomes very large. To counteract this, we define[21]

$$(62) \qquad \widetilde{v}_t \equiv E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} \widetilde{\varepsilon}_{t+1} - \bar{v} \right] \right) \right]$$

where $\bar{v}$ denotes the deterministic steady state value of the stationary value function (61) and can be computed as follows

$$(63) \qquad \bar{v} = \frac{1}{1-\beta} \left[\ln c + \psi \ln(1-N) + \frac{\beta}{1-\beta} \bar{a} \right]$$

Substituting v_{t+1} in (61) for $\widetilde{v}_t$ yields the normalized, stationary value function

$$(64) \qquad v_t = \ln c_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \left[\ln \widetilde{v}_t + \frac{\gamma}{2} \left(\frac{1}{1-\beta} \bar{a} + \bar{v} \right) \right]$$

With the stationary value function in hand, we reexpress the pricing kernel in terms of stationary variables

$$(65) \qquad m_{t+1} = \beta \frac{c_t e^{Z_t}}{c_{t+1} e^{Z_{t+1}}} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right) \right]}$$

Multiplying both the denominator and numerator of the foregoing with $\exp \left(-\frac{\gamma}{2} \frac{1}{1-\beta} Z_t \right)$, and rearranging yields

$$(66) \qquad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \varepsilon_{a,t+1})} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right]}$$

Writing out the definition of $\varepsilon_{a,t+1}$ yields (13) in the text. Recognizing the expectational term in the previous equation can be replaced by the product $\widetilde{v}_t \exp \left(\frac{\gamma}{2} \left[\bar{v} + \frac{1}{1-\beta} \bar{a} \right] \right)$, we substitute it for this

---

[21]Rudebusch and Swanson (2012) adopt, in their companion Mathematica codes, a very similar procedure to improve numerical stability.

product and collect terms

$$(67) \qquad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a}+\varepsilon_{a,t+1})} \frac{\exp\left(\frac{\gamma}{2}\left[v_{t+1}+\frac{1}{1-\beta}\varepsilon_{a,t+1}-\bar{v}\right]\right)}{\tilde{v}_t}$$

The period t counterpart of the foregoing follows

$$(68) \qquad m_t = \beta \frac{c_{t-1}}{c_t} e^{-(\bar{a}+\varepsilon_{a,t})} \frac{\exp\left(\frac{\gamma}{2}\left[v_t+\frac{1}{1-\beta}\varepsilon_{a,t}-\bar{v}\right]\right)}{\tilde{v}_{t-1}}$$

A.2 Proof of Proposition 5.1

Rearrange the definition of the second order increment to express the second order approximation as the sum of the first order approximation, the second order increment, and the second order constant risk adjustment

$$(69) \qquad y_t^{(2)} = y_t^{(1)} + dy_t^{(2)} + \frac{1}{2}y_{\sigma^2}$$

Applying the expectations operator to the foregoing yields the mean of the second order approximation

$$(70) \qquad Ey_t^{(2)} = Ey_t^{(1)} + Edy_t^{(2)} + \frac{1}{2}y_{\sigma^2}$$

Centering the second order approximation (69) around its mean by subtracting (70) from (69) yields

$$(71) \qquad y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - Ey_t^{(1)}\right) + \left(dy_t^{(2)} - Edy_t^{(2)}\right)$$

Noting that the mean of the first order approximation is the deterministic steady state of y_t, i.e., $Ey_t^{(1)} = \bar{y}$, the foregoing can be rewritten as

$$(72) \qquad y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - \bar{y}\right) + \left(dy_t^{(2)} - Edy_t^{(2)}\right)$$

Using the definition of the first order increment $dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$, the foregoing is

$$(73) \qquad y_t^{(2)} - Ey_t^{(2)} = dy_t^{(1)} + \left(dy_t^{(2)} - Edy_t^{(2)}\right)$$

Multiplying the foregoing with its transposition at $t-j$ and noting that $Ey_t^{(2)} = Ey_{t-j}^{(2)}$ and $Edy_t^{(2)} = Edy_{t-j}^{(2)}$ yields

$$(74) \qquad \left(y_t^{(2)} - Ey_t^{(2)}\right)\left(y_{t-j}^{(2)} - Ey_t^{(2)}\right)'$$
$$= \left[dy_t^{(1)} + \left(dy_t^{(2)} - Edy_t^{(2)}\right)\right]\left[dy_{t-j}^{(1)} + \left(dy_{t-j}^{(2)} - Edy_t^{(2)}\right)\right]'$$

$$= dy_t^{(1)} dy_{t-j}^{(1)\prime} + \left(dy_t^{(2)} dy_{t-j}^{(1)\prime} - E dy_t^{(2)} dy_{t-j}^{(1)\prime} \right)$$

$$+ \left(dy_t^{(1)} dy_{t-j}^{(2)\prime} - dy_t^{(1)} E dy_t^{(2)\prime} \right) + \left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_t^{(2)} \right)^{\prime}$$

Applying the expectations operator to the foregoing delivers

(75)
$$E\left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_t^{(2)} \right)^{\prime} \right]$$

$$= E\left(dy_t^{(1)} dy_{t-j}^{(1)\prime} \right) + E\left(dy_t^{(2)} dy_{t-j}^{(1)\prime} \right) - E dy_t^{(2)} E dy_{t-j}^{(1)\prime}$$

$$+ E\left(dy_t^{(1)} dy_{t-j}^{(2)\prime} \right) - E dy_t^{(1)} E dy_t^{(2)\prime} + E\left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_t^{(2)} \right)^{\prime} \right]$$

To simplify the foregoing, apply the expectations operator to the definition of the first order incre-

ment, yielding its mean

(76)
$$E dy_t^{(1)} = E y_t^{(1)} - \bar{y}$$

As $E y_t^{(1)} = \bar{y}$, the foregoing implies that the mean of the first order increment is zero

(77)
$$E dy_t^{(1)} = 0$$

Using the this result and noting that $E dy_t^{(1)} = E dy_{t-j}^{(1)}$, (75) reduces to

(78)
$$E\left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_t^{(2)} \right)^{\prime} \right]$$

$$= E\left(dy_t^{(1)} dy_{t-j}^{(1)\prime} \right) + E\left(dy_t^{(2)} dy_{t-j}^{(1)\prime} \right) + E\left(dy_t^{(1)} dy_{t-j}^{(2)\prime} \right)$$

$$+ E\left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_t^{(2)} \right)^{\prime} \right]$$

It then remains to show that

(79)
$$E\left(dy_t^{(2)} dy_{t-j}^{(1)\prime} \right) = 0, \quad E\left(dy_t^{(1)} dy_{t-j}^{(2)\prime} \right) = 0$$

One way is to use the moving average representation of the order increments. I.e., inserting the

moving average representation of the first and second order approximations in the definition of the

order increments yields

(80)
$$dy_t^{(1)} = \sum_{i=0}^{\infty} y_i \varepsilon_{t-i}$$

(81)
$$dy_t^{(2)} = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

Therefore the product of the two order increments, when set in expectation, takes the form of the

third moments of the shocks, which is equal to zero under normality.

A.3 Proof of Proposition 5.2

First note that $E y_t^{(3)} = E y_t^{(2)}$ under normality[22]. Given this result, applying the expectations operator to the definition of the third order increment $d y_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$ immediately implies $E d y_t^{(3)} = 0$.

Next, rearranging the definition of the third order increment delivers

$$(82) \qquad y_t^{(3)} = y_t^{(2)} + d y_t^{(3)}$$

Applying the expectations operator to the foregoing yields

$$(83) \qquad E y_t^{(3)} = E y_t^{(2)}$$

Centering (82) around its mean by subtracting (83) from (82) gives

$$(84) \qquad y_t^{(3)} - E y_t^{(3)} = y_t^{(2)} - E y_t^{(2)} + d y_t^{(3)}$$

Multiplying the foregoing with its transposition at $t - j$ and noting $E y_t^{(3)} = E y_{t-j}^{(3)}$ and $E y_t^{(2)} = E y_{t-j}^{(2)}$ delivers

$$
\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_t^{(3)} \right)' = d y_t^{(3)} d y_{t-j}^{(3)'} + \left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_t^{(2)} \right)'
$$
$$
+ d y_t^{(3)} y_{t-j}^{(2)'} - d y_t^{(3)} E y_t^{(2)'} + y_t^{(2)} d y_{t-j}^{(3)'} - E y_t^{(2)} d y_{t-j}^{(3)'}
$$

Applying the expectations operator to the foregoing, noting $E d y_t^{(3)} = 0$, gives

$$(85) \quad E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_t^{(3)} \right)' \right] = E_t \left[d y_t^{(3)} d y_{t-j}^{(3)'} \right] + E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_t^{(2)} \right)' \right]$$
$$
+ E \left(d y_t^{(3)} y_{t-j}^{(2)'} \right) + E \left(y_t^{(2)} d y_{t-j}^{(3)'} \right)
$$

Rewrite the definition of the second order increment $d y_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2} y_{\sigma^2}$ as

$$(86) \qquad y_t^{(2)} = d y_t^{(2)} + y_t^{(1)} + \frac{1}{2} y_{\sigma^2} = d y_t^{(2)} + d y_t^{(1)} + \bar{y} + \frac{1}{2} y_{\sigma^2}$$

Given the foregoing expression and noting $E d y_t^{(3)} = 0$, $E \left(y_t^{(2)} d y_{t-j}^{(3)'} \right)$ on the right hand side of (85) can be rewritten as

$$(87) \qquad E \left(y_t^{(2)} d y_{t-j}^{(3)'} \right) = E \left[\left(d y_t^{(2)} + d y_t^{(1)} + \bar{y} + \frac{1}{2} y_{\sigma^2} \right) d y_{t-j}^{(3)'} \right] = E \left(d y_t^{(1)} d y_{t-j}^{(3)'} \right)$$

---

[22]To see this, applying the expectations operator to the second order approximation (26) and comparing the resulting expression with the mean of the third order approximation (21)

Noting that $E\left(dy_t^{(2)}dy_{t-j}^{(3)'}\right)$ is zero under normality[23]. Analogously, $E\left(dy_t^{(3)}y_{t-j}^{(2)'}\right)$ on the right hand side of (85) can be written as

(89) $$E\left[dy_t^{(3)}y_{t-j}^{(2)'}\right] = E\left(dy_t^{(3)}dy_{t-j}^{(1)'}\right)$$

Inserting the last two equations in (85) yields

$$E\left[\left(y_t^{(3)} - Ey_t^{(3)}\right)\left(y_{t-j}^{(3)} - Ey_t^{(3)}\right)'\right] = E_t\left[dy_t^{(3)}dy_{t-j}^{(3)'}\right] + E\left[\left(y_t^{(2)} - Ey_t^{(2)}\right)\left(y_{t-j}^{(2)} - Ey_t^{(2)}\right)'\right]$$
$$+ E\left(dy_t^{(1)}dy_{t-j}^{(3)'}\right) + E\left(dy_t^{(3)}dy_{t-j}^{(1)'}\right)$$

A.4 Second Moments of $dy_t^{(2)}$

The second order increment $dy_t^{(2)}$ can be expressed recursively as

(90) $$dy_t^{(2)} = \alpha dy_{t-1}^{(2)state} + \frac{1}{2}\left[\beta_{22}dy_{t-1}^{state\otimes[2]} + 2\beta_{20}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t\right) + \beta_{00}\varepsilon_t^{\otimes[2]}\right]$$

If the previous equation can be cast as a linear recursion, then standard linear methods can be applied to the computation of the second moments. Note $dy_t^{(2)}$, besides being linearly autoregressive in the state variable block of itself $dy_{t-1}^{(2)state}$, is a linear function of all the second order permutations of products of the first order increment $dy_{t-1}^{(1)state}$ and the shocks. This relationship guides the calculations, and we therefore compute the second moments of $dy_t^{(2)state}$ first, then recover the second moments of variables of interest[24].

The state variable block of (90) takes the form

(91) $$dy_t^{(2)state} = \alpha^{state}dy_{t-1}^{(2)state} + \frac{1}{2}\beta_{22}^{state}dy_{t-1}^{(1)state\otimes[2]} + \beta_{20}^{state}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t\right) + \frac{1}{2}\beta_{00}^{state}\varepsilon_t^{\otimes[2]}$$

To cast the foregoing in a linear recursion, we take the state variable block of the first order increment $dy_t^{(1)state}$ and raise it to the second Kronecker power, noting throughout we use (ns) to

---

[23]Again consider the moving average representation of the third order increment

(88) $$dy_t^{(3)} = \frac{1}{2}\sum_{i=0}^{\infty}y_{\sigma^2,i}\varepsilon_{t-i} + \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}y_{k,j,i}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i})$$

When multiplying with the moving average representation of the second order increment, the result, in expectation, is a sum of the third and fifth moments of shocks, and equal to zero under normality.

[24]This procedure is widely adopted to minimize the dimension and improve the speed of the computation. See, e.g., Uhlig's (1999) toolkit, Schmitt-Grohé and Uribe's (2004) software package and Dynare.

denote the number of state variables

$$(92) \qquad dy_t^{(1)state\otimes[2]} = \alpha^{state\otimes[2]} dy_{t-1}^{(1)state\otimes[2]} + (K_{ns,ns} + I_{ns^2}) \left(\alpha^{state} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right)$$

$$+ \beta_0^{state\otimes[2]} \varepsilon_t^{\otimes[2]}$$

where $K_{ns,ns}$ is a $ns^2 \times ns^2$ commutation matrix (See Magnus and Neudecker (1979)). Combing (91) and (92) yields the following linear recursion containing the linear recursion of $dy_t^{(2)state}$

$$(93) \qquad X_t^{(2)} = \Theta^{(2)X} X_{t-1}^{(2)} + \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} \\ \beta_0^{state\otimes[2]} \end{bmatrix} E\left(\varepsilon_t^{\otimes[2]} \right) + \Phi^{(2)X} \Xi_t^{(2)}$$

where

$$(94) \qquad X_t^{(2)} = \begin{bmatrix} dy_t^{(2)state} \\ dy_t^{(1)state\otimes[2]} \end{bmatrix}$$

$$(95) \qquad \Theta^{(2)X} = \begin{bmatrix} \alpha^{state} & \frac{1}{2}\beta_{22}^{state} \\ 0 & \alpha^{state\otimes[2]} \end{bmatrix}$$

$$(96) \qquad \Phi^{(2)X} = \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} & \beta_{20}^{state} \\ \beta_0^{state\otimes[2]} & (K_{ns,ns} + I_{ns^2})\left(\alpha^{state} \otimes \beta_0^{state} \right) \end{bmatrix}$$

$$(97) \qquad \Xi_t^{(2)} = \begin{bmatrix} \varepsilon_t^{\otimes[2]} - E\varepsilon_t^{\otimes[2]} \\ dy_{t-1}^{(1)state} \otimes \varepsilon_t \end{bmatrix}$$

While the second term on the right hand side of (93) vanishes after centering (93) around its mean, it ensures, by compensating the subtraction of $E\left(\varepsilon_t^{\otimes[2]} \right)$ in $\Xi_t^{(2)}$, that $\Xi_t^{(2)}$ is orthogonal[25] to $X_{t-1}^{(2)}$

$$(98) \qquad E\left(X_{t-1}^{(2)} \Xi_t^{(2)\prime} \right) = 0$$

With the linear recursion of $X_t^{(2)}$, the second order increment (90) can be recast as the following linear recursion

$$(99) \qquad dy_t^{(2)} = \Theta^{(2)} X_{t-1}^{(2)} + \begin{bmatrix} \frac{1}{2}\beta_{00} \\ \beta_0^{\otimes[2]} \end{bmatrix} E\left(\varepsilon_t^{\otimes[2]} \right) + \Phi^{(2)} \Xi_t^{(2)}$$

$$\text{where } \Theta^{(2)} = \begin{bmatrix} \alpha & \frac{1}{2}\beta_{22} \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} \frac{1}{2}\beta_{00} & \beta_0^{\otimes[2]} \end{bmatrix}$$

Noting $E\left(\Xi_t^{(2)} \right) = 0$ by construction, and the mean of the foregoing writes

$$(100) \qquad E dy_t^{(2)} = \Theta^{(2)} E X_t^{(2)} + \begin{bmatrix} \frac{1}{2}\beta_{00} \\ \beta_0^{\otimes[2]} \end{bmatrix} E\left(\varepsilon_t^{\otimes[2]} \right)$$

---

[25]This orthogonality condition significantly simplifies the calculation of the autocovariances that followed.

A.4.1 Contemporaneous Covariance

Centering (99) around its mean—by subtracting (100) from (99)—yields the following centered linear recursion of the second order increment

$$(101) \qquad \left(dy_t^{(2)} - Edy_t^{(2)}\right) = \Theta^{(2)}\left(X_{t-1}^{(2)} - EX_t^{(2)}\right) + \Phi^{(2)}\Xi_t^{(2)}$$

Multiplying the foregoing with its transposition and applying the expectations operator to the resulting expression yields the contemporaneous variance of the second order increment

$$(102) \qquad \Gamma_0^{(2)} = \Theta^{(2)}\Gamma_0^{(2)X}\Theta^{(2)'} + \Omega^{(2)}$$

where

$$(103) \qquad \Gamma_0^{(2)X} = E\left[\left(X_t^{(2)} - EX_t^{(2)}\right)\left(X_t^{(2)} - EX_t^{(2)}\right)'\right]$$

$$(104) \qquad \Gamma_0^{(2)} = E\left[\left(dy_t^{(2)} - Edy_t^{(2)}\right)\left(dy_t^{(2)} - Edy_t^{(2)}\right)'\right]$$

$$(105) \qquad \Omega^{(2)} = \Phi^{(2)}E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right)\Phi^{(2)'}$$

This requires the contemporaneous variance of $X_t^{(2)}$, i.e., $\Gamma_0^{(2)X}$, as well as $E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right)$. Starting with $\Gamma_0^{(2)X}$, we can proceed by applying the expectations operator to (93) to yield

$$(106) \qquad EX_t^{(2)} = \Theta^{(2)X}EX_t^{(2)} + \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} \\ \beta_0^{state\otimes[2]} \end{bmatrix} E\left(\varepsilon_t^{\otimes[2]}\right)$$

Centering the foregoing around its mean yields

$$(107) \qquad X_t^{(2)} - EX_t^{(2)} = \Theta^{(2)X}\left(X_{t-1}^{(2)} - EX_t^{(2)}\right) + \Phi^{(2)X}\Xi_t^{(2)}$$

Multiplying the foregoing with its transposition and applying the expectations operator, it follows the unknown contemporaneous variance of $X_t^{(2)}$ solves the following Lyapunov equation[26]

$$(108) \qquad \Gamma_0^{(2)X} = \Theta^{(2)X}\Gamma_0^{(2)X}\Theta^{(2)X'} + \Omega^{(2)X}$$

where

$$(109) \qquad \Omega^{(2)X} = \Phi^{(2)X}E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right)\Phi^{(2)X'}$$

---

[26]Note $\Gamma_0^{(2)X}$ is of dimension $(ns + ns^2) \times (ns + ns^2)$. For models with a large number of state variables, splitting (108) into four Sylvester equations of smaller size by exploiting the triangularity of $\Theta^{(2)X}$ and solving them one by one is computationally a lot less expensive than solving (108) as a whole. This division also enables exploitation of the symmetry of $\Gamma_0^{(2)X}$ and therefore can avoid redundant computations.

(110)

Thus, $\Gamma_0^{(2)X}$ can be calculated given $E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right)$ and, therefore, $\Gamma_0^{(2)X}$ in (102) too. We requires this variance, which is given by

(111) $$E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right) = \begin{bmatrix} (I_{ne^2} + K_{ne,ne})\left[E\left(\varepsilon_t\varepsilon_t'\right) \otimes E\left(\varepsilon_t\varepsilon_t'\right)\right] & 0 \\ 0 & \Gamma_0^{(1)X} \otimes E\left(\varepsilon_t\varepsilon_t'\right) \end{bmatrix}$$

In the right hand side of (111), $\Gamma_0^{(1)X}$ is the state variable block of the contemporaneous variance of the first order approximation (or of the first order increment), and therefore already known from calculations at the first order.

The upper left entry of the right hand side of (111) contains the fourth moment of the shocks and can be computed using Tracy and Sultan's (1993, p. 344) formula. The two zero entries in (111) are due to the fact that the third moments of the shocks are zero under normality, and $dy_{t-1}^{(1)state}$ is uncorrelated with current shocks.

A.4.2   Autocovariances

Now we turn to the autocovariances of $dy_t^{(2)}$. To start, note that under normality, $\Xi_t^{(2)}$ is serially uncorrelated

(112) $$E\left(\Xi_t^{(2)}\Xi_{t-j}^{(2)'}\right) = 0 \ \forall \ j > 0$$

Given the contemporaneous variance $\Gamma_0^{(2)X}$, multiplying (107) with the transposition of (101) and taking expectation yields the contemporaneous variance between the $X_t^{(2)}$ and $dy_t^{(2)}$

(113) $$\Gamma_0^{(2)X,dy} = \Theta^{(2)X}\Gamma_0^{(2)X}\Theta^{(2)'} + \Omega^{(2)X,dy}$$

where

(114) $$\Gamma_0^{(2)X,dy} = E\left[\left(X_t^{(2)} - EX_t^{(2)}\right)\left(dy_t^{(2)} - Edy_t^{(2)}\right)'\right]$$

(115) $$\Omega^{(2)X,dy} = \Phi^{(2)X}E\left(\Xi_t^{(2)}\Xi_t^{(2)'}\right)\Phi^{(2)'}$$

With all the three contemporaneous variances in hand, the orthogonality (98) and (112) ensures the autocovariance of $dy_t^{(2)}$ can be computed with the following recursive formulae

(116) $$\Gamma_j^{(2)} = \Theta^{(2)}\Gamma_{j-1}^{(2)X,dy}$$

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$$(117) \qquad \Gamma_j^{(2)X,dy} = \Theta^{(2)X}\Gamma_{j-1}^{(2)X,dy}$$

where

$$(118) \qquad \Gamma_j^{(2)} = E\left[\left(dy_t^{(2)} - Edy_t^{(2)}\right)\left(dy_{t-j}^{(2)} - Edy_t^{(2)}\right)'\right]$$

$$(119) \qquad \Gamma_j^{(2)X,dy} = E\left[\left(X_t^{(2)} - EX_t^{(2)}\right)\left(dy_{t-j}^{(2)} - Edy_t^{(2)}\right)'\right]$$

A.5 Second Moments of $dy_t^{(3)}$

The third order increment can be expressed recursively as

$$(120) \quad dy_t^{(3)} = \alpha dy_{t-1}^{(3)state} + \frac{1}{6}\left[\beta_{333,1}dy_{t-1}^{(1)state\otimes[3]} + \beta_{000}\varepsilon_t^{\otimes[3]}\right]$$
$$+ \beta_{22}\left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state}\right) + \beta_{20}\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t\right)$$
$$+ \frac{1}{2}\left[\beta_{300}\left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes[2]}\right) + \beta_{330,1}\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t\right) + \beta_{\sigma^2 0}\varepsilon_t + \beta_{\sigma^2 1}dy_{t-1}^{(1)state}\right]$$

Its state variable block takes the form

$$(121) \qquad dy_t^{(3)state} = \alpha^{state}dy_{t-1}^{(3)state} + \frac{1}{6}\left[\beta_{333,1}^{state}dy_{t-1}^{(1)state\otimes[3]} + \beta_{000}^{state}\varepsilon_t^{\otimes[3]}\right]$$
$$+ \beta_{22}^{state}\left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state}\right) + \beta_{20}^{state}\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t\right)$$
$$+ \frac{1}{2}\left[\beta_{300}^{state}\left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes[2]}\right) + \beta_{330,1}^{state}\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t\right)\right]$$
$$+ \beta_{\sigma^2 0}^{state}\varepsilon_t + \beta_{\sigma^2 1}^{state}dy_{t-1}^{(1)state}\right]$$

From the terms on the left hand side of the foregoing, we need to build up two additional recursions, the first in the Kronecker product of the first and second order increments and the second in the triple Kronecker product of the first order increment, to construct the linear recursion containing $dy_t^{(3)state}$ that can be used for calculating moments

$$(122)$$
$$dy_t^{(2)state} \otimes dy_t^{(1)state} = \alpha^{state\otimes[2]}\left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state}\right) + \left[\left(\frac{1}{2}\beta_{22}^{state}\right) \otimes \alpha^{state}\right]dy_{t-1}^{(1)state\otimes[3]}$$
$$+ \left(\alpha^{state} \otimes \beta_0^{state}\right)\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t\right) + \left[\left(\frac{1}{2}\beta_{00}^{state}\right) \otimes \beta_0^{state}\right]\varepsilon_t^{state\otimes[3]}$$
$$+ \left[\left(\beta_{20}^{state} \otimes \alpha^{state}\right)K_{ns*ne,ns} + \left(\frac{1}{2}\beta_{22}^{state}\right) \otimes \beta_0^{state}\right]\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t\right)$$

$$+\left(\left[\left(\frac{1}{2}\beta_{00}^{state}\right)\otimes\alpha^{state}\right]K_{ne^2,ns}+\beta_{20}^{state}\otimes\beta_{0}^{state}\right)\left(dy_{t-1}^{(1)state}\otimes\varepsilon_{t}^{\otimes[2]}\right)$$

(123) $\quad dy_{t}^{(1)state\otimes[3]}=\alpha^{state\otimes[3]}dy_{t-1}^{(1)state\otimes[3]}+\beta_{0}^{state\otimes[3]}\varepsilon_{t}^{\otimes[3]}$

$$+\left[(K_{ns,ns}\otimes I_{ns}+I_{ns^3})K_{ns^2,ns}+I_{ns^3}\right]\left(\alpha^{state\otimes[2]}\otimes\beta_{0}^{state}\right)\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_{t}\right)$$

$$+\left[K_{ns^2,ns}+(K_{ns,ns}\otimes I_{ns}+I_{ns^3})\right]\left(\alpha^{state}\otimes\beta_{0}^{state\otimes[2]}\right)\left(dy_{t-1}^{(1)state}\otimes\varepsilon_{t}^{\otimes[2]}\right)$$

Given the foregoing two equations, along with the state variable block of the first order increment

(124) $$dy_{t}^{(1)state}=\alpha^{state}dy_{t-1}^{(1)state}+\beta_{0}^{state}\varepsilon_{t}$$

we construct the following linear recursion

(125) $$X_{t}^{(3)}=\Theta^{(3)X}X_{t-1}^{(3)}+\Phi^{(3)X}\Xi_{t}^{(3)}$$

where[27]

(126) $$X_{t}^{(3)}=\begin{bmatrix}dy_{t}^{(3)state}\\dy_{t}^{(2)state}\otimes dy_{t}^{(1)state}\\dy_{t}^{(1)state\otimes[3]}\\dy_{t}^{(1)state}\end{bmatrix},\qquad\Xi_{t}^{(3)}=\begin{bmatrix}\varepsilon_{t}^{\otimes[3]}\\dy_{t-1}^{(2)state}\otimes\varepsilon_{t}\\dy_{t-1}^{(1)state}\otimes\left(\varepsilon_{t}^{\otimes[2]}-E\varepsilon_{t}^{\otimes[2]}\right)\\dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_{t}\\\varepsilon_{t}\end{bmatrix}$$

Note there is no need to center $X_{t}^{(3)}$ before computing its contemporaneous variance as its mean is zero under normality, i.e., $EX_{t}^{(3)}=0$. In the third entry of $\Xi_{t}^{(3)}$, $\varepsilon_{t}^{\otimes[2]}$ is adjusted using its mean, such that $\Xi_{t}^{(3)}$ is orthogonal to $X_{t-1}^{(3)}$

(127) $$E\left(X_{t-1}^{(3)}\Xi_{t}^{(3)\prime}\right)=0$$

and it is can be shown that $\Xi_{t}^{(3)}$ is serially uncorrelated

(128) $$E\left(\Xi_{t}^{(3)}\Xi_{t-j}^{(3)\prime}\right)=0\ \forall\ j>0$$

A.5.1 Contemporaneous Covariance

With linear recursion (125), the third order increment (120) can be cast in a linear recursion[28]

(129) $$dy_{t}^{(3)}=\Theta^{(3)}X_{t-1}^{(3)}+\Phi^{(3)}\Xi_{t}^{(3)}$$

Multiplying the foregoing with its transposition and applying the expectations operator to the

---

[27] $\Theta^{(3)X}$ and $\Phi^{(3)X}$ are specified in section A.8.

[28] $\Theta^{(3)}$ and $\Phi^{(3)}$ are specified in section A.8.

resulting expression yields the contemporaneous variance of the third order increment

(130) $$\Gamma_0^{(3)} = \Theta^{(3)}\Gamma_0^{(3)X}\Theta^{(3)'} + \Omega^{(3)}$$

where

(131) $$\Gamma_0^{(3)} = E\left(dy_t^{(3)}dy_t^{(3)'}\right)$$

(132) $$\Omega^{(3)} = \Phi^{(3)}E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)\Phi^{(3)'}$$

To compute the yet known contemporaneous variance of $X_t^{(3)}$, i.e., $\Gamma_0^{(3)X}$, we multiply (125) with its transposition and apply the expectations operator to the resulting expression. It follows that $\Gamma_0^{(3)X}$ solves the following Lyapunov equation[29]

(133) $$\Gamma_0^{(3)X} = \Theta^{(3)X}\Gamma_0^{(3)X}\Theta^{(3)X'} + \Omega^{(3)X}$$

where

(134) $$\Gamma_0^{(3)X} = E\left(X_t^{(3)}X_t^{(3)'}\right)$$

(135) $$\Omega^{(3)X} = \Phi^{(3)X}E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)\Phi^{(3)X'}$$

with $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$ as specified in section A.8.

Given $\Gamma_0^{(3)X}$, multiplying (125) with the transposition of (129) and applying the expectations operator yields the contemporaneous variance between $X_t^{(3)}$ and $dy_t^{(3)}$

(136) $$\Gamma_0^{(3)X,dy} = \Theta^{(3)X}\Gamma_0^{(3)X}\Theta^{(3)'} + \Omega^{(3)X,dy}$$

where

(137) $$\Gamma_0^{(3)X,dy} = E\left(X_t^{(3)}dy_t^{(3)'}\right)$$

(138) $$\Omega^{(3)X,dy} = \Phi^{(3)X}E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)\Phi^{(3)'}$$

---

[29]Note that (133) is a Lyapunov equation of dimension $\left(ns + ns^2 + ns^3 + ns\right) \times \left(ns + ns^2 + ns^3 + ns\right)$. By exploiting the triangularity of $\Theta^{(3)X}$ and the symmetry of $\Gamma_0^{(3)X}$, that large Lyapunov equation can be split and reduced to 10 Sylvester equations of dimension up to $ns^3 \times ns^3$.

A.5.2 Autocovariances

For the autocovariance of the third order increment, the orthogonality (127) and $\Xi_t^{(3)}$ being serially uncorrelated, i.e., (128), ensure that it can be computed with the following recursive formulae

$$(139) \qquad \Gamma_j^{(3)} = \Theta^{(3)} \Gamma_{j-1}^{(3)X,dy}$$

$$(140) \qquad \Gamma_j^{(3)X,dy} = \Theta^{(3)X} \Gamma_{j-1}^{(3)X,dy}$$

where

$$(141) \qquad \Gamma_j^{(3)} = E\left(dy_t^{(3)} dy_{t-j}^{(3)'} \right)$$

$$(142) \qquad \Gamma_j^{(3)X,dy} = E\left(X_t^{(3)} dy_{t-j}^{(3)'} \right)$$

A.6 Second Moments between $dy_t^{(1)}$ and $dy_t^{(3)}$

First rewrite the linear recursion of the first order increment (22) using $X_t^{(3)}$

$$(143) \qquad dy_t^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} X_{t-1}^{(3)} + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} \Xi_t^{(3)}$$

Multiplying the foregoing with the transposition of the linear recursion of the third order increment (129), and applying the expectations operator to the resulting expression yields the contemporaneous covariance between $dy_t^{(1)}$ and $dy_t^{(3)}$

$$(144) \qquad \Gamma_0^{(1),(3)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_0^{(3)X} \Theta^{(3)'} + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} E\left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'}$$

where

$$(145) \qquad \Gamma_0^{(1),(3)} = E\left(dy_t^{(1)} dy_t^{(3)'} \right)$$

The autocovariance, $\Gamma_j^{(1),(3)}$, can be computed using the following recursive formula

$$(146) \qquad \Gamma_j^{(1),(3)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_{j-1}^{(3)X,dy}$$

A.7 Variance Decomposition

The decomposition the variance of the third order approximation follows directly from the decomposition of the third order increment. Defining

$$(147) \qquad dy_t^{(3)} \equiv dy_t^{(3)amp} + dy_t^{(3)risk}$$

Multiplying the foregoing with its transposition and applying the expectations operator, a variance decomposition immediately follows

$$(148) \qquad \Gamma_0^{(3)} = \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk}\right)'$$

where

$$(149) \qquad \Gamma_0^{(3)amp} = E\left(dy_t^{(3)amp} dy_t^{(3)amp'}\right)$$

$$(150) \qquad \Gamma_0^{(3)risk} = E\left(dy_t^{(3)risk} dy_t^{(3)risk'}\right)$$

$$(151) \qquad \Gamma_0^{(3)amp,risk} = E\left(dy_t^{(3)amp} dy_t^{(3)risk'}\right)$$

Proposition (5.2) in the text implies the contemporaneous variance of the variables of interest takes the form

$$(152) \qquad \Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)} + \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)}\right)'$$

Inserting the decomposed $\Gamma_0^{(3)}$, i.e., (148), in the previous equation yields the decomposition of the contemporaneous variance of the variables of interest

$$(153) \qquad \begin{aligned} \Gamma_0^{y^{(3)}} =& \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk}\right)' \\ &+ \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)}\right)' \end{aligned}$$

Note the decomposition (153) is not yet complete as the cross-contemporaneous variance $\Gamma_0^{(1),(3)}$ can be further broken down into two parts[30]

$$(154) \qquad \begin{aligned} \Gamma_0^{(1),(3)} =& E\left(dy_t^{(1)} dy_t^{(3)'}\right) \\ =& E\left[dy_t^{(1)}\left(dy_t^{(3)amp} + dy_t^{(3)risk}\right)'\right] \end{aligned}$$

---

[30]In (154), $\Gamma_0^{(1)amp,(3)amp}$ is used to denote $E\left(dy_t^{(1)} dy_t^{(3)amp'}\right)$ as there is only amplification effects in the first order increment $dy_t^{(1)}$.

$$=E\left(dy_t^{(1)}dy_t^{(3)amp\prime}\right)+E\left(dy_t^{(1)}dy_t^{(3)risk\prime}\right)$$

$$=\Gamma_0^{(1)amp,(3)amp}+\Gamma_0^{(1)amp,(3)risk}$$

Inserting the foregoing in (153) yields the complete variance decomposition

(155)
$$\Gamma_0^{y^{(3)}}=\Gamma_0^{y^{(2)}}+\Gamma_0^{(3)amp}+\Gamma_0^{(3)risk}+\Gamma_0^{(3)amp,risk}+\left(\Gamma_0^{(3)amp,risk}\right)'$$

$$+\Gamma_0^{(1)amp,(3)amp}+\Gamma_0^{(1)amp,(3)risk}$$

$$+\left(\Gamma_0^{(1)amp,(3)amp}+\Gamma_0^{(1)amp,(3)risk}\right)'$$

Letting $\Gamma_0^{y^{(3)amp}}$ collect the contribution from all amplification channels of all three orders, $\Gamma_0^{y^{(3)risk,amp}}$ collects all interaction between amplification and time-varying risk adjustment channels and $\Gamma_0^{y^{(3)risk}}$ collects the contribution from the time-varying risk adjustment channel

(156)
$$\Gamma_0^{y^{(3)amp}}=\Gamma_0^{y^{(2)}}+\Gamma_0^{(3)amp}+\Gamma_0^{(1)amp,(3)amp}+\left(\Gamma_0^{(1)amp,(3)amp}\right)'$$

(157)
$$\Gamma_0^{y^{(3)risk,amp}}=\Gamma_0^{(3)amp,risk}+\left(\Gamma_0^{(3)amp,risk}\right)'+\Gamma_0^{(1)amp,(3)risk}+\left(\Gamma_0^{(1)amp,(3)risk}\right)'$$

(158)
$$\Gamma_0^{y^{(3)risk}}=\Gamma_0^{(3)risk}$$

Inserting the foregoing in (155) yields (38) in the text. Note the first order amplification effect reported in Table 5 is included in (156). In particular, it is included in $\Gamma_0^{y^{(2)}}$. As implied by proposition 5.1, the contemporaneous variance of the second order approximation takes the form

(159)
$$\Gamma_0^{y^{(2)}}=\Gamma_0^{y^{(1)}}+\Gamma_0^{(2)}$$

where $\Gamma_0^{y^{(1)}}$ captures the first order amplification effect.

To compute the individual terms in (155), first note $dy_t^{(3)amp}$ collects all amplification effects and $dy_t^{(3)risk}$ collects the time-varying risk adjustment effect in the third order increment

(160)
$$dy_t^{(3)amp}=\alpha dy_{t-1}^{(3)amp,state}+\frac{1}{6}\left[\beta_{333,1}dy_{t-1}^{(1)state\otimes[3]}+\beta_{000}\varepsilon_t^{\otimes[3]}\right]$$

$$+\beta_{22}\left(dy_{t-1}^{(2)state}\otimes dy_{t-1}^{(1)state}\right)+\beta_{20}\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)$$

$$+\frac{1}{2}\left[\beta_{300}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t^{\otimes[2]}\right)+\beta_{330,1}\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\right]$$

(161)
$$dy_t^{(3)risk}=\alpha dy_{t-1}^{(3)risk,state}+\frac{1}{2}\beta_{\sigma^20}\varepsilon_t+\frac{1}{2}\beta_{\sigma^21}dy_{t-1}^{(1)state}$$

We start with constructing an auxiliary vector $X_{t-1}^{(3D)}$ for this decomposition

(162)
$$X_t^{(3D)} = \begin{bmatrix} dy_t^{(3)amp,state} \\ dy_t^{(3)risk,state} \\ dy_t^{(2)state} \otimes dy_t^{(1)state} \\ dy_t^{(1)state\otimes[3]} \\ dy_t^{(1)state} \end{bmatrix}$$

With the foregoing auxiliary vector, $dy_t^{(3)amp}$ and $dy_t^{(3)risk}$ can be cast as linear recursions

(163)
$$dy_t^{(3)amp} = \Theta^{(3)amp} X_{t-1}^{(3D)} + \Phi^{(3)amp} \Xi_t^{(3)}$$

(164)
$$dy_t^{(3)risk} = \Theta^{(3)risk} X_{t-1}^{(3D)} + \Phi^{(3)risk} \Xi_t^{(3)}$$

where

(165)
$$\Theta^{(3)amp} = \begin{bmatrix} \alpha & 0 & \beta_{22} & \frac{1}{6}\beta_{333,1} & \frac{1}{2}\beta_{300}^{state} \left(I_{ns} \otimes E\varepsilon_t^{\otimes[2]} \right) \end{bmatrix}$$

(166)
$$\Theta^{(3)risk} = \begin{bmatrix} 0 & \alpha & 0 & 0 & \frac{1}{2}\beta_{\sigma^2 1} \end{bmatrix}$$

(167)
$$\Phi^{(3)amp} = \begin{bmatrix} \frac{1}{6}\beta_{000} & \frac{1}{2}\beta_{330,1} & \frac{1}{2}\beta_{300} & \beta_{20} & 0 \end{bmatrix}$$

(168)
$$\Phi^{(3)risk} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}\beta_{\sigma^2 0} \end{bmatrix}$$

Multiplying (163) with its transposition and applying the expectations operator yields the contemporaneous variance $\Gamma_0^{(3)amp}$, which collects the contribution of amplification channels to the total variance of the third order increment

(169)
$$\Gamma_0^{(3)amp} = \Theta^{(3)amp} E\left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) \Theta^{(3)amp'} + \Phi^{(3)amp} E\left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$$

where $E\left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ is as calculated in section A.4. $E\left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right)$ can be computed using the following relationship

(170)
$$X_t^{(3)} = A^D X_t^{(3D)}$$

where

(171)
$$A^D = \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

therefore

(172)
$$E\left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) = A^{D+} E\left(X_{t-1}^{(3)} X_{t-1}^{(3)'} \right) A^{D+'} = A^{D+} \Gamma_0^{(3)X} A^{D+'}$$

where A^{D+} denotes the Moore-Penrose inverse of A^D and $\Gamma_0^{(3)X}$ is already known. Then $\Gamma_0^{(3)amp}$ can be computed using

(173) $\qquad \Gamma_0^{(3)amp} = \left(\Theta^{(3)amp} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' + \Phi^{(3)amp} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$

Likewise, the contemporaneous variance $\Gamma_0^{(3)risk}$ collects the contribution of the time-varying risk adjustment channel to the total variance of the third order increment, and can be computed using

(174) $\qquad \Gamma_0^{(3)risk} = \left(\Theta^{(3)risk} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)risk} A^{D+} \right)' + \Phi^{(3)risk} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)risk'}$

$\Gamma_0^{(3)amp,risk}$ and its transposition collects the contribution of the interaction between the amplification and time-vary risk adjustment channels to the total variance of the third order increment, and can be computed using

(175) $\qquad \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' = \Gamma_0^{(3)} - \Gamma_0^{(3)amp} - \Gamma_0^{(3)risk}$

To compute $\Gamma_0^{(1)amp,(3)amp}$, multiply (143) with the transposition of (163) and apply the expectations operator to the resulting expression to yield

(176)
$\Gamma_0^{(1)amp,(3)amp} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$

As $\Gamma_0^{(1),(3)}$ was already computed in section A.6, $\Gamma_0^{(1)amp,(3)risk}$ can be obtained by subtracting the foregoing from $\Gamma_0^{(1),(3)}$.

A.8 Coefficient Matrices

This section contains explicit expressions for several coefficient matrices left implicit above.

$$\Theta^{(3)} = \begin{bmatrix} \alpha & \beta_{22} & \frac{1}{6}\beta_{333,1} & \frac{1}{2}\beta_{300} \left(I_{ns} \otimes E\varepsilon_t^{\otimes[2]} \right) + \frac{1}{2}\beta_{\sigma^2 1} \end{bmatrix}$$

$$\Phi^{(3)} = \begin{bmatrix} \frac{1}{6}\beta_{000} & \frac{1}{2}\beta_{330,1} & \frac{1}{2}\beta_{300} & \beta_{20} & \frac{1}{2}\beta_{\sigma^2 0} \end{bmatrix}$$

$$\Theta^{(3)}X = \begin{bmatrix} \alpha^{state} & \beta_{22}^{state} & \frac{1}{6}\beta_{333,1}^{state} & \frac{1}{2}\left[\beta_{300}^{state}\left(I_{ns}\otimes E\varepsilon_t^{\otimes[2]}\right)+\beta_{\sigma^2 1}^{state}\right] \\[4pt] 0 & \alpha^{state\otimes[2]} & \left(\frac{1}{2}\beta_{22}^{state}\right)\otimes\alpha^{state} & \left(\left[\left(\frac{1}{2}\beta_{00}^{state}\right)\otimes\alpha^{state}\right]K_{ne^2,ns}+\beta_{20}^{state}\otimes\beta_0^{state}\right)\left(I_{ns}\otimes E\varepsilon_t^{\otimes[2]}\right) \\[4pt] 0 & 0 & \alpha^{state\otimes[3]} & \left[K_{ns^2,ns}+\left(K_{ns,ns}\otimes I_{ns}+I_{ns^3}\right)\right]\left(\alpha^{state}\otimes\beta_0^{state\otimes[2]}\right)\left(I_{ns}\otimes E\varepsilon_t^{\otimes[2]}\right) \\[4pt] 0 & 0 & 0 & \alpha^{state} \end{bmatrix}$$

$$\Phi^{(3)}X = \begin{bmatrix} \frac{1}{6}\beta_{000}^{state} & \left(\beta_{20}^{state}\otimes\alpha^{state}\right)K_{ns*ne,ns}+\left(\frac{1}{2}\beta_{22}^{state}\right)\otimes\beta_0^{state} & \frac{1}{2}\beta_{330,1}^{state} & \left[\left(\frac{1}{2}\beta_{00}^{state}\right)\otimes\alpha^{state}\right]K_{ne^2,ns}+\beta_{20}^{state}\otimes\beta_0^{state} & \frac{1}{2}\beta_{300}^{state} & \beta_{20}^{state} & \frac{1}{2}\beta_{\sigma^2 0}^{state} & \cdots \\[4pt] \beta_0^{state\otimes[3]} & \left[\left(K_{ns,ns}\otimes I_{ns}+I_{ns^3}\right)K_{ns^2,ns}+I_{ns^3}\right]\left(\alpha^{state\otimes[2]}\otimes\beta_0^{state}\right) & \multicolumn{1}{c}{} & \alpha^{state}\otimes\beta_0^{state} & 0 & 0 & \cdots \\[4pt] 0 & \multicolumn{2}{c}{\left[K_{ns^2,ns}+\left(K_{ns,ns}\otimes I_{ns}+I_{ns^3}\right)\right]\left(\alpha^{state}\otimes\beta_0^{state\otimes[2]}\right)} & 0 & 0 & \cdots \\[4pt] 0 & & & 0 & \beta_0^{state} & \cdots \end{bmatrix}$$

$$E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right) =$$

$$\begin{bmatrix}
E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t^{\otimes[3]'}\right) & E\left[\varepsilon_t^{\otimes[3]}\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)'\right] & \cdots \\[4pt]
E\left[\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\varepsilon_t^{\otimes[3]'}\right] & E\left[\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)'\right] & \cdots \\[4pt]
0 & 0 & \cdots \\[4pt]
E\left[\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)\varepsilon_t^{\otimes[3]'}\right] & E\left[\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)'\right] & \cdots \\[4pt]
E\left(\varepsilon_t\varepsilon_t^{\otimes[3]'}\right) & E\left[\varepsilon_t\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)'\right] & \cdots
\end{bmatrix}$$

$$\begin{bmatrix}
\cdots & 0 & \cdots \\[4pt]
\cdots & 0 & \cdots \\[4pt]
\cdots & E\left[\left(dy_{t-1}^{(1)state}\otimes\left(\varepsilon_t^{\otimes[2]}-E\varepsilon_t^{\otimes[2]}\right)\right)\left(dy_{t-1}^{(1)state}\otimes\left(\varepsilon_t^{\otimes[2]}-E\varepsilon_t^{\otimes[2]}\right)\right)'\right] & \cdots \\[4pt]
\cdots & 0 & \cdots \\[4pt]
\cdots & 0 & \cdots
\end{bmatrix}$$

$$\begin{bmatrix}
\cdots & E\left[\varepsilon_t^{\otimes[3]}\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)'\right] & E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t'\right) \\[4pt]
\cdots & E\left[\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)'\right] & E\left[\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\varepsilon_t'\right] \\[4pt]
\cdots & 0 & 0 \\[4pt]
\cdots & E\left[\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)'\right] & E\left[\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)\varepsilon_t'\right] \\[4pt]
\cdots & E\left[\varepsilon_t\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)'\right] & E\left(\varepsilon_t\varepsilon_t'\right)
\end{bmatrix}$$

A.9 Computing Elements in $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$

For every nonzero entry of $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$ in section A.8, the terms inside the expectations operator are either i) second, fourth, or sixth moments of the shocks, or ii) the product of these moments with the state variable block of the order increments, i.e., $dy_{t-1}^{(2)state}$ and $dy_{t-1}^{(1)state}$. The fourth and sixth moments of the shocks can be computed using Tracy and Sultan's (1993, p. 344-345) formulae. E.g., for sixth moments in the form $E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t^{\otimes[3]'}\right)$, applying the mixed Kronecker product rule yields

(177)
$$E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t^{\otimes[3]'}\right) = E\left(\varepsilon_t\varepsilon_t' \otimes \varepsilon_t\varepsilon_t' \otimes \varepsilon_t\varepsilon_t'\right)$$

then Tracy and Sultan's (1993) Theorem 3 (repeated here) can be applied directly

(178)
$$E\left(\varepsilon_t\varepsilon_t' \otimes \varepsilon_t\varepsilon_t' \otimes \varepsilon_t\varepsilon_t'\right) = \left[E\left(\varepsilon_t\varepsilon_t'\right)\right]^{\otimes[3]}\left[K + (K_{ne} \otimes K_{ne,ne}) + (K_{ne,ne} \otimes K_{ne}) + K_{ne,ne^2}(K_{ne,ne} \otimes K_{ne})\right]$$

$$+ K\left(\left[vec\left(E\left(\varepsilon_t\varepsilon_t'\right)\right) vec'\left(E\left(\varepsilon_t\varepsilon_t'\right)\right)\right] \otimes E\left(\varepsilon_t\varepsilon_t'\right)\right)K$$

where

(179)
$$K = K_{ne^3} + K_{ne,ne^2} + K_{ne^2,ne}$$

is a sum of commutation matrices (See Magnus and Neudecker (1979)).

For the fourth moment in the form $E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t'\right)$, Jinadasa and Tracy's (1986, p. 404) formula (repeated here) can likewise be applied directly

(180)
$$E\left(\varepsilon_t^{\otimes[3]}\varepsilon_t'\right) = E(\varepsilon_t\varepsilon_t') \otimes vec\left(E(\varepsilon_t\varepsilon_t')\right) + vec\left(E(\varepsilon_t\varepsilon_t')\right) \otimes E(\varepsilon_t\varepsilon_t') + (I_{ne} \otimes K_{ne,ne})\left[vec\left(E(\varepsilon_t\varepsilon_t')\right) \otimes E(\varepsilon_t\varepsilon_t')\right]$$

For the entries in the form of a product between the moments and the state variable block of order increments, use the property of the Kronecker product of column vectors and the mixed Kronecker product rule to rearrange until they are in the form of a (Kronecker) product of two clusters: one cluster contains the state variable block of the order increments only, and the other contains (the product of) shocks only. As all the order increments of the last period are uncorrelated with the current shocks, the expected value of the two clusters can be computed separately. E.g.

(181)
$$E\left[\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t\right)\varepsilon_t^{\otimes[3]'}\right] = E\left[dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t \otimes \varepsilon_t^{\otimes[3]'}\right] = E\left[dy_{t-1}^{(1)state\otimes[2]} \otimes \left(\varepsilon_t \otimes \varepsilon_t^{\otimes[3]'}\right)\right]$$

$$=E\left[dy_{t-1}^{(1)state\otimes[2]}\otimes\left(\varepsilon_t\varepsilon_t^{\otimes[3]'}\right)\right]=E\left(dy_{t-1}^{(1)state\otimes[2]}\right)\otimes E\left(\varepsilon_t\varepsilon_t^{\otimes[3]'}\right)$$

where $E\left(dy_{t-1}^{(1)state\otimes[2]}\right)$ was computed in section A.4 and $E\left(\varepsilon_t\varepsilon_t^{\otimes[3]'}\right)$ can be computed using the transposed version of (180).

In fact, many nonzero entries in $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$ can be recycled from the calculations in section A.4 and therefore need not to be computed again. E.g., the block entry in the second row and second column of $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$ can be written as

$$(182)\quad E\left[\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right)'\right]=E\left(dy_{t-1}^{(1)state\otimes[2]}dy_{t-1}^{(1)state\otimes[2]'}\right)\otimes E\left(\varepsilon_t\varepsilon_t'\right)$$

The first term on the right hand side of the foregoing can be recycled from $\Gamma_0^{(2)X}$ as the lower right entry (the block entry in the second row and second column) of $\Gamma_0^{(2)X}$ takes the form

$$(183)\qquad \Gamma_{0,22}^{(2)X}=E\left[\left(dy_{t-1}^{(1)state\otimes[2]}-Edy_{t-1}^{(1)state\otimes[2]}\right)\left(dy_{t-1}^{(1)state\otimes[2]}-Edy_{t-1}^{(1)state\otimes[2]}\right)'\right]$$

$$(184)\qquad =E\left(dy_{t-1}^{(1)state\otimes[2]}dy_{t-1}^{(1)state\otimes[2]'}\right)-E\left(dy_{t-1}^{(1)state\otimes[2]}\right)E\left(dy_{t-1}^{(1)state\otimes[2]}\right)'$$

therefore

$$(185)\qquad E\left(dy_{t-1}^{(1)state\otimes[2]}dy_{t-1}^{(1)state\otimes[2]'}\right)=\Gamma_{0,22}^{(2)X}+E\left(dy_{t-1}^{(1)state\otimes[2]}\right)E\left(dy_{t-1}^{(1)state\otimes[2]}\right)'$$

Some entries of $E\left(\Xi_t^{(3)}\Xi_t^{(3)'}\right)$ are zero as they contain one or some of terms equal to zero under normality: the odd moments of the exogenous shocks, $E\left(dy_t^{(1)state}\right)$, $E\left(dy_t^{(1)state\otimes[3]}\right)$ and $E\left(dy_t^{(1)state\otimes[5]}\right)$.

Table 1: Parameter Values: Common to All Three Calibrations

Parameter	β	ψ	χ	α	δ	$\bar{a}$	ρ_σ	τ
Value	0.9926	2.9869	25.8	0.331	0.021	0.004	0.9	0.15

See Tallarini (2000) and the main text.

Table 2: Parameter Values: Calibrating Homoskedastic Volatility

Calibration	Baseline	Constant Volatility	Expected Utility
$\overline{\sigma_a}$	0.009824769	0.011588754	0.0115

$\overline{\sigma_a}$ calibrated to keep the standard deviation of $\Delta \ln(c) = 0.0055$

Table 3: Mean Comparison

Variable	$Det.S.S.^*$	Baseline Calibration	Tallarini (2000)
$\log(k)$	2.084	2.137	2.158
i	0.200	0.211	0.216
$\log(c)$	-0.567	-0.554	-0.549
$\log(y)$	-0.265	-0.242	-0.232
$\log(N)$	-1.467	-1.460	-1.456
R^f	1.149	1.047	1.011
R	1.149	1.053	1.022

* The deterministic steady state value
See Table 5, Tallarini (2000).

Table 4: Standard Deviation Comparison

Variable	Baseline Calibration	Tallarini (2000)	Data
$\Delta\log(c)$	0.0055	0.0055	0.0055
$\Delta\log(y)$	0.0096	0.0095	0.0104
$\Delta\log(i)$	0.0240	0.0224	0.0279
$\log(c) - \log(y)$	0.0154	0.0147	0.0377
$\log(i) - \log(y)$	0.0425	0.0403	0.0649

See Table 7, Tallarini (2000).

Table 5: Variance Decomposition in Percentage

	Constant Volatility Calibration		Baseline (Stochastic Volatility) Calibration	
	1st order amp.	time-varying risk adj.	1st order amp.	time-varying risk adj.
MPR_t	0	100	0	100
ERP	0	100	0	100
RP	106.50	0.30	80.76	8.52
$\log(k)$	97.34	0.01	75.07	1.37
i	96.63	0.01	57.21	30.26
$\log(c)$	97.58	0.01	75.88	2.62
$\log(y)$	96.31	0.02	44.52	36.97
$\log(N)$	98.46	0.01	66.26	18.62

For each calibration, the columns may not add up to 100 due to the omission of 2nd and 3rd order amplification and cross effects.
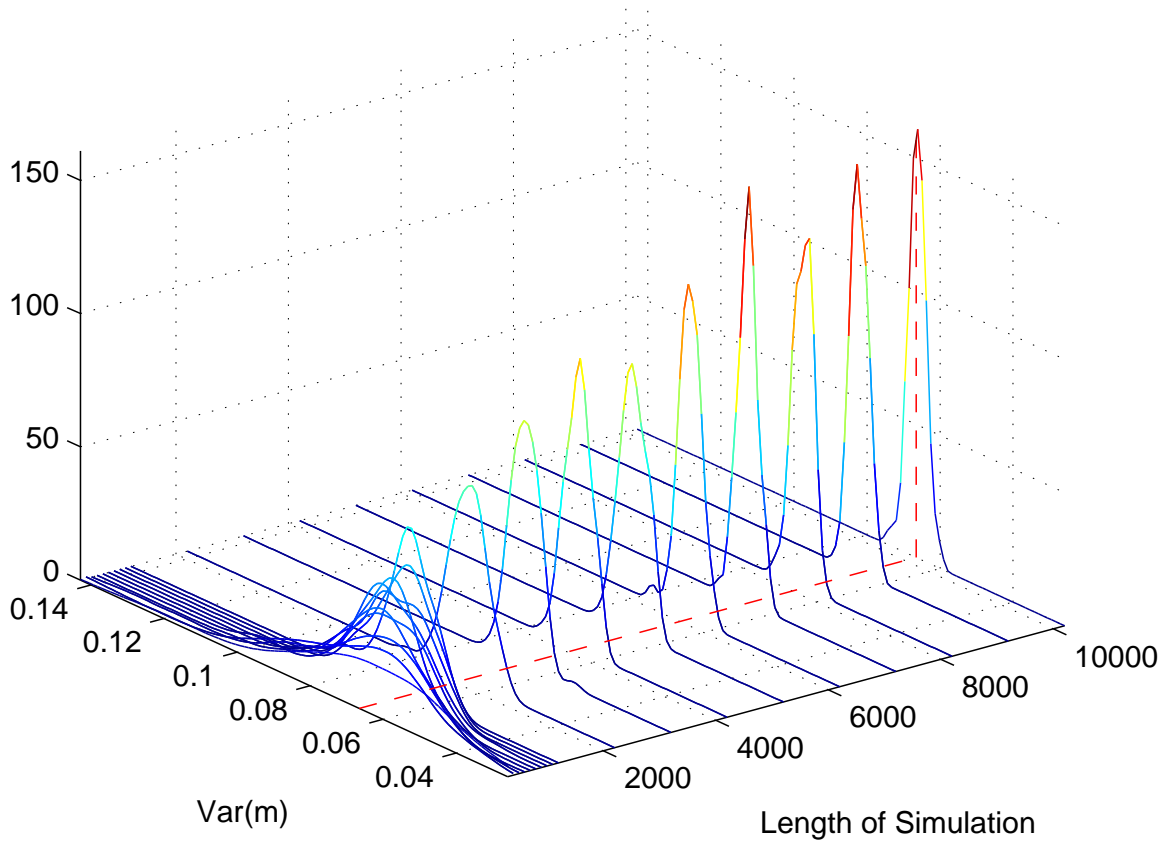
Figure 1: Monte Carlo Consistency
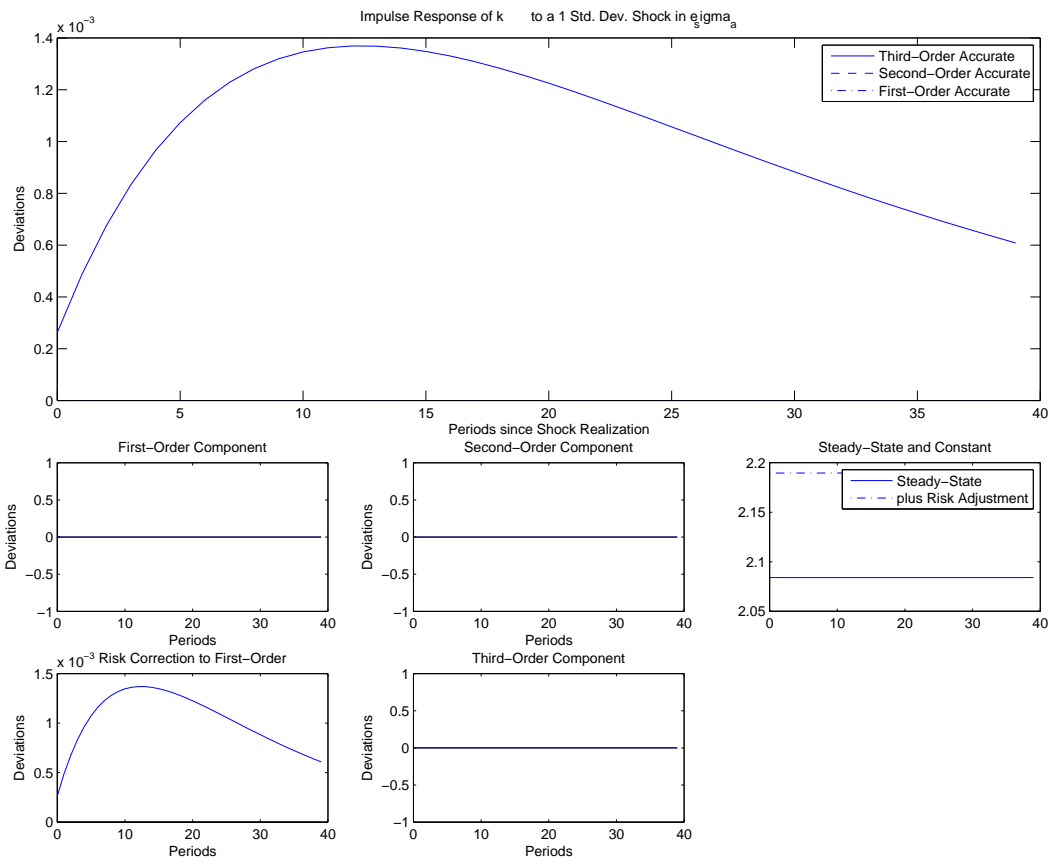
Figure 2: Capital IRF: Volatility shock
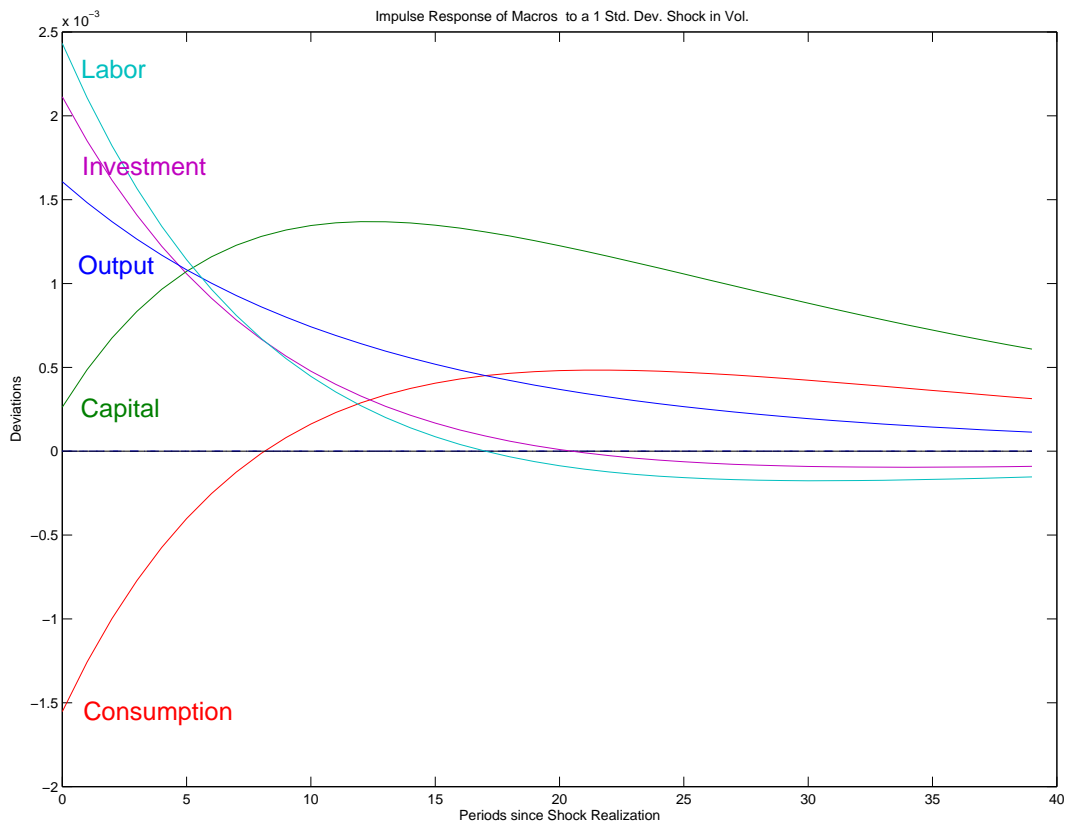
Figure 3: Macro IRFs: Volatility Shock and Precautionary Reaction
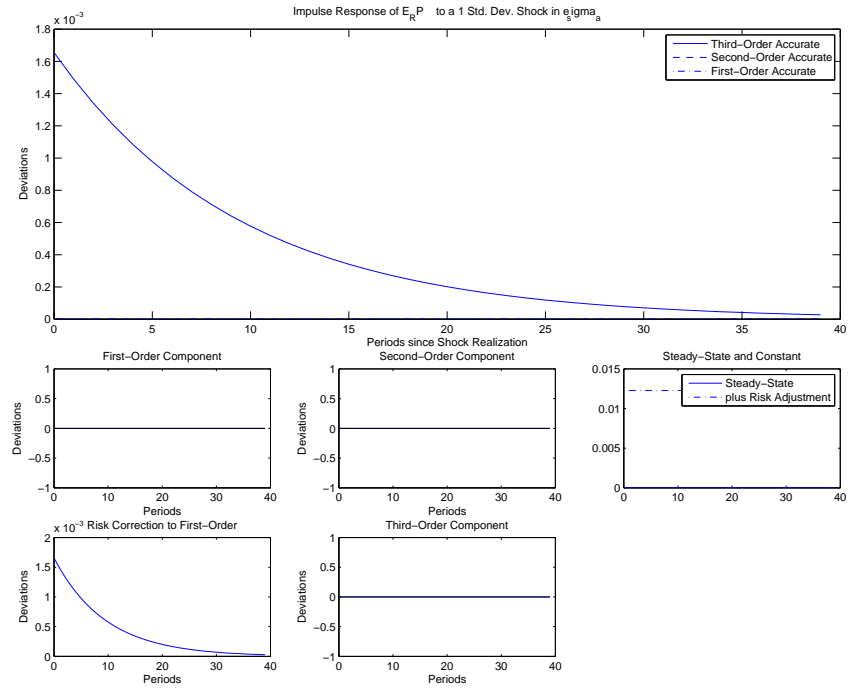
(a) Volatility Shock
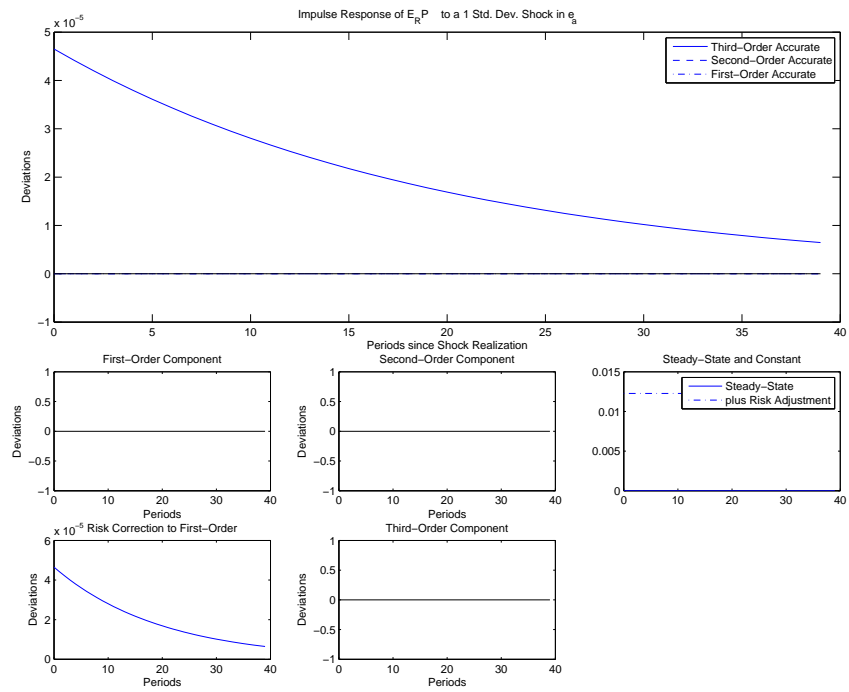
(b) Growth Shock

Figure 4: Expected Risk Premium IRF: Volatility and Growth Shock
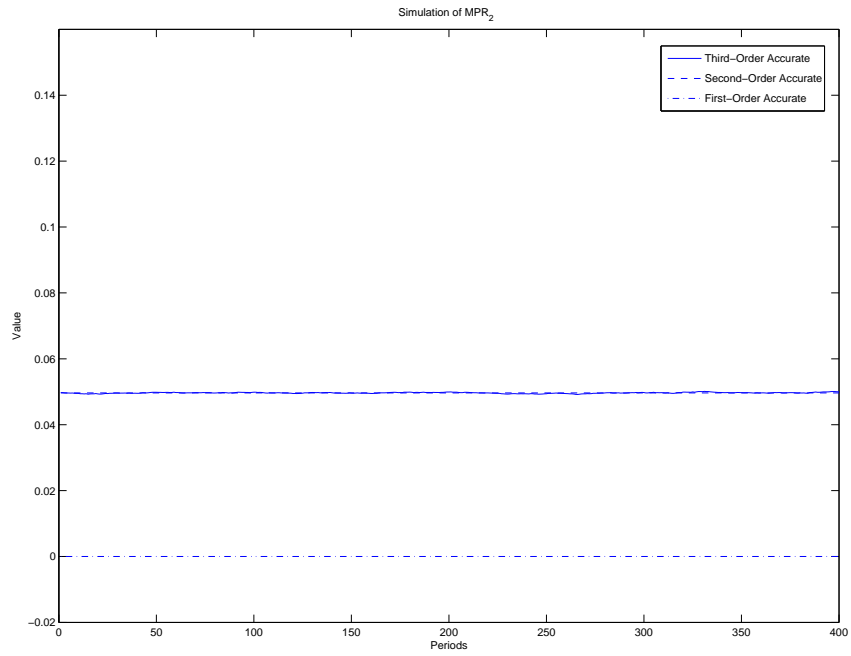
(a) Constant Volatility Calibration
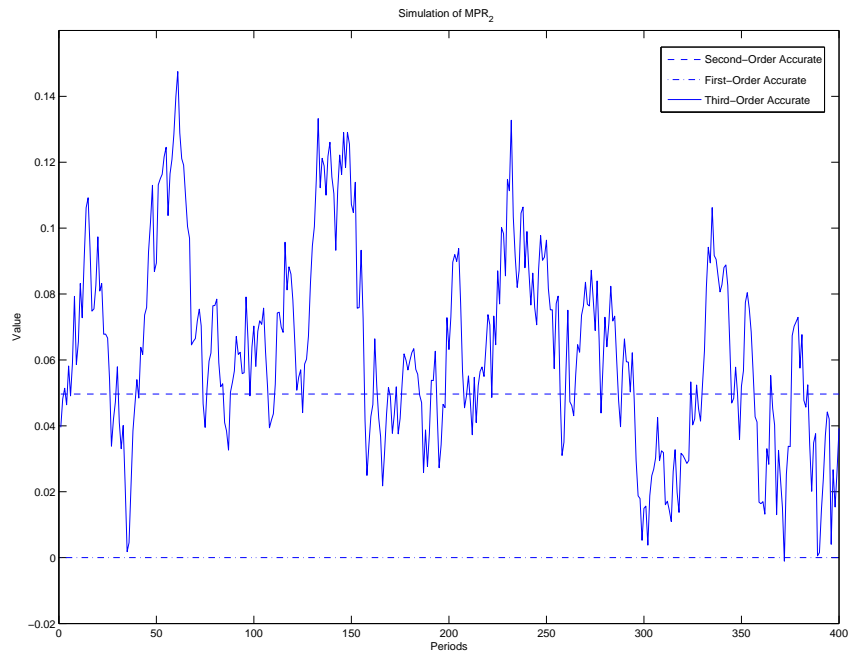
(b) Baseline (Stochastic Volatility) Calibration

Figure 5: Stochastic Volatility and Squared Conditional Market Price of Risk
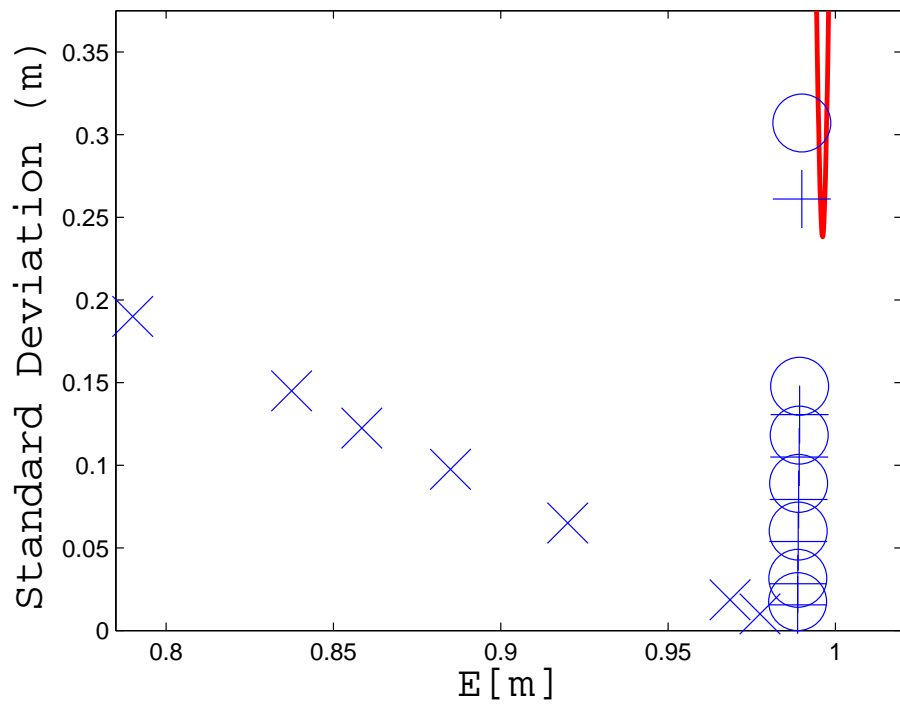
(a) $\overline{\sigma}_a = 0.0115$ for All Three Calibrations
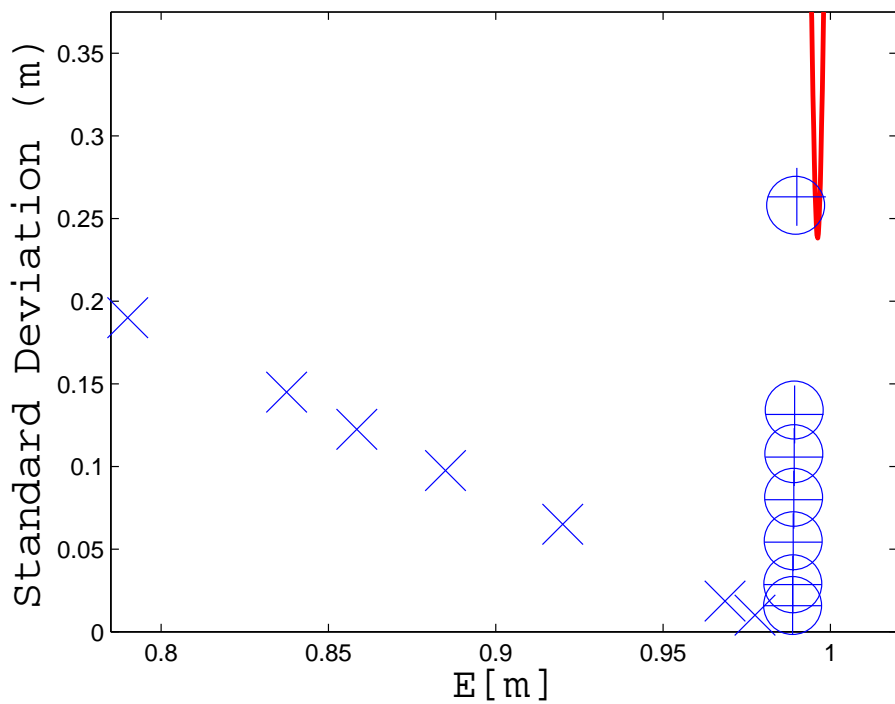
(b) $\overline{\sigma}_a$ Calibrated to Match $\Delta \log(c) 0.0055$

Figure 6: Stochastic Volatility and the Hansen-Jagannathan Bounds
$\times$: Expected Utility; $+$: Constant Volatility; $\bigcirc$: Baseline (Stochastic Volatility)

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