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Tail Conditional Expectation for vector-valued Risks

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Tail Conditional Expectation for vector valued risks

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PRELIMINARY VERSION

Abstract

In his paper we introduce a *quantile-based* risk measure for multivariate financial positions: the vector-valued Tail-conditional-expectation ($\mathcal{TCE}$). We adopt the framework proposed by Jouini, Meddeb, and Touzi [9] to deal with multi-assets portfolios when one accounts for frictions in the financial market. In this framework, the space of risks formed by essentially bounded random vectors, is endowed with some partial vector preorder $\succeq$ accounting for market frictions. In a first step we provide a definition for *quantiles of vector-valued risks* which is compatible with the preorder $\succeq$. The $\mathcal{TCE}$ is then introduced as a natural extension of the "classical" real-valued tail-conditional-expectation. Our main result states that for continuous distributions $\mathcal{TCE}$ is equal to a

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coherent vector-valued risk measure. We also provide a numerical algorithm for computing vector-valued quantiles and $\mathcal{TCE}$.

Key words : Risk measures, vector-valued risk measures, coherent risk-measures, quantiles, tail-conditional-expectation.

MSC Classification (2000): 91B28, 49L25, 35B05.

1 Introduction

In their seminal paper [2], Artzner et al. adopt an axiomatic approach to characterize economically *coherent* risk measures. The authors consider the resulting net worth of a financial position, at the end of a given investment period, and describe it by a real-valued random variable X on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Their definition of a risk measure relies on the following basic approach. An investor (or regulator) defines among the set of all possible financial positions, a subset $\mathcal{A}$ of *acceptable* positions regarded as risk-free. Then, the risk measure $\rho(X)$ of a position $X \in \mathbb{R}^{\Omega}$, corresponds to the "extra" capital requirement that has to be invested at the beginning of the period in some "secure" instrument so that the resulting position is acceptable, i.e. $X + \rho(X) \in \mathcal{A}$. A set of axioms, namely: (i) subadditivity, (ii) monotonicity, (iii) positive homogeneity, and (iv) translation invariance, guarantees the economic coherence of a risk measure ρ. The notion of coherent risk measure has been extended to convex risk measure [6], and has been generalized to more complex spaces of risk, which allows to take into consideration financial positions with different types of cash streams structures. For instance, Delbaen [5] considers a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and identifies the space of risks with the space of essentially bounded random variables $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$. Cheridito, Delbaen and Kupper [4] extend the definition and the dual representation of coherent and convex risk measures to the space of càdlàg processes. Jaschke and Küchler [8] consider coherent risk measures on abstract spaces of risk including deterministic, stochastic, single or multi-periodic cash-streams structures. The above mentioned generalizations, however, do not take into account the problem of portfolio aggregation. Indeed, if we consider realistic situations where investors have access to different markets and form multi-assets portfolios, in the presence of frictions such as : transaction costs, liquidity problems, irreversible transfers,... etc., a position cannot be merely described by a real-valued random net worth, or a real-valued process of cash-streams. Focusing on this problem, Jouini, Meddeb, and Touzi [9] describe financial positions through $\mathbb{R}^d$-valued random variables $X = (X^1, ..., X^d)$, where each component X^i corresponds to net worth at the end of the investment period of an investor's position on the i^{th} market. They restrict the space of risks to essentially bounded $\mathbb{R}^d$-valued random portfolios X, $X \in L_d^\infty$, and assume that the portfolios in L_d^∞ are ordered according to a partial ordering relation $\preceq$ which accounts for frictions on the financial market. By analogy to [2], the authors define a vector-valued risk measure as set-valued function R which associates to each risky portfolio X, deterministic portfolios $\bar{x} = (\bar{x}^1, ..., \bar{x}^d)$ where $\bar{x}^i$ corresponds to some "extra" capital invested in a secure instrument from market i, and such that the position $X + \bar{x}$ is *acceptable*. Jouini et al. extend the axiomatic characterization of real-valued coherent risk measures to the multi-dimensional case, and they provide a dual representation result for coherent vector-valued risk measures, which is consistent with the representation theorem for coherent real-valued risk measures.

Our main concern in this paper is to define a distribution-based vector-valued risk measure, to verify its coherency, and to propose a procedure to compute it.
In the one-dimensional context, risk measures of investment strategies are often defined in terms of a quantile of a given distribution. This is a natral procedure since the worst realizations of a financial position are concentrated on the left tail of its disribution. A typical example is the Value-at-Risk (VaR) which is a distribution tail related measure identifying the loss that is likely to be exceeded by a specified probability, over a given time horizon. A second important example is the *Tail Conditional Expectation* (TCE), which has been suggested as an alternative to VaR [2]. Indeed, while VaR fails to be a coherent risk measure (VaR is not subadditive), the risk mesure TCE results to coincide, under some conditions on the distribution of the risks, with a coherent risk measure, see [2], [3].

The main difficulty regarding the generalization of quantile-based measures to this framework is the fact that vector preorders are, in general, partial preorders. Then, what can be considered in a context of multidimensional portfolios as the analogous of "worst cases" or "tail distributions" ? This is the first question we shall address by suggesting a suitable definition of quantiles for multi-dimensional portfolios. We shall then introduce Vector-valued Tail conditional Expectation as a natural extension to the "classical" real-valued TCE.
This paper is organized as follows. In Section 2 we recall the notion of vector-valued risk measures and acceptance sets, then we present our definition of the α-quantile of the distribution of an $\mathbb{R}^d$-valued portfolio and the corresponding definition of vector valued Tail Conditional Expectation. Our main result, provided in Section 3, states that for continuous distributions, the vector-valued Tail Conditional Expectation is equal the vector valued Generalized Worst Conditional Expectation ($\mathcal{G}WEC$) which is a coherent risk measure . Finally, we detail in Section 4 a numerical procedure to compute quantiles and Tail Conditional Expectations for vector-valued risks.

Notations : We first introduce the main notations of the paper.
Given an element x of a finite-dimensional vector space, we shall denote by x^i its $i-th$ component.
For $i \leq d \in \mathbb{N}$, we shall denote by $\mathbf{1}_i$ the element of the canonical basis of R^d defined by : $\mathbf{1}_i^j = 0$ if $j \neq i$, and $\mathbf{1}_i^i = 1$.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall denote by L^d_∞ the set of essentially bounded $\mathbb{R}^d$-valued random variables, and $||X||_\infty$ denotes the essential supremum of $X \in L_d^\infty$
Finally, for a set A, $\mathbf{1}_A$ states for its indicator function.

2 Definitions

We consider investors having access to $d \geq 1$ different markets. A financial position is then described by a d-dimensional random vector $X := \left(X^1, ..., X^d\right)'$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each component X^i represents the resulting wealth in the market i of the investor's positions or strategies. Following Jouini, Meddeb, and Touzi [9], we restrict our attention to financial positions in L_d^∞, and we assume that they are ordered according to the following rule :

$$X \ \succeq \ Y \quad \text{if and only if} \quad X - Y \ \in \ K \ \ \mathbb{P} - \text{a.s.}$$

where K is a closed convex cone of $\mathbb{R}^d$ satisfying the following conditions

$$\mathbb{R}_+^d \ \subset \ K \quad \text{and} \quad K \ \neq \ \mathbb{R}^d \ . \tag{2.1}$$

Observe that the condition $\mathbb{R}_+^d \subset K$ implies that portfolios with nonnegative entries are nonnegative w.r.t. the partial order relation $\succeq$.

2.1 Vector-valued risk measures

For the convenience of the reader, we give hereafter a short remainder of the notions of vector-valued risk measures, and acceptance sets. We refer the interested reader to [9] for a thourough treatment.

A basic approach for describing the risk incurred by some agent consists of defining, within the positions he can take, a subset of desirable or *acceptable* positions. Thus a portfolio is considered to be risky or not, whether it lies or not in the acceptance set.

Definition 2.1 *A coherent acceptance set is a subset $\mathcal{A}$ of L_d^∞ satisfying*

C0– $\mathcal{A}$ *is a closed and convex cone of* L_d^∞ *containing* 0 .

C1– *For all* $X \in \mathcal{A}$ *, for all* $Y \in L_d^\infty$ *,* $Y \succeq X \ \Rightarrow \ Y \in \mathcal{A}$.

C2– $\mathbb{R}^d \not\subset \mathcal{A}$.

A vector valued risk measure associates with each risk X a set, $R(X)$, of deterministic portfolios $\bar{x} \in \mathbb{R}^d$ such that: the modified position $X + \bar{x}$ is "accetable".

Definition 2.2 *A coherent risk measure is a set-valued map $R : L^\infty_d \twoheadrightarrow \mathbb{R}^d$ satisfying the following axioms*

A0– For all $X \in L^\infty_d, R(X)$ is closed, and $0 \in R(0) \neq \mathbb{R}^d$.

A1– For all $X, Y \in L^\infty_d,\ X \succeq Y \Rightarrow R(Y) \subset R(X)$.

A2– For all $X, Y \in L^\infty_d,\ R(X) + R(Y) \subset R(X+Y)$.

A3– For all $X \in L^\infty_d$, for all $t > 0,\ R(tX) = tR(X)$.

A4– For all $X \in L^\infty_d$, for all $x \in \mathbb{R}^d$, $R(X + \bar{x}) = \{-\bar{x}\} + R(X)$.

The following Proposition states the relation between coherent acceptance sets and coherent risk measures.

Proposition 2.1 *Let $\mathcal{A}$ be some subset of L^∞_d and define the set-valued map $R_{\mathcal{A}} : L^\infty_d \twoheadrightarrow \mathbb{R}^n$ as follows*

$$R_{\mathcal{A}}(X) := \left\{ \bar{x} \in \mathbb{R}^d \ : \ X + \bar{x} \in \mathcal{A} \right\}$$

Then $\mathcal{A}$ is a coherent acceptance set if and only if $R_{\mathcal{A}}$ is a coherent risk measure.

Example 2.1 Fix some $\alpha \in (0,1)$. The (vector-valued) *Worst Conditional Expectation* at level α is the set-valued map defined on L^∞_d by

$$\mathcal{WCE}_\alpha(X) := \left\{ \bar{x} \in \mathbb{R}^d \ : \ \mathbb{E}\left[X + \bar{x}|B\right] \succeq 0 \text{ for all } B \in \mathcal{F} \text{ with } \mathbb{P}(B) > \alpha \right\} .$$

The *vector-valued worst conditional expectation*, $\mathcal{WCE}_\alpha$, has been introduced by Jouini et al. [9] as a natural extension of the coherent *real-valued worst conditional expectation*, WCE_α defined, for a real-valued random variable $\mathcal{X}$, by

$$WCE_\alpha(\mathcal{X}) := -\inf \left\{ \mathbb{E}\left[\mathcal{X} \,|\, B\right] ,\ B \in \mathcal{F}, \mathbb{P}(B) \geq \alpha \right\} .$$

It can be easily checked, that $\mathcal{WCE}_\alpha$ is a coherent vector-valued risk measure.

Remark 2.1 Actually, in [9], the authors introduce the notion of (d, n)-coherent risk measure: by assuming that the convex cone K satisfies the additional property of *substitutability*

$$\text{for all } i = n+1, ..., d \ : \ -\mathbf{1}_i + \delta \mathbf{1}_1 \text{ and} \mathbf{1}_i - \gamma \mathbf{1}_1 \in K \qquad \text{for some } \delta, \gamma > 0 \ .$$

it possible to regulate risks in L^∞_d by using deterministic portfolios of the form $\bar{x} = (x, 0, ..., 0)$, where $x \in \mathbb{R}^n$. In this paper we restrict our analysis to (d, d)-coherent risk measures in order to simplify our presentation. Our work adapts without any difficulty to more general (d, n)-coherent risk measure.

We end this subsection by introducing a new vector-valued coherent risk measure based on the $\mathcal{W}CE_\alpha$. We call this risk measure the *Generalized Worst Conditional Expectation* at level α, we denote it by $\mathcal{G}WCE_\alpha$, and we defined by

$$\mathcal{G}WCE_\alpha(X) \ = \ \bigcup_{\tilde{X}} \mathcal{W}CE_\alpha(\tilde{X}) \tag{2.2}$$

where the union is taken over all random variables $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ having the same distribution as X.

Proposition 2.2 *$\mathcal{G}WCE_\alpha$ is a coherent distribution-based risk measure.*

Proof. We only prove that $\mathcal{G}WCE_\alpha$ satisfies the subadditivity axiom $A2$, the other properties $(A1, A3-4)$ being easy to check.
Let X, Y be in L^∞_d, and $\bar{x}$, $\bar{y}$ respectively in $\mathcal{G}WCE_\alpha(X)$ and $\mathcal{G}WCE_\alpha(Y)$. This means that $\bar{x} \in \mathcal{W}CE_\alpha(\tilde{X}')$ (reps. $\bar{y} \in \mathcal{W}CE_\alpha(\tilde{Y}')$), where the random vector $\tilde{X}'$ (resp. $\tilde{Y}'$) defined on $(\tilde{\Omega}^x, \tilde{\mathcal{F}}^x, \tilde{\mathbb{P}}^x)$ (resp. $(\tilde{\Omega}^y, \tilde{\mathcal{F}}^y, \tilde{\mathbb{P}}^y)$) has the same distribution as X (resp. Y). We have to show that $\bar{x} + \bar{y} \in \mathcal{G}WCE_\alpha(X + Y)$.
Define the product probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) := (\tilde{\Omega}^x \times \tilde{\Omega}^y, \tilde{\mathcal{F}}^x \otimes \tilde{\mathcal{F}}^y, \tilde{\mathbb{P}}^x \otimes \tilde{\mathbb{P}}^y)$, and the random variables

$$\tilde{X}(\omega, \omega') = \tilde{X}'(\omega) \quad \text{and} \quad \tilde{Y}(\omega, \omega') = \tilde{Y}'(\omega') \ \text{ for } (\omega, \omega') \in \tilde{\Omega} \ .$$

For all $\tilde{B} = B^x \times B^y \in \tilde{\mathcal{F}}^x \otimes \tilde{\mathcal{F}}^y$ such that $\tilde{\mathbb{P}}(\tilde{B}) \geq \alpha$ we have necessarily that $\min\left\{\tilde{\mathbb{P}}^x(B^x)\,;\,\tilde{\mathbb{P}}^x(B^x)\right\} \geq \alpha$, and we easily verify, using the definition of $\mathcal{W}CE_\alpha$

$$\tilde{\mathbb{E}}\left[\bar{x} + \tilde{X}|\tilde{B}\right] = \tilde{\mathbb{E}}^x\left[\bar{x} + \tilde{X}'|B^x\right] \in K \quad \text{and} \quad \tilde{\mathbb{E}}\left[\bar{y} + \tilde{Y}|\tilde{B}\right] = \tilde{\mathbb{E}}^y\left[\bar{y} + \tilde{Y}'|B^y\right] \in K\ ,$$

hence $\tilde{\mathbb{E}}\left[(\bar{x} + \bar{y})\tilde{X} + \tilde{Y}|\tilde{B}\right] \in K$, and by arbitrariness of $\tilde{B}$ we deduce that $\bar{x} + \bar{y} \in \mathcal{W}CE_\alpha(\tilde{X} + \tilde{Y}) \subset \mathcal{G}XCE_\alpha(X + Y)$. □

2.2 Quantiles of vector-valued risks

Given a real-valued random variable $\mathcal{X}$, and a confidence level $\alpha \in (0,1)$, the $\alpha\%$ worst realisations of $\mathcal{X}$, situated at the left tail of its distribution, are described by the lower and upper α quantiles defined by

$$q_\alpha(\mathcal{X}) := \inf\{\xi \in \mathbb{R} :\ \mathbb{P}(\mathcal{X} \leq \xi) \geq \alpha\} \quad \text{and} \quad q^\alpha(X) := \inf\{\xi \in \mathbb{R} :\ \mathbb{P}(X \leq \xi) > \alpha\}$$

The main difficulty in the multidimensional framework is that the vector ordering $\succeq$ is a partial ordering. Hence, the possible realizations of a d-dimensional portfolio $X \in L^\infty_d$ are not comparable and speaking about "worst realizations" does not make sense. In order to find an analogous, in this context, to tails of distribution, we consider subsets A of $\mathbb{R}^d$, which may contain non-comparable possible values of X, but satisfy the following requirement :

$$X(\omega) \in A \ \Rightarrow\ X(\omega) - K \subset A\ ,$$

i.e., if a realization $X(w)$ is among the set A, then all positions which are more risky then $X(w)$ are also contained in A.

Such subsets are eligible for defining the analogous of quantiles for random vectors. We introduce the set :

$$\mathcal{Q} := \left\{\ A \in \mathcal{B}(\mathbb{R}^d)\ :\ A - K = A\ \right\} \tag{2.3}$$

Definition 2.3 *Let X be in L^∞_d. The lower α-quantile of X is the set*

$$\mathcal{Q}_\alpha(X) := \{A \in \mathcal{Q} \ : \ \mathbb{P}(X \in A) \ \geq \ \alpha\}$$

and the upper α-quantile of X is the set

$$\mathcal{Q}^\alpha(X) := \{A \in \mathcal{Q} \ : \ \mathbb{P}(X \in A) \ > \ \alpha\}$$

2.3 Tail Conditional Expectation

Let X be in L^∞_d, and A be some set in $\mathcal{Q}_\alpha(X)$, the set :

$$\left\{ \, \eta \in \mathbb{R}^d \, , \ \ \mathbb{E}[\eta + X \,|\, X \in A] \ \succeq \ 0 \, \right\} \ \ = \ \ \{-\mathbb{E}\,[X \,|\, X \in A]\} \, + \, K$$

represents the set of deterministic portfolios η such that the conditional expected value of the modified position $X+\eta$, when X falls in the set A of "bad" scenarii, is admissible. An immediate proposal for the generalization to the vector-valued framework of TCE_α and TCE^α, which are defined for a real-valued random variable $\mathcal{X}$ by

$$TCE_\alpha(\mathcal{X}) \ := \ \mathbb{E}[-\mathcal{X}|\mathcal{X} \leq q_\alpha(\mathcal{X})] \quad \text{and} \quad TCE^\alpha(\mathcal{X}) \ := \ \mathbb{E}[-\mathcal{X}|\mathcal{X} \leq q^\alpha(\mathcal{X})]$$

is the following.

Definition 2.4 *Let X be in L^∞_d and $\alpha \in (0,1)$.*
The lower vector valued tail conditional expectation of X at level α is defined by

$$\begin{aligned} \mathcal{TCE}_\alpha(X) \ &:= \ \left\{ \, \eta \in \mathbb{R}^d \ : \ \forall A \in \mathcal{Q}_\alpha(X)\,, \mathbb{E}\,[\eta + X \,|\, X \in A] \succeq 0 \, \right\} \\ &= \bigcap_{A \in \mathcal{Q}_\alpha(X)} \left(\{-\mathbb{E}\,[X \,|\, X \in A]\} \, + \, K \right) \, . \end{aligned} \tag{2.4}$$

The upper vector valued tail conditional expectation of X at level α is defined by

$$\begin{aligned} \mathcal{TCE}^\alpha(X) \ &:= \ \left\{ \, \eta \in \mathbb{R}^d \ : \ \forall A \in \mathcal{Q}^\alpha(X)\,, \mathbb{E}\,[\eta + X \,|\, X \in A] \succeq 0 \, \right\} \\ &= \bigcap_{A \in \mathcal{Q}^\alpha(X)} \left(\{-\mathbb{E}\,[X \,|\, X \in A]\} \, + \, K \right) \, . \end{aligned} \tag{2.5}$$

Proposition 2.3 *Let α be in $(0,1)$. $\mathcal{TCE}_\alpha$ and $\mathcal{TCE}^\alpha$ define on L_d^∞ set valued maps satisfying axioms $A0$, $A1$, $A3$ and $A4$.*

In general $\mathcal{TCE}_\alpha$, resp. $\mathcal{TCE}^\alpha$, does not satisfy axiom $A2$, i.e. it is not subadditive, and hence is not a coherent risk measure.

2.4 Consistency with the one-dimensional framework

We end this Section by verifying that the definitions given above are consistent with the usual definitions of quantiles and Tail conditional expectation of real-valued risk. Clearly, in the one-dimensional context

$$\mathcal{Q} \;=\; \{(-\infty, a]\;,\; a \in \mathbb{R}\} \cup \{(-\infty, a)\;,\; a \in \mathbb{R}\}$$

Proposition 2.4 *For all $\mathcal{X}$ in L^∞, and α in $(0,1)$,*

$$q_\alpha(\mathcal{X}) \;=\; \sup \bigcap_{A \in \mathcal{Q}_\alpha(\mathcal{X})} A \quad \textit{and} \quad q^\alpha(\mathcal{X}) \;=\; \sup \bigcap_{A \in \mathcal{Q}^\alpha(\mathcal{X})} A$$

Proof. By the definitions of $q_\alpha(\mathcal{X})$, and $\mathcal{Q}_\alpha(\mathcal{X})$, for all $\varepsilon > 0$, and $A \in \mathcal{Q}_\alpha(\mathcal{X})$

$$(-\infty, q_\alpha(\mathcal{X}) - \varepsilon] \subset A\,, \quad \text{and} \quad (-\infty, q_\alpha(\mathcal{X}) + \varepsilon] \in \mathcal{Q}_\alpha(\mathcal{X})\;.$$

It follows that

$$(-\infty, q_\alpha(\mathcal{X})) = \bigcap_{\varepsilon>0} (-\infty, q_\alpha(\mathcal{X}) - \varepsilon] \subset \bigcap_{A \in \mathcal{Q}_\alpha(\mathcal{X})} A\,,$$

$$\text{and} \quad \bigcap_{A \in \mathcal{Q}_\alpha(\mathcal{X})} A \subset \bigcap_{\varepsilon>0} (-\infty, q_\alpha(\mathcal{X}) + \varepsilon] \;=\; (-\infty, q_\alpha(\mathcal{X})]\;,$$

hence : $q_\alpha(\mathcal{X}) = \sup \bigcap_{A \in \mathcal{Q}_\alpha(\mathcal{X})} A$.
Similarly, we have that $(-\infty, q^\alpha(\mathcal{X})) \subset \bigcap_{A \in \mathcal{Q}^\alpha(\mathcal{X})} A \subset (-\infty, q^\alpha(\mathcal{X})]$, hence : $q^\alpha(\mathcal{X})$ $= \sup \bigcap_{A \in \mathcal{Q}^\alpha(\mathcal{X})} A$. □

The following Proposition states the relation between the (real-valued) Tail conditional expectation, and the vector-valued Tail conditional expectation.

Proposition 2.5 *For all $\mathcal{X}$ in L^∞, and for all α in $(0,1)$*

$$TCE_\alpha(\mathcal{X}) \ = \ \min \ \mathcal{T}CE_\alpha(\mathcal{X}) \quad \textit{and} \quad TCE^\alpha(\mathcal{X}) \ = \ \min \ \mathcal{T}CE^\alpha(\mathcal{X})$$

Proof. $\mathcal{T}CE_\alpha(\mathcal{X})$ is given by

$$\mathcal{T}CE_\alpha(\mathcal{X}) \ = \ \bigcap_{A \in \mathcal{Q}_\alpha(\mathcal{X})} \left[-E\left[\mathcal{X} \,|\, \mathcal{X} \in A\right], \infty\right) \ ,$$

1. Notice that for all $n \geq 1$, $\left(-\infty, q_\alpha(\mathcal{X}) + \frac{1}{n}\right] \in \mathcal{Q}_\alpha(\mathcal{X})$, we deduce that for all $n \geq 1$

$$-\mathbb{E}\left[\mathcal{X} \,|\, \mathcal{X} \leq q_\alpha(\mathcal{X}) + \frac{1}{n}\right] \ \leq \ \min \mathcal{T}CE_\alpha(\mathcal{X})$$

By a dominated convergence argument the left hand side of the last inequality converges to $TCE_\alpha(\mathcal{X})$ as n goes to ∞, then $TCE_\alpha(\mathcal{X}) \leq \min \mathcal{T}CE_\alpha(\mathcal{X})$.

2. Notice that $\mathcal{Q}_\alpha(X) \subset \bar{\mathcal{Q}}_\alpha(\mathcal{X})$ where the set is given by

$$\bar{\mathcal{Q}}_\alpha(\mathcal{X}) \ := \ \left\{(-\infty, a], a \geq q_\alpha(X)\right\} \cup \left\{(-\infty, a), a > q_\alpha(X)\right\} \ .$$

Consequently, to verify that $TCE_\alpha(\mathcal{X}) \geq \min \mathcal{T}CE_\alpha(\mathcal{X})$, it is sufficient to check that for all $A \in \bar{\mathcal{Q}}_\alpha(\mathcal{X})$, $TCE_\alpha(X) \geq E\left[-X|X \in A\right]$.

2.1. Let $a \geq q_\alpha(\mathcal{X})$, set $p_\alpha := \mathbb{P}(\mathcal{X} \leq q_\alpha(\mathcal{X}))$, and $p_a := \mathbb{P}(\mathcal{X} \leq a)$. We have $p_a \geq p_\alpha$, and

$$\begin{aligned}
TCE_\alpha(\mathcal{X}) - \left(-E\left[\mathcal{X} \,|\, \mathcal{X} \leq a\right]\right) \ &= \ E\left[\mathcal{X} \,|\, \mathcal{X} \leq a\right] - E\left[\mathcal{X} \,|\, \mathcal{X} \leq q_\alpha(\mathcal{X})\right] \\
&= \ \frac{1}{p_\alpha p_a}\left\{p_\alpha \mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq a}\right] - p_a \mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_\alpha(\mathcal{X})}\right]\right\} \\
&= \ \frac{1}{p_\alpha p_a}\left\{p_\alpha \mathbb{E}\left[\mathcal{X}\mathbf{1}_{q_\alpha(\mathcal{X}) < \mathcal{X} \leq a}\right]\right. \\
&\qquad \left. - \left(p_a - p_\alpha\right) \mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_\alpha(\mathcal{X})}\right]\right\} \\
&\geq \ \frac{1}{p_a p_\alpha}\left(p_a - p_\alpha\right)\left\{p_\alpha \, q_\alpha(\mathcal{X}) - \mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_\alpha(\mathcal{X})}\right]\right\} \\
&\geq \ 0 \ .
\end{aligned}$$

2.2 A similar computation shows that if $a > q_\alpha(X)$, then

$$TCE_\alpha(\mathcal{X}) - (-E\,[\mathcal{X} \,|\, \mathcal{X} \leq a]) \;\; \geq \;\; 0\,.$$

We prove similarly that $TCE^\alpha(\mathcal{X}) = \min \mathcal{TCE}^\alpha(\mathcal{X})$. $\square$

3 Coherency of Tail Conditional Expectation

In general, the (real valued) tail conditional expectation is not subadditive, hence does not define a coherent risk measure. In the previous literature equality between the (real-valued) tail conditional expectation and the coherent (real-valued) worst conditional expectation has been established under two sets of assumptions.

Proposition 3.1 *[2] Assume that Ω is finite and that the probability on Ω is uniform. If X is a risk such that no two values of X in different states are ever equal, then $TCE_\alpha(X) = WCE_\alpha(X)$*

We refer the reader to [2] for the proof of this result.

Proposition 3.2 *[3] Let $\alpha \in (0,1)$ and X a real valued random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X^-] < \infty$. Then :*

$$\left\{ \begin{array}{l} \mathbb{P}(X < q_\alpha(X)) > 0 \quad \textit{and} \quad \mathbb{P}(X \leq q^\alpha(X)) = \alpha \\ \qquad\qquad\qquad\qquad \textit{or} \\ \mathbb{P}(X < q_\alpha(X)) = 0 \quad \textit{and} \quad \mathbb{P}(X = q^\alpha(X)) = 0 \end{array} \right.$$
$$\textit{iff} \;\; WCE_\alpha(X) = TCE_\alpha(X) = TCE^\alpha(X)\,. \tag{3.1}$$

In particular, (3.1) *holds if the distribution of X is continuous.*

We refer the reader to [3].

The main results of this paper extends Proposition 3.1 to the multidimensional framework and state that, in the case of a general probability space, if the distribution of X is continuous, and the condition $\mathbb{P}[X \in X(B)] = \mathbb{P}[B]$, for all B in $\mathcal{F}$ holds, then

$$\mathcal{W}CE_\alpha(X) = \mathcal{T}CE_\alpha(X) \ .$$

The inclusion $\mathcal{W}CE_\alpha(X) \subset \mathcal{T}CE_\alpha(X)$ is straightforward. The following paragraphs are dedicated to the proof of the second inclusion.

3.1 Finite probability space

Theorem 3.1 *Assume that Ω is finite and that the probability on Ω is uniform. If X is a risk such that no two values of X in different states are ever equal, then $\mathcal{W}CE_\alpha(X) = \mathcal{T}CE_\alpha(X)$.*

The proof of this Proposition is based on the following Proposition .

Proposition 3.3 *Assume that Ω is finite and that the probability on Ω is uniform. Let X be a risk such that no two values of X in different states are ever equal. Then, for all B in $\mathcal{F}$, there exists $A \subset \mathbb{R}^d$ such that:*

$$\begin{array}{ll} \textit{(i)} & A \in \mathcal{Q}_\alpha(X) \\ \textit{(ii)} & \mathbb{P}[X \in A] = \mathbb{P}[B] \\ \textit{(iii)} & \mathbb{E}[X \mid B] \succeq \mathbb{E}[X \mid X \in A] . \end{array}$$

In the proof of Proposition 3.3, we are going to use the notation :

$$\mathcal{E}(D) := (D - K) \setminus D \ , \quad D \subset \mathbb{R}^d \ . \tag{3.2}$$

The set $\mathcal{E}(D)$ is such that :

1. if $d \in D$, $e \notin D$, and $d \succeq e$, then $e \in \mathcal{E}(D)$,
2. if $e \in \mathcal{E}(D)$ then there exists some $d \in D$ such that $d \succeq e$.

We can qualify the subset $\mathcal{E}(D)$ as the "bad positions induced by D" and not contained in D.

Proof of Proposition 3.3 Let B be in $\mathcal{F}$. Notice that, since $X(\omega) \neq X(\omega')$ whenever $\omega \neq \omega'$,

$$\mathbb{P}(B) \;=\; \mathbb{P}\left(X \in X(B)\right) \;, \quad \text{and} \quad \mathbb{E}\left[X\mathbf{1}_B\right] \;=\; \mathbb{E}\left[X\mathbf{1}_{X\in X(B)}\right] \;.$$

If $\mathbb{P}\left[X \in \mathcal{E}(X(B))\right] = 0$ then we verify immediately that the set $A := X(B) \cup \mathcal{E}(X(B))$ satisfies (i), (ii), and (iii).

Assume that $\mathbb{P}\left[X \in \mathcal{E}(X(B))\right] \neq 0$. The idea is to replace elements from $X(B)$ by elements from $\mathcal{E}(X(B))$ in a convenient way, in order to obtain a set A with the desired properties.

Set $N := Card(\Omega)$. Then

$$\mathbf{p} \;:=\; \mathbb{P}\left[X = X(\omega)\right] \;=\; \frac{1}{N} \;, \quad \text{for all} \quad \omega \in \Omega \;.$$

Consider the (finite) sequence $(A_n)_{0\leq n\leq N}$ of subsets of $\mathbb{R}^d$ defined as follows:

1. $A_0 := X(B)$,
2. for n, $0 \leq n < N$:

   – if $\mathbb{P}\left[X \in \mathcal{E}(A_n)\right] = 0$: we set $A_{n+1} := A_n$,

   – if $\mathbb{P}\left[X \in \mathcal{E}(A_n)\right] > 0$: $A_{n+1} := (A_n \setminus \{a_n\}) \cup \{e_n\}$, where (a_n, e_n) can be chosen such that

$$(a_n, e_n) \;\in\; A_n \times \mathcal{E}(A_n) \;\cap\; X(\Omega)^2 \;, \; a_n \succeq e_n \;, \quad \text{and } a_n \notin \mathcal{E}(A_n \setminus \{a_n\}) \;.$$

Indeed, since $\mathbb{P}\left[X \in \mathcal{E}(A_n)\right] > 0$, we can chose some $e_n \in \mathcal{E}(A_n) \cap X(\Omega)$. By definition of $\mathcal{E}(A_n)$, there exists some $a^0 \in A_n$ such that $e_n \in \{a^0\} - K$. Then, consider the sequence $(a^k)_{k\geq 0}$ defined by induction as follows

$$\begin{aligned} a^{k+1} &:= a \in \{a \in A_n \cap X(\Omega) \ : \ a \neq a^k \ , \ a \succeq a^k\} \text{ if this set is non-empty} \\ a^{k+1} &:= a^k \ \text{ if } \{a \in A_n \cap X(\Omega) \ : \ a \neq a^k \ , \ a \succeq a^k\} \ = \emptyset \ . \end{aligned}$$

Since $X(\Omega)$ is finite, it is easy to see that for some $k^* \geq 0$, $a^k = a^{k^*}$ for all $k \geq k^*$, and that $a_n := a^{k^*}$ satisfies : $a_n \succeq e_n$, and $a_n \notin \mathcal{E}(A_n \setminus \{a_n\})$.

In this procedure, while $\mathcal{E}(A_n)$ is nonempty, A_{n+1} is obtained by replacing an element of A_n by a worst position contained in $\mathcal{E}(A_n)$. Notice that the property $a_n \notin \mathcal{E}(A_n \setminus \{a_n\})$ implies that $a_n \notin A_{n+k}$ for all $k \geq 1$.

Since the probability $\mathbb{P}$ is uniform, the sequence (A_n) is such that $\mathbb{P}[X \in A_n] = \mathbb{P}[X \in X(B)]$ for $n = 0, ..., N$, in particular

$$\mathbb{P}(B) \ = \ P[X \ \in \ A_N] \ .$$

We also verify for $0 \leq n < N$ that :

$$\begin{aligned} \mathbb{E}\left[X\mathbf{1}_{X\in A_{n+1}}\right] &= \mathbb{E}\left[X\mathbf{1}_{X\in A_n}\right] \quad \text{or} \\ \mathbb{E}\left[X\mathbf{1}_{X\in A_{n+1}}\right] &= \mathbb{E}\left[X\mathbf{1}_{X\in A_n}\right] - \mathbf{p}\,(a_n - e_n) \quad \text{where} \ \ a_n \succeq e_n \ . \end{aligned}$$

Hence $\mathbb{E}\left[X\mathbf{1}_{X\in A_n}\right] \succeq \mathbb{E}\left[X\mathbf{1}_{X\in A_{n+1}}\right]$. In particular

$$\mathbb{E}\left[X\mathbf{1}_B\right] \ = \ \mathbb{E}\left[X\mathbf{1}_{X\in X(B)}\right] \ \succeq \ \mathbb{E}\left[X\mathbf{1}_{X\in A_N}\right] \ .$$

It remains to verify that $\mathbb{P}[X \in \mathcal{E}(A_N)] = 0$ to conclude that $\tilde{A}_N := A_N \cup \mathcal{E}(A_N)$ satisfies (i), (ii), and (iii).

Assume to the contrary that $\mathbb{P}[X \in \mathcal{E}(A_N)] \neq 0$. Recall that, by construction, the sequence (A_n), for each $n \geq 0$, $a_n \notin A_{n+k}$, for all $k \geq 1$. Hence $\{a_0, ..., a_N\}$ is a set of $N+1$ distinct elements from $X(\Omega)$ which contradicts the fact that $Card(\Omega) = N$. □

Proof of Theorem ?? 1. Since $\{X^{-1}(A)\,, A\in \mathcal{Q}_\alpha(X)\}\subset\{B\in\mathcal{F}\,,\mathbb{P}(B)\,\geq\,\alpha\}$, we have that $\mathcal{W}CE_\alpha(X)\subset\mathcal{T}CE_\alpha(X)$.

2. To get the reverse inclusion, we have to show that for all x in $\mathcal{T}CE_\alpha(X)$, and for all B in $\mathcal{F}$ with $\mathbb{P}(B)\geq\alpha$, $\mathbb{E}\,[x+X\,|\,B]\succeq 0$.
Let x be in $\mathcal{T}CE_\alpha(X)$, and B in $\mathcal{F}$ with $\mathbb{P}(B)\geq\alpha$. From Proposition 3.3, there exists a subset A of $\mathbb{R}^d$ such that $A-K=A$, A closed, $\mathbb{P}(A)=\mathbb{P}(B)$, and $\mathbb{E}\,[X\mathbf{1}_B]$ $\succeq\mathbb{E}\,[X\mathbf{1}_{X\in A}]$. In particular :

$$\begin{aligned}
&\mathbb{E}\,[X\,|\,B]-\mathbb{E}\,[X\,|\,X\in A] \;=\; \frac{1}{\mathbb{P}(B)}\left(\mathbb{E}\,[X\mathbf{1}_B]-\mathbb{E}\,[X\mathbf{1}_{X\in A}]\right)\;\succeq\; 0\;,\\
&A\;\in\;\mathcal{Q}_\alpha(X)\;\;\text{hence}\;\;\mathbb{E}\,[x+X\,|\,X\in A]\;\succeq\;0\;.
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathbb{E}\,[x+X\,|\,B] \;&=\; \mathbb{E}\,[x+X\,|\,X\in A]+\left(\mathbb{E}\,[X\,|\,B]-\mathbb{E}\,[X\,|\,X\in A]\right)\\
&=\; \mathbb{E}\,[x+X\,|\,X\in A]+\frac{1}{\mathbb{P}(B)}\left(\mathbb{E}\,[X\mathbf{1}_B]-\mathbb{E}\,[X\mathbf{1}_{X\in A}]\right)\;\succeq\;0\,.
\end{aligned}$$

□

3.2 General probability space

Theorem 3.2 *Let X be a risk in L^∞_d having a continuous probabiliy density f. Then*

$$\mathcal{G}WCE_\alpha(X)=\mathcal{T}CE_\alpha(X)$$

The proof of this Theorem relies on the following Proposition.

Proposition 3.4 *Let X be in L^∞_d. Assume that X has a continuous probability density f. Then , for all B in $\mathcal{F}^x:=\sigma(X)$, there exists $A\subset\mathbb{R}^d$ such that:*

$$\begin{aligned}
&\textit{(i)}\quad A\;\in\;\mathcal{Q}_\alpha(X)\\
&\textit{(ii)}\quad \mathbb{P}\,[X\in A]\;=\;\mathbb{P}\,[B]\\
&\textit{(iii)}\quad \mathbb{E}\,[X\,|\,B]\;\succeq\;\mathbb{E}\,[X\,|\,X\in A]\;\;.
\end{aligned}$$

We defer the proof of the Proposition 3.4, and start by proving Theorem 3.2

Proof of Theorem 3.2 1. Since $\{X^{-1}(A)\,, A \in \mathcal{Q}_\alpha(X)\} \subset \{B \in \mathcal{F}\,, P(B) \geq \alpha\}$, it is easy to check that $\mathcal{G}WCE_\alpha(X) \subset \mathcal{T}CE_\alpha(X)$.

2. To get the reverse inclusion, we are going to show that for all x in $\mathcal{T}CE_\alpha(X)$, and for all B in $\mathcal{F}^x = \sigma(X)$ with $\mathbb{P}(B) \geq \alpha$, $\mathbb{E}\,[x + X \,|\, B] \succeq 0$. Indeed this implies that

$$\mathcal{T}CE_\alpha(X) \subset \mathcal{W}CE_{(\Omega,\mathcal{F}^x,\mathbb{P}),\alpha}(\tilde{X}) \subset \mathcal{G}WCE_\alpha(X)$$

Let x be in $\mathcal{T}CE_\alpha(X)$, and B in $\mathcal{F}^x$ with $\mathbb{P}(B) \geq \alpha$. Proposition 3.4 shows that there exists a subset A of $\mathbb{R}^d$ such that $A - K = A$, A closed, $\mathbb{P}(A) = \mathbb{P}(B)$, and $\mathbb{E}\,[X\mathbf{1}_B]$ $\succeq \mathbb{E}\,[X\mathbf{1}_{X\in A}]$. In particular :

$$\begin{array}{l} \mathbb{E}\,[X \,|\, B] - \mathbb{E}\,[X \,|\, X \in A] \;=\; \dfrac{1}{\mathbb{P}(B)}\,(\mathbb{E}\,[X\mathbf{1}_B] - \mathbb{E}\,[X\mathbf{1}_{X\in A}]) \;\succeq\; 0\,, \\ A \;\in\; \mathcal{Q}_\alpha(X) \;\text{ hence }\; \mathbb{E}\,[x + X \,|\, X \in A] \;\succeq\; 0\,. \end{array}$$

Consequently

$$\begin{aligned} \mathbb{E}\,[x + X \,|\, B] &= \mathbb{E}\,[x + X \,|\, X \in A] + (\mathbb{E}\,[X \,|\, B] - \mathbb{E}\,[X \,|\, X \in A]) \\ &= \mathbb{E}\,[x + X \,|\, X \in A] + \frac{1}{\mathbb{P}(B)}\,(\mathbb{E}\,[X\mathbf{1}_B] - \mathbb{E}\,[X\mathbf{1}_{X\in A}]) \;\succeq\; 0\,. \end{aligned}$$

□

We now turn to the proof of Proposition 3.4. The proof consists in two steps. The first one is

Lemma 3.1 *We introduce the set $\mathcal{S}$ of pairs of subsets (δ, β) such that:*

1. *β and δ are open subsets of $\mathbb{R}^d$,*

2. *$\delta \cap \beta = \emptyset$,*

3. *$\delta \cap [(D \setminus \delta) - K] = \emptyset$,*

4. $\beta - K \ = \ \beta$*,*

5. $d \succeq b$ *for all* (d, b) *in* $\delta \times \beta$ *,*

6. $\mathbb{P}_X(\beta \cap \mathcal{E}(D)) = \mathbb{P}_X(\delta \cap D)$ *.*

Elements of $\mathcal{S}$ *are ordered as follows :*

$$(\delta, \beta) \propto (\bar{\delta}, \bar{\beta}) \quad \textit{iff} \ \ \delta \subset \bar{\delta} \ \textit{and} \ \beta \subset \bar{\beta} \ . \tag{3.3}$$

Then $\mathcal{S}$ *admits a maximal element w.r.t the order relation* $\propto$*.*

Proof. Notice that each totally ordered finite sequence $\{(\delta_{i_0}, \beta_{i_0}), ..., (\delta_{i_N}, \beta_{i_N})\}$ of elements of $\mathcal{S}$, admits a maximum. We shall denote by $m(i_0, ..., i_N)$ the index of the maximum of $\{(\delta_{i_0}, \beta_{i_0}), ..., (\delta_{i_N}, \beta_{i_N})\}$.

Let $(\delta_i, \beta_i)_{i \in I}$ be a totally ordered family of elements of $\mathcal{S}$. we claim that $(\delta_i, \beta_i)_{i \in I}$ has an upperbound .

$$\text{Set} \ \ \delta_\infty \ := \bigcup_{i \in I} \delta_i \ \ \text{and} \ \ \beta_\infty \ := \bigcup_{i \in I} \beta_i \ .$$

It is clear that $(\delta_\infty, \beta_\infty)$ is an upperbound for $(\delta_i, \beta_i)_{i \in I}$, provided it is contained in $\mathcal{S}$. This is what we proof through the Steps 1-6.

1. δ_∞ and β_∞ are open subsets of $\mathbb{R}^d$.

2. By definition of δ_∞, β_∞, the order relation $\propto$, and using property (2) of the elements of $\mathcal{S}$:

$$\delta_\infty \cap \beta_\infty \ \subset \ \bigcup_{i,j \in I} \delta_i \cap \beta_j \ \subset \ \bigcup_{i,j \in I} \delta_{m(i,j)} \cap \beta_{m(i,j)} \ = \ \bigcup_{i,j \in I} \emptyset \ = \ \emptyset \ .$$

3. For all i in I, $(D \setminus \delta_\infty) \subset (D \setminus \delta_i)$, then

$$\begin{aligned} \delta_\infty \cap [(D \setminus \delta_\infty) - K] \ &= \ \bigcup_{i \in I} \delta_i \cap [(D \setminus \delta_\infty) - K] \\ &\subset \ \bigcup_{i \in I} \delta_i \cap [(D \setminus \delta_i) - K] \ = \ \bigcup_{i \in I} \emptyset \ = \ \emptyset \ . \end{aligned}$$

4. $\beta_\infty - K = \bigcup_{i\in I} \beta_i - K = \bigcup_{i\in I} \beta_i = \beta_\infty$.

5. Let d be in δ_∞, and b be in β_∞. Let i , j be in I such that $d \in \delta_i$, and $b \in \beta_j$. Then $(d,b) \in \delta_{m(i,j)} \times \beta_{m(i,j)}$, and it follows from property (5) of elements of $\mathcal{S}$ that $d \succeq b$.
Consequently, for all (d,b) in $\delta_\infty \times \beta_\infty$, $d \succeq b$.

6. It remains to show that $\mathbb{P}[X \in \beta_\infty \cap \mathcal{E}(D)] = \mathbb{P}[X \in \delta_\infty \cap D]$ to conclude that $(\delta_\infty, \beta_\infty)$ is an element of $\mathcal{S}$. In order to establish this property we use the following Lemma which proof is reported later on.

Lemma 3.2 *Let G and H be two subsets of $\mathbb{R}^d$, then*

$$\mathbb{P}(X \in G \cap H) = \sup_{C \in \mathcal{C}(G)} \mathbb{P}(X \in C \cap H) ,$$

where $\mathcal{C}(G)$ denotes the set of all compact subsets of G.

By Lemma (3.2) : $\mathbb{P}[X \in \delta_\infty \cap D] = \sup_{C\in\mathcal{C}(\delta_\infty)} \mathbb{P}[X \in C \cap D]$.
Let C be a compact subset of $\cup_{i\in I}\delta_i$ $\mathcal{C}(\delta_\infty)$. Since δ_i, $i \in I$ are open subsets of $\mathbb{R}^d$, there exists a finite sequence $\{\delta_{i_1}, ..., \delta_{i_N}\}$ such that $C \subset \cup_{k=1}^N \delta_{i_k}$.
The finite sequence $\{(\delta_{i_1}, \beta_{i_1}), ..., (\delta_{i_N}, \beta_{i_N})\}$ is totally ordered in $\mathcal{S}$, hence it admits a maximum. Let $i_0 := m(i_1, ..., i_N)$ be the index of its maximum. Then by definition of the order relation $\propto$, and by property (6) of the elements of $\mathcal{S}$:

$$\cup_{k=1}^N \delta_{i_k} \subset \delta_{i_0} \quad \text{and} \quad \mathbb{P}[X \in \delta_{i_0} \cap D] = \mathbb{P}[X \in \beta_{i_0} \cap \mathcal{E}(D)]$$

it follows:

$$\mathbb{P}[X \in C \cap D] \leq \mathbb{P}[X \in \delta_{i_0} \cap D] = \mathbb{P}[X \in \beta_{i_0} \cap \mathcal{E}(D)] \leq \mathbb{P}[X \in \beta_\infty \cap \mathcal{E}(D)] .$$

Taking the supremum over $\mathcal{C}(\delta_\infty)$, we get : $\mathbb{P}[X \in \delta_\infty \cap D] \leq \mathbb{P}[X \in \beta_\infty \cap \mathcal{E}(D)]$.

A similar argument shows that the reverse inequality holds. Then

$$\mathbb{P}\left[X \in \delta_\infty \cap D\right] \;=\; \mathbb{P}\left[X \in \beta_\infty \cap \mathcal{E}(D)\right] \ .$$

In $\mathcal{S}$, each totally ordered family admits an upperbound, then, by Zorn's Lemma, $\mathcal{S}$ admits a maximal element. $\square$

Proof of Lemma 3.2 Since the distribution of X is absolutely continuous w.r.t to the Lebesgue measure, for each set S in $\mathbb{R}^d$

$$\mathbb{P}\left(X \in S\right) \;=\; \sup_{C \in \mathcal{C}(S)} \mathbb{P}\left(X \in C\right) \ ,$$

where $\mathcal{C}(S)$ denotes the set of all compact subsets of S (see [**?**]).
Let C be in $\mathcal{C}(G \cap H)$, then C is in $\mathcal{C}(G)$, it follows that

$$\mathbb{P}\left(X \in G \cap H\right) \;=\; \sup_{C \in \mathcal{C}(G \cap H)} \mathbb{P}\left(X \in C \cap H\right) \;\leq\; \sup_{C \in \mathcal{C}(G)} \mathbb{P}\left(X \in C \cap H\right) \ .$$

The reverse inequality is straightforward, since for all C in $\mathcal{C}(G)$, $C \cap H \subset G \cap H$, hence $\mathbb{P}\left(X \in C \cap H\right) \leq \mathbb{P}\left(X \in G \cap H\right)$. $\square$

Now we are ready for the proof of the second step of Proposition 3.4.

Proof of Proposition 3.4

Let B be in $\mathcal{F}$, and set $D := X(B)$. X is such that

$$\mathbb{P}(B) \;=\; \mathbb{P}(X \in D) \quad \text{and} \quad \mathbb{E}\left[X \mathbf{1}_B\right] \;=\; \mathbb{E}\left[X \mathbf{1}_{X \in D}\right] \ .$$

Assume that $\mathbb{P}\left(X \in \mathcal{E}(D)\right) = 0$. Then, clearly the set $A := D \cup \mathcal{E}(D) = D - K$ satisfies each of the requirement (i), (ii), and (iii).
Now we concentrate on the case where $\mathbb{P}\left(X \in Ec(D)\right) > 0$. We are going to prove that it is possible to obtain a subset A with the required properties by substituting elements from D by "worst" elements w.r.t to the preoerder $\succeq$, i.e. elements taken from $\mathcal{E}(D)$.

Lemma 3.1 states that the set $\mathcal{S}$ admits a maximal element. Let $(\delta^\star, \beta^\star)$ be such an element.

1. If $\mathbb{P}[X \in D \setminus \delta^\star] = 0$, then the set $A := \beta^\star \cap \mathcal{E}(D)$ satisfies (i), (ii), and (iii). Indeed, by definition of $\mathcal{E}(D)$, $\mathcal{E}(D) - K = \mathcal{E}(D)$. The property (4) of elements of $\mathcal{S}$ implies $\beta^\star - K = \beta^\star$. Hence

$$A - K \ = \ [\beta^\star \cap \mathcal{E}(D)] - K \ = \ \beta^\star \cap \mathcal{E}(D) \ = \ A \ .$$

Using $\mathbb{P}[X \in \ D \setminus \delta^\star] = 0$, and the property (6) of elements on $\mathcal{S}$

$$\mathbb{P}[X \in A] \ = \ \mathbb{P}[X \in \beta^\star \cap \mathcal{E}(D)] \ = \ \mathbb{P}[X \in D \cap \delta^\star] \ = \ \mathbb{P}[X \in D] \ .$$

Since $P[X \in D] = \mathbb{P}(B) \geq \alpha$, it follows that

$$A \ \in \ \mathcal{Q}_\alpha(X) \quad \text{and} \quad \mathbb{P}(X \in A) \ = \ P(B)$$

Using $\mathbb{P}[D \setminus \delta^\star] = 0$, and $p := \mathbb{P}[X \in \delta^\star \cap D] = \mathbb{P}[X \in \beta^\star \cap \mathcal{E}(D)]$

$$\begin{aligned}
& \mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X\in A}] \\
= \ & \mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X\in A}] \\
= \ & \mathbb{E}[X\mathbf{1}_{X\in D\cap\delta^\star}] - \mathbb{E}\left[X\mathbf{1}_{X\in\beta^\star\cap\mathcal{E}(D)}\right] \\
= \ & \frac{1}{p}\int\int (X(\omega) - X(\omega'))\, \mathbf{1}_{X\in D\cap\delta^\star}(\omega)\mathbf{1}_{X\in\beta^\star\cap\mathcal{E}(D)}(\omega')dP(\omega)dP(\omega') \\
\succeq \ & 0
\end{aligned}$$

where the last inequality follows from property (5) of elements of $\mathcal{S}$.

2. Now we consider the case where $\mathbb{P}[X \in D \setminus \delta^\star] > 0$, and we set

$$D^\star \ := \ (D \setminus \delta^\star) \cup (\mathcal{E}(D) \cap \beta^\star) \ .$$

We shall prove that in this case $\mathbb{P}[X \in \mathcal{E}(D^\star)] \ = \ 0$, it then follows that the set $A \ := \ D^\star \cup \mathcal{E}(D^\star)$ satisfies (i), (ii) and (iii).

Assume to the contrary that $\mathbb{P}\,[X \in \mathcal{E}(D^\star)] > 0$. We are going to show that, then, it is possible to find $(\bar{\delta}, \bar{\beta})$ in $\mathcal{S}$ such that $(\delta^\star, \beta^\star) \propto (\bar{\delta}, \bar{\beta})$, which is in contradiction with the fact that $(\delta^\star, \beta^\star)$ is a maximal element.

2.1. Notice that since f is contiuous, and the cone K contains $\mathbb{R}^d_+$, then $\mathbb{P}\,[X \in D \setminus \delta^*]$ implies that there exists some $a \in \mathcal{E}(D^\star) \cap int\,[D \setminus \delta^\star - K]$, with $f(a) > 0$.
Let $r > 0$ such that $B(a, r) \subset int\,[D \setminus \delta^\star - K]$, and $z \in B(a, r) \cap int[a + K]$, b in $\in D \setminus \delta^\star$ such that $z = b - k$ for some k in K. Then $u := b - a$ is in $int[K]$.
2.2. Now consider the function Ψ defined on $\mathbb{R}$ by :

$$\Psi(t) \;=\; \mathbb{P}\,(X \in [(a + t.u) + K] \cap [D \setminus \delta^\star]) - \mathbb{P}\,[X \in [(a + t.u) - K] \cap \mathcal{E}(D \setminus \delta^\star)]$$

We verify that

$$\begin{cases} \lim_{t \to +\infty} \Psi(t) = -\mathbb{P}\,(X \in \mathcal{E}(D \setminus \delta^\star)) \;<\; 0 \\ \lim_{t \to -\infty} \Psi(t) = \mathbb{P}\,(X \in D \setminus \delta^\star)) \;>\; 0 \end{cases}$$

In fact, since a is in $b + int(K)$, there exists some $\eta > 0$ such that $B(a, \eta) \subset b + int(K)$. Let $\lambda := \frac{3\|X\|_\infty}{\eta}$, then

$$B(a', 3\|X\|_\infty) \;\subset\; b + int(K)\;, \quad \text{with} \quad a' := b + \lambda(a - b)\;.$$

By translation: $b + (a - a') - K$, which is equal to $a + \lambda(b - a) - K$ contains $B(a', 3\|X\|_\infty)$. We deduce that

$$\forall \lambda' > \lambda, B(a', 3\|X\|_\infty) \subset a + \lambda(b - a) - K\;.$$

It follows that

$$\forall \lambda' > \lambda\,,\; \mathbb{P}\,[X \in [(a + \lambda'.u) + K] \cap [D \setminus \delta^\star]] = 0$$
$$\text{and}\;\; \mathbb{P}\,[X \in [(a + \lambda'.u) - K] \cap \mathcal{E}(D \setminus \delta^\star)] = \mathbb{P}\,[X \in \mathcal{E}(D \setminus \delta^\star)]\;.$$

Then

$$
\begin{aligned}
\lim_{t\to\infty}\Psi(t) &= \lim_{t\to\infty}\mathbb{P}\left[X\in[(a+\lambda'.u)+K]\cap[D\setminus\delta^\star]\right] \\
&\qquad -\mathbb{P}\left[X\in[(a+\lambda'.u)-K]\cap\mathcal{E}(D\setminus\delta^\star)\right] \\
&= -\mathbb{P}\left[X\in\mathcal{E}(D\setminus\delta^\star)\right]
\end{aligned}
$$

Similar arguments provide the limit at $-\infty$.
We state the following Lemma which is proved later on.

Lemma 3.3 *The function* Ψ *is continuous.*

Since Ψ is continuous, there exists $t^\star$ such that $\Psi(t^\star)=0$.
Consider the subsets $[(a+t^\star.u)-int(K)]$ and $[(a+t^\star.u)+int(K)]$, they are open, disjoint and their union is nonempty as it contains either a or b. We obtain by setting

$$
\bar{\delta}:=\delta^\star\cup[(a+t^\star.u)+int(K)] \quad \text{and} \quad \bar{\beta}:=\beta^\star\cup[(a+t^\star.u)-int(K)]
$$

an element $(\bar{\delta},\bar{\beta})$ of $\mathcal{S}$ which majorates strictly $(\delta^\star,\beta^\star)$. This is in contradiction with the fact that $(\delta^\star,\beta^\star)$ is maximal.
We then conclude that if $\mathbb{P}[X\in D\setminus\delta^\star]>0$ then $\mathcal{E}(D^\star)$ is $\mathbb{P}_X$-null, which ends the proof. $\square$

Proof of Lemma 3.3. It is sufficient to prove that for any subset A and any closed convex cone K with nonempty interior, the mapping $\Psi_1: t\mapsto\mathbb{P}_X(A\cap[(a+t.u)+K])$ is continuous; then Ψ is continuous as the sum of two continuous applications.
Notice that

$$
K(t'):=(a+t')+K\subset(a+t.u)-K=:K(t) \quad \text{for} \quad t\le t'\in\mathbb{R}\,.
$$

Then:

$$\begin{aligned} |\Psi(t)+\Psi(t')| &= \Psi(t)-\Psi(t') \\ &= \mathbb{P}\left[X\in A\cap (K(t)\setminus K(t'))\right] \\ &\leq \|f\|_{B(0,\|X\|_\infty)} L(B(0,\|X\|_\infty)\cap (K(t)\setminus K(t'))) \end{aligned}$$

where f is the density of X and L denotes the Lebesgue measure on $\mathbb{R}^d$

The result then follows from the continuity of the Lebesgue measure on the space of bounded convex bodies equipped with the Pompeiu-Hausdorff-Blaschke metric (see [10]). $\square$

Remark 3.1 Let X be a risk in L_d^∞ having a continuous probabiliy density f. Then

$$\mathcal{TCE}_\alpha(X) \;=\; \bigcap_{A\in\mathcal{Q}_\alpha(X), A \text{ closed}} -\{\mathbb{E}\left[X|X\in A\right]\}+K\ .$$

Indeed, since f is continuous for each borel set A contained in $\mathcal{Q}_\alpha(X)$, for each $n\geq 1$ there exists some closed set A_n such that $A_n\subset A$ and $\mathbb{P}(X\in A_n)\geq \mathbb{P}(X\in A)-1/n$. Then the sequence $\left(\bar{A}_n := A_n - K\right)$ is a sequence of closed subsets contained in $Q_\alpha(X)$ and satisfying $\mathbb{E}\left[X|X\in\bar{A}_n\right]\xrightarrow[n\to\infty]{}\mathbb{E}\left[X|X\in A\right]$, and the required result follows from the closedness of $\mathcal{TCE}_\alpha(X)$.

4 Numerical computation of $\mathcal{TCE}$

4.1 The discrete distribution case

In the case of a discrete distribution, the $\mathcal{TCE}$ can be computed through a rather simple algorithm since the Quantile set is a finite collection of finite subsets of $\mathbb{R}^d$. The main idea of the algorihm is to compute recursively the quantiles of the distribution :

- assume that the random vector X has N realisations denoted by x_i, $i=1,...,N$,

- and denote by $\mathbb{Q}_alpha$ the list of α quantiles, inizialized to an empty list

$Algorithm: \ \ \alpha - QuantilesSet$

Begin

var List A ;

var integer i, c ;

$for\ i = 1\ to\ N\ \ :$

$for\ c = 1\ to\ N - i + 1\ \ :\ \ A\ := \ empty\ list$;

$Quantile(A, i, c)$;

end

where

$Algorithm: \ \ Quantile(A, i, c)$

Begin

var integer j ;

$A\ :=\ A + \{x_i\}$;

$if\ (c > 0) then$

$for\ j = i + 1\ to\ N\ \ :\ \ Quantile(A, j, c - 1)$;

$else$

$if\ (\mathbb{P}(X \in A) \geq \alpha)\ then\ \ Q\ :=\ Q + \{A\}$;

$A\ :=\ A -$ (last element in A);

$end\ if$

$A\ :=\ A -$ (last element in A);

end

Example 4.1 We consider the example of an investor having access to two financial markets indexed by i, $i = 1, 2$. Each maket i consists of a non risky asset B^i, and a risky asset S^i, and the currency in markets i is denoted by m_i. For each i, transfers between the B^i's account and the S^i's account are free of charges, hence the position of the investor in the market i can be described by a single account X^i. However transfers between the two markts are subject to proportional transaction costs. Taking the currency m^1 as reference, when the investor transfers 1 m_1 from account i, he receives $1 - \lambda^{ij}$ m_1 in the account j.
Let $\Omega := \{\omega_k, 1 \leq k \leq 10\}$, and $\mathbb{P}(\omega_k) = \frac{1}{10}$ for $k = 1, ..., 10$. Denote by $X = (X^1, X^2)$ the random vector representing the Profit&Loss (P&L) (expressed in m^1) at a future date T of some investor's position. The realisations of X are given in the following table.

	X^1	X^2
ω_1	35	10
ω_2	20	5
ω_3	25	30
ω_4	10	20
ω_5	15	-10

	X^1	X^2
ω_6	5	0
ω_7	8	-2
ω_8	-8	8
ω_9	-10	6
ω_{10}	-15	-10

Figures 4.1, 4.1, 1 and 1, report $\mathcal{TCE}_\alpha(X)$ for diffrent values of α. On each of these figures, $\mathcal{TCE}_\alpha(X)$ corresponds to the hatched rectangle at the top right corner. It is obtained as the intersection of the sets $\{-\mathbb{E}[X|X \in A]\} + K$, $A \in \mathcal{Q}_\alpha(X)$. We also reported on these figures the realizaions of X : $\circ$, and the points $\{-\mathbb{E}[X|X \in A]\}, A \in \mathcal{Q}_\alpha(X)$: $\circ$.

Application : In order to secure his position the investor can procede as follows

- choose $\bar{x} = (\bar{x}^1, \bar{x}^2)$ in $\mathcal{TCE}_\alpha(X)$,

- place $(x^1 + x^2 * K)e^{-r^1 T}\ m_1$ in B^1,

- buy x^2 unities of the exchange options paying 1 m_2 for $K\ m_1$.

The resulting $P\&L$ of modified position at the end of the investment period is equal to $X + \bar{x}$, and satisfies, according to the defition of $\mathcal{TCE}_\alpha$: $\mathbb{E}\left[X + \bar{x} | X \in A\right] \succeq 0$ for all $A \in \mathcal{Q}_\alpha(X)$.

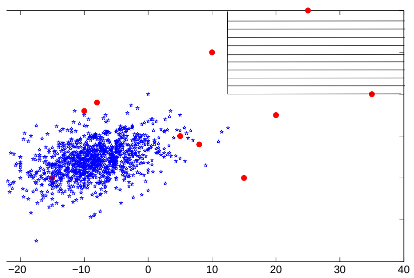

Figure 1: $\alpha = 2\%$

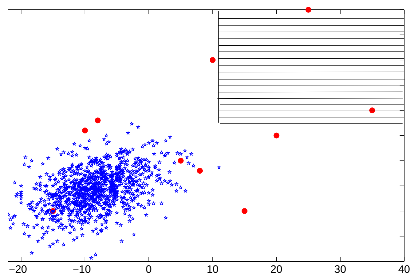

Figure 3: $\alpha = 3\%$

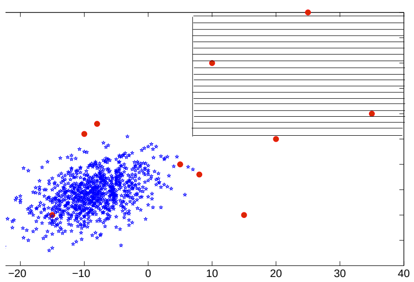

Figure 2: $\alpha = 4\%$

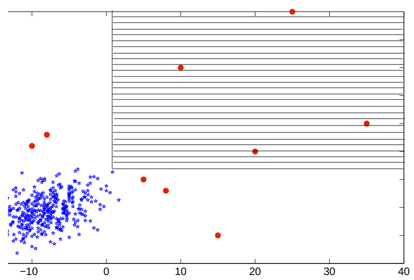

Figure 4: $\alpha = 6\%$

4.2 The continuous case : a numerical approximation

Let X be in L_d^∞, and $L > 0$ such that $\mathbb{P}\left(X \in \prod_{k=1}^{d}]-L, L[\right) = 1$. For each $n \geq 1$ we consider the grid

$$G_n \;:=\; \prod_{k=1}^{d} \left\{ -L + i\frac{2L}{n} \;,\; 0 \leq i \leq n \right\} \;,$$

then Ω can be partitioned into $\Omega = \cup_{\ell=1}^{n^d} A_\ell$, where

$$A_\ell \;=\; \left\{ X \in \prod_{k=1}^{d} [-L + (j_k - 1)\frac{2L}{n}, -L + j_k\frac{2L}{n}[\right\}$$
$$\text{for each} \;\; \ell = 1 + \textstyle\sum_{k=1}^{d} n^{k-1}(j_k - 1) \quad \text{with} \;\; 1 \leq j_1, ..., j_d \leq n \; .$$

For each $n \geq 1$, we define the random vector X_n by

$$X_n \;=\; \left(-L * (j_k - 1)\tfrac{2L}{n}, 1 \leq k \leq d\right)' \quad \text{on } A_\ell$$
$$\text{for each} \;\; 1 \leq j_1, ..., j_d \leq n \;, \quad \text{and} \;\; \ell = 1 + \textstyle\sum_{k=1}^{d} n^{k-1}(j_k - 1) \; .$$

For each $n \geq 1$, $\mathcal{T}CE_\alpha(X_n)$ can be computed by the algorithm described in the previous subsection. The following Proposition shows that the sequence $(\mathcal{T}CE_\alpha(X_n))$ allows to approximates elements of $\mathcal{T}CE_\alpha(X)$.

Proposition 4.1 *The sequence $(X_n)_{n\geq 1}$ converges to X in probability and satisfies*

$$\liminf_n \mathcal{T}CE_\alpha(X_n) \;:= \bigcup_n \bigcap_{k \geq n} \mathcal{T}CE_\alpha(X_k) \qquad \subset \mathcal{T}CE_\alpha(X)$$

Proof. By definition of the sequence (X_n), $\mathbb{P}\left(|X_n - X| > \frac{2}{n}\right) = 0$ for all $n \geq 1$, hence (X_n) converges to X in probability.
Let η be in $\liminf_n \mathcal{T}CE_\alpha(X_n)$. Observe that it is sufficient to prove that $\mathbb{E}\left[X | X \in A\right]$ is in K for all A in $\mathcal{Q}_\alpha(X)$ with A closed (see Remark 3.1).

Let A be a closed subset of $\mathbb{R}^d$ contained in $\mathcal{Q}_\alpha(X)$. By the definition of the sequence $(X_n)_{n\geq 1}$, $X_n(\omega) \preceq X(\omega)$ for all $n \geq 1$ and $\omega \in \Omega$. Consequently, $\mathbb{P}(X_n \in A) \geq P(X \in A)$, and A is in $\mathcal{Q}_\alpha(X_n)$ for all $n \geq 1$. Since $\eta \in \liminf_n \mathcal{T}CE_\alpha(X_n)$, we have for n sufficiently large

$$\eta + \mathbb{E}\left[X_n | X_n \in A\right] \in K$$

On the other hand

$$\mathbb{E}\left[X_n | X_n \in A\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[X | X \in A\right] \ .$$

Indeed, by the definition of the sequence (X_n), and the closedness of the set A we have that $\liminf_n \{X_n \in A\} = \{X \in A\}$, then

$$\lim_n \mathbb{P}(X_n \in A) \ = \ \mathbb{P}(\liminf_n \{X_n \in A\}) \ = \ P(X \in A) \ ,$$

and

$$\begin{aligned} \left|\mathbb{E}\left[X_n \mathbf{1}_{X_n \in A}\right] - \mathbb{E}\left[X \mathbf{1}_{X \in A}\right]\right| \ &\leq \ \mathbb{E}\left[|X_n - X|\right] + ||X||_\infty \left|\mathbb{P}(X_n \in A) - \mathbb{P}(X \in A)\right| \\ &\xrightarrow[n\to\infty]{} \ 0 \ . \end{aligned}$$

It follows from the closedness of the cone K, that $\eta + \mathbb{E}\left[X | X \in A\right]$ is in K. By arbitrariness of A in $\{A' \in \mathcal{Q}_\alpha(X), A' \ \textit{closed}\}$ we onclude to the required result. □

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