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Compactness in Spaces of Inner Regular Measures and a General Portmanteau Lemma

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Compactness in spaces of inner regular measures and a general Portmanteau lemma

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Abstract This paper may be understood as a continuation of Topsøe's seminal paper ([16]) to characterize, within an abstract setting, compact subsets of finite inner regular measures w.r.t. the weak topology. The new aspect is that neither assumptions on compactness of the inner approximating lattices nor nonsequential continuity properties for the measures will be imposed. As a providing step also a generalization of the classical Portmanteau lemma will be established. The obtained characterizations of compact subsets w.r.t. the weak topology encompass several known ones from literature. The investigations rely basically on the inner extension theory for measures which has been systemized recently by König ([8], [10],[12]).

KEYWORDS: Inner premeasures, weak topology, generalized Portmanteau lemma.

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0 Introduction

The most influential result concerning compactness of spaces of measures has been presented by Prokhorov ([15]) for probability Borel-measures on Polish spaces. His equivalent characterization of relative compactness by uniform tightness has turned out to be an important tool to check convergence in law for many stochastic processes. Therefore nowadays this criterion may be found in most of standard textbooks of probability theory (see e.g. [14], [2], [5], [3]). Another characterization of compact sets of Borel Probability measures on Polish spaces has been shown by Huber and Strassen ([6]) in terms of a continuity property that the upper envelopes of these sets satisfy. Prokhorov as well as Huber and Strassen used the so called topology of weak convergence which is derived from the weak $*$ topology on the algebraic dual of the space of bounded continuous mappings. Of course this topology may be extended to spaces of finite Baire-measures on general topological spaces. This has been done by Varadarajan ([19]) who has also found an equivalent characterization for compact sets. However the topology of weak convergence rely on hidden regularity properties. Finite Baire-measures are inner regular w.r.t. the functionally closed sets, and in the special context of metrizable spaces they coincide with the finite Borel-measures, being inner regular w.r.t. the closed subsets. But in general finite Borel-measures are not inner regular w.r.t. the closed subsets. So for spaces of such measures the topology of weak convergence is not a reasonable concept because the measures are not uniquely determined by the restrictions of the integrals to bounded continuous mappings. Furthermore it seems to be necessary to impose regularity for the measures to find tractable extensions of the topology of weak convergence.

Fresh ideas had been presented by Topsøe in two seminal publications (cf. [16], [17]). The framework is based on a pair $\mathcal{S}, \mathcal{G}$ of lattices on an abstract set Ω, where $\mathcal{S}$ is stable under countable intersections. One may think of $\mathcal{G}$ as a topology and $\mathcal{S}$ as the set of closed or closed compact sets. It is known from extension results (e.g. [8], Theorem 6.31) that a finite measure on the $\sigma-$algebra $\sigma(\mathcal{S})$ generated by $\mathcal{S}$ which is inner regular w.r.t. $\mathcal{S}$ may be extended to the $\sigma-$algebra $\sigma(\mathcal{S}\top\mathcal{S})$

generated by the transporter $\mathcal{S}\top\mathcal{S} := \{A \subseteq \Omega \mid A \cap B \in \mathcal{S} \text{ for every } B \in \mathcal{S}\}$. In particular for $\mathcal{S}$ containing the closed compact subsets, we obtain extensions to Radon measures. So, assuming that the complements of the members of $\mathcal{G}$ are contained in $\mathcal{S}\top\mathcal{S}$, Topsøe considered the space of finite measures on $\sigma(\mathcal{S}\top\mathcal{S})$ which are inner regular w.r.t. $\mathcal{S}$, and he equipped this space with the coarsest topology such that $\mathrm{Q} \mapsto \mathrm{Q}(G)$ is lower semicontinuous for $G \in \mathcal{G} \cup \{\Omega\}$, and even continuous in the case of $G = \Omega$. It is a generalization of the topology of weak convergence in view of the classical Portmanteau lemma, and so he called it weak topology. Under the assumption that disjoint sets from $\mathcal{S}$ may be separated by disjoint sets from $\mathcal{G}$, he succeeded in giving a general characterization of relatively compact subsets in two cases. Firstly, if $\mathcal{S}$ is semicompact, and secondly for subsets in the topological subspace of finite measures P satisfying $\inf\limits_{A \in M} P(A) = \mathrm{P}(B)$ for every downward directed family M in $\mathcal{S}$ with $\bigcap\limits_{A \in M} A = B \in \mathcal{S}$ if $\Omega \in \mathcal{S}$ (cf. [17]).

This paper takes up the investigations by Topsøe. The aim is to characterize the relatively compact sets of finite measures which are inner regular w.r.t. the lattice $\mathcal{S}$ without further assumptions. So $\mathcal{S}$ need not to be semicompact, and the nonsequential continuity property of the measures will be not assumed.

The paper is organized as follows. In the next section some basic concepts and results from abstract measure and integration theory will be recalled. Besides a useful general Daniell-Stone representation theorem some inner extension results will be reviewed, and a new one will be established, which will be crucial for the following investigations. Afterwards we shall introduce in section 2 the weak topology on spaces $\mathcal{M}_f(\Omega, \mathcal{S})$ of finite measures which are inner regular w.r.t. lattices $\mathcal{S}$ containing the sample space Ω and being stable under countable intersections. However, we shall propose a slightly different approach which does not rely on an additional lattice $\mathcal{G}$ in general. It coincides with Topsøe's suggestion in many relevant cases, in particular under his separation property. The advantage of the version used throughout this paper is that it enables us to weaken Topsøe's separation property. The rationale is as follows. We shall assume to have a second lattice

$\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ also stable under countable intersections and containing Ω such that finite measures on $\sigma(\widetilde{\mathcal{S}})$ which are inner regular w.r.t. $\widetilde{\mathcal{S}}$ may be extended uniquely to a measure from $\mathcal{M}_f(\Omega, \mathcal{S})$. Sufficient conditions are provided by the extension result from the first section. Then we can endow the space $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ of the finite inner regular measures on $\sigma(\widetilde{\mathcal{S}})$ also with a weak topology. The crucial idea is to find conditions that the topological spaces $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ and $\mathcal{M}_f(\Omega, \mathcal{S})$ are homeomorphic. Then it will turn out that the adaption of Topsøe's separation property for $\widetilde{\mathcal{S}}$ instead of $\mathcal{S}$ ensures that the $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ and $\mathcal{M}_f(\Omega, \mathcal{S})$ are regular Hausdorff spaces. Afterwards we shall assume in section 3 that $\widetilde{\mathcal{S}}$ is generated by weak upper level sets of special kind of function systems. Such function systems suggest to define topologies on $\mathcal{M}_f(\Omega, \mathcal{S})$ analogously to the classical topology of weak convergence. The subject of section 3 is to compare it with the weak topology. The investigations lead to a generalized version of the Portmanteau lemma, in particular $\mathcal{M}_f(\Omega, \mathcal{S})$ is a completely regular Hausdorff space without imposing the adaption of Topsøe's separation property. The results from sections 2, 3 will be used to show several characterizations of compact subsets w.r.t. the weak topology on $\mathcal{M}_f(\Omega, \mathcal{S})$. Suitable specializations retain the contributions by Varadarajan as well as Huber and Strassen.

1 Notations and preliminaries

Let us begin with recalling some basic notions from abstract measure and integration theory. The reader is referred to the monograph by König (cf. [8], overview in [12]) for a comprehensive account. Let Ω be a nonvoid set. Nonvoid collections of subsets of Ω are called **lattices** if they are stable under finite unions and intersections. For a lattice $\mathcal{S}$ on Ω, we define the set system $\mathcal{S}\perp$ consisting of all $\Omega \setminus A$ with $\mid A \in \mathcal{S}$ and the **transporter** $\mathcal{S}\top\mathcal{S} := \{B \subseteq \Omega \mid A \cap B \in \mathcal{S} \text{ for all } A \in \mathcal{S}\}$. The symbol $\sigma(\mathcal{S})$ stands for the $\sigma-$algebra on Ω generated by the set system $\mathcal{S}$. Furthermore we define $\mathcal{S}^\sigma/\mathcal{S}_\sigma$ to consist of all at most countable unions/intersections of sets from the lattice $\mathcal{S}$.

A set function $\phi : \mathcal{S} \to [0, \infty]$ on a lattice $\mathcal{S}$ is said to be **isotone** if $\phi(A) \leq \phi(B)$ holds for every pair $A, B \in \mathcal{S}$ with $A \subseteq B$, and it is defined to be **modular** if $\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B)$ for $A, B \in \mathcal{S}$. We shall call an isotone set function ϕ on the lattice $\mathcal{S}$ to be **upward/downward continuous at** A if $A \in \mathcal{S}$, and $\sup\limits_n \phi(A_n) = \phi(A)/ \inf\limits_n \phi(A_n) = \phi(A)$ whenever $(A_n)_n$ is an isotone/antitone sequence in $\mathcal{S}$ with $\bigcup\limits_{n=1}^{\infty} A_n = A/ \bigcap\limits_{n=1}^{\infty} A_n = A$. If it is upward/downward continuous at each $A \in \mathcal{S}$, we shall say that it is **upward/downward continuous**.

Another important concept within measure theory is regularity. Setting $\inf \emptyset := \infty$, $\sup \emptyset := 0$ an isotone set function ϕ on a lattice $\mathcal{S}$ is said to be **inner/outer regular w.r.t.** $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{S}$, and

$$\phi(A) = \sup_{A \supseteq T \in \mathcal{T}} \phi(T) \text{ resp. } \phi(A) = \inf_{A \subseteq T \in \mathcal{T}} \phi(T)$$

for all $A \in \mathcal{S}$. An isotone set function ϕ on a lattice $\mathcal{S}$ with $\emptyset \in \mathcal{S}$ and $\phi(\emptyset) = 0$ is defined to be an **inner premeasure** if it can be extended to a measure on a $\sigma-$algebra which is inner regular w.r.t. $\mathcal{S}_\sigma$, a so called **inner extension** of ϕ. An inner extension μ of an inner premeasure ϕ will be named **maximal** if every inner extension of ϕ is a restriction of μ. It is known that every inner premeasure ϕ has a unique maximal inner extension (cf. [8], Theorems 6.18, 6.31). We shall need the following extension results by König.

Proposition 1.1 *Let ϕ be a bounded isotone, modular set function ϕ on a lattice $\mathcal{S}$ with $\emptyset \in \mathcal{S}$ and $\phi(\emptyset) = 0$. Then we have:*

.1 Let $\mathcal{T} \subseteq (\mathcal{S} \top \mathcal{S})\perp$ be a lattice with $\emptyset \in \mathcal{T}$ such that two disjoint sets from $\mathcal{S}$ may be separated by two disjoint sets from $\mathcal{T}$. Then the mapping

$$\widehat{\phi} : \mathcal{S} \to \mathbb{R}, \ A \mapsto \inf_{A \subseteq G \in \mathcal{T}} \sup_{G \supseteq B \in \mathcal{S}} \phi(B)$$

is an inner premeasure if it is downward continuous at $\emptyset$.

.2 Let ϕ be an inner premeasure with maximal inner extension $\mu : \mathcal{F} \to \mathbb{R}$, and let $\mathcal{T} \supseteq \mathcal{S}$ be a lattice such that every set from $\mathcal{T}$ is enclosed in some set from $\mathcal{S}$. Furthermore let the

mapping $\varphi : \mathcal{T}_\sigma \to \mathbb{R}$ *be defined by* $\varphi(G) = \inf\{\mu(B) \mid G \subseteq B \in \mathcal{F}\}$. *Then* $\varphi|\mathcal{T}$ *is the unique inner premeasure on* $\mathcal{T}$ *with* $\varphi|\mathcal{S} = \phi$ *if* $\varphi|\mathcal{T}$ *is downward continuous at* $\emptyset$, *and if* $\varphi(G \cup H) \geq \varphi(G) + \varphi(H)$ *holds for disjoint* G, H *from* $\mathcal{T}_\sigma$.

Statement .1 follows from Theorem 6.31 in [8] combined with Proposition 3.3 in [11] and statement .2 is just Theorem 19.11 in [8].

The following application of Proposition 1.1, .2 is basic for the investigations later on.

Theorem 1.2 *Let* $\widetilde{\mathcal{S}}$ *be a lattice on an abstract set* Ω *with* $\emptyset, \Omega \in \widetilde{\mathcal{S}}$, *and let* P *be a finite measure on* $\sigma(\widetilde{\mathcal{S}})$ *which is inner regular w.r.t.* $\widetilde{\mathcal{S}}_\sigma$. *Furthermore let* $\mathcal{S}$ *denote a lattice on* Ω, *enclosing* $\widetilde{\mathcal{S}}$ *as well as satisfying*

(1) for every antitone sequence $(A_n)_n$ *in* $\mathcal{S}$ *with* $\bigcap_{n=1}^{\infty} A_n = \emptyset$ *there exists an antitone sequence* $(B_n)_n$ *in* $\sigma(\widetilde{\mathcal{S}})$ *with* $A_n \subseteq B_n$ *for each* n *and* $\bigcap_{n=1}^{\infty} B_n = \emptyset$,

(2) disjoint sets A_1, A_2 *from* $\mathcal{S}_\sigma$ *may be separated by disjoint* $B_1, B_2 \in \sigma(\widetilde{\mathcal{S}})$.

Then there exists a unique finite measure Q *on* $\sigma(\mathcal{S})$ *which is inner regular w.r.t.* $\mathcal{S}_\sigma$, *and extends* P. *Furthermore we have* $\mathrm{Q}(A) = \inf\{\mathrm{P}(G) \mid A \subseteq G \in \widetilde{\mathcal{S}}_\sigma\perp\}$ *for every* $A \in \mathcal{S}_\sigma$.

The proof of Theorem 1.2 will be worked out in appendix A.

We shall make also use of a general inner Daniell-Stone representation result. Let us recall that a function system $E \subseteq [0, \infty[^\Omega$ is called a **Stonean lattice cone** if for $X, Y \in E$ and $\lambda \geq 0, t > 0$ the mappings $X + Y, \min\{X, Y\}, \max\{X, Y\}$ as well as $\lambda X, \min\{X, t\}, \max\{X - t, 0\}$ belong to E. A functional $I : E \to \mathbb{R}$ is defined to be **isotone and positive-linear** if $I(X) \leq I(Y)$ for $X \leq Y$ and $I(\lambda X) = \lambda I(X)$ as well as $I(X + Y) = I(X) + I(Y)$ for $X, Y \in E,\ \lambda \geq 0$. The announced inner Daniell-Stone theorem may be found in [10] (Theorem 2.5 with Theorem 1.3 and Theorem 4.2 in [9]).

Proposition 1.3 *Let $I : E \to \mathbb{R}$ be an isotone and positive-linear functional on a Stonean lattice cone $E \subseteq [0,\infty[^{\Omega}$ which contains the nonnegative constants. Furthermore let $\mathcal{S}\perp \subseteq (\mathcal{S}_\sigma)^\sigma$ be valid, where $\mathcal{S}$ consists of all $X^{-1}([x,\infty[)$ with $X \in E$ and $x > 0$.*
Then there exists a finite measure P on $\sigma(\mathcal{S})$ which is inner regular w.r.t. $\mathcal{S}_\sigma$ and satisfies $\int X \, d\mathrm{P} = I(X)$ for every $X \in E$ if and only if $\inf_n I(X_n) = 0$ for $X_n \searrow 0$ and $\sup_n I(Y_n) = I(Y)$ for $Y_n \nearrow Y$. In this case all representing finite measures are inner regular w.r.t. $\mathcal{S}_\sigma$ and coincide.

2 Weak topologies on spaces of inner regular finite measures

Let $\mathcal{S}$ be a lattice on a nonvoid set Ω with

(2.1) $\emptyset, \Omega \in \mathcal{S}$;

(2.2) $\mathcal{S}_\sigma = \mathcal{S}$.

We shall consider the set $\mathcal{M}_f(\Omega,\mathcal{S})$ gathering all finite measures on $\sigma(\mathcal{S})$ which are inner regular w.r.t. $\mathcal{S}$. It will be equipped with the coarsest topology τ_w such that for each $A \in \mathcal{S} \cup \{\Omega\}$ the mapping $\psi_A : \mathcal{M}_f(\Omega,\mathcal{S}) \to \mathbb{R}$, $\mathrm{Q} \mapsto \mathrm{Q}(A)$, is upper semicontinuous, and such that ψ_Ω is continuous. We may describe τ_w also by the basic neigbourhood system consisting of

$$N_w(\mathrm{P}, A_1, ..., A_n, \varepsilon) := \{\mathrm{Q} \in \mathcal{M}_f(\Omega,\mathcal{S}) \mid |\,\mathrm{P}(\Omega) - \mathrm{Q}(\Omega)| < \varepsilon, \ \ \mathrm{Q}(A_i) < \mathrm{P}(A_i) + \varepsilon, i = 1, ..., n\}$$

for $\mathrm{P} \in \mathcal{M}_f(\Omega,\mathcal{S})$, $n \in \mathbb{N}$, $A_1, ..., A_n \in \mathcal{S}, \varepsilon > 0$. In the following we shall call τ_w the **weak topology**. This is in accordance with Topsøe's suggestion to define weak topologies for finite inner regular measures within an abstract framework, using the additional lattice $\mathcal{G}$ defined to consist of all complements of sets from $\mathcal{S}$ (cf. [16] and introduction).

Historically, for finite Baire-measures Alexandroff (cf. [1]) introduced the topology induced by the weak convergence. To recall weak convergence means that a net $(\mathrm{Q})_{j\in J}$ of finite Baire-measures converges to a finite Baire-measure Q if $(\int X \; d\,\mathrm{Q}_j)_{j\in J}$ converges to $\int X \; dQ$ for every bounded continuous X. This topology coincides with the usual topology used for finite Borel-measures in the context of metrizable topologies. Recall that the functionally closed subsets are exactly the subsets of the form $X^{-1}(\{0\})$, where X denotes a real-valued continuous mapping. The functionally open subsets are the complements of funcionally closed ones. Since finite Baire-measures are inner regular w.r.t. the functionally closed subsets (cf. [8], Addendum 8.5), and since finite Borel-measures on metric spaces are inner regular the closed subsets, we can recognize by classical Portmanteau lemma (e.g. [1], p. 180) that in the topological context this classical approach coincides with the weak topology.

Obviously for different P, Q from $\mathcal{M}_f(\Omega, \mathcal{S})$ the restrictions to $\mathcal{S}$ differ too. Nevertheless this does not imply that the weak topology is Hausdorff. In order to show this property we shall need some suitable separation properties. For an exposition as general as possible let us assume that there is some additional lattice $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ with

(2.3) $\emptyset, \Omega \in \widetilde{\mathcal{S}}$

(2.4) $\widetilde{\mathcal{S}}_\sigma = \widetilde{\mathcal{S}}$.

The set $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ gathers all the finite measures on $\sigma(\widetilde{\mathcal{S}})$ which are inner regular w.r.t. $\widetilde{\mathcal{S}}$. The weak topology $\tau_{\tilde{w}}$ on $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ is defined analogously to the weak topology τ_w on $\mathcal{M}_f(\Omega, \mathcal{S})$, with respective basic neighbourhood system $\{N_{\tilde{w}}(\mathrm{Q}, B_1, ..., B_n, \varepsilon) \mid \mathrm{Q} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}),\ n \in \mathbb{N},\ B_1, ..., B_n \in \widetilde{\mathcal{S}}, \varepsilon > 0\}$. According to Theorem 1.2 we have a well-defined mapping from $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ into $\mathcal{M}_f(\Omega, \mathcal{S})$ if

(2.5) for every antitone sequence $(A_n)_n$ in $\mathcal{S}$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ there exists an antitone sequence $(B_n)_n$ in $\sigma(\widetilde{\mathcal{S}})$ with $A_n \subseteq B_n$ for each n and $\bigcap_{n=1}^{\infty} B_n = \emptyset$, (2.6) disjoint sets A_1, A_2 from $\mathcal{S}$ may be separated by disjoint $B_1, B_2 \in \sigma(\widetilde{\mathcal{S}})$

are satisfied. We want to investigate the continuity of this mapping, additionally assuming

(2.7) for disjoint $A_1 \in \mathcal{S}$ and $A_2 \in \widetilde{\mathcal{S}}$ there exists some $B \in \widetilde{\mathcal{S}}$ with $A_1 \subseteq B \subseteq \Omega \setminus A_2$.

Proposition 2.1 *Under (2.5) - (2.7) the mapping* $F : \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}) \to \mathcal{M}_f(\Omega, \mathcal{S})$, *which is defined by* $F(\mathrm{P})|\sigma(\widetilde{\mathcal{S}}) = \mathrm{P}$, *is a homeomorphism between* $\tau_{\tilde{w}}$ *and* τ_w.

Proof:

The mapping is well defined by Theorem 1.2, and its injectivity is trivial. For the surjectivity let us fix some $\mathrm{Q} \in \mathcal{M}_f(\Omega, \mathcal{S})$. Applying usual arguments, one obtains

$$\mathcal{A} := \left\{B \in \sigma(\widetilde{\mathcal{S}}) \mid \mathrm{Q}(B) = \inf\{\mathrm{Q}(G) \mid B \subseteq G \in \widetilde{\mathcal{S}}\perp\} = \sup\{\mathrm{Q}(A) \mid B \supseteq A \in \widetilde{\mathcal{S}}\}\right\}$$

as a $\sigma-$algebra. Moreover, in view of (2.7), $\widetilde{\mathcal{S}}\perp$ is enclosed because Q is inner regular w.r.t. $\mathcal{S}$. That means that $\mathrm{Q}\,|\sigma(\widetilde{\mathcal{S}}) \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$, which shows that F is surjective.
Furthermore every net $(\mathrm{Q}_j)_{j \in J}$ in $\mathcal{M}_f(\Omega, \mathcal{S})$ which converges to some Q w.r.t. τ_w implies the convergence of $(F^{-1}(\mathrm{Q}_j))_{j \in J}$ to $F^{-1}(\mathrm{Q})$ w.r.t. $\tau_{\tilde{w}}$ since $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$. Thus F^{-1} is continuous.
Conversely, let $(\mathrm{P}_j)_{j \in J}$ denote any net in $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ which converges to some P w.r.t. $\tau_{\tilde{w}}$.
By virtue of (2.5), (2.6) the application of Theorem 1.2 yields that for every $A \in \mathcal{S}$ and arbitrary $\varepsilon > 0$ there is some $G \in \widetilde{\mathcal{S}}\perp$ enclosing A such that $F(\mathrm{P})(A) + \varepsilon > \mathrm{P}(G)$. Then (2.7) guarantees some $B \in \widetilde{\mathcal{S}}$ with $A \subseteq B \subseteq G$. Since $\psi_B : \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}) \to \mathbb{R},\ \mathrm{Q} \mapsto \mathrm{Q}(B)$, is upper semicontinuous w.r.t. $\tau_{\tilde{w}}$, we obtain

$$F(\mathrm{P})(A) + \varepsilon > \mathrm{P}(B) \geq \limsup_j \mathrm{P}_j(B) = \limsup_j F(\mathrm{P}_j)(B) \geq \limsup_j F(\mathrm{P}_j)(A).$$

This shows the continuity of F and completes the proof. ■

First properties of the weak topology τ_w may be obtained by the additional assumption

(2.8) Disjoint sets A_1, A_2 from $\widetilde{\mathcal{S}}$ may be separated by disjoint $G_1, G_2 \in \widetilde{\mathcal{S}}\perp$.

Proposition 2.2 *Let the assumptions (2.5) - (2.8) be satisfied. Then $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_w)$ is a regular Hausdorff space.*

Proof:

According to Proposition 2.1 it suffices to prove that $(\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}), \tau_{\tilde{w}})$ is a regular Hausdorff space. Let $(\mathrm{P}_j)_{j\in J}$ be a net in $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ which converges to P and Q w.r.t. $\tau_{\tilde{w}}$. The net $(\mathrm{P}_j)_{j\in J}$ has a universal subnet $(\mathrm{P}_{j(i)})_{i\in I}$, which induces the universal subnet $(\mathrm{P}_{j(i)}(\Omega))_{i\in I}$ that converges to $\mathrm{P}(\Omega)$ by assumption. In particular it has a bounded subnet $(\mathrm{P}_{j(i(k))}(\Omega))_{k\in K}$. Since $(\mathrm{P}_{j(i(k))})_{k\in K}$ is a universal net too, $(\mathrm{P}_{j(i(k))}(A))_{k\in K}$ is a universal net in the compact set $[0, \sup_{k\in K} \mathrm{P}_{j(i(k))}(\Omega)]$ for every $A \in \widetilde{\mathcal{S}}$. Hence we obtain an isotone, modular set function $\phi : \widetilde{\mathcal{S}} \to \mathbb{R}$ with $\phi(\emptyset) = 0$ and $\phi(A) = \lim_k \mathrm{P}_{j(i(k))}(A)$ for each $A \in \widetilde{\mathcal{S}}$. Assumption (2.8) allows us to apply Proposition 1.1, .1. Drawing on this result the set function

$$\widehat{\phi}:\ \widetilde{\mathcal{S}} \to \mathbb{R},\ A \mapsto \inf_{A\subseteq G\in\ \widetilde{\mathcal{S}}\perp}\ \sup_{G\supseteq B\in\ \widetilde{\mathcal{S}}} \phi(B),$$

is an inner premeasure. Indeed $(\mathrm{P}_{j(i(k))})_{k\in K}$ converges to P, which is inner regular w.r.t. $\widetilde{\mathcal{S}}$ and outer regular w.r.t. $\widetilde{\mathcal{S}}\perp$, and thus

$$\widehat{\phi}(A) \le \inf_{A\subseteq G\in\widetilde{\mathcal{S}}\perp}\ \sup_{G\supseteq B\in\widetilde{\mathcal{S}}} \mathrm{P}(B) = \mathrm{P}(A)$$

holds for every $A \in \mathcal{S}$. Hence there exists some $\mathrm{Q}_1 \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ which extends $\widehat{\phi}$. In particular $\mathrm{Q}_1 \le \mathrm{P}$ with $\mathrm{Q}_1(\Omega) = \mathrm{P}(\Omega)$. The same line of reasoning leads to $\mathrm{Q}_1 \le \mathrm{Q}$ with $\mathrm{Q}_1(\Omega) = \mathrm{Q}(\Omega)$. Thus $\mathrm{P} = \mathrm{Q}_1 = \mathrm{Q}$, which shows that $\tau_{\tilde{w}}$ is Hausdorff.

Next let us fix some $\mathrm{P} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ and a neigbourhood $N_{\tilde{w}}(\mathrm{P}, A_1, ..., A_n, \varepsilon)$. Since P is outer regular w.r.t. $\widetilde{\mathcal{S}}\perp$ there exists for each A_i a set $G_i \in \widetilde{\mathcal{S}}\perp$ which encloses A_i and satisfies $|\,\mathrm{P}(G_i) - \mathrm{P}(A_i)| < \frac{\varepsilon}{2}$.

By (2.8) we may find for every A_i sets $H_i \in \widetilde{\mathcal{S}}\perp$ and $B_i \in \widetilde{\mathcal{S}}$ with $A_i \subseteq H_i \subseteq B_i \subseteq G_i$. Now let $(\mathrm{Q}_j)_{j\in J}$ denote a net in $N_{\tilde{w}}(\mathrm{P}, B_1, ..., B_n, \frac{\varepsilon}{2})$ which converges to some Q w.r.t. $\tau_{\tilde{w}}$. Then we may conclude for each $i \in \{1, ..., n\}$

$$\mathrm{Q}(A_i) \leq \mathrm{Q}(H_i) \leq \liminf_j \mathrm{Q}_j(H_i) \leq \liminf_j \mathrm{Q}_j(B_i) \leq \mathrm{P}(B_i) + \frac{\varepsilon}{2} < \mathrm{P}(A_i) + \varepsilon$$

Hence the closure of $N_{\tilde{w}}(\mathrm{P}, B_1, ..., B_n, \frac{\varepsilon}{2})$ is contained in $N_{\tilde{w}}(\mathrm{P}, A_1, ..., A_n, \varepsilon)$, which completes the proof.

■

Remark:

In the case of $\mathcal{S} = \widetilde{\mathcal{S}}$ the assumptions (2.5) - (2.8) reduce to the condition that disjoint sets from $\mathcal{S}$ may be separated by disjoint sets from $\mathcal{S}\perp$. This is just the initial separation property that Topsøe used in [16].

Next we want to avoid separation condition (2.8). Instead we consider the case that $\widetilde{\mathcal{S}}$ consists of countable intersections of level sets $X^{-1}([x, \infty[)$ of mappings X from a Stonean lattice cone $E \subseteq [0, \infty[^\Omega$. In order to obtain properties for τ_w in this situation we shall provide us in the following section with a general Portmanteau lemma.

3 A general Portmanteau lemma

Throughout this section we shall assume that there is some Stonean lattice cone $E \subseteq [0, \infty[^\Omega$ with $1 \in E$ such that $\widetilde{\mathcal{S}} = \{\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[) \mid X_n \in E, x_n > 0\}$. Additionally all members of E should be bounded, and the further condition

(3.1) $\sup X - X \in E$ for every $X \in E$

should be fulfilled. Notice that $\widetilde{\mathcal{S}}\perp \subseteq \widetilde{\mathcal{S}}^\sigma$ holds under (3.1).

The function system E induces a topology $\tau_{w,E}$ on $\mathcal{M}_f(\Omega, \mathcal{S})$ defined by the basic neigbourhood system consisting of

$$N_E(\mathrm{P}, X_1, ..., X_n, \varepsilon) := \{\mathrm{Q} \in \mathcal{M}_f(\Omega, \mathcal{S}) \mid |\int X_i \, d\,\mathrm{Q} - \int X_i \, d\,\mathrm{P} \,| < \varepsilon, i = 1, ..., n\}$$

for $\mathrm{P} \in \mathcal{M}_f(\Omega, \mathcal{S})$, $n \in \mathbb{N}$ and $X_1, ..., X_n \in E$. In view of the inner Daniell-Stone theorem 1.3 measures from $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ whose integrals coincide on E are identical. Therefore, due to Theorem 1.2, measures from $\mathcal{M}_f(\Omega, \mathcal{S})$ are uniquely determined by the restrictions of their integrals to E provided that (2.5) and (2.6) are satisfied. In this case $\tau_{w,E}$ is obviously Hausdorff. Moreover, transferring the proof of Theorem II.1 in [19] verbatim, we obtain that under (2.5) and (2.6) $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_{w,E})$ is even a completely regular Hausdorff space. In the next step we want to compare the topologies τ_w and $\tau_{w,E}$. The investigations lead to a general Portmanteau lemma.

Theorem 3.1 *Let* $\mathrm{Q} \in \mathcal{M}_f(\Omega, \mathcal{S})$, *let* $(\mathrm{Q}_j)_{j \in J}$ *be a net in* $\mathcal{M}_f(\Omega, \mathcal{S})$, *and consider the following statements:*

.1 $\lim_j \mathrm{Q}_j(\Omega) = \mathrm{Q}(\Omega)$ *and* $\limsup_j \mathrm{Q}_j(A) \leq \mathrm{Q}(A)$ *for all* $A \in \mathcal{S}$.

.2 $\lim_j \mathrm{Q}_j(\Omega) = \mathrm{Q}(\Omega)$ *and* $\liminf_j \mathrm{Q}_j(G) \geq \mathrm{Q}(G)$ *for all* $G \in \mathcal{S}\perp$.

.3 $\lim_j \mathrm{Q}_j(\Omega) = \mathrm{Q}(\Omega)$ *and* $\limsup_j \mathrm{Q}_j(A) \leq \mathrm{Q}(A)$ *for all* $A \in \widetilde{\mathcal{S}}$.

.4 $\lim_j \mathrm{Q}_j(\Omega) = \mathrm{Q}(\Omega)$ *and* $\liminf_j \mathrm{Q}_j(G) \geq \mathrm{Q}(G)$ *for all* $G \in \widetilde{\mathcal{S}}\perp$.

.5 $\lim_j \int X \, d\,\mathrm{Q}_j = \int X \, d\,\mathrm{Q}$ *for all* $X \in E$.

Then under assumptions (2.5), (2.6), (3.1) .1 $\Rightarrow$ .3 $\Leftrightarrow$ .5 *and the equivalences* .1 $\Leftrightarrow$ .2 *as well as* .3 $\Leftrightarrow$ .4 *hold. If in addition assumption (2.7) is valid, then all statements are equivalent.*

Proof:

Let (2.5), (2.6), (3.1) be satisfied. Then the equivalences .1 $\Leftrightarrow$ .2 and .3 $\Leftrightarrow$ .4 are obvious, also implication .1 $\Rightarrow$ .3 by $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$.

proof of .5 $\Rightarrow$ .3:

Let $A \in \widetilde{\mathcal{S}}$ with indicator mapping 1_A. Since $1 \in E$, statement .5 implies $\lim\limits_j \mathrm{Q}_j(\Omega) = \mathrm{Q}(\Omega)$. Moreover, it is known that there exists some antitone sequence $(X_n)_n$ in E with $1_A = \inf\limits_n X_n$ (cf. [9], Proposition 3.2). Therefore $\psi_A = \inf\limits_n \psi_{X_n}$, where

$$\psi_A : \mathcal{M}_f(\Omega, \mathcal{S}) \to \mathbb{R},\ \mathrm{Q} \mapsto \mathrm{Q}(A),\ \text{and}\ \psi_{X_n} : \mathcal{M}_f(\Omega, \mathcal{S}) \to \mathbb{R},\ \mathrm{Q} \mapsto \int X_n \, d\,\mathrm{Q}\ \ (n \in \mathbb{N}).$$

In particular ψ_A is upper semicontinuous w.r.t. $\tau_{w,E}$, and thus $\limsup\limits_j \mathrm{Q}_j(A) \leq \mathrm{Q}(A)$ follows immediately from statement .5.

proof of .3 $\Rightarrow$ .5:

Let $M_b(\Omega)$ denote the space of all bounded real-valued mappings on Ω. It will be equipped with the supremum norm $\|\cdot\|_\infty$.

Let $X \in E$. Since $X \in M_b(\Omega)$ with $X^{-1}([x, \infty[) \in \widetilde{\mathcal{S}}$ for $x > 0$ we may approximate it uniformly by an isotone sequence $(X_n)_n$ of nonnegative functions with finite range and level sets $X_n^{-1}([x, \infty[)$ $(x > 0, n \in \mathbb{N})$ belonging to $\widetilde{\mathcal{S}}$ (cf. [8], Proposition 22.1). We may describe for each $n \in \mathbb{N}$ and $\mathrm{P} \in \{\mathrm{Q}, \mathrm{Q}_j \mid j \in J\}$ the integral $\int X_n \, d\,\mathrm{P}$ by $\sum\limits_{i=1}^{r_n} \lambda_i \,\mathrm{P}(A_i)$ for some $\lambda_1, ..., \lambda_{r_n} > 0$ and $A_1, ...A_{r_n} \in \widetilde{\mathcal{S}}$ (cf. [8], Properties 11.8). Thus statement .3 implies

$$\limsup_j \int X \; d\,\mathrm{Q}_j \leq \limsup_j (\mathrm{Q}_j(\Omega)\|X - X_n\|_\infty + \int X_n \; d\,\mathrm{Q}_j) \leq \mathrm{Q}(\Omega)\|X - X_n\|_\infty + \int X_n \; d\,\mathrm{Q}$$

for every $n \in \mathbb{N}$. Hence $\limsup\limits_j \int X \; d\,\mathrm{Q}_j \leq \int X \; d\,\mathrm{Q}$ by monotone convergence. Since $\sup X - X \in E$ due to (3.1), we may employ the same line of reasoning to obtain $\limsup\limits_j \int (\sup X - X) \; d\,\mathrm{Q}_j \leq \int (\sup X - X) \; d\,\mathrm{Q}$. This shows $\lim\limits_j \int X \; d\,\mathrm{Q}_j = \int X \; d\,\mathrm{Q}$.

Finally, additional assumption (2.7) forces implication .3 $\Rightarrow$ .1 due to Proposition 2.1. This means that all statements are equivalent, and the proof is complete. ■

As a consequence of the Portmanteau lemma and the discussion on $\tau_{w,E}$ we can emphasize the following property of τ_w.

Proposition 3.2 *$(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_w)$ is a completely regular Hausdorff space if the conditions (2.5), (2.6), (2.7) and (3.1) are valid.*

Remark 3.3 *In order to find for the lattice $\mathcal{S}$ a lattice $\widetilde{\mathcal{S}}$ and the Stonean lattice cone E in the Portmanteau lemma 3.1, a first attempt might be to choose the set system E defined to consist of the bounded nonnegative $X \in \mathbb{R}^\Omega$ with $X^{-1}([x, \infty[), X^{-1}(]-\infty, x]) \in \mathcal{S}$ for $x > 0$. It is indeed a Stonean lattice cone which satisfies (3.1). Then one has to look whether in addition the assumptions (2.5), (2.6) and (2.7) are satisfied with $\widetilde{\mathcal{S}} := \{X^{-1}([x, \infty[) \mid X \in E, x > 0\}$. For prominent applications of this line of reasoning let (Ω, τ_Ω) be a topological space:*

1) *If $\mathcal{S}$ is the set of the functionally closed subsets, then E is the set of nonnegative bounded continuous mappings on Ω with $\widetilde{\mathcal{S}} = \mathcal{S}$. Then all the assumptions (2.5) - (2.7) are satisfied. Therefore Theorem 3.1 retains the classical Portmanteau lemma for finite Baire measures, and for finite Borel-measures in the case of metrizable τ_Ω. Moreover, the classical Portmanteau lemma may be extended to finite Borel-measures if τ_Ω is perfectly normal (notice 1.5.19 in [4]).*

2) *If $\mathcal{S}$ is the set of the closed subsets, then again E is the set of nonnegative bounded continuous mappings on Ω, but with $\widetilde{\mathcal{S}}$ gathering all the functionally closed subsets. Then in view of Urysohn's lemma the assumptions (2.5) - (2.7) are fulfilled for (Ω, τ_Ω) being normal and countably paracompact. Thus the classical Portmanteau lemma may be extended to finite Borel-measures which are inner regular w.r.t. the closed subsets if (Ω, τ_Ω) is normal and countably paracompact.*

4 Compactness in spaces of inner regular measures

Let $\mathcal{S}, \widetilde{\mathcal{S}}$ be lattices on a nonvoid set Ω with $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$, and satisfying the conditions (2.1) - (2.4). Furthermore let us retake the further notations from section 2. We want to investigate necessary and sufficient conditions for compactness w.r.t. the weak topology τ_w on $\mathcal{M}_f(\Omega, \mathcal{S})$. Let us begin with the considerations under the assumptions (2.5) - (2.8).

Theorem 4.1 *Let $cl(\Delta)$ be the closure of some $\Delta \subseteq \mathcal{M}_f(\Omega, \mathcal{S})$ w.r.t. τ_w, let $\nu := \sup_{\mathrm{Q} \in cl(\Delta)} \mathrm{Q}$. Additionally, let the assumptions (2.5) - (2.8) be satisfied, and consider the following statements:*

.1 Δ is relatively compact w.r.t. τ_w.

.2 ν is real-valued, and $\nu|\widetilde{\mathcal{S}}$ is downward continuous.

.3 ν is real-valued with $\nu(A) = \inf_{A \subseteq G \in \widetilde{\mathcal{S}}\perp} \sup_{G \supseteq B \in \widetilde{\mathcal{S}}} \nu(B)$ for each $A \in \widetilde{\mathcal{S}}$, and $\nu|\widetilde{\mathcal{S}}$ is downward continuous at $\emptyset$.

Then the implications .3 $\Rightarrow$ .1 $\Rightarrow$ .2 are valid. Moreover, in the case of $\widetilde{\mathcal{S}}\perp \subseteq \widetilde{\mathcal{S}}^{\sigma}$ all the statements .1 - .3 are equivalent.

Remark:

It is already known that the implication .3 $\Rightarrow$ .2 is even valid when Δ is not relatively compact (cf. [13], Lemma 1.4).

Proof:

<u>proof of .1 $\Rightarrow$ .2</u>:

By definition the mapping $\psi_\Omega : cl(\Delta) \to \mathbb{R},\ \mathrm{Q} \mapsto \mathrm{Q}(\Omega)$, is continuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Then, due to compactness of $cl(\Delta)$, the set $\{\mathrm{Q}(\Omega) \mid \mathrm{Q} \in cl(\Delta)\}$ is compact. In particular ν is real-valued.

Next, let $(A_n)_n$ denote an antitone sequence in $\widetilde{\mathcal{S}}$ with $\bigcap\limits_{n=1}^{\infty} A_n =: A \in \widetilde{\mathcal{S}}$. By definition of the weak topology the mappings

$$\psi_B : cl(\Delta) \to \mathbb{R}, \mathrm{Q} \mapsto \mathrm{Q}(B) \ (B \in \{A, A_n \mid n \in \mathbb{N}\})$$

are upper semicontinuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Since $cl(\Delta)$ is assumed to be a compact Hausdorff space w.r.t. the relative topology of τ_w, we may apply the general Dini lemma (cf. [7], Theorem 3.7), and we obtain

$$\inf_n \nu(A_n) = \inf_n \sup_{\mathrm{Q}\in cl(\Delta)} \psi_{A_n}(\mathrm{Q}) = \sup_{\mathrm{Q}\in cl(\Delta)} \inf_n \psi_{A_n}(\mathrm{Q}) = \sup_{\mathrm{Q}\in cl(\Delta)} \psi_A(\mathrm{Q}) = \nu(A)$$

<u>proof of .3 $\Rightarrow$ .1</u>:

Let $F : \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}) \to \mathcal{M}_f(\Omega, \mathcal{S})$ be the homeomorphism according to Proposition 2.1. It suffices to prove that $F^{-1}(\Delta)$ is relatively compact w.r.t. $\tau_{\tilde{w}}$. Since $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_{\tilde{w}})$ is a regular Hausdorff space, it is known that $F^{-1}(\Delta)$ is relatively compact if and only if every universal net in $F^{-1}(\Delta)$ converges (cf. [18], Lemma 2.3). So let us fix a universal net $(\mathrm{Q}_j)_{j\in J}$ in $F^{-1}(\Delta)$. It induces for each $A \in \widetilde{\mathcal{S}}$ the universal net $(\mathrm{Q}_j(A))_{j\in J}$ in $\mathbb{R}$ and the relatively compact subset $\{\mathrm{Q}_j(A) \mid j \in J\}$ since ν is real-valued. Therefore, we obtain some mapping $\phi : \widetilde{\mathcal{S}} \to \mathbb{R}$ such that $\phi(A) = \lim\limits_j \mathrm{Q}_j(A)$ for every $A \in \widetilde{\mathcal{S}}$. Routine procedures yield that ϕ is an isotone modular set function with $\phi(\emptyset) = 0$ and $\phi \le \nu|\widetilde{\mathcal{S}}$.

By assumption on $\nu|\widetilde{\mathcal{S}}$ we have $\widehat{\phi} \le \nu|\widetilde{\mathcal{S}}$ for the isotone set function

$$\widehat{\phi} : \widetilde{\mathcal{S}} \to \mathbb{R}, \ A \mapsto \inf_{A\subseteq G\in \widetilde{\mathcal{S}}\perp} \ \sup_{G\supseteq B\in \widetilde{\mathcal{S}}} \phi(B),$$

which is even downward continuous at $\emptyset$ because $\nu|\widetilde{\mathcal{S}}$ satisfies this property. Thus, drawing on Proposition 1.1, .1, we obtain $\widehat{\phi}$ as an inner premeasure with $\phi(\Omega) = \widehat{\phi}(\Omega)$. This means that there is some $\mathrm{Q} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ with $\widehat{\phi} = \mathrm{Q}\,|\widetilde{\mathcal{S}}$. Moreover, $\limsup\limits_j \mathrm{Q}_j(A) \le \mathrm{Q}(A)$ for $A \in \widetilde{\mathcal{S}}$ due to $\phi \le \widehat{\phi}$, and $\lim\limits_{j} \mathrm{Q}_j(\Omega) = \phi(\Omega) = \widehat{\phi}(\Omega)$. Therefore $(\mathrm{Q}_j)_{j\in J}$ converges to Q w.r.t. $\tau_{\tilde{w}}$ with $\mathrm{Q} \leq \nu$. Thus $F^{-1}(\Delta)$ is relatively compact w.r.t. $\tau_{\tilde{w}}$.

Now let $\widetilde{\mathcal{S}}\perp \subseteq \widetilde{\mathcal{S}}^\sigma$. It remains to prove the implication $.2 \Rightarrow .3$.

<u>proof of $.2 \Rightarrow .3$:</u>

Let us fix $A \in \widetilde{\mathcal{S}}$. By assumption there exists an isotone sequence $(A_n)_n$ in $\widetilde{\mathcal{S}}$ with $\Omega \setminus A = \bigcup\limits_{n=1}^{\infty} A_n$. Moreover, for each n we may find by (2.8) disjoint $G_{1n}, G_{2n} \in \widetilde{\mathcal{S}}\perp$ with $A \subseteq G_{1n}$ and $A_n \subseteq G_{2n}$. Then we can define by $G_n := \bigcap\limits_{m=1}^{n} G_{1m}$ and $B_n := \bigcap\limits_{m=1}^{n} \Omega \setminus G_{2m}$ antitone sequences $(G_n)_n$ and $(B_n)_n$ in $\widetilde{\mathcal{S}}\perp$ and $\widetilde{\mathcal{S}}$ respectively with $A \subseteq G_n \subseteq B_n \subseteq \Omega \setminus A_n$, implying $A = \bigcap\limits_{n=1}^{\infty} G_n = \bigcap\limits_{n=1}^{\infty} B_n$. Hence we may conclude from statement .2

$$\nu(A) = \lim_{n\to\infty} \nu(B_n) \geq \lim_{n\to\infty} \sup_{G_n \supseteq B \in \widetilde{\mathcal{S}}} \nu(B) \geq \inf_{A \subseteq G \in \widetilde{\mathcal{S}}\perp} \sup_{G \supseteq B \in \widetilde{\mathcal{S}}} \nu(B) \geq \nu(A).$$

Since $\nu|\widetilde{\mathcal{S}}$ is downward continuous at $\emptyset$ by statement .2 again, statement .3 is shown, which completes the proof. ■

Drawing on Topsøe's investigations in [16] we may give a further characterization of relatively compact subsets in the topological subspace $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ consisting of all $\mathrm{Q} \in \mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ with $\inf\limits_{A\in M} \mathrm{Q}(A) = \mathrm{Q}(B)$ for every nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap\limits_{A\in M} A = B \in \mathcal{S}$.

Theorem 4.2 *Let $\Delta \subseteq \mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ and $\bar{\nu} := \sup\limits_{\mathrm{Q}\in\Delta} \mathrm{Q}$. If for any $A \in \mathcal{S}$ and every $\omega \in \Omega \setminus A$ there is some $B \in \widetilde{\mathcal{S}}$ with $\omega \in B \subseteq \Omega \setminus A$, then under the assumptions (2.5) - (2.8) the following statements are equivalent*

.1 Δ is relatively compact w.r.t. the relative topology of τ_w to $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$.

.2 $\bar{\nu}(\Omega) < \infty$, and $\inf\limits_{A\in M} \bar{\nu}(A) = 0$ for each nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap\limits_{A\in M} A = \emptyset$.

Proof:

Let the assumptions (2.5) - (2.8) be valid, and let $F : \mathcal{M}_f(\Omega, \mathcal{S}) \to \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ be the homeomorphism according to Proposition 2.1. Furthermore let $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau) \subseteq \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ be the counterpart of $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$. Obviously $F^{-1}(\Delta) \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau)$.

Topsøe showed (cf. Theorem 5 in [16]) that statement .2 holds if and only if $F^{-1}(\Delta)$ is relatively compact w.r.t. the relative topology of $\tau_{\tilde{w}}$ to $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau)$. So it remains to prove that statement .1 is fulfilled if and only if $F^{-1}(\Delta)$ is relatively compact w.r.t. the relative topology of $\tau_{\tilde{w}}$ to $\mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau)$. The only if part is trivial. For the if part it suffices to show that $F(\mathrm{Q}) \in \mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ for every $\mathrm{Q} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau)$. So let $\mathrm{Q} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}}, \tau)$, and let $M \subseteq \mathcal{S}$ be downward directed with $\bigcap\limits_{B \in M} B =: A \in \mathcal{S}$.

For $\varepsilon > 0$ Theorem 1.2 and condition (2.7) yield that there is some $\hat{A} \in \widetilde{\mathcal{S}}$ with $A \subseteq \hat{A}$ and $F(\mathrm{Q})(A) + \varepsilon > \mathrm{Q}(\hat{A})$. Then $\widehat{M} := \{B \cup \hat{A} \mid B \in M\} \subseteq \mathcal{S}$ is downward directed with $\bigcap\limits_{D \in \widehat{M}} D = \hat{A}$.

Now let N consist of all $B \in \widetilde{\mathcal{S}}$ with $D \subseteq B$ for some $D \in \widehat{M}$, at least Ω belongs to N. Furthermore, N is downward directed since $\widehat{M}$ satisfies this property. Moreover, for any $\omega \in \Omega \setminus \hat{A}$ there is some $D \in \widehat{M}$ with $\omega \in \Omega \setminus D$. By assumption we may find a set $B \in \widetilde{\mathcal{S}}$ with $\omega \in B \subseteq \Omega \setminus D$, which implies by condition (2.7) that there is a set $\hat{B} \in \widetilde{\mathcal{S}}$ with $D \subseteq \hat{B} \subseteq \Omega \setminus B$. Thus $\hat{B} \in N$ as well as $\omega \in B \subseteq \Omega \setminus \hat{B}$, and therefore $\Omega \setminus \hat{A} = \bigcup\limits_{B \in N} \Omega \setminus B$.

Finally

$$F(\mathrm{Q})(A) \leq \inf_{B \in M} F(\mathrm{Q})(B) \leq \inf_{D \in \hat{M}} F(\mathrm{Q})(D) \leq \inf_{B \in N} \mathrm{Q}(B) = \mathrm{Q}(\hat{A}) < F(\mathrm{Q})(A) + \varepsilon,$$

which means $F(\mathrm{Q})(A) = \inf\limits_{B \in M} F(\mathrm{Q})(B)$, and the proof is complete. ∎

In the following we want to use the general Portmanteau lemma 3.1 to obtain a result concerning compactness w.r.t. the weak topology.

Theorem 4.3 *Let $E \subseteq \{X \in [0, \infty[^{\Omega} \mid X$ bounded $\}$ denote a Stonean lattice cone with $1 \in E$, let further $L := E - E$ and $\widetilde{\mathcal{S}} := \{X^{-1}([x, \infty[) \mid X \in E, x > 0\}$. Furthermore $cl(\Delta)$ denotes the*

closure of some subset Δ of $\mathcal{M}_f(\Omega, \mathcal{S})$ w.r.t. τ_w, and induces the mappings $\nu := \sup\limits_{\mathrm{Q} \in cl(\Delta)} \mathrm{Q}$ as well as $I : L \to]-\infty, \infty]$, which is defined by $I(X) = \sup\limits_{\mathrm{Q} \in cl(\Delta)} \int X \, d\mathrm{Q}$.
Then under the assumptions (2.5) - (2.7), (3.1) $I(X) = \sup\limits_{\mathrm{Q} \in \Delta} X \, d\mathrm{Q}$ for $X \in L$, and the following statements are equivalent:

.1 Δ is relatively compact w.r.t. τ_w.

.2 I is real-valued, and $I(X_n) \searrow I(X)$ whenever $(X_n)_n$ is an antitone sequence in L with $X_n \searrow X \in L$.

Furthermore each of the statements .1, .2 implies

.3 ν is real-valued, and $\nu|\widetilde{\mathcal{S}}$ is downward continuous.

If in addition assumption (2.8) is satisfied, then the statements .1 - .3 are equivalent, and each of them is equivalent with

.4 ν is real-valued, and $\nu|\widetilde{\mathcal{S}}$ is downward continuous at $\emptyset$ with $\nu(A) = \inf\limits_{A \subseteq G \in \widetilde{\mathcal{S}}\perp} \sup\limits_{G \supseteq B \in \widetilde{\mathcal{S}}} \nu(B)$ for each $A \in \widetilde{\mathcal{S}}$.

Proof:

Firstly, Portmanteau lemma 3.1 means that $\tau_w = \tau_{w,E}$ under (2.5) - (2.7) and (3.1). Hence in this situation $I(X) = \sup\limits_{\mathrm{Q} \in \Delta} \int X \, d\mathrm{Q}$ holds for each $X \in L$.
Since (3.1) induces $\widetilde{\mathcal{S}}\perp \subseteq \widetilde{\mathcal{S}}^{\sigma}$, the equivalence of .1, . 3, .4 in the case of (2.5) - (2.8) follows immediately from Theorem 4.1, whereas under (2.5) - (2.7), (3.1) the implication .1 $\Rightarrow$ .3 can be shown as in the proof of Theorem 4.1. Therefore it remains to prove equivalence of .1 and .2 if (2.5) - (2.7) and (3.1) are valid.
<u>proof of .1 $\Rightarrow$ .2</u>:
For every $X \in L$ the mapping $\psi_X : \mathcal{M}_f(\Omega, \mathcal{S}) \to \mathbb{R}, \ \mathrm{Q} \mapsto \int X \, d\mathrm{Q}$ is continuous w.r.t. the weak

topology due to the general Portmanteau lemma. Therefore I is real-valued by statement .1. Let $(X_n)_n$ be an antitone sequence in L with $X_n \searrow X$ for some $X \in L$. Then the general Dini lemma (cf. [7], Theorem 3.7) yields

$$I(X) = \sup_{\mathrm{Q}\in cl(\Delta)} \psi_X(\mathrm{Q}) = \sup_{\mathrm{Q}\in cl(\Delta)} \inf_n \psi_{X_n}(\mathrm{Q}) = \inf_n \sup_{\mathrm{Q}\in cl(\Delta)} \psi_{X_n}(\mathrm{Q}) = \inf_n I(X_n),$$

which shows statement .2.

<u>proof of .2 $\Rightarrow$ .1:</u>

Let L^* denote the space of real linear forms on L. It will be equipped with the so called weak $*$ topology, i.e. the relative topology of the product topology on $\mathbb{R}^L$ to L^*.

By assumption I is a real sublinear form on L, and it is therefore associated with the nonvoid set $\Delta(I)$ consisting of all $\Lambda \in L^*$ with $\Lambda \leq I$. It is known that $\Delta(I)$ is a compact subset w.r.t. the weak $*$ topology. This follows from $\Delta(I) = L^* \cap \mathop{\times}_{X\in L} [-I(-X), I(X)]$. This description also ensures that every linear form from $\Lambda(I)$ is positive, in particular the restrictions to E are isotone and positive-linear.

Let $(\mathrm{Q}_j)_{j\in J}$ be a net in Δ. Then, defining $\Lambda_{\mathrm{Q}_j} \in L^*$ by $\Lambda_{\mathrm{Q}_j}(X) = \int X \, d\,\mathrm{Q}_j$, we obtain by $(\Lambda_{\mathrm{Q}_j})_{j\in J}$ a net in $\Lambda(I)$. Compactness of $\Lambda(I)$ implies that there is a subnet $(\Lambda_{\mathrm{Q}_{j(k)}})_{k\in K}$ which converges to some $\Lambda \in \Lambda(I)$ w.r.t. the weak $*$ topology. Moreover, statement .2 ensures that $\lim_n \Lambda(X_n) = 0 = \lim_n \Lambda(Y - Y_n)$ holds for any antitone sequence $(X_n)_n$ in E with $X_n \searrow 0$ and every isotone sequence $(Y_n)_n$ in E with $Y_n \nearrow Y \in E$. Thus we may apply the inner Daniell-Stone theorem 1.3 to $\Lambda|E$. Hence we can find some $\mathrm{P} \in \mathcal{M}_f(\Omega, \widetilde{\mathcal{S}})$ with $\Lambda(X) = \int X \, d\,\mathrm{P}$ for $X \in E$. Drawing on Theorem 1.2, P can be extended uniquely to some finite measure $\mathrm{Q} \in \mathcal{M}_f(\Omega, \mathcal{S})$, so that $\Lambda(X) = \int X \, d\,\mathrm{Q}$ holds for $X \in E$. In particular $\lim_k \int X \, d\,\mathrm{Q}_{j(k)} = \int X \, d\,\mathrm{Q}$ for every $X \in E$. That means that $(\mathrm{Q}_{j(k)})_{k\in K}$ converges to Q w.r.t. the weak topology due to the general Portmanteau lemma. Thus Δ is relatively compact, which completes the proof. ∎

In the following we shall present a criterion to replace condition (2.8), which implies the following variant of Theorem 4.3.

Corollary 4.4 *Let us retake notations and assumptions from Theorem 4.3. If* $\frac{X}{Y} \in E$ *holds for* $X, Y \in E$ *with* $Y > 0$*, then under the assumptions (2.5) - (2.7) and (3.1) all the four statements from Theorem 4.3 are equivalent.*

Proof:

It remains to prove that assumption (2.8) is valid. For this purpose let $A_1, A_2 \in \widetilde{\mathcal{S}}$ be disjoint. By definition there exist $X_1, X_2 \in E$ and positive numbers x_1, x_2 with $A_i = X_i^{-1}([x_i, \infty[)$ for $i = 1, 2$. Then $Z_i := 1 - \frac{\min\{X_i, x_i\}}{x_i}$ belongs to E with $0 \leq Z_i \leq 1$ and $A_i = Z_i^{-1}(\{0\})$ for $i = 1, 2$. Since A_1, A_2 are disjoint, $Z_1 + Z_2 \in E$ with $Z_1 + Z_2 > 0$. Hence by assumption $Y_i := \frac{Z_i}{Z_1 + Z_2}$ is a member of E for $i = 1, 2$. Thus, $Y_i^{-1}([0, \frac{1}{4}[)$ $(i = 1, 2)$ are disjoint elements of $\widetilde{\mathcal{S}}\perp$ with $A_i \subseteq Y_i^{-1}([0, \frac{1}{4}[)$ for $i = 1, 2$, which completes the proof. ■

Corollary 4.4 is useful to characterize relatively compact subsets of finite Baire measures, and finite Borel-measures which are inner regular w.r.t. closed subsets. Let us start with relatively compact subsets of finite Baire measures on Hausdorff spaces. We may apply Corollary 4.4 directly.

Corollary 4.5 *Let* τ_Ω *be a Hausdorff topology on* Ω*, and let* $\mathcal{S}, \mathcal{T}$ *be respectively the sets of functionally closed and functionally open subsets w.r.t.* τ_Ω*. Furthermore* $cl(\Delta)$ *denotes the closure of a set* Δ *of finite Baire-measures w.r.t. the topology generated by weak convergence, and induces the mapping* $\nu := \sup\limits_{\mathrm{Q} \in cl(\Delta)} \mathrm{Q}$*. Additionally let* L *consist of all bounded real-valued continuous mappings on* Ω*, and let* $I : L \to]-\infty, \infty]$ *be defined by* $I(X) = \sup\limits_{\mathrm{Q} \in cl(\Delta)} \int X \, d\mathrm{Q}$*.*

Then the following statements are equivalent:

.1 Δ *is relatively compact w.r.t. the topology induced by weak convergence.*

.2 ν *is real-valued, and* $\nu|\mathcal{S}$ *is downward continuous.*

.3 ν *is real-valued, and* $\nu|\mathcal{S}$ *is downward continuous at* $\emptyset$ *with* $\nu(A) = \inf_{A \subseteq G \in \mathcal{T}} \sup_{G \supseteq B \in \mathcal{S}} \nu(B)$ *for each* $A \in \mathcal{S}$.

.4 I *is real-valued, and* $\lim_{n \to \infty} I(X_n) = I(X)$ *if* $(X_n)_n$ *is an antitone sequence in* L *with* $X_n \searrow X \in L$.

Remark:

Varadarajan has also shown the equivalence of the statements .1, .4 in Corollary 4.5 (cf. [19], Theorem 25).

In order to apply Corollary 4.4 to finite Borel-measures let us consider a normal and countably paracompact topology on Ω. For instance perfectly normal and metrizable topologies satisfy these properties (cf. [4], 5.2.5, 4.1.13). Then we may choose for $\mathcal{S}$ the lattice of all closed subsets, and for $\widetilde{\mathcal{S}}$ the set of all functionally closed subsets. Additionally E is defined to consist of all nonnegative bounded real-valued continuous mappings on Ω. Noticing Urysohn's lemma, $\mathcal{S}, \widetilde{\mathcal{S}}$ and E satisfy the requirements of Corollary 4.4 to guarantee the equivalence of all the statements there. Then Theorem 4.3 reads as follows.

Corollary 4.6 *Let* (Ω, τ_Ω) *be a normal and countably paracompact space, and let* $\mathcal{S}, \widetilde{\mathcal{S}}, \mathcal{T}$ *be respectively the set of closed, functionally closed and functionally open subsets w.r.t.* τ_Ω. *Furthermore* Δ *denotes a set of finite Borel-measures which are inner regular w.r.t. the closed subsets, and let* $cl(\Delta)$ *be the closure of* Δ *w.r.t. the topology generated by weak convergence. Additionally, let* L *consist of all bounded real-valued continuous mappings on* Ω, *and let* $I : L \to]-\infty, \infty]$ *be defined by* $I(X) = \sup_{\mathrm{Q} \in cl(\Delta)} \int X \, d\mathrm{Q}$.
Then, setting $\nu := \sup_{\mathrm{Q} \in cl(\Delta)} \mathrm{Q}$, *the following statements are equivalent:*

.1 Δ *is relatively compact w.r.t. the topology induced by weak convergence.*

.2 ν *is real-valued, and* $\nu|\widetilde{\mathcal{S}}$ *is downward continuous.*

.3 ν is real-valued, and $\nu|\widetilde{\mathcal{S}}$ is downward continuous at $\emptyset$ with $\nu(A) = \inf_{A\subseteq G\in\mathcal{T}} \sup_{G\supseteq B\in\widetilde{\mathcal{S}}} \nu(B)$ for each $A \in \widetilde{\mathcal{S}}$.

.4 I is real-valued, and $\lim_{n\to\infty} I(X_n) = I(X)$ if $(X_n)_n$ is an antitone sequence in L with $X_n \searrow X \in L$.

Remark:

Corollary 4.6 encompasses the case that (Ω, τ_Ω) is perfectly normal. Since in perfectly normal spaces all closed subsets are functionally closed, and each open subset is functionally open (cf. [4], 1.5.19), we may replace then in Corollary 4.6 $\widetilde{\mathcal{S}}$ by $\mathcal{S}$ and $\mathcal{T}$ by τ_Ω. Moreover, Corollary 4.6 generalizes also a result by Huber and Strassen who showed the equivalence of statements .1, .2 in Corollary 4.6 for probability measures on Polish spaces ([6]). Note that metrizable topologies are perfectly normal.

A Appendix

Proof of Theorem 1.2:

$\phi := \mathrm{P}\,|\widetilde{\mathcal{S}}$ is an inner premeasure, and P is the restriction of the maximal extension $\mu_\phi : \mathcal{F}_\phi \to \mathbb{R}$ of ϕ. Additionally, let the mapping $\varphi_\phi : \mathcal{S}_\sigma \to \mathbb{R}$ be defined by $\varphi_\phi(A) = \inf\{\mu_\phi(B) \mid A \subseteq B \in \mathcal{F}_\phi\}$. Obviously, $\varphi_\phi(A) = \inf\{\mathrm{P}(B) \mid A \subseteq B \in \widetilde{\mathcal{S}}_\sigma\perp\}$ holds for every $A \in \mathcal{S}_\sigma$ since μ_ϕ is outer regular w.r.t. $\widetilde{\mathcal{S}}_\sigma\perp$ and extends P .

The restriction $\varphi_\phi|\mathcal{S}$ is downward continuous at $\emptyset$ due to assumption (1) of Theorem 1.2.

Next, let $A_1, A_2 \in \mathcal{S}_\sigma$ be disjoint. Under assumption (2) of Theorem 1.2 we may find disjoint sets B_1, B_2 from $\sigma(\widetilde{\mathcal{S}})$ with $A_i \subseteq B_i$ for $i = 1, 2$. Furthermore there exists for arbitrary $\varepsilon > 0$ some $B \in \sigma(\widetilde{\mathcal{S}})$ with $A_1 \cup A_2 \subseteq B$ and $\varphi_\phi(A_1 \cup A_2) + \varepsilon > \mu_\phi(B)$. Then

$$\varphi_\phi(A_1 \cup A_2) + \varepsilon > \mu_\phi(B) \geq \mu_\phi(B \cap B_1) + \mu_\phi(B \cap B_2) \geq \varphi_\phi(A_1) + \varphi_\phi(A_2).$$

Hence the statement of Theorem 1.2 follows from Proposition 1.1, .2. ■

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