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Notes on an Endogenous Growth Model with two Capital Stocks II: The Stochastic Case

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Abstract

This paper extends the class of stochastic AK growth models with a closed-form solution to the case where there are two capital goods in the model. To be precise, we consider the Uzawa-Lucas model of endogenous growth with human and physical capital. The extension holds, even if an external effect in the use of human capital in goods production occurs. Using the "guess and verify" method, we determine the value function of the social planner in the centralized economy and the value function of the representative agent in the decentralized case. We show that the introduction of income taxes on wages and of a subsidy on physical capital earnings is able to help the decentralized economy in reaching the social optimum, while keeping the policy maker's budget balanced. Then the time series implications of the model's solution are derived. In Appendix to the paper the uniqueness of the value functions is proved by using an alternative method.

Key words: closed-form solution, value function, saddle path stability, endogenous growth

JEL Classifications: C61, C62

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1 Introduction

In this chapter, we present the value function of a stochastic version of the Uzawa (1965) and Lucas (1988) model of endogenous growth in discrete time. The externality of human capital in goods production inherent to the model causes a difference between the social planner's solution and the market outcome. We pay attention to this fact by treating both cases separately and by presenting both the social planner's and the representative agent's value function. The chapter generalizes the results of Bethmann (2002) where a deterministic version of this model was examined. Furthermore, we show that the inefficiency of the decentralized economy can be overcome by introducing taxes and subsidies on factor compensations.

The main feature of the Uzawa-Lucas model is the fact that the agents have to allocate their human capital between two production sectors. On the one hand, there is a goods sector where a single good usable for consumption and physical capital investment is produced. This sector exhibits a production technology that uses human and physical capital. On the other hand, there is a schooling sector where agents augment their stock of human capital. Here, human capital is the only input factor. In short, agents have to "learn or to do" (Chamley, 1993). In his seminal paper, Lucas (1988) argues that the economy's average level of human capital contributes to total factor productivity in goods production. In a decentralized economy the individual's accumulation of human capital has no appreciable influence on this average level and agents are only compensated for their respective factor supplies. This incentive structure leads to non-efficient equilibria. Since agents are not able to coordinate their actions, their discounted utility could be higher without making a single agent worse off. As a result, the solution for the centralized economy deviates from that of the decentralized case.

The theoretical model considered here differs from that studied by Lucas (1988) in two ways. First, there is our choice of the utility function. We assume logarithmic preferences which imply that the constant intertemporal elasticity of substitution is equal to one. This assumption reduces the number of parameters by one and simplifies the calculations. Second, we assume discrete time where the two capital stocks depreciate fully at the end of the period. This way the closed-form solution of the stochastic one sector growth model with logarithmic preferences and full depreciation of physical capital (cf. McCallum, 1989) is extended to the case with two capital goods.

The chapter is organized as follows. Section 2 introduces the model. Section 3 presents the value function in closed form as the solution to the social planner's dynamic optimization problem (DOP). In Section 4 we present the value function of the representative agent. Section 5 shows that the solutions are saddle path stable and determines their time-series implications. Section 6 summarizes our results and concludes. Appendix proves the uniqueness of the value functions found in the third and fourth section by using an alternative method.

2 The model

We consider a closed economy populated by an infinite number of homogeneous, infinitely-lived agents. The representative agent enters every period t with predetermined endowments of human and physical capital, h_t and k_t, respectively. Furthermore, there are two sectors in the economy. Firms produce a single homogeneous good and a schooling sector provides educational services.

2.1 The household

The population is assumed to be constant and normalized to one. The representative agent has logarithmic preferences over sequences of consumption:

$$U(c_0, c_1, ..) = \mathrm{E}_t \left[\sum_{t=0}^{\infty} \beta^t \ln(c_t) \right], \tag{1}$$

where c_t is the level of consumption in period $t \in \mathbb{N}_0$ and $\beta \in (0,1)$ is the subjective discount factor. Expectations are formed over the sequence of shocks $\{\varepsilon_t\}_{t=0}^{\infty}$ entering goods production. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. In each period, agents have a fixed endowment of time, which is normalized to one unit. The variable u_t denotes the fraction of time allocated to goods production in period t. Furthermore, as agents do not benefit from leisure, the whole time budget is allocated to the two production sectors. The fraction $1-u_t$ of time is spent in the schooling sector. Note that in any solution the condition

$$u_t \in [0,1] \tag{2}$$

has to be fulfilled. The variables c_t and u_t are the two control variables of the agent. When maximizing her discounted stream of utility, the agent has to pay attention to the following budget constraint:

$$\tau_r r_t k_t + \tau_w w_t u_t h_t = c_t + k_{t+1}, \qquad \forall t \in \mathbb{N}_0, \qquad k_0 > 0, \tag{3}$$

where k_t is the agent's physical capital stock in period t. The terms $\tau_r r_t k_t$ and $\tau_w w_t u_t h_t$ are, respectively, the net returns on physical capital and work effort after taxation. We assume that both parameters τ_r and τ_w are positive. If the parameter τ_r is smaller than 1, we have a tax on physical capital, if it is larger than 1, we have a subsidy. The same is true for the parameter τ_w. If $\tau_w < 1$, work effort is taxed, if $\tau_w > 1$, work effort is subsidized. Hence, the rates of taxation are given by $\tau_r - 1$ and $\tau_w - 1$, respectively. The above constraint implies full depreciation of physical capital. The variables r_t and w_t are market-clearing factor prices. Prices and tax rates are endogenous to the model. The former via the market clearing mechanism and the latter via the government's balanced budget condition. Despite this fact, prices and tax rates are taken as given by the representative agent. The left-hand side describes her income derived from physical capital plus the income stemming from effective work, which is determined by the worker's level of human capital h_t multiplied by the fraction of time spent in the goods sector in period t, i.e. $h_t u_t$. We assume that the initial values k_0 and h_0 are strictly positive. On the right-hand side, the spending of the agent's earnings appears, which she can either consume or invest. Another constraint the agent has to keep in mind is the evolution of her stock of human capital when allocating $1-u_t$ to the schooling sector.

2.2 The schooling sector

The creation of human capital is determined by a linear technology in human capital only:

$$h_{t+1} = B(1-u_t) h_t, \qquad \forall t \in \mathbb{N}_0, \qquad h_0 > 0, \tag{4}$$

where we assume that B is positive[1]. If we set u_t in equation (4) equal to zero, we get the potential stock of next period's human capital. If we set u_t equal to one, tomorrow's

[1] The case when B equals 0 corresponds to the neoclassical growth model.

stock of human capital is equal to zero. The schooling technology implies that the potential marginal and average product of human capital coincide and are equal to B, whereas the realized marginal and average products are equal to $B(1-u_t)$. Note that the depreciation rate of human capital is 100 percent per period.

2.3 The goods sector

We assume an infinitely large number of profit-maximizing firms producing a single good. They are using a Cobb-Douglas technology in physical capital k_t and effective work $h_t u_t$. Furthermore, the average skill of workers $h_{a,t}$ has a positive influence on total factor productivity. Hence, output y_t is determined by:

$$y_t = A_t k_t^{\alpha} (u_t h_t)^{1-\alpha} h_{a,t}^{\gamma}. \tag{5}$$

The parameter α is the output elasticity of physical capital and we assume $\alpha \in (0,1)$. The parameter γ is non-negative and measures the degree of the external effect of human capital. If we set u_t equal to one, we get the potential output in the goods sector. The homogeneity of the agents implies that:

$$h_{a,t} = h_t, \qquad \forall t \in \mathbb{N}_0. \tag{6}$$

The state variable A_t denotes total factor productivity. Throughout this chapter, we assume that $\ln A_t$ follows a first-order autoregressive process, i.e.:

$$\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}, \qquad \forall t \in \mathbb{N}_0 \qquad \text{and} \qquad \varepsilon \sim N\left(0, \sigma^2\right). \tag{7}$$

This assumption is a generalization of Bethmann (2002), where A was taken as fixed. The firm has to rent physical and human capital on perfectly competitive factor markets. In the decentralized economy, the representative firm's profit Π in period t is given by:

$$\Pi(k_t, h_t; A_t, h_{a,t}) = A_t k_t^{\alpha} (u_t h_t)^{1-\alpha} h_{a,t}^{\gamma} - r_t k_t - w_t u_t h_t,$$

where the semicolon indicates that the whole paths of $h_{a,t}$ and A_t are treated as exogenous by the representative firm. The first-order necessary conditions for the profit-maximizing factor demands are:

$$r_t \equiv \tfrac{\partial y_t}{\partial k_t} = \tfrac{\alpha y_t}{k_t} \quad \text{and} \quad w_t \equiv \tfrac{\partial y_t}{\partial (u_t h_t)} = \tfrac{(1-\alpha) y_t}{u_t h_t}. \tag{8}$$

These market-clearing factor prices ensure that the zero-profit condition holds. Inserting the prices into the agent's budget constraint (3) yields:

$$\tau_r \alpha y_t + \tau_w (1-\alpha) y_t = c_t + k_{t+1}, \qquad \forall t \in \mathbb{N}_0. \tag{9}$$

2.4 The state sector in the decentralized economy

In each period t, we require the state's budget to be balanced. Therefore:

$$(\tau_r - 1) r_t k_t = (1 - \tau_w) w_t u_t h_t \tag{10}$$

must hold for all $t \in \mathbb{N}_0$. This means that if we consider a tax on physical capital returns, we are subsidizing work effort at the same time and vice versa. This remark ends the presentation of the model. In Section 3, we solve the centralized version of this model.

3 The centralized solution of the model

In the centralized economy, the social planner internalizes the contribution of the economy's average level of human capital to goods production. That is, the planner is able to reach the efficient allocation of resources without the instrument of taxation. Therefore, we assume $\tau_r = \tau_w = 1$ throughout this section.

The central planner internalizes the social returns of human capital when choosing his optimal controls. This means that he exploits the symmetry condition stated in equation (6) and writes his DOP as follows:

$$U = \sup_{\{c_t, u_t\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

with respect to the state dynamics

$$\begin{aligned}
k_{t+1} &= A k_t^{\alpha} u_t^{1-\alpha} h_t^{1-\alpha+\gamma} - c_t, && \forall\, t \in \mathbb{N}_0, && (11)\\
h_{t+1} &= B\,(1-u_t)\,h_t, && \forall\, t \in \mathbb{N}_0, && (12)\\
\ln A_{t+1} &= \rho \ln A_t + \varepsilon_{t+1}, && \forall\, t \in \mathbb{N}_0, && (13)\\
k_t \geq 0 \quad &\text{and} \quad h_t \geq 0, && \forall\, t \in \mathbb{N}_0. &&
\end{aligned}$$

Since the social planner uses the symmetry from (6), he has simply dropped the index a. Furthermore, the initial values k_0, h_0, and $A_0 > 0$ are assumed to be given and the social planner has to ensure that

$$c_t > 0 \qquad \text{and} \qquad 0 \leq u_t \leq 1$$

hold for all $t \in \mathbb{N}_0$. He defines the value function as the solution to his optimization problem from time t onwards:

$$V(k_t, h_t, A_t) \equiv \sup_{\{c_s, u_s\}_{s=t}^{\infty}} \mathrm{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} \ln c_s \right] \quad \text{s.t. (11), (12), and (13).}$$

The Bellman equation associated with the planner's DOP is given by:

$$V(k_t, h_t, A_t) = \sup_{c_t, u_t} \left\{ \ln c_t + \beta \mathrm{E}_t \left[V(k_{t+1}, h_{t+1}, A_{t+1}) \right] \right\}. \tag{14}$$

The first-order necessary conditions for the optimal consumption choice and the optimal allocation of human capital between the two sectors are given by:

$$c_t \quad : \qquad \frac{1}{c_t^*} = \beta \mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}} \right], \tag{15}$$

$$u_t \quad : \qquad u_t^* = \left(\frac{\mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}} \right] (1-\alpha) A_t}{\mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial h_{t+1}} \right] B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t^{\frac{\alpha-\gamma}{\alpha}}}, \tag{16}$$

where V_t stands for $V(k_t, h_t, A_t)$ and the asterisk denotes optimality. Equation (15) describes the behavior along the optimal consumption path. When shifting a marginal unit of today's output from consumption to investment, today's marginal change in utility should equal the expected discounted marginal change of wealth with respect to tomorrow's capital stock. Equation (16) states that the weighted expected marginal change of wealth with respect to physical capital equals the weighted expected marginal change of wealth with respect to human capital. The first weight is the marginal product of human capital in goods production, given a certain choice of u_t. The second weight is the potential marginal product of human capital when the remaining fraction of human capital is allocated to the educational sector.

We now turn to the Euler Equations. The envelope property with respect to physical capital is straightforward. Together with the above first-order necessary conditions (15) and (16), it gives rise to the following Euler equation in consumption:

$$\frac{1}{c_t} = \beta \mathrm{E}_t \left[\frac{1}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}} \right]. \tag{17}$$

This is the Lucas asset pricing equation (cf. Lucas, 1978) with the constant of relative risk aversion being equal to one. Along the optimal consumption path, marginal utility of consumption in time t must be equal to the discounted expected marginal utility from the return on investment in the next period. The envelope condition for the stock of human capital is given by:

$$\frac{\partial V_t}{\partial h_t} = \beta \mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial h_{t+1}} \frac{B(1-\alpha+\gamma u_t)}{1-\alpha} \right]. \tag{18}$$

The last term on the right-hand side, γu_t, indicates that the expected marginal social gain of exploiting the external effect in goods production has an impact on the evolution of the shadow price of human capital. To be precise, today's shadow price of human capital is positively influenced by the degree of the external effect of human capital in goods production. This is the mechanism by which the external effect enters the second Euler equation along the optimal allocation of human capital between the two sectors:

$$u_t = \left(\frac{\mathrm{E}_t \left[\frac{1}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}} \right] (1-\alpha) A_t}{\mathrm{E}_t \left[\frac{(1-\alpha+\gamma u_{t+1}) y_{t+1}}{c_{t+1} h_{t+1} u_{t+1}} \right] B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t^{\frac{\alpha-\gamma}{\alpha}}}. \tag{19}$$

The transversality conditions with respect to the two capital stocks that establish the sufficiency of the two Euler equations (17) and (19) are given by:

$$\lim_{T\to\infty} \beta^T \mathrm{E}_0 \left[\frac{\alpha y_T}{c_T k_T} k_T \right] = 0 \quad \text{and} \quad \lim_{T\to\infty} \beta^T \mathrm{E}_0 \left[\frac{(1-\alpha+\gamma u_T) y_T}{c_T u_T h_T} h_T \right] = 0. \tag{20}$$

The conditions (20) assert that the intertemporal budget constraints are met by the planner's decisions. Since the social planner exploits the external effect of human capital, the derivative of the production function with respect to human capital looks different from that in a decentralized economy below. This derivative is the sum of the private marginal return from $u_t h_t$ and the marginal social gain of the average stock of human capital h_t.

Using the guess and verify method, it is possible to generalize Robinson Crusoe's value function V and the planer's value function found in Bethmann (2002) as follows:

$$V = \theta + \theta_B \ln B + \theta_A \ln A_t + \theta_k \ln k_t + \theta_h \ln h_t, \tag{21}$$

where the θ_i's, with $i \in \{k, h, B, A\}$, are defined as follows[2]:

$$\theta_B := \frac{(1-\alpha+\gamma)\beta}{(1-\beta)^2(1-\alpha\beta)}, \quad \theta_A := \frac{1}{(1-\rho\beta)(1-\alpha\beta)}, \quad \theta_k := \frac{\alpha}{1-\alpha\beta}, \quad \theta_h := \frac{1-\alpha+\gamma}{(1-\alpha\beta)(1-\beta)}.$$

[2]The constant term is given by: $\theta := \frac{\ln[1-\alpha\beta]}{1-\beta} + \frac{(1-\alpha)\ln[1-\beta]}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha\beta\ln\alpha}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha\beta+\gamma)\ln\beta}{(1-\beta)^2(1-\alpha\beta)} + \frac{(1-\alpha)\ln[1-\alpha]}{(1-\alpha\beta)(1-\beta)} + \frac{\beta(1-\alpha+\gamma)\ln[1-\alpha+\gamma]}{(1-\alpha\beta)(1-\beta)^2} - \frac{(1-\alpha+\beta\gamma)\ln[1-\alpha+\beta\gamma]}{(1-\alpha\beta)(1-\beta)^2}$.

The function V implies the following controls along the welfare-maximizing consumption and human capital allocation paths:

$$c_t = (1-\alpha\beta)\, y_t \quad \text{and} \quad u_t = \tfrac{(1-\alpha)(1-\beta)}{1-\alpha+\beta\gamma} := u\ . \tag{22}$$

Note that $0 \leq u \leq 1$ is satisfied even in the strict sense. Furthermore, the allocation of human capital is constant regardless of the respective endowments of human and physical capital. The central planner simply devotes a constant share of goods production to consumption. Surely, findings (22) must also hold in period T such that it is easy to see that the Euler equations (17) and (19) and the transversality conditions (20) are satisfied. This remark closes the discussion of the centralized case. In the next section we turn to the decentralized economy.

4 The decentralized solution of the model

In the decentralized case, we assume a representative agent with rational behavior. The agent knows that her stock of human capital equals the average level of human capital in the economy. Furthermore, she knows that the external effects of human capital in goods production, captured by the term $h^{\gamma}_{a,t}$, may increase her and all the other agents' wealth. But here, in the decentralized case, the market mechanism prevents a coordination of agents' actions. This can be understood as a Nash game producing the prisoner's dilemma. For this reason, we introduce a government that taxes and subsidizes the respective factor compensations. In the first subsection we write down the representative agent's optimization problem. Then, the second subsection characterizes the agent's optimal behavior. Finally, the third subsection determines the government's optimal taxation policy.

4.1 The representative agent's optimization problem

Although the external effect of the economy's average human capital stock in period t may be not exploited, the whole path of $h_{a,t}$ is predictable and is therefore treated as given by the agents. The representative agent's DOP is given by:

$$U = \sup_{\{c_t,u_t\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

with respect to the state dynamics

$$\begin{aligned}
k_{t+1} &= \tau_r r_t k_t + \tau_w w_t u_t h_t - c_t, && \forall\, t \in \mathbb{N}_0, && (23)\\
h_{t+1} &= B\,(1-u_t)\, h_t, && \forall\, t \in \mathbb{N}_0, && (24)\\
h_{a,t+1} &= B\,(1-u_{a,t})\, h_{a,t}, && \forall\, t \in \mathbb{N}_0, && (25)\\
\ln A_{t+1} &= \rho \ln A_t + \varepsilon_{t+1}, && \forall\, t \in \mathbb{N}_0, && (26)\\
k_t \geq 0 \quad &\text{and} \quad h_t \geq 0, && \forall\, t \in \mathbb{N}_0.
\end{aligned}$$

The variable u_a stands for the average human capital allocation in the decentralized economy, the value of which cannot be influenced by the representative agent.

We start the analysis of the decentralized economy with the definition of the value function as the solution to the representative agent's problem:

$$V\,(k_t, h_t; A_t, h_{a,t}) \equiv \sup_{\{c_s,u_s\}_{s=t}^{\infty}} E_t \left[\sum_{s=t}^{\infty} \beta^{s\text{-}t} \ln c_s \right] \quad \text{s.t. (2) - (8).}$$

Note that the value function of the representative agent is restricted to a given path of $h_{a,t}$. The corresponding Bellman equation is given by:

$$V\left(k_t, h_t; A_t, h_{a,t}\right) \equiv \sup_{c_t, u_t} \left\{\ln c_t + \beta \mathrm{E}_t \left[V\left(k_{t+1}, h_{t+1}; A_{t+1}, h_{a,t+1}\right)\right]\right\}. \tag{27}$$

Taking the derivatives with respect to the two controls and inserting the market-clearing factor prices (8) gives us the following first-order necessary conditions:

$$c_t \ : \ \frac{1}{c_t^*} = \beta \mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}}\right], \tag{28}$$

$$u_t \ : \ u_t^* = \left(\frac{\mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}}\right] (1-\alpha) \tau_w A_t}{\mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial h_{t+1}}\right] B}\right)^{\frac{1}{\alpha}} \frac{k_t h_{a,t}^{\frac{\gamma}{\alpha}}}{h_t}, \tag{29}$$

where V_{t+1} is a shortcut for $V\left(k_{t+1}, h_{t+1}; A_{t+1}, h_{a,t+1}\right)$. Equation (28) is very standard and characterizes the effect of shifting one unit of today's output from consumption to investment. Today's marginal change in utility should equal the expected discounted marginal change in tomorrow's wealth with respect to tomorrow's capital stock. Equation (29) considers the shifting of a marginal unit of human capital from the goods production sector to the schooling sector, or vice versa. The condition states that the marginal change in goods production due to this shifting, weighted by the expected shadow price of physical capital, should equal the marginal change in the schooling sector weighted by the expected shadow price of human capital. Using the envelope property of the optimal decision rules:

$$c_t^* = c\left(k_t, h_t; A_t, h_{a,t}\right) \quad \text{and} \quad u_t^* = u\left(k_t, h_t; A_t, h_{a,t}\right), \tag{30}$$

leads us to the following envelope conditions:

$$\frac{\partial V_t}{\partial k_t} = \beta \mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}}\right] \frac{\alpha \tau_r y_t}{k_t}, \quad \text{and} \quad \frac{\partial V_t}{\partial h_t} = \beta \mathrm{E}_t \left[\frac{\partial V_{t+1}}{\partial h_{t+1}}\right] B.$$

These conditions together with the above first-order necessary conditions along the optimal consumption path (28) and for the optimal allocation of human capital (29) imply the following Euler equations:

$$\frac{1}{c_t} = \beta \mathrm{E}_t \left[\frac{1}{c_{t+1}} \frac{\tau_r \alpha y_{t+1}}{k_{t+1}}\right], \tag{31}$$

$$u_t = \left(\frac{\mathrm{E}_t \left[\frac{1}{c_{t+1}} \frac{\alpha \tau_r y_{t+1}}{k_{t+1}}\right]}{\mathrm{E}_t \left[\frac{1}{c_{t+1}} \frac{y_{t+1}}{u_{t+1} h_{t+1}}\right]} \frac{A_t}{B}\right)^{\frac{1}{\alpha}} \frac{k_t h_{a,t}^{\frac{\gamma}{\alpha}}}{h_t}. \tag{32}$$

The two Euler equations (31) and (32) are necessary for a policy to attain the optimum. Together with the following transversality conditions they are also sufficient:

$$\lim_{T \to \infty} \beta^T \mathrm{E}_t \left[\frac{1}{c_T} \frac{\tau_r \alpha y_T}{k_T} k_T\right] = 0 \quad \text{and} \quad \lim_{T \to \infty} \beta^T \mathrm{E}_t \left[\frac{1}{c_T} \frac{\tau_w (1-\alpha) y_T}{u_T h_T} h_T\right] = 0 \tag{33}$$

Note that the first fraction in both conditions is the derivative of the utility function and the second fraction is the derivative of the goods sector production function with respect to the inputs of physical and human capital. To be more precise, the last derivative is taken with respect to the fraction of human capital that is allocated to the goods sector, i.e. $u_t h_t$. These derivatives are multiplied by the respective state variable. The transversality conditions tell us that the expected discounted marginal utility of an additional unit of the capital stocks in the "last period" is equal to zero. These requirements rule out that the agent plays Ponzi games.

4.2 The representative agent's optimal decisions

Again, it is easy to check that a generalized version of Robinson Crusoe's value function and of the representative agent's value function V found in Bethmann (2002) satisfies the Bellman equation (27) and the first order necessary conditions (28) and (29) simultaneously:

$$V\left(k_t, h_t; h_{a,t}\right) = \varphi + \varphi_B \ln B + \varphi_A \ln A_t + \varphi_k \ln k_t + \varphi_h \ln h_t + \varphi_{h_a} \ln h_{a,t}, \qquad (34)$$

where the φ_i's, with $i \in \{k, h, h_a, B, A\}$, are defined as follows[3]:

$$\begin{array}{lll} \varphi_k := \frac{\alpha}{1-\alpha\beta}, & \varphi_h := \frac{1-\alpha}{(1-\beta)(1-\alpha\beta)}, & \varphi_{h_a} := \frac{\gamma}{(1-\beta)(1-\alpha\beta)}, \\ \varphi_B := \frac{(1-\alpha+\gamma)\beta}{(1-\beta)^2(1-\alpha\beta)}, & \varphi_A := \frac{1}{(1-\rho\beta)(1-\alpha\beta)}. & \end{array}$$

The optimal controls implied by V are the following:

$$c_t = (1-\alpha\beta)\, y_t \qquad \text{and} \qquad u_t = \tfrac{\tau_w(1-\beta)}{\beta+\tau_w(1-\beta)}. \qquad (35)$$

If the government sets τ_w and τ_r equal to 1, these results correspond exactly to the deterministic case examined in Bethmann (2002). V implies a constant allocation of human capital between the two production sectors, i.e. the evolution of the average stock of human capital h_a does not enter the first-order necessary condition for u_t in (35). Hence, there is no linkage between the representative agent's decision and the economy-wide average decision. Therefore the solution strategy of determining the evolution of the agent's stock of capital and then exploiting the symmetry condition (6) is equivalent to the strategy of finding a fixed point where the representative agent's policy rules coincide with the economy-wide average decisions. Hence, the equation:

$$h_{a,t+1} = B\tfrac{\beta}{\beta+\tau_w(1-\beta)} h_{a,t}.$$

determines the path of the economy-wide average level of human capital in the decentralized economy. Together with the agent's optimal controls, this result implies that the Euler equations (31) and (32) and the transversality conditions (33) are met.

4.3 The government's optimal policy

The government wants to reach the social planner's solution by taxing, respectively subsidizing the agent's factor compensations. Note that the absence of τ_r in the first order conditions (35) implies that the planner's solution can be reached by simply requiring u_t to be socially optimal. On the other hand, assumption (10) requires that the state has to ensure that its budget is balanced in each period. These two requirements lead us to the following two conditions:

$$\tfrac{(1-\alpha)(1-\beta)}{1-\alpha+\beta\gamma} = \tfrac{\tau_w(1-\beta)}{\beta+\tau_w(1-\beta)} \quad \text{and} \quad (\tau_r - 1)\,\alpha = (1-\tau_w)\,(1-\alpha)\,. \qquad (36)$$

This implies the following optimal values of τ_w and τ_r:

$$\tau_w = \tfrac{1-\alpha}{1-\alpha+\gamma} \quad \text{and} \quad \tau_r = \tfrac{\alpha-\alpha^2+\gamma}{(1-\alpha+\gamma)\alpha}.$$

[3]The constant is given by: $\varphi \equiv \frac{\ln[1-\alpha\beta]}{1-\beta} + \frac{(1-\alpha)\ln[1-\beta]}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha\beta\ln\alpha}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha\beta+\gamma)\beta\ln\beta}{(1-\beta)^2(1-\alpha\beta)} + \frac{(1-\alpha)\ln[\tau_w]}{(1-\alpha\beta)(1-\beta)} - \frac{(1-\alpha+\beta\gamma)\ln[\beta+\tau_w(1-\beta)]}{(1-\alpha\beta)(1-\beta)^2}$.

Hence, the compensation of work effort is reduced by the ratio of the output elasticities of human capital in the decentralized and centralized economy, i.e. by the ratios of private and social marginal returns of human capital in goods production. These tax revenues are then distributed to the owners of the physical capital stock. This result is very intuitive and leads to an increased goods production in the decentralized economy.[4] Note that the planner's and the representative agent's value functions are identical if we apply the above condition on τ_w and use (6).

In the last two sections, we have studied both the centralized as well as the decentralized version of the Uzawa-Lucas Model of Endogenous Growth. We have found the two value functions and shown that the implied controls satisfy the Euler equations and the transversality conditions. In the next section, we show that the solutions are saddle path stable and determine their time-series implications.

5 Stability properties and time series implications of the solutions

In this section, the aim is twofold. First, we want to determine the stability properties of the two solutions. Second, we want to characterize the time-series properties. Lucas (1988) points out that the growth rate of human capital along the balanced growth path is given by $B(1-u_{bgp})$. Furthermore, he shows that the growth rates of physical capital, output, and consumption are $\frac{1-\alpha+\gamma}{1-\alpha}$ times the growth rate of human capital. Mulligan and Sala-i-Martin (1993) use this property in order to introduce transformed state-like and control-like variables. These new variables remain constant along the balanced growth path. This stationarity together with the fact that the number of state variables is reduced by one makes the analysis of growth models much simpler. Benhabib and Perli (1994) follow this strategy and define the state-like variable x_t and the control-like variable q_t. In principle, we apply the same strategy and argue that the DOP is homogeneous in the initial conditions $h_0 = h_{a,0}$ and k_0. However, as in Bethmann (2002) and Bethmann and Reiß (2003), our consideration leads us to a different definition of the control-like variable q_t[5].

Because of the homogeneity in the initial conditions of the central planner's DOP, we define the state-like variable x_t and the control-like variable q_t as follows:

$$x_t := \frac{k_t}{h_t^{\frac{1-\alpha+\gamma}{1-\alpha}}} \quad \text{and} \quad q_t := \frac{c_t}{h_t^{\frac{1-\alpha+\gamma}{1-\alpha}}}.$$

Similarly, the representative agent's DOP is homogeneous in its initial conditions. The only difference is that we must distinguish between the representative agent's stock of human capital h and the economy-wide average stock of human capital h_a. Therefore we redefine the state-like variable x_t and the control-like variable q_t as:

$$x_t := \frac{k_t}{h_t h_{a,t}^{\frac{\gamma}{1-\alpha}}} \quad \text{and} \quad q_t := \frac{c_t}{h_t h_{a,t}^{\frac{\gamma}{1-\alpha}}}.$$

[4] Uhlig and Yanagawa (1996) present an opposite result. They study a two period OLG model with endogenous growth where lower labor income taxes correspond to higher capital income taxes. Thereby the young generation is able to generate higher savings which in turn lead to higher growth.

[5] The first paper studies a discrete time version of the deterministic Uzawa Lucas Model of Endogenous Growth with full depreciation of human and physical capital while the second refers to continuous time and no depreciation. In both papers, we apply the same definition of q as we do here. On the other hand, Benhabib and Perli (1994) use $q = c/k$.

The state-like variable can be interpreted as a weighted ratio of the two capital stocks. In the deterministic model, the state-like variable x_t remains constant along the balanced growth path. Here, we consider a stochastic model such that x_t may sometimes be above or below its balanced growth path where the dynamics stem from the physical capital stock since our solutions imply that h evolves deterministically both in the centralized as well as in the decentralized case. In Section 4 we have shown that the government is able to force agents to make socially optimal decisions, i.e. to internalize the external effects stemming from the economy-wide average stock of human capital. Therefore this section focuses on the decentralized case. The representative agent's solution is fully described by the policy rules (35) together with the laws of motion for k_t, h_t, $h_{a,t}$, and A_t. Using our results, the dynamics of total factor productivity, of the state-like variable, and of the control-like variable are described by the following equations:

$$\begin{aligned}
\ln A_{t+1} &= \rho \ln A_t + \varepsilon_{t+1} \\
x_{t+1} &= \frac{\alpha\beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} x_t^{\alpha} A_t \\
q_t &= \frac{(1-\alpha\beta) u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} x_t^{\alpha} A_t.
\end{aligned}$$

Taking logarithms and using small letters with a hat in order to indicate this transformation, we arrive at:

$$\begin{aligned}
\hat{a}_{t+1} &= \rho \hat{a}_t + \varepsilon_{t+1}, \\
\hat{x}_{t+1} &= \ln\left[\frac{\alpha\beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}}\right] + \alpha\hat{x}_t + \hat{a}_t, \\
\hat{q}_t &= \ln\left[\frac{(1-\alpha\beta) u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}}\right] + \alpha\hat{x}_t + \hat{a}_t.
\end{aligned}$$

The law of motion of total factor productivity is a first-order autoregressive process with stable root ρ:

$$\hat{a}_t = \frac{\varepsilon_t}{1-\rho L}.$$

The evolution of the logged state-like variable $\hat{x}$ is described by a stochastic first-order difference equation with stable root α and stochastic disturbance $\hat{a}$. Hence the logged state-like variable $\hat{x}$ follows an AR(2) process:

$$\hat{x}_{t+1} = \frac{1}{1-\alpha} \ln\left[\frac{\alpha\beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}}\right] + \frac{\varepsilon_t}{(1-\rho L)(1-\alpha L)},$$

where the constant term on the right-hand side is the unconditional mean of the log state-like variable $\hat{x}$. Since the control-like variable q_t is non-ambiguously determined by A_t and x_t, we conclude that the whole system is saddle-path stable. Furthermore, the control-like variable q_t follows an AR(2) process:

$$\hat{q}_t = \ln[1-\alpha\beta] + \ln u + \frac{\alpha}{1-\alpha} \ln\left[\frac{\alpha\beta}{B^{\frac{1-\alpha+\gamma}{1-\alpha}}(1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}}\right] + \frac{\varepsilon_{t-1}}{(1-\rho L)(1-\alpha L)}.$$

We conclude that the detrended output $\hat{s}_t := \hat{y}_t - \hat{h}_t - \frac{\gamma}{1-\alpha}\hat{h}_{a,t}$ is also AR(2). Note that $B(1-u_{bgp})$ in the decentralized case is equal to $B\frac{\beta}{\beta+\tau_w(1-\beta)}$, such that optimal taxation induces a human capital growth rate of $B\beta\frac{1-\alpha+\gamma}{1-\alpha+\beta\gamma}$, whereas a laissez-faire policy implies $B\beta$, such that the growth rates in the centralized case or in the decentralized case with optimal taxation are indeed higher than in the decentralized economy with suboptimal or no taxation. This concludes the discussion of the time-series implications of our solutions. In the next section we formally prove the uniqueness of the value functions found before.

6 Summary and concluding remarks

We have proven that the functions (21) and (34) are the value functions of the social planner and of the representative agent, respectively. We can use these functions and the first order necessary conditions along the optimal consumption paths in order to find the optimal level of consumption. The result is the typical consumption rule for the standard AK model with logarithmic preferences, Cobb-Douglas technology, and full depreciation of physical capital. It is easy to check that this result does fit the Euler equation in consumption (31). Similarly, we can use (34) and the first order necessary condition for the optimal human capital allocation (29). We find that the optimal way to shift human capital between the two production sectors is to hold u_t constant, once we have found the optimal allocation. Similar to the consumption rule, it can be shown that this policy rule fulfills the Euler equation (32). Furthermore, the restriction $u_t \in [0, 1]$ holds. The transversality conditions in (33) ensure that the policy rules (35) of the representative agent are necessary and sufficient for a utility maximizing path. In the centralized case, the optimal stock of human capital employed in the goods sector u_t is a little bit smaller than in the decentralized case without taxation, although $u_t \in [0, 1]$ still holds. Hence, the path of human capital in the centralized economy lies above the human capital path in the decentralized economy given the same initial stocks of capital.

Finally, we have shown that the time series properties of the model are similar to those of the standard neoclassical growth model when looking at the detrended time series. This is due to the fact that the optimal human capital allocation is a constant and thus unaffected by the state variables. As a consequence, the growth rate of human capital is always equal to $B(1 - u_{bgp})$. Hence, the introduction of the schooling sector does not change the dynamics of the model.

A Appendices

A.1 Finding the value function by iteration

In Sections A.2 and A.3 of this appendix, we will use an iterative method to find the value functions that attain the suprema of the two DOPs considered in Sections 3 and 4. We introduce some basic concepts of stochastic dynamic programming from the textbook by Stokey and Lucas (1989) and finally formulate Theorem 9.12. This verification theorem states that under certain conditions a solution to the Bellman equation is necessary and sufficient even in the stochastic unbounded returns case.

Let $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ be any measurable spaces, and let $(S, \mathcal{S}) := (X \times Z, \mathcal{X} \times \mathcal{Z})$ be the product space. The set X is the set of possible values for the vector of endogenous state variables, Z is the set of possible values for the exogenous shock, and S is the set of possible states of the system. The evolution of the stochastic shocks is described by a stationary transition function Q on $(Z, \mathcal{Z})$.

In each period t, the decision-maker chooses the vector of endogenous states in the subsequent period. The constraints on this choice are described by a correspondence $\Gamma : X \times Z \to X$; that is, $\Gamma(x, z)$ is the set of feasible values for next period's state variables if the current state is (x, z). Let A be the graph of Γ:

$$A = \left\{(x, y, z) \in X \times X \times Z : y \in \Gamma(x, z)\right\}.$$

Let $F : A \mapsto \mathbb{R}$ be the per-period return function. Hence $F(x, y, z)$ gives us the current period return if the current state is (x, z) and $y \in \Gamma(x, z)$ is chosen as next period's vector of endogenous state variables. The constant one-period discount factor is denoted by β and we assume $\beta \in (0, 1)$. The givens for the problem at hand are $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, Γ, F, and β.

In period 0, with the current state (x_0, z_0) known, the decision maker chooses a value for x_1. In addition, he makes contingency plans for periods $t \in \mathbb{N}$. He realizes that the decision to be carried out in period t depends on the information that will be available at that time. Thus he chooses a sequence of functions, one for each period $t \in \mathbb{N}$. The t-th function in this sequence specifies a value for x_{t+1} as a function of the information that will be available in period t. For $t \geq 1$, this information is the sequence of shocks $(z_1, z_2, .., z_t)$. The decision maker chooses this sequence of functions to maximize the expected discounted sum of returns, where the expectation is over realizations of shocks. We define the following product spaces:

$$(Z^t, \mathcal{Z}^t) = (\underbrace{Z \times .. \times Z}_{t \text{ times}}, \underbrace{\mathcal{Z} \times .. \times \mathcal{Z}}_{t \text{ times}}),$$

for all $t \in \mathbb{N}$. Furthermore let $z^t = (z_1, .., z_t) \in Z^t$ denote a partial history of shocks in periods 1 through t.

Definition 1. *A **plan** is a value $\pi_0 \in X$ and a sequence of measurable functions $\pi_t : Z^t \to X, t \in \mathbb{N}$.*

Hence, in period t with the partial history of shocks z^t, the function $\pi_t(z^t)$ tells us the value of next period's states x_{t+1}.

Definition 2. *A plan π is **feasible** from $(x_0, z_0) \in S$ if*

(1a) $\pi_0 \in \Gamma(x_0, z_0)$,

(1b) $\pi_t(z^t) \in \Gamma[\pi_{t\text{-}1}(z^{t\text{-}1}), z_t]$.

Let $\Pi(x_0, z_0)$ denote the set of plans that are feasible from (x_0, z_0). This set is nonempty if the correspondence Γ is nonempty and a certain measurability constraint is met.

Assumption 1. *Γ is nonempty-valued and the graph of Γ is $(\mathcal{X} \times \mathcal{X} \times \mathcal{Z})$-measurable. In addition, Γ has a measurable selection; that is, there exists a measurable function $h : (X, Z) \mapsto X$ such that $h(x, z) \in \Gamma(x, z)$ for all $(x, z) \in S$.*

Under this assumption, the set $\Pi(x_0, z_0)$ is nonempty for all $(x_0, z_0) \in S$.[6] A plan π constructed by using the same measurable selection h from Γ in every period t is said to be *stationary* or *Markov*, since the action it prescribes for each period t depends only on the state $[\pi_{t\text{-}1}(z^{t\text{-}1}), z_t]$ in that period. Nonstationary plans can be constructed by using different measurable selections h_t in each period. Let a feasible plan and the transition function Q on $(Z, \mathcal{Z})$ be given. We want to calculate the total, discounted, expected returns associated with this plan. Given the initial state $(x_0, z_0) \in S$, we define the following probability measures $\mu^t(z_0, \cdot) : \mathcal{Z}^t \mapsto [0, 1]$:

$$\mu^t(z_0, Z) = \int_{Z_1} .. \int_{Z_{t\text{-}1}} \int_{Z_t} \mathcal{Q}(z_{t\text{-}1}, dz_t)\mathcal{Q}(z_{t\text{-}2}, dz_{t\text{-}1})..\mathcal{Q}(z_0, dz_1), \quad \forall\, t \in \mathbb{N}.$$

The domain of the per-period return function F is the set A, the graph of Γ. Then we can define the set $\mathcal{A}$ as:

$$\mathcal{A} = \big\{C \in \mathcal{X} \times \mathcal{X} \times \mathcal{Z} : C \subseteq A\big\}.$$

Under Assumption 1, $\mathcal{A}$ is a σ-algebra. Furthermore, if F is $\mathcal{A}$-measurable, then for any $(x_0, z_0) \in S$ and any $\pi \in \Pi(x_0, z_0)$,

$$F[\pi_{t\text{-}1}(z^{t\text{-}1}), \pi_t(z^t), z_t] \text{ is } \mathcal{Z}^t\text{-measurable}, \ \forall\, t \in \mathbb{N}.$$

This rationalizes our next assumption.

Assumption 2. *$F : A \mapsto \mathbb{R}$ is $\mathcal{A}$-measurable, and either (a) or (b) holds.*

(a) *$F \geq 0$ or $F \leq 0$*

(b) *For each $(x_0, z_0) \in S$ and each plan $\pi \in \Pi$,*

$$F[\pi_{t\text{-}1}(z^{t\text{-}1}), \pi_t(z^t), z_t] \ \textit{is } \mu^t\textit{-integrable}, \ \forall\, t \in \mathbb{N},$$

and the limit

$$F[x_0, \pi_0, z_0] + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F[\pi_{t\text{-}1}(z^{t\text{-}1}), \pi_t(z^t), z_t]\, \mu^t(x_0, dz^t)$$

exists (although it may be plus or minus infinity).

Assumption 2 ensures that, for each $(x_0, z_0) \in S$, we can define the functions $u_n(\cdot, x_0, z_0) : \Pi(x_0, z_0) \mapsto \mathbb{R}, n \in \mathbb{N}_0$, by:

$$\begin{aligned} u_0(\pi, x_0, z_0) &= F[x_0, \pi_0, z_0], \\ u_n(\pi, x_0, z_0) &= F[x_0, \pi_0, z_0] + \sum_{t=1}^{n} \int_{Z^t} \beta^t F[\pi_{t\text{-}1}(z^{t\text{-}1}), \pi_t(z^t), z_t]\, \mu^t(x_0, dz^t). \end{aligned}$$

[6]A proof of this result can be found in Lucas and Stokey (1989), page 243.

The value of $u_n(\pi, x_0, z_0)$ is the sum of expected discounted returns in periods 0 through n from plan π if the initial state is (x_0, z_0). Assumption 2 also ensures that for each $(x_0, z_0) \in S$ we can define $u(\cdot, x_0, z_0) : \Pi(x_0, z_0) \mapsto \bar{\mathbb{R}}$ to be the limit of the series as the horizon recedes:

$$u(\pi, x_0, z_0) = \lim_{n\to\infty} u_n(\pi, x_0, z_0).$$

Thus $u(\pi, x_0, z_0)$ is the infinite sum of expected discounted returns from the plan π if the initial state is (x_0, z_0). Under Assumptions 1 and 2, the function $u(\cdot, x, z)$ is well defined on the nonempty set $\Pi(x, z)$, for each $(x, z) \in S$. In this case we can define the supremum function $v^* : S \mapsto \bar{\mathbb{R}}$ by:

$$v^*(x, z) = \sup_{\pi\in\Pi(x,z)} u(\pi, x, z).$$

That is v^* is the unique function satisfying the following two conditions:

$$\begin{aligned} v^* &\geq u(\pi, x, z), \quad \text{all } \pi \in \Pi(x, z); \\ v^* &= \lim_{k\mapsto\infty} u(\pi^k, x, z), \quad \text{for some sequence } \{\pi^k\} \text{ in } \Pi(x, z). \end{aligned}$$

In the bounded returns case, a solution v to the functional equation must have the property that the expected discounted value of the implied policy in the very far future is equal to zero, that is we exclude for example sustained overinvestment. The difficulty with the unbounded returns case is that there may be some $(x_0, z_0) \in S$ and $\pi \in \Pi(x_0, z_0)$ for which the condition:

$$\lim_{t\to\infty} \beta^t \int_{Z^t} v[\pi_{t\text{-}1}(z^{t\text{-}1}), z_t]\mu^t(z_0, dz^t) = 0, \quad \forall\pi \in \Pi(x_0, z_0), \quad \forall(x_0, z_0) \in S \tag{37}$$

does not hold. For each $(x_0, z_0) \in S$, however, we can define $\hat{\Pi}(x_0, z_0)$ to be the subset of $\Pi(x_0, z_0)$ on which this condition holds. Then define $\hat{v} : S \mapsto \mathbb{R}$ by

$$\hat{v}(x, z) = \sup_{\pi\in\hat{\Pi}(x,z)} u(\pi, x, z)$$

Clearly $\hat{v} \leq v^*$. The following theorem provides sufficient conditions for the two functions to be equal.

Theorem 3. *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, $\mathcal{Q}$, Γ, F, and β satisfy Assumptions 1 and 2. Let Π, $\hat{\Pi}$, u, v^*, and $\hat{v}$ be as defined above. Suppose v is a measurable function satisfying the functional equation*

$$v(x, z) = \sup_{y\in\Gamma(x,z)} \Big[F(x, y, z) + \beta \int v(y, z')\mathcal{Q}(z, dz')\Big],$$

and that the associated policy correspondence G is nonempty and permits a measurable selection. For each $(x, z) \in S$, let $\pi^*(\cdot, x, z)$ be a plan generated by G from (x, z). Suppose in addition that

(a) *$\pi^*(\cdot, x, z) \in \hat{\Pi}(x, z)$; and*

(b) *for any $(x_0, z_0) \in S$ and $\pi \in \Pi(x_0, z_0)$, there exists $\hat{\pi} \in \hat{\Pi}(x, z)$ such that $u(\hat{\pi}, x, z) \geq u(\pi, x, z)$.*

Then $v^*(x, z) = \hat{v}(x, z) = v(x, z) = u(\pi^*(\cdot; x, z), x, z), \ \forall(x, z) \in S$.

Proof. See Stokey and Lucas (1989), page 274. □

In the next sections we will apply this theorem to our model.

A.2 The centralized economy

The social planner's optimization problem can be rewritten such that in every period t the states k_t and h_t are given and next period's states k_{t+1} and h_{t+1} have to be chosen, i.e. we want to replace the variables c_t and u_t. Equation (9) can be solved for c_t and the resulting expression can be substituted into the utility function. Similarly, we solve equation (4) for $u_t h_t$ and insert the result into the production function. In terms of the state variables, the planner's maximization problem is now given by:

$$\sup_{\{k_{t+1},h_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t F\left(k_t, h_t, k_{t+1}, h_{t+1}, A_t\right) \right]$$

such that

$$\begin{aligned}
&F\left(k_t, h_t, k_{t+1}, h_{t+1}, A_t\right) = \ln\left[A_t k_t^{\alpha} \left(h_t - \tfrac{h_{t+1}}{B} \right)^{1-\alpha} h_t^{\gamma} - k_{t+1} \right],\\
&0 < h_{t+1} < B h_t,\\
&0 < k_{t+1} < A_t k_t^{\alpha} h_t^{1-\alpha+\gamma},\\
&\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}.
\end{aligned}$$

Hence, let $(h_t, k_t)^T \in X = \mathbb{R}^2_{++}$ and $A_t \in Z = \mathbb{R}_{++}$ with the Borel sets $\mathcal{X}$ and $\mathcal{Z}$. Let $\beta \in (0,1)$ and let:

$$\Gamma(k_t, h_t, A_t) = \left\{ (k_{t+1}, h_{t+1}) \middle| A_t k_t^{\alpha} \left(h_t\text{-}\tfrac{h_{t+1}}{B} \right)^{1\text{-}\alpha} h_t^{\gamma}\text{-}k_{t+1} \in \mathbb{R}_{++};\ k_{t+1}, h_{t+1} > 0 \right\}$$

and

$$F(k_t, h_t, k_{t+1}, h_{t+1}, A_t) = \ln\left[A_t k_t^{\alpha} \left(h_t\text{-}\tfrac{h_{t+1}}{B} \right)^{1\text{-}\alpha} h_t^{\gamma}\text{-}k_{t+1} \right],$$

where $\alpha \in (0,1)$ and $\gamma \in [0,\alpha)$. Let the exogenous shocks be serially correlated with $E[\ln A_{t+1}] = \rho \ln A_t$. In order to apply Theorem 3, we want verify that Assumptions 1 and 2 hold; find (v, G) and construct the plan $\pi^*(\cdot; x, z)$, for all $(x,z) \in S$, and show that the hypotheses (a) and (b) hold.

Clearly Assumption 1 holds: $\Gamma(k, h, A) \neq \emptyset$ and there are lots of measurable selections, for example,

$$h(h_t, k_t, A_t) = \left(\tfrac{1}{2} B h_t, \tfrac{1}{2} A_t k_t^{\alpha} h_t^{1-\alpha+\gamma}\right) \in \Gamma(h_t, k_t, A_t).$$

To establish that Assumption 2 holds, note first that the per-period return function $F\left[\pi^1_{t\text{-}1}(A^{t\text{-}1}), \pi^2_{t\text{-}1}(A^{t\text{-}1}), \pi^1_t(A^t), \pi^2_t(A^t), A_t\right]$ is $\mu^t(A_0, \cdot)$-integrable and second that for any (h_t, k_t, A_t) and any $\pi \in \Pi(h_t, k_t, A_t)$ for all $t \in \mathbb{N}$:

$$\begin{aligned}
\ln \pi^1_{t\text{-}1}(A^{t\text{-}1}) &< t \ln B + \ln h_0 \qquad (38)\\
\ln \pi^2_{t\text{-}1}(A^{t\text{-}1}) &< \sum_{i=0}^{t-1} \alpha^i \ln A_{t\text{-}1\text{-}i} + (1-\alpha+\gamma) \sum_{i=0}^{t-1} \alpha^i \ln \pi^1_{t\text{-}2\text{-}i}(A^{t\text{-}2\text{-}i}) + \alpha^t \ln k_0
\end{aligned}$$

holds. Using the first inequality (38), we may further simplify the second and finally arrive at the following condition:[7]

$$\begin{aligned}
\ln \pi^2_{t\text{-}1}(z^{t\text{-}1}) &< \sum_{i=0}^{t-1} \alpha^i \ln A_{t\text{-}1\text{-}i} + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^t - 1}{(1-\alpha)^2} \right) (1-\alpha+\gamma) \ln B + \alpha^t \ln k_0\\
&\quad + \tfrac{(1-\alpha+\gamma)(1-\alpha^t)}{1-\alpha} \ln h_0.
\end{aligned}$$

[7] Note, that $\sum_{s=0}^{t} s\alpha^s = \alpha \frac{1-\alpha^t}{(1-\alpha)^2} - \frac{\alpha^{t+1} t}{1-\alpha}$ holds.

Applying the expectations operator with respect to the information set available in period 0 to all $A_{t\text{-}1\text{-}i}$ gives:

$$
\begin{aligned}
E_0\left[\ln \pi^2_{t\text{-}1}(z^{t\text{-}1})\right] \quad < \quad & \tfrac{\rho^t-\alpha^t}{\rho-\alpha}\ln A_0 + \tfrac{(1-\alpha+\gamma)(1-\alpha^t)}{1-\alpha}\ln h_0 + \alpha^t \ln k_0 \\
& + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^t-1}{(1-\alpha)^2}\right)(1-\alpha+\gamma)\ln B
\end{aligned} \tag{39}
$$

Since $F(h_t, k_t, h_{t+1}, k_{t+1}, A_t) \leq F(h_t, k_t, 0, 0, A_t)$ holds for the per-period return function, we know that:

$$
\begin{aligned}
& F\left[\pi^1_{t\text{-}1}(A^{t\text{-}1}), \pi^2_{t\text{-}1}(A^{t\text{-}1}), \pi^1_t(A^t), \pi^2_t(A^t), A_t\right] \\
\leq \quad & \ln A_t + (1-\alpha+\gamma)\ln \pi^1_{t\text{-}1}(A^{t\text{-}1}) + \alpha \ln \pi^2_{t\text{-}1}(A^{t\text{-}1})
\end{aligned} \tag{40}
$$

must also hold. Hence for any triple (h_0, k_0, A_0) and for any feasible plan π, the sequence of expected one period returns satisfies:

$$
\begin{aligned}
E_0\left[F(\cdot, t)\right] \quad < \quad & \rho^t \ln A_0 + \alpha\left(\tfrac{\rho^t-\alpha^t}{\rho-\alpha}\right)\ln A_0 + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^{t+1}-\alpha}{(1-\alpha)^2}\right)(1-\alpha+\gamma)\ln B \\
& \tfrac{(1-\alpha+\gamma)(1-\alpha^{t+1})}{1-\alpha}\ln h_0 + \alpha^{t+1}\ln k_0,
\end{aligned}
$$

where $F(\cdot, t) := F(k_t, h_t, k_{t+1}, h_{t+1}, A_t)$. Then for any feasible plan, the expected total returns are bounded from above:

$$
\lim_{n\to\infty} \mathrm{E}_0\left[\sum_{t=0}^{n} \beta^t F(\cdot, t)\right] \leq \tfrac{\beta(1-\alpha+\gamma)\ln B}{(1-\beta)^2(1-\alpha\beta)} + \tfrac{\ln A_0}{(1-\rho\beta)(1-\alpha\beta)} + \tfrac{(1-\alpha+\gamma)\ln h_0}{(1-\alpha\beta)(1-\beta)} + \tfrac{\alpha \ln k_0}{1-\alpha\beta}.
$$

We know from Section 3 that:

$$
v(h, k, A) = \theta + \theta_h \ln h + \theta_k \ln k + \theta_A \ln A + \theta_B \ln B \tag{41}
$$

is a solution to the functional equation. The coefficients θ_i, with $i \in \{h, k, A, B\}$, are defined as follows:

$$
\theta_h := \tfrac{1-\alpha+\gamma}{(1-\alpha\beta)(1-\beta)}, \quad \theta_k := \tfrac{\alpha}{1-\alpha\beta}, \quad \theta_A := \tfrac{1}{(1-\rho\beta)(1-\alpha\beta)}, \quad \text{and} \quad \theta_B := \tfrac{(1-\alpha+\gamma)\beta}{(1-\beta)^2(1-\alpha\beta)}.
$$

Indeed these coefficients imply that the function $v(h, k, A)$ is below the upper bound. The policy functions associated with v are given by:

$$
h_{t+1} \quad = \quad B\left(\tfrac{\beta(1-\alpha+\gamma)}{1-\alpha+\beta\gamma}\right)h_t, \tag{42}
$$

$$
k_{t+1} \quad = \quad \alpha\beta A_t k_t^{\alpha} h_t^{1-\alpha+\gamma}\left(\tfrac{(1-\alpha)(1-\beta)}{1-\alpha+\beta\gamma}\right)^{1-\alpha}. \tag{43}
$$

Hence, given any initial state (h_0, k_0, A_0), the plan $\pi^{1*}[\cdot, h_0, k_0, A_0]$ generated by the first policy rule can be calculated explicitly. Using this plan we can also calculate the second plan $\pi^{2*}[\cdot, h_0, k_0, A_0]$; in logs, they are:

$$
\begin{aligned}
\ln \pi^{1*}_{t\text{-}1}[\cdot, h_0, k_0, A_0] \quad &= \quad t\ln B + t\ln\tfrac{\beta(1-\alpha+\gamma)}{1-\alpha+\beta\gamma} + \ln h_0, \\
\ln \pi^{2*}_{t\text{-}1}[\cdot, h_0, k_0, A_0] \quad &= \quad \sum_{i=0}^{t-1}\alpha^i\left(\ln[\alpha\beta] + (1-\alpha)\ln\left[\tfrac{(1-\alpha)(1-\beta)}{1-\alpha+\beta\gamma}\right] + (1-\alpha+\gamma)\ln h_0\right) \\
+\sum_{i=0}^{t-1}\alpha^i i\,(1-\alpha+\gamma) \quad & \left(\ln\left[\tfrac{\beta(1-\alpha+\gamma)}{1-\alpha+\beta\gamma}\right] + \ln B\right) + \alpha^t \ln k_0 + \sum_{i=0}^{t-1}\alpha^{t\text{-}1\text{-}i}A_i.
\end{aligned}
$$

It only remains to be shown that conditions (a) and (b) of Theorem 3 hold. In order to verify that the plans $\pi^{1*}[\cdot, h_0, k_0, A_0]$ and $\pi^{2*}[\cdot, h_0, k_0, A_0]$ satisfy (a), we have to show that (37) holds for all (h_0, k_0, A_0) when applying $\pi^{1*}[\cdot, h_0, k_0, A_0]$ and $\pi^{2*}[\cdot, h_0, k_0, A_0]$, where v is given by (41). In our case we consider:

$$\begin{aligned}
& v\left[\pi^{1*}_{t\text{-}1}\left(A^{t\text{-}1}\right), \pi^{2*}_{t\text{-}1}\left(A^{t\text{-}1}\right), A_t\right] \\
= \; & \theta + \theta_B \ln B + \theta_h \left(t \ln B + t \ln \tfrac{\beta(1-\alpha+\gamma)}{1-\alpha+\beta\gamma} + \ln h_0\right) + \theta_k \alpha^t \ln k_0 \\
& + \theta_k \sum_{i=0}^{t-1} \alpha^i \left(\ln\left[\alpha\beta\right] + (1-\alpha)\ln\left[\tfrac{(1-\alpha)(1-\beta)}{1-\alpha+\beta\gamma}\right] + (1-\alpha+\gamma)\ln h_0\right) \\
& + \theta_k \sum_{i=0}^{t-1} \alpha^i i\,(1-\alpha+\gamma)\left(\ln\left[\tfrac{\beta(1-\alpha+\gamma)}{1-\alpha+\beta\gamma}\right] + \ln B\right) + \theta_k \sum_{i=0}^{t-1} \alpha^{t\text{-}1\text{-}i} A_i + \theta_A \ln A_t.
\end{aligned}$$

Using the fact that $0 < \beta < 1$, $0 < \alpha\beta < 1$, and $E_0\left[\ln A_t\right] = \rho^t \ln A_0$, it is straightforward to show that condition (37) indeed holds.[8]

In order to verify condition (b), we need to show that for any initial state (h_0, k_0, A_0) in S, any plan in $\Pi(h_0, k_0, A_0)$ is weakly dominated by a plan in $\hat{\Pi}(h_0, k_0, A_0)$. Let $(h_0, k_0, A_0) \in S$ and $\pi \in \Pi(h_0, k_0, A_0)$ be arbitrary. By definition $\pi \in \hat{\Pi}(h_0, k_0, A_0)$ if and only if (37) holds. With v given by (41), the condition (37) reads as follows:

$$\lim_{t\to\infty} \beta^t \mathrm{E}_0 \Big[\theta + \theta_h \ln \pi^1_{t\text{-}1}[A^{t\text{-}1}] + \theta_k \ln \pi^2_{t\text{-}1}[A^{t\text{-}1}] + \theta_A \ln A_t + \theta_B \ln B\Big] = 0.$$

It follows from the assumptions on the A_t's that:

$$\lim_{t\to\infty} \beta^t \mathrm{E}_0 \Big[\theta_A \ln A_t\Big] = \lim_{t\to\infty} \left(\rho\beta\right)^t \theta_A \ln A_0 = 0.$$

Hence (37) holds if and only if:

$$\lim_{t\to\infty} \beta^t \mathrm{E}_0 \Big[\theta_h \ln \pi^1_{t\text{-}1}[A^{t\text{-}1}] + \theta_k \ln \pi^2_{t\text{-}1}[A^{t\text{-}1}]\Big] = 0. \tag{44}$$

That is, $\pi \in \hat{\Pi}(h_0, k_0, A_0)$ if and only if condition (44) holds. In addition, we know from (38) and (39) that for all (h_t, k_t, A_t) and any $\pi \in \Pi(h_t, k_t, A_t)$ for all $t \in \mathbb{N}$:

$$\lim_{t\to\infty} \beta^t \mathrm{E}_0 \Big[\ln \pi^1_{t\text{-}1}[A^{t\text{-}1}]\Big] \leq 0 \quad \text{and} \quad \lim_{t\to\infty} \beta^t \mathrm{E}_0 \Big[\ln \pi^2_{t\text{-}1}[A^{t\text{-}1}]\Big] \leq 0 \tag{45}$$

must hold. Now suppose that $\pi \notin \hat{\Pi}(h_0, k_0, A_0)$, i.e. (44) fails to hold. It follows from the inequality in (40) that:

$$u\left(\pi, h_0, k_0, A_0\right) \leq \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \Big(\ln A_t + (1-\alpha+\gamma)\ln \pi^1_{t\text{-}1}[A^{t\text{-}1}] + \alpha \ln \pi^2_{t\text{-}1}[A^{t\text{-}1}]\Big)\right].$$

Since (44) fails, the conditions in (45) imply that this series must diverge to minus infinity: $u\left(\pi, h_0, k_0, A_0\right) = -\infty$; in this case π^{1*} and π^{2*} dominate π^1 and π^2. Thus condition (b) is satisfied, and Theorem 3 applies. That is, v is indeed the value function and the policy rules are given by (42) and (43). This ends the discussion of the centralized case. In the next subsection we turn to the decentralized economy.

[8]Note, that for $\beta \in (0, 1)$, $\lim_{t\to\infty} t\beta^t = 0$ holds.

A.3 The decentralized economy

We start the analysis of the decentralized case by rewriting the representative agent's optimization problem. In every period t the states A_t, k_t, h_t, and $h_{a,t}$ are given and next period's states k_{t+1} and h_{t+1} have to be chosen. The agent also knows the government's balanced budget restriction (36). This means that the agent's earnings generated by human and physical capital income can not exceed the economy's per capita production. Then the maximization problem is given by:

$$\sup_{\{k_{t+1},h_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t F\left(k_t, h_t, k_{t+1}, h_{t+1}; A_t, h_{a,t}\right) \right],$$

such that

$$\begin{aligned}
&F\left(k_t, h_t, k_{t+1}, h_{t+1}; A_t, h_{a,t}\right) = \ln\left[A_t k_t^{\alpha} \left(h_t - \tfrac{h_{t+1}}{B} \right)^{1-\alpha} h_{a,t}^{\gamma} - k_{t+1} \right],\\
&0 < h_{t+1} < B h_t,\\
&0 < k_{t+1} < A_t k_t^{\alpha} h_t^{1-\alpha} h_{a,t}^{\gamma},\\
&0 < h_{a,t+1} < B h_{a,t},\\
&\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1},\\
&h_t = h_{a,t}.
\end{aligned}$$

We have argued in Section 4 that the representative agent does not exploit the external effect, because the market mechanism prevents agents from coordinating their actions. However, the path of $h_{a,t}$ is predictable and the representative agent treats this path as given.

Let $(h_t, k_t) \in X = \mathbb{R}_{++}^2$ and $(A_t, h_{a,t}) \in Z = \mathbb{R}_{++}^2$ with the Borel sets $\mathcal{X}$ and $\mathcal{Z}$.[9] Let us now turn to the policy correspondence Γ, which is given by:

$$\Gamma\left(h_t, k_t; A_t, h_{a,t}\right) = \left\{ (h_{t+1}, k_{t+1}) \;\middle|\; \begin{array}{l} A_t k_t^{\alpha} \left(h_t - \frac{h_{t+1}}{B} \right)^{1-\alpha} h_{a,t}^{\gamma} - k_{t+1} \in \mathbb{R}_{++};\\ k_{t+1}, h_{t+1} > 0;\ h_{a,t} = h_t \end{array} \right\}$$

for all $t \geq 0$. The exogenous shocks are serially correlated with $\mathrm{E}_t\left[\ln A_{t+1}\right] = \rho \ln A_t$. Again we want to verify that Assumptions 1 and 2 hold; find (v, G) and construct the plan $\pi^*(\cdot; x, z)$ all $(x, z) \in S$, and show that the two hypotheses (a) and (b) of Theorem 3 hold.

First, note that Assumption 1 holds: $\Gamma(h_t, k_t; A_t, h_{a,t})$ is non-empty and there are lots of measurable selections, for example:

$$h(h_t, k_t; A_t, h_{a,t}) = \left(\tfrac{1}{2} B h_t, \tfrac{1}{2} A_t k_t^{\alpha} h_t^{1-\alpha} h_{a,t}^{\gamma}\right) \in \Gamma(h_t, k_t; A_t, h_{a,t}).$$

In order to show that Assumption 2 holds, note first that the per-period return function $F\left[\pi_{t\text{-}1}^1(z^{t\text{-}1}), \pi_{t\text{-}1}^2(z^{t\text{-}1}), \pi_t^1(z^t), \pi_t^2(z^t), z_t\right]$ is $\mu^t(z_0, \cdot)$-integrable and second that for any (x_t, z_t) and any $\pi \in \Pi(x_t, z_t)$ for all $t \in \mathbb{N}$:

$$\begin{aligned}
\ln \pi_{t\text{-}1}^1(z^{t\text{-}1}) &< t \ln B + \ln h_0,\\
\ln \pi_{t\text{-}1}^2(z^{t\text{-}1}) &< \sum_{i=0}^{t-1} \alpha^i \left\{ \ln A_{t\text{-}1\text{-}i} + (1-\alpha) \ln \pi_{t\text{-}2\text{-}i}^1(z^{t\text{-}2\text{-}i}) + \gamma \ln h_{a,t\text{-}1\text{-}i} \right\} + \alpha^t \ln k_0
\end{aligned}$$

[9] Note that this Borel set differs from that in the previous section.

hold. Using the first inequality, we may further simplify the second and finally arrive at the following condition:[10]

$$\begin{aligned}\ln \pi^2_{t\text{-}1}(z^{t\text{-}1}) \quad < \quad & \sum_{i=0}^{t-1} \alpha^i \ln A_{t\text{-}1\text{-}i} + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^t - 1}{(1-\alpha)^2}\right)(1-\alpha+\gamma)\ln B \\ & + \tfrac{\gamma}{1-\alpha}(1-\alpha^t)\ln h_{a,0} + (1-\alpha^t)\ln h_0 + \alpha^t \ln k_0.\end{aligned}$$

Applying the expectations operator with respect to the information set available in period 0 to all $A_{t\text{-}1\text{-}i}$ gives:

$$\begin{aligned}E_0\left[\ln \pi^2_{t\text{-}1}(z^{t\text{-}1})\right] \quad < \quad & \tfrac{\rho^t - \alpha^t}{\rho - \alpha}\ln A_0 + \tfrac{\gamma}{1-\alpha}(1-\alpha^t)\ln h_0 + (1-\alpha^t)\ln h_0 \\ & + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^t-1}{(1-\alpha)^2}\right)(1-\alpha+\gamma)\ln B + \alpha^t \ln k_0.\end{aligned}$$

Since $F(h_t, k_t, h_{t+1}, k_{t+1}, A_t; h_{a,t}) \leq F(h_t, k_t, 0, 0, A_t; h_{a,t})$ holds for the per-period return function we know that:

$$\begin{aligned}& F\left[\pi^1_{t\text{-}1}(z^{t\text{-}1}), \pi^2_{t\text{-}1}(z^{t\text{-}1}), \pi^1_t(z^t), \pi^2_t(z^t), z_t\right] \\ \leq \quad & \ln A_t + \gamma \ln h_{a,t} + (1-\alpha)\ln \pi^1_{t\text{-}1}(z^{t\text{-}1}) + \alpha \ln \pi^2_{t\text{-}1}(z^{t\text{-}1})\end{aligned} \tag{46}$$

must also hold. Hence for any pair (h_0, k_0, A_0) and for any feasible plan π, the sequence of expected one period returns satisfies:

$$\begin{aligned}E_0\left[F(\cdot, t)\right] \quad < \quad & \rho^t \ln A_0 + \alpha\left(\tfrac{\rho^t-\alpha^t}{\rho-\alpha}\right)\ln A_0 + \left(\tfrac{t}{1-\alpha} + \tfrac{\alpha^{t+1}-\alpha}{(1-\alpha)^2}\right)(1-\alpha+\gamma)\ln B \\ & \tfrac{(1-\alpha+\gamma)(1-\alpha^{t+1})}{1-\alpha}\ln h_0 + \alpha^{t+1}\ln k_0,\end{aligned}$$

where $F(\cdot, t) := F(k_t, h_t, k_{t+1}, h_{t+1}, A_t)$. Then for any feasible plan, the expected total returns are bounded from above:

$$\begin{aligned}& \lim_{n\to\infty} \mathrm{E}_0\Big[\sum_{t=0}^{n} \beta^t F(\cdot, t)\Big] \\ \leq \quad & \tfrac{\alpha \ln k_0}{1-\alpha\beta} + \tfrac{(1-\alpha)\ln h_0}{(1-\alpha\beta)(1-\beta)} + \tfrac{\gamma \ln h_{a,0}}{(1-\alpha\beta)(1-\beta)} + \tfrac{\beta(1-\alpha+\gamma)\ln B}{(1-\beta)^2(1-\alpha\beta)} + \tfrac{\ln A_0}{(1-\rho\beta)(1-\alpha\beta)}\end{aligned}$$

This concludes our search for an upper bound of the value function, i.e. we have shown that the limit in Assumption 2 exists although it may be minus infinity.

We know from Section 4 that:

$$v(h, k, A) = \varphi + \varphi_k \ln k + \varphi_h \ln h + \varphi_{h_a} \ln h_a + \varphi_A \ln A + \varphi_B \ln B \tag{47}$$

solves the functional equation. The coefficients φ_i, with $i \in \{k, h, h_a, A, B\}$, were defined as follows[11]:

$$\begin{aligned}&\varphi_k := \tfrac{\alpha}{1-\alpha\beta}, \qquad && \varphi_h := \tfrac{1-\alpha}{(1-\beta)(1-\alpha\beta)}, \qquad && \varphi_{h_a} := \tfrac{\gamma}{(1-\beta)(1-\alpha\beta)}, \\ &\varphi_B := \tfrac{(1-\alpha+\gamma)\beta}{(1-\beta)^2(1-\alpha\beta)}, && \varphi_A := \tfrac{1}{(1-\rho\beta)(1-\alpha\beta)}.\end{aligned}$$

[10]Note that $\sum_{s=0}^t s\alpha^s = \alpha\frac{1-\alpha^t}{(1-\alpha)^2} - \frac{\alpha^{t+1}t}{1-\alpha}$ holds.

[11]The constant φ is given by: $\varphi := \frac{\ln[1-\alpha\beta]}{1-\beta} + \frac{(1-\alpha)\ln[1-\beta]}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha\beta\ln\alpha}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha\beta+\gamma)\beta\ln\beta}{(1-\beta)^2(1-\alpha\beta)}$.

Indeed these coefficients imply that the function $v(h,k,A)$ is below the upper bound. The policy functions associated with v are given by:

$$\begin{aligned} h_{t+1} &= B\tfrac{\beta}{\beta+\tau_w(1-\beta)}h_t, & (48)\\ k_{t+1} &= \alpha\beta A_t k_t^{\alpha} h_t^{1-\alpha+\gamma}\left(\tfrac{\tau_w(1-\beta)}{\beta+\tau_w(1-\beta)}\right)^{1-\alpha}. & (49)\end{aligned}$$

Hence, given any initial state $s_0 = (k_0, h_0, h_{a,0}, A_0)$, the plan $\pi^{1*}[\cdot, s_0]$ generated by the first policy rule can be calculated explicitly. Using this plan we also can calculate the second plan $\pi^{2*}[\cdot, s_0]$; in logs, they are:

$$\begin{aligned} \ln \pi^{1*}_{t\text{-}1}[\cdot, s_0] &= t\ln B + t\ln\left[\tfrac{\beta}{\beta+\tau_w(1-\beta)}\right] + \ln h_0,\\ \ln \pi^{2*}_{t\text{-}1}[\cdot, s_0] &= \sum_{i=0}^{t-1}\alpha^i\Big(\ln[\alpha\beta] + (1-\alpha)\ln\left[\tfrac{\tau_w(1-\beta)}{\beta+\tau_w(1-\beta)}\right] + (1-\alpha)\ln h_0 + \gamma\ln h_{a,0}\Big)\\ &\quad+\sum_{i=0}^{t-1}\alpha^i i(1-\alpha+\gamma)\left(\ln\left[\tfrac{\beta}{\beta+\tau_w(1-\beta)}\right]+\ln B\right) + \alpha^t\ln k_0 + \sum_{i=0}^{t-1}\alpha^{t\text{-}1\text{-}i}A_i. \end{aligned}$$

It remains only to be shown that conditions (a) and (b) of Theorem 3 hold. In order to verify that the plans $\pi^{1*}[\cdot, s_0]$ and $\pi^{2*}[\cdot, s_0]$ satisfy condition (a), we have to show that (37) holds for all (s_0) when applying $\pi^{1*}[\cdot, s_0]$ and $\pi^{2*}[\cdot, s_0]$, where v is given by (47). In our case we consider:

$$\begin{aligned} &v\left[\pi^{1*}_{t\text{-}1}\left(A^{t\text{-}1}, h_a^{t\text{-}1}\right), \pi^{2*}_{t\text{-}1}\left(A^{t\text{-}1}, h_a^{t\text{-}1}\right), A_t, h_{a,t}\right]\\ =\ &\varphi + \varphi_B\ln B + (\varphi_h+\varphi_{h_a})\Big(t\ln B + t\ln\left[\tfrac{\beta}{\beta+\tau_w(1-\beta)}\right]\Big) + \varphi_h\ln h_0 + \varphi_{h_a}\ln h_{a,0}\\ &+\varphi_k\alpha^t\ln k_0 + \varphi_k\sum_{i=0}^{t-1}\alpha^i\Big(\ln[\alpha\beta]+(1-\alpha)\ln\left[\tfrac{\tau_w(1-\beta)}{\beta+\tau_w(1-\beta)}\right]+(1-\alpha)\ln h_0+\gamma\ln h_{a,0}\Big)\\ &+\varphi_k\sum_{i=0}^{t-1}\alpha^i i\,(1-\alpha+\gamma)\left(\ln\left[\tfrac{\beta}{\beta+\tau_w(1-\beta)}\right]+\ln B\right) + \varphi_k\sum_{i=0}^{t-1}\alpha^{t\text{-}1\text{-}i}A_i + \varphi_A\ln A_t. \end{aligned}$$

Using the fact that $0 < \beta < 1$, $0 < \alpha\beta < 1$, and $E_0[\ln A_t] = \rho^t \ln A_0$, it is straightforward to show that condition (37) indeed holds[12].

In order to verify condition (b) we need to show that for any initial state $(s_0) \in S$, any plan in $\Pi(s_0)$ is weakly dominated by a plan in $\hat{\Pi}(s_0)$. Let $(s_0) \in S$ and $\pi \in \Pi(s_0)$ be arbitrary. By definition $\pi \in \hat{\Pi}(s_0)$ if and only if (37) holds. With v given by (47), condition (37) reads as follows:

$$\lim_{t\to\infty}\beta^t \mathrm{E}_0\Big[\varphi' + \varphi_A\ln A_t + \varphi_{h_a}\ln h_{a_t} + \varphi_h\ln\pi^1_{t\text{-}1}[A^{t\text{-}1}, h_a^{t\text{-}1}] + \varphi_k\ln\pi^2_{t\text{-}1}[A^{t\text{-}1}, h_a^{t\text{-}1}]\Big] = 0,$$

where $\varphi' := \varphi + \varphi_B \ln B$. It follows from the assumptions on the A_t's that

$$\lim_{t\to\infty}\beta^t \mathrm{E}_0\Big[\varphi_A\ln A_t\Big] = \lim_{t\to\infty}(\rho\beta)^t\varphi_A\ln A_0 = 0.$$

Furthermore, note that the path of $\ln h_{a,t}$ is bounded by $t\ln B + \ln h_{a,0}$. Hence condition (37) holds if and only if:

$$\lim_{t\to\infty}\beta^t \mathrm{E}_0\Big[\varphi_h\ln\pi^1_{t\text{-}1}[A^{t\text{-}1}] + \varphi_k\ln\pi^2_{t\text{-}1}[A^{t\text{-}1}]\Big] = 0. \tag{50}$$

[12] Note, that for $\beta \in (0,1)$ $\lim_{t\to\infty} t\beta^t = 0$ holds.

That is, $\pi \in \hat{\Pi}(s_0)$ if and only if condition (50) holds. In addition, we know from (38) and (39) that for all s_t and any $\pi \in \Pi(s_t)$ for all $t \in \mathbb{N}$

$$\lim_{t\to\infty} \beta^t \mathrm{E}_0\Big[\ln \pi^1_{t\text{-}1}[A^{t\text{-}1}, h_a^{t\text{-}1}]\Big] \le 0 \quad \text{and} \quad \lim_{t\to\infty} \beta^t \mathrm{E}_0\Big[\ln \pi^2_{t\text{-}1}[A^{t\text{-}1}, h_a^{t\text{-}1}]\Big] \le 0 \qquad (51)$$

must hold. Now suppose that $\pi \notin \hat{\Pi}(s_0)$, i.e. (44) fails. It follows from the inequality in (46) that

$$u(\pi, h_0, k_0, A_0) \le \mathrm{E}_0\left[\sum_{t=0}^{\infty} \beta^t \Big(\ln A_t + (1-\alpha+\gamma)\ln \pi^1_{t\text{-}1}[A^{t\text{-}1}] + \alpha \ln \pi^2_{t\text{-}1}[A^{t\text{-}1}]\Big)\right].$$

Since (44) fails, the conditions in (45) imply that this series must diverge to minus infinity: $u(\pi, h_0, k_0, A_0) = -\infty$; in this case π^{1*} and π^{2*} dominate π^1 and π^2. Thus condition (b) is satisfied, and Theorem 3 applies. That is, v is indeed the value function and the policy rules are given by (48) and (49).

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