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Conditional and Dynamic Convex Risk Measures

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Abstract. We extend the definition of a convex risk measure to a conditional framework where additional information is available. We characterize these risk measures through the associated acceptance sets and prove a representation result in terms of conditional expectations. As an example we consider the class of conditional entropic risk measures. A new regularity property of conditional risk measures is defined and discussed. Finally we introduce the concept of a dynamic convex risk measure as a family of successive conditional convex risk measures and characterize those satisfying some natural time consistency properties.

Key words: Conditional convex risk measure, robust representation, regularity, entropic risk measure, dynamic convex risk measure, time consistency

JEL Classification: D81

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1 Introduction

In recent years a growing attention has been devoted to an axiomatic treatment of the quantification of financial risks. Artzner et al. proposed in their seminal work [1] a set of desirable axioms that every risk measure should satisfy, defining in such a way the class of coherent risk measures. Delbaen [4] proved that, under a mild continuity assumption, every coherent risk measure can be represented as worst expected loss with respect to a given set of probabilistic models. Föllmer and Schied [8] and Frittelli and Rosazza Gianin [9] introduced independently the more general class of convex risk measures weakening the axioms of positive homogeneity and subadditivity by replacing them with convexity. They also extended Delbaen's representation result allowing for the occurrence of a penalty function defined on a set of probabilistic models. A common feature of all these approaches is the static temporal setting, i.e. the risk measures do not accommodate for intermediate payoffs or additional information. These two issues have been addressed simultaneously in T.Wang [15], Riedel [11] and Weber [16]. We concentrate on the information aspect and investigate on a general probability space conditional risk measures.

This paper aims at giving a possible axiomatic foundation to the risk assessment of final payoffs when additional information is available. This is the case, for example, when the riskiness of a payoff occurring at time T is quantified at an intermediate date $t \in (0, T)$. We define conditional convex risk measures as maps, satisfying some natural axioms, which associate to every payoff, represented by a random variable X, its riskiness $\rho(X)$ which is itself a random variable, depending on the available information. Furthermore, under a mild technical assumption, we give a characterization of these maps as worst conditional expected loss with respect to a given set of probabilistic models, maybe corrected by some random penalty function. A new regularity property is introduced and several equivalent formulations are presented; this property, which is economically plain, states that $\rho(X)$ should not depend on that part of the future which is ruled out by the additional information. As an example for conditional convex risk measures, the class of entropic risk measures, as defined in [7], is generalized to the conditional setting. These risk measures are first defined as capital requirements with respect to an utility-based acceptability criterion. Then their penalty functions are identified as the conditional relative entropy between the considered probabilistic models and a reference model. The last part of the paper is devoted to a study of dynamic convex risk measures, i.e. families of conditional convex risk measures, describing the risk assessment of a final payoff at successive dates. We introduce two economically motivated properties of time consistency that relate different components of a dynamic convex risk measure. Finally, we provide some characterizations of these properties in terms of the family of penalty functions of their components.

The paper is organized as follows. In Section 2 we define conditional convex risk measures by generalizing the translation invariance and convexity axioms. We also provide a characterization of these risk measures as conditional capital requirements with respect to suitable acceptance sets. In Section 3 we show that, under a continuity assumption, every conditional convex risk measure can be represented as worst conditional expected loss with a random penalty function defined on a set of probabilistic models. Section 4 contains a discussion of a regularity property which is shared by every conditional convex risk measure. In Section 6 the class of conditional entropic risk measures is introduced as an example for conditional convex risk measures. Section 6 is devoted to dynamic convex risk measures and contains a discussion of natural time consistency properties. Finally, the Appendix collects definitions and some useful results about extended valued random variables and essential suprema.

2 Conditional convex risk measures

We denote with L^0, resp. L^∞, the space of random variables, resp. bounded random variables, defined on some fixed probability space $(\Omega, \mathcal{F}, P)$. A good feature of these spaces is their invariance with respect to the probability measure, provided it is chosen in the equivalence class of P. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra and define the two subspaces

$$\begin{aligned} L^0_{\mathcal{G}} &\triangleq \{X \in L^0 \mid X \text{ is } \mathcal{G}\text{-measurable}\}, \\ L^\infty_{\mathcal{G}} &\triangleq L^\infty \cap L^0_{\mathcal{G}}. \end{aligned}$$

Finally, we define the following two sets of probability measures:

$$\begin{aligned} \mathcal{P} &\triangleq \{Q \text{ probability measure on } (\Omega, \mathcal{F}) \mid Q \ll P \text{ on } \mathcal{F}\} \\ \mathcal{P}_{\mathcal{G}} &\triangleq \{Q \in \mathcal{P} \mid Q \equiv P \text{ on } \mathcal{G}\}. \end{aligned}$$

The probability measures in $\mathcal{P}$ can be interpreted as *probabilistic models*. An element $X \in L^\infty$ describes a random net payoff to be delivered to an agent at a fixed future date. The σ-algebra $\mathcal{G}$ collects the information available to the agent who is assessing the riskiness of the payoff X. As a consequence the risk measurement of X leads to a random variable $\rho(X)$ which is measurable with respect to $\mathcal{G}$, i.e. an element of the space $L^0_{\mathcal{G}}$. We are thus studying maps of the type $\rho : L^\infty \to L^0_{\mathcal{G}}$, to be called *conditional risk measures*. Plainly, we interpret $\rho(X)(\omega)$ as the degree of riskiness of X when the state ω prevails.

Remark 2.1 The σ-algebra $\mathcal{G}$ can be interpreted in different ways. It can model additional information available at date $t = 0$ to the agent. Alternatively, it can be interpreted as information available at a future date $t > 0$, resulting from the observation of some variables related to the payoff X in the time interval $[0, t]$. In both cases, the sources of information can be public, i.e. shared by all agents, or private. Hence, conditional risk measures open a way to the analysis of the consequences of asymmetric information for risk measurement. $\diamondsuit$

Consider the following three properties to be shared by a conditional risk measure ρ:

- **(Conditional) Translation Invariance** For any $X \in L^\infty$ and $Z \in L^\infty_{\mathcal{G}}$:

$$\rho(X + Z) = \rho(X) - Z$$

- **Monotonicity**. For any $X, Y \in L^\infty$:

$$X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$$

- **(Conditional) Convexity**. For any $X, Y \in L^\infty$ and $\Lambda \in L^\infty_{\mathcal{G}}$ with $0 \leq \Lambda \leq 1$:

$$\rho(\Lambda X + (1 - \Lambda)Y) \leq \Lambda\rho(X) + (1 - \Lambda)\rho(Y).$$

Definition 2.2 A map $\rho : L^\infty \to L^\infty_\mathcal{G}$ is called a *conditional convex risk measure* if it is translation invariant, monotone, convex and satisfies $\rho(0) = 0$.

If we have no initial information, i.e. $\mathcal{G}$ is the trivial σ-algebra, then the definition of a conditional convex risk measure coincides with the unconditional one.

Remark 2.3 Some economic considerations to be discussed in Section 4 suggest to assume that $\rho(0)$ is a constant random variable. The choice $\rho(0) = 0$ has no particular economic relevance, but it allows mathematical simplification as it implies $\rho(\alpha) = -\alpha$ for every $\alpha \in \mathbb{R}$. However, from the mathematical viewpoint, the assumption $\rho(0) \in L^\infty_\mathcal{G}$ is sufficient to ensure the validity of the following results. $\diamondsuit$

Remark 2.4 We do not loose any generality by assuming that the values of a conditional convex risk measure are bounded. Indeed, if $X \in L^\infty$, then $-||X||_\infty \leq X \leq ||X||_\infty$, so that

$$-\infty < -||X||_\infty = \rho(||X||_\infty) \leq \rho(X) \leq \rho(-||X||_\infty) = ||X||_\infty < +\infty$$

and as a consequence $\rho(X) \in L^\infty_\mathcal{G}$. $\diamondsuit$

The economic rationale behind the properties characterizing conditional convex risk measures is the same as in the unconditional case (see [1], [8], [9]). In particular, translation invariance provides the interpretation of a convex risk measure ρ as (conditional) capital requirement. Indeed, it is easy to show that the conditional risk measure ρ is translation invariant if and only if

$$\rho(X) = \text{ess.inf}\,\{Y \in L^\infty_\mathcal{G} \mid X + Y \in \mathcal{A}_\rho\}, \tag{1}$$

where

$$\mathcal{A}_\rho \triangleq \{X \in L^\infty_\mathcal{G} \mid \rho(X) \leq 0\}$$

is called the *acceptance set* of ρ. The following proposition states some important relations between conditional risk measures and acceptance sets. We refer to the Appendix for the definition and some properties of the essential infimum.

Proposition 2.5 *If ρ is a conditional convex risk measure, then its acceptance set $\mathcal{A}_\rho$ is:*

a. conditionally convex, i.e. $\Lambda\mathcal{A}_\rho + (1-\Lambda)\mathcal{A}_\rho \subseteq \mathcal{A}_\rho$ for every $\Lambda \in L^\infty_\mathcal{G}$ with $0 \leq \Lambda \leq 1$,

b. solid, i.e. $X \geq Y \in \mathcal{A}_\rho \Rightarrow X \in \mathcal{A}_\rho$,

c. such that $\text{ess.inf}\,\mathcal{A}_\rho = 0$ *and* $0 \in \mathcal{A}_\rho$.

Conversely, if a set $\mathcal{A} \subset L^\infty$ satisfies the previous properties, then the map

$$\rho_\mathcal{A}(X) \triangleq \text{ess.inf}\,\{Y \in L^\infty_\mathcal{G} \mid X + Y \in \mathcal{A}\}, \quad X \in L^\infty,$$

is a conditional convex risk measure.

Proof. The properties of acceptance sets follow directly from the definition of conditional risk measures. Concerning the second claim, it is straightforward to verify that $\rho_{\mathcal{A}}$ satisfies translation invariance, monotonicity and the normalization property $\rho_{\mathcal{A}}(0) = 0$; it follows in particular that $\rho_{\mathcal{A}}$ takes values in $L^\infty_{\mathcal{G}}$. We show that $\rho_{\mathcal{A}}$ is conditionally convex. Suppose that $X, Y \in L^\infty$ and that $Z_X, Z_Y \in L^\infty_{\mathcal{G}}$ are such that $Z_X + X \in \mathcal{A}$ and $Z_Y + Y \in \mathcal{A}$. For $\Lambda \in L^\infty_{\mathcal{G}}$ with $0 \leq \Lambda \leq 1$, the conditional convexity of $\mathcal{A}$ implies $\Lambda(Z_X + X) + (1-\Lambda)(Z_Y + Y) \in \mathcal{A}$. Hence, by the translation invariance of $\rho_{\mathcal{A}}$,

$$\begin{aligned} 0 &\geq \rho_{\mathcal{A}}(\Lambda(Z_X + X) + (1-\Lambda)(Z_Y + Y)) \\ &= \rho_{\mathcal{A}}(\Lambda X + (1-\Lambda)Y) - (\Lambda Z_X + (1-\Lambda)Z_Y) \end{aligned}$$

and the conditional convexity of $\rho_{\mathcal{A}}$ follows. $\square$

3 A robust representation result

In the unconditional case, we remind that every convex risk measure $\rho : L^\infty \to \mathbb{R}$ which is continuous in a mild sense, admits the representation:

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{-E_Q X - \alpha(Q)\}, \quad X \in L^\infty, \tag{2}$$

in terms of a so called *penalty function* $\alpha : \mathcal{P} \to [0, +\infty]$; see [6] or [9] for a proof. We prove below that a similar characterization holds as well for conditional convex risk measures which are continuous in a sense to be specified. In this more general representation formula, the expectations are conditional on the available information $\mathcal{G}$, the penalty function is random-valued and the supremum is understood in the essential sense. Moreover, in the conditional case the additional information allows to exclude a-priori some probabilistic models. In fact, we show that only the models in $\mathcal{P}_{\mathcal{G}} \subseteq \mathcal{P}$ may enter the representation. This fact can be interpreted in economic terms as caution: the smaller the information $\mathcal{G}$, the larger is the subset $\mathcal{P}_{\mathcal{G}}$ of probabilistic models which can be considered in the worst case representation. For ease of presentation we give the following definition where $L^0_{\mathcal{G}}(\overline{\mathbb{R}}_+) \triangleq \{X \in L^0(\overline{\mathbb{R}}) \mid X \text{ is } \mathcal{G}\text{-measurable}, X \geq 0\}$ (see the Appendix).

Definition 3.1 A map $\rho : L^\infty \to L^\infty_{\mathcal{G}}$ is said to be *representable* if

$$\rho(X) = \operatorname*{ess.sup}_{Q \in \mathcal{P}_{\mathcal{G}}} \{-E_Q(X \mid \mathcal{G}) - \alpha(Q)\}, \quad X \in L^\infty \tag{3}$$

for a map $\alpha : \mathcal{P}_{\mathcal{G}} \to L^0_{\mathcal{G}}(\overline{\mathbb{R}}_+)$. In this case, α is called a *(random) penalty function* for ρ.

It is immediate to check that any representable map with a penalty function α satisfying

$$\operatorname*{ess.inf}_{Q \in \mathcal{P}_{\mathcal{G}}} \alpha(Q) = 0$$

is a conditional convex risk measure. Under a mild continuity condition the converse holds as well as the following theorem shows.

Theorem 3.2 *Let $\rho : L^\infty \to L^\infty_{\mathcal{G}}$ be a conditional convex risk measure. Then the following are equivalent:*

a. ρ is continuous from above, i.e. $X_n \searrow X$ P-a.s. implies $\rho(X_n) \nearrow \rho(X)$ P-a.s.;

b. ρ is representable;

c. ρ is representable in terms of

$$\alpha^*(Q) \triangleq \underset{X \in L^\infty}{\text{ess.sup}} \{-E_Q(X \mid \mathcal{G}) - \rho(X)\}, \quad Q \in \mathcal{P}_{\mathcal{G}}.$$

Proof. $c \Longrightarrow b$ This implication follows immediately.
$b \Longrightarrow a$. Suppose that ρ is representable with a penalty function α and that $X_n \searrow X$ P-a.s. By monotone convergence we have that

$$-E_Q(X_n \mid \mathcal{G}) - \alpha(Q) \nearrow -E_Q(X \mid \mathcal{G}) - \alpha(Q)$$

for every $Q \in \mathcal{P}_{\mathcal{G}}$. Hence, the robust representation yields

$$\begin{aligned}
\rho(X) &= \underset{Q \in \mathcal{P}_{\mathcal{G}}}{\text{ess.sup}} \{\lim_{n\to\infty} [-E_Q(X_n \mid \mathcal{G}) - \alpha(Q)]\} \\
&\leq \liminf_{n\to\infty} \underset{Q \in \mathcal{P}_{\mathcal{G}}}{\text{ess.sup}} \{-E_Q(X_n \mid \mathcal{G}) - \alpha(Q)\} \\
&= \liminf_{n\to\infty} \rho(X_n).
\end{aligned}$$

On the other hand, the monotonicity of ρ implies $\liminf_{n\to\infty} \rho(X_n) \leq \rho(X)$.
$a \Longrightarrow c$. The inequality

$$\rho(X) \geq \underset{Q \in \mathcal{P}_{\mathcal{G}}}{\text{ess.sup}} \{-E_Q(X \mid \mathcal{G}) - \alpha^*(Q)\}$$

easily follows from the definition of α^*. Indeed, for every $Q \in \mathcal{P}_{\mathcal{G}}$ and $X \in L^\infty$ it holds $\rho(X) \geq -E_Q(X \mid \mathcal{G}) - \alpha^*(Q)$ and thus the inequality is recovered taking the supremum over all $Q \in \mathcal{P}_{\mathcal{G}}$.
Hence, in order to prove the representability of ρ it suffices to show that

$$E_P[\rho(X)] \leq E_P[\underset{Q \in \mathcal{P}_{\mathcal{G}}}{\text{ess.sup}} \{-E_Q(X \mid \mathcal{G}) - \alpha^*(Q)\}].$$

To this end, consider the map $\rho_0 : L^\infty \to \mathbb{R}$ defined by $\rho_0(X) \triangleq E_P[\rho(X)]$, $X \in L^\infty$. It is simple to check that ρ_0 is a (unconditional) convex risk measure; furthermore, if $X_n \searrow X$ P-a.s., then $\rho(X_n) \nearrow \rho(X)$ P-a.s. and, by monotone convergence

$$\rho_0(X_n) = E_P[\rho(X_n)] \nearrow E_P[\rho(X)] = \rho_0(X),$$

so that ρ_0 is continuous from above. Hence, Theorem 4.26 in [8] implies that ρ_0 has the following representation:

$$\rho_0(X) = \sup_{Q \in \mathcal{P}} \{-E_Q X - \alpha_0^*(Q)\},$$

where

$$\alpha_0^*(Q) \triangleq \sup_{X \in L^\infty} \{-E_Q X - \rho_0(X)\}.$$

We now prove that if $\alpha_0^*(Q) < +\infty$, then $Q \in \mathcal{P}_\mathcal{G}$. To this end, note that if $X \in L^\infty_\mathcal{G}$, then by translation invariance of ρ we have $\rho_0(X) = E_P[\rho(X)] = -E_P X$. Suppose now that $Q(A) \neq P(A)$ for some $A \in \mathcal{G}$, then

$$\begin{aligned} \alpha_0^*(Q) &\geq \sup_{\lambda \in \mathbb{R}} \{-E_Q(\lambda I_A) - \rho_0(\lambda I_A)\} \\ &= \sup_{\lambda \in \mathbb{R}} \{-\lambda Q(A) + \lambda P(A)\} = +\infty. \end{aligned}$$

Since $E_Q X = E_Q[E_Q(X \mid \mathcal{G})] = E_P[E_Q(X \mid \mathcal{G})]$ for every $Q \in \mathcal{P}_\mathcal{G}$, it follows that

$$\rho_0(X) = \sup_{Q \in \mathcal{P}_\mathcal{G}} \{-E_P[E_Q(X \mid \mathcal{G})] - \alpha_0^*(Q)\}.$$

Next we prove that $E_P[\alpha^*(Q)] = \alpha_0^*(Q)$ for every $Q \in \mathcal{P}_\mathcal{G}$. We claim that for every $Q \in \mathcal{P}_\mathcal{G}$ the set $\mathcal{B}_Q \triangleq \{-E_Q(X \mid \mathcal{G}) - \rho(X) \mid X \in L^\infty\}$ is upward directed (see the Appendix). In fact, if $X, Y \in L^\infty$ we can define $Z \triangleq X I_A + Y I_{A^c} \in L^\infty$, where $A \triangleq \{-E_Q(X \mid \mathcal{G}) - \rho(X) \geq -E_Q(Y \mid \mathcal{G}) - \rho(Y)\} \in \mathcal{G}$. Since both I_A and I_{A^c} are $\mathcal{G}$-measurable, $0 \leq I_A \leq 1$ and $I_{A^c} = 1 - I_A$, the conditional convexity of ρ yields:

$$\rho(Z) = \rho(X I_A + Y I_{A^c}) \leq I_A \rho(X) + I_{A^c} \rho(Y).$$

As a consequence

$$\begin{aligned} -E_Q(Z \,|\mathcal{G}) - \rho(Z) = \\ &= -\, E_Q(X I_A + Y I_{A^c} \mid \mathcal{G}) - \rho(X I_A + Y I_{A^c}) \\ &\geq [-E_Q(X \mid \mathcal{G}) - \rho(X)] I_A + [-E_Q(Y \mid \mathcal{G}) - \rho(Y)] I_{A^c} \\ &\geq \max(-E_Q(X \mid \mathcal{G}) - \rho(X), -E_Q(Y \mid \mathcal{G}) - \rho(Y)), \end{aligned}$$

thanks to the definition of A, and therefore $\mathcal{B}_Q$ is upward directed.
Then it follows by Lemma A.2, for any $Q \in \mathcal{P}_\mathcal{G}$,

$$\begin{aligned} E_P[\alpha^*(Q)] &= E_P[\operatorname*{ess.sup}_{X \in L^\infty} \{-E_Q(X \mid \mathcal{G}) - \rho(X)\}] \\ &= \sup_{X \in L^\infty} \{-E_P E_Q(X \mid \mathcal{G}) - E_P[\rho(X)]\} \\ &= \sup_{X \in L^\infty} \{-E_Q X - \rho_0(X)\} = \alpha_0^*(Q). \end{aligned}$$

Hence, we get

$$\begin{aligned} E_P[\rho(X)] = \rho_0(X) &= \sup_{Q \in \mathcal{P}_\mathcal{G}} \{-E_P[E_Q(X \mid \mathcal{G})] - E_P[\alpha^*(Q)]\} \\ &\leq E_P[\operatorname*{ess.sup}_{Q \in \mathcal{P}_\mathcal{G}} \{-E_Q(X \mid \mathcal{G}) - \alpha^*(Q)\}], \end{aligned}$$

and the proof is complete. $\square$

Remark 3.3 Analogue to the unconditional case, the penalty function α^* in the preceding theorem is the *minimal* penalty function which may enter a robust representation for ρ, i.e. $\alpha^* \leq \alpha$ for all penalty functions α for ρ. The following useful equality holds

$$\alpha^*(Q) = \operatorname*{ess.sup}_{X \in \mathcal{A}_\rho} \{-E_Q(X \mid \mathcal{G})\}, \quad Q \in \mathcal{P}_\mathcal{G},$$

as in the unconditional case. $\Diamond$

The following lemma will be useful in Section 6.

Lemma 3.4 *If α^* is the minimal penalty function of a representable conditional risk measure ρ and $\mathcal{H} \subseteq \mathcal{G}$ is a sub-σ-algebra, then*

$$E_P(\rho(X) \mid \mathcal{H}) = \operatorname*{ess.sup}_{Q \in \mathcal{P}_\mathcal{G}} \{-E_Q(X \mid \mathcal{H}) - E_P(\alpha^*(Q) \mid \mathcal{H})\}, \ X \in L^\infty.$$

Proof. First we prove that the set

$$\mathcal{C}_X \triangleq \{-E_Q(X \mid \mathcal{G}) - \alpha^*(Q) \mid Q \in \mathcal{P}_\mathcal{G}\}$$

is upward directed. Indeed, for any $Q', Q'' \in \mathcal{P}_\mathcal{G}$ define the probability measure Q on $\mathcal{F}$ by

$$Q(B) \triangleq Q'(A \cap B) + Q''(A^c \cap B), \quad B \in \mathcal{F},$$

where

$$A \triangleq \{-E_{Q'}(X \mid \mathcal{G}) - \alpha^*(Q') \geq -E_{Q''}(X \mid \mathcal{G}) - \alpha^*(Q'')\} \in \mathcal{G}.$$

It is immediate to observe that $Q \in \mathcal{P}_\mathcal{G}$ and that $E_Q(X \mid \mathcal{G}) = I_A E_{Q'}(X \mid \mathcal{G}) + I_{A^c} E_{Q''}(X \mid \mathcal{G})$ for any $X \in L^\infty$. By applying Lemma A.3 we obtain

$$\begin{aligned}
\alpha^*(Q) &= \operatorname*{ess.sup}_{X \in L^\infty} \{-E_Q(XI_A \mid \mathcal{G}) - I_A\rho(X)\} + \operatorname*{ess.sup}_{X \in L^\infty} \{-E_Q(XI_{A^c} \mid \mathcal{G}) - I_{A^c}\rho(X)\} \\
&= \operatorname*{ess.sup}_{X \in L^\infty} \{-I_A E_{Q'}(X \mid \mathcal{G}) - I_A\rho(X)\} + \operatorname*{ess.sup}_{X \in L^\infty} \{-I_{A^c} E_{Q''}(X \mid \mathcal{G}) - I_{A^c}\rho(X)\} \\
&= I_A\alpha^*(Q') + I_{A^c}\alpha^*(Q'').
\end{aligned}$$

As a consequence

$$\begin{aligned}
&-E_Q(X \mid \mathcal{G}) - \alpha^*(Q) = \\
&\qquad = I_A(-E_{Q'}(X \mid \mathcal{G}) - \alpha^*(Q')) + I_{A^c}(-E_{Q''}(X \mid \mathcal{G}) - \alpha^*(Q'')) \\
&\qquad \geq \max(-E_{Q'}(X \mid \mathcal{G}) - \alpha^*(Q'), -E_{Q''}(X \mid \mathcal{G}) - \alpha^*(Q'')),
\end{aligned}$$

by definition of A. Hence, the set $\mathcal{C}_X$ is upward directed.
Then Lemma A.2 in its conditional form yields

$$\begin{aligned} E_P(\rho(X) \mid \mathcal{H}) &= \operatorname*{ess.sup}_{Q\in\mathcal{P}_\mathcal{G}} \{E_P(-E_Q(X \mid \mathcal{G}) - \alpha^*(Q) \mid \mathcal{H})\} \\ &= \operatorname*{ess.sup}_{Q\in\mathcal{P}_\mathcal{G}} \{-E_Q(X \mid \mathcal{H}) - E_P(\alpha^*(Q) \mid \mathcal{H})\}, \end{aligned}$$

as we desired. □

Remark 3.5 Note that the restriction to $\mathcal{P}_\mathcal{G}$ in the choice of the probabilistic models gives sense to expressions like ess.sup $\{-E_Q(X \mid \mathcal{G}) \mid Q \in \mathcal{Q}\}$. Indeed, if $Q \in \mathcal{P}_\mathcal{G}$, then $L^0_\mathcal{G}(P) = L^0_\mathcal{G}(Q)$; therefore $E_Q(X \mid \mathcal{G}) \in L^0_\mathcal{G}(P)$ for every Q and the supremum is well-defined. In the proof of the previous theorem, we also made use of the natural equality $E_P[E_Q(X \mid \mathcal{G})] = E_P X$, which holds if and only if $Q \in \mathcal{P}_\mathcal{G}$. Moreover, it is important to observe that if Q and P are not even equivalent on $\mathcal{G}$, then it may happen that $E_Q(X \mid \mathcal{G}) \notin L^0(P)$ and in that case the expectation of X with respect to P is meaningless. ◊

Remark 3.6 A *conditional coherent risk measure* can be defined as a conditional convex risk measure that is positively homogeneous, i.e. $\rho(\Lambda X) = \Lambda\rho(X)$ for any $X \in L^\infty$ and any $\Lambda \in L^\infty_\mathcal{G}$ with $\Lambda \geq 0$. As in the unconditional case, it can easily be shown that the minimal penalty function of a representable conditional coherent risk measure ρ vanishes on the convex set

$$\mathcal{Q}^* \triangleq \{Q \in \mathcal{P}_\mathcal{G} \mid E_Q(X \mid \mathcal{G}) \geq -\rho(X)\ \forall X \in L^\infty\}$$

and takes otherwise the value $+\infty$. Therefore it can be represented as

$$\rho(X) = \operatorname*{ess.sup}_{Q^*\in\mathcal{Q}} \{-E_Q(X \mid \mathcal{G})\}.$$

Conversely, any map with such a representation is a conditional coherent risk measure. ◊

Example 3.7 For a parameter $\Lambda \in L^\infty_\mathcal{G}$ with $0 < \Lambda < 1$ we consider the set of probabilistic models

$$\mathcal{Q}_\Lambda \triangleq \{Q \in \mathcal{P}_\mathcal{G} \mid \frac{dQ}{dP} \leq \Lambda^{-1}\}.$$

The corresponding conditional coherent risk measure

$$\mathrm{AVaR}_\Lambda(X) \triangleq \operatorname*{ess.sup}_{Q\in\mathcal{Q}_\Lambda} \{-E_Q(X \mid \mathcal{G})\}, \quad X \in L^\infty$$

is called *Average Value at Risk* at level Λ. It generalizes the unconditional risk measure AVaR_λ (see e.g. Definition 4.36 in [8]). ◊

For an example of a conditional convex risk measure which is not coherent we refer to the entropic risk measures to be introduced in Section 5.

4 A regularity property

In a sense, the additional information available to the agent has to be fully used when assessing the riskiness of a payoff X. This means, in particular, that if we know that an event $A \in \mathcal{G}$ is prevailing, then the riskiness of X should depend only on what is really possible to happen, i.e. on the restriction of X to A. This simple requirement is captured by the following property.

Definition 4.1 A conditional risk measure $\rho : L^\infty \to L^\infty_\mathcal{G}$ is said to be *regular* if for every $A \in \mathcal{G}$ and $X, Y \in L^\infty$

$$XI_A = YI_A \Longrightarrow \rho(X)I_A = \rho(Y)I_A$$

Some equivalent definitions of regularity are stated in the next proposition.

Proposition 4.2 *The following are equivalent for a conditional risk measure ρ:*

a. ρ is regular;
b. $\rho(XI_A) = \rho(X)I_A$ for every $A \in \mathcal{G}$ and $X \in L^\infty$;
c. $\rho(XI_A + YI_{A^c}) = \rho(X)I_A + \rho(Y)I_{A^c}$ for every $A \in \mathcal{G}$ and $X, Y \in L^\infty$;
d. $\rho(\sum_{n=1}^N X_n I_{A_n}) = \sum_{n=1}^N \rho(X_n)I_{A_n}$ for pairwise disjoint $A_n \in \mathcal{G}$, $X_n \in L^\infty$ and $N \geq 1$.

Proof. *a.* $\Longrightarrow$ *b.* Since $XI_A = (XI_A)I_A$ and $0I_{A^c} = (XI_A)I_{A^c}$, by regularity we have $\rho(X)I_A = \rho(XI_A)I_A$ and $0 = \rho(0)I_{A^c} = \rho(XI_A)I_{A^c}$. Summing up we obtain

$$\rho(XI_A) = \rho(XI_A)I_A + \rho(XI_A)I_{A^c} = \rho(X)I_A.$$

b. $\Longrightarrow$ *c.* We have

$$\begin{aligned}
\rho(XI_A + YI_{A^c}) &= \rho(XI_A + YI_{A^c})I_A + \rho(XI_A + YI_{A^c})I_{A^c} \\
&= \rho((XI_A + YI_{A^c})I_A) + \rho((XI_A + YI_{A^c})I_{A^c}) \\
&= \rho(XI_A) + \rho(YI_{A^c}) \\
&= \rho(X)I_A + \rho(Y)I_{A^c}.
\end{aligned}$$

c. $\Longrightarrow$ *d.* The proof is an induction on N. When $N = 1$ just take $Y = 0$ in c. Suppose for induction that c. holds for every index less or equal to N. If $(A_n)_{n=1}^{N+1}$ is a family of pairwise disjoint events in $\mathcal{G}$, define $B \triangleq \cup_{n=1}^N A_n \in \mathcal{G}$. Using c. we obtain

$$\begin{aligned}
\rho(\sum_{n=1}^{N+1} X_n I_{A_n}) &= \rho(\sum_{n=1}^{N} X_n I_{A_n} I_B + X_{N+1} I_{A_{N+1}} I_{B^c}) \\
&= \rho(\sum_{n=1}^{N} X_n I_{A_n}) I_B + \rho(X_{N+1} I_{A_{N+1}}) I_{B^c} \\
&= \sum_{n=1}^{N} \rho(X_n) I_{A_n} + \rho(X_{N+1}) I_{A_{N+1}}.
\end{aligned}$$

d. $\implies$ *a.* From d. follows b. directly. If $XI_A = YI_A$ then b. implies $\rho(X)I_A = \rho(XI_A) = \rho(YI_A) = \rho(Y)I_A$ and therefore ρ is regular. $\square$

In the unconditional case, regularity holds trivially for any map $\rho : L^\infty \to L^\infty_{\mathcal{G}} = \mathbb{R}$. When $\mathcal{G}$ is not trivial, then this is not true in general. For example, the simple map $\rho(X) \triangleq E_P X,\ X \in L^\infty$ is not regular if $\mathcal{G}$ is not trivial.

Proposition 4.3 *Every conditional convex risk measure is regular.*

Proof. From the conditional convexity of $\mathcal{A}_\rho$ follows directly

$$I_A \mathcal{A}_\rho + I_{A^c} \mathcal{A}_\rho = \mathcal{A}_\rho \tag{4}$$

for all $A \in \mathcal{G}$.
Then for $X, Y \in L^\infty$ and $A \in \mathcal{G}$, we have

$$\begin{aligned}
\rho(I_A X + I_{A^c} Y) &= \text{ess.inf}\,\{Z \in L^\infty_{\mathcal{G}} \mid I_A X + I_{A^c} Y + Z \in \mathcal{A}_\rho\} \\
&= \text{ess.inf}\,\left\{L^\infty_{\mathcal{G}} \cap (\mathcal{A}_\rho - I_A X - I_{A^c} Y)\right\} \\
&= \text{ess.inf}\,\left\{L^\infty_{\mathcal{G}} \cap (I_A(\mathcal{A}_\rho - X) + I_{A^c}(\mathcal{A}_\rho - Y))\right\},
\end{aligned}$$

where the third equality follows from (4). It is immediate to see that if $\mathcal{A}, \mathcal{A}' \subseteq L^\infty$ and $A \in \mathcal{G}$, then $L^\infty_{\mathcal{G}} \cap (I_A \mathcal{A} + I_{A^c} \mathcal{A}') = I_A(L^\infty_{\mathcal{G}} \cap \mathcal{A}) + I_{A^c}(L^\infty_{\mathcal{G}} \cap \mathcal{A}')$. Hence, we obtain

$$\begin{aligned}
\rho(I_A X + I_{A^c} Y) &= \text{ess.inf}\,\left\{I_A(L^\infty_{\mathcal{G}} \cap (\mathcal{A}_\rho - X)) + I_{A^c}(L^\infty_{\mathcal{G}} \cap (\mathcal{A}_\rho - Y))\right\} \\
&= I_A\, \text{ess.inf}\,\left\{L^\infty_{\mathcal{G}} \cap (\mathcal{A}_\rho - X)\right\} + I_{A^c}\, \text{ess.inf}\,\left\{L^\infty_{\mathcal{G}} \cap (\mathcal{A}_\rho - Y)\right\} \\
&= I_A \rho(X) + I_{A^c} \rho(Y),
\end{aligned}$$

where we applied Lemma A.3 in the second equality. We conclude by applying Proposition 4.2. $\square$

Remark 4.4 Regularity of a conditional risk measure implies in particular the following natural property: if a final payoff X is constant on an event $A \in \mathcal{G}$, that is $XI_A = \gamma I_A$ P-a.s. for a constant $\gamma \in \mathbb{R}$, then $\rho(X)$ should be constant as well on that event. Indeed, if $XI_A = \gamma I_A$ P-a.s. for $\gamma \in \mathbb{R}$ and $A \in \mathcal{G}$, then according to Proposition 4.2 regularity implies:

$$\rho(X)I_A = \rho(XI_A) = \rho(\gamma I_A) = \rho(\gamma)I_A = -\gamma I_A,$$

so that ρ is constant on A as well. $\Diamond$

5 The class of conditional entropic risk measures

In the unconditional case, the notion of entropic risk measure has been introduced in [8] (see example 4.60). In the definition of this class of risk measures, it is assumed that an agent has an exponential utility $u_\gamma(x) = 1 - \exp(-\gamma x)$, with $\gamma > 0$ as risk aversion coefficient. His acceptance set is then naturally defined to be

$$\mathcal{A}_\gamma \triangleq \{X \in L^\infty \mid E_P u_\gamma(X) \geq E_P u_\gamma(0) = 0\},$$

which is solid and convex; the resulting convex risk measure $\rho_\gamma(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_\gamma\}$ is called the *entropic risk measure* associated with the risk aversion γ. It has been proved (see [8]) that the risk measure ρ_γ is continuous from above and that its minimal penalty function in the robust representation is

$$\alpha_0^*(Q) \triangleq \frac{1}{\gamma} H(Q|P),$$

where

$$H(Q|P) \triangleq E_P\left(\frac{dQ}{dP}\log\frac{dQ}{dP}\right)$$

is the relative entropy of Q with respect to P.

Remark 5.1 If we replace in the previous construction the exponential utility with another, increasing and concave but otherwise general, utility u we obtain the larger class of utility-based convex risk measures. An interesting issue is the comparison between the initial preference structure, $X \succeq Y \Leftrightarrow E_P u(X) \geq E_P u(Y)$, and the derived one, $X \succeq' Y \Leftrightarrow \rho_u(X) \leq \rho_u(Y)$, where ρ_u is the risk measure induced by u. In [10] the entropic risk measures have been characterized as the only ones - apart from those induced by linear utilities - for which $\succeq$ and $\succeq'$ coincide. $\Diamond$

We now pass to the conditional case in which a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ is fixed. We assume, as before, that the agent is characterized by the exponential utility $u_\gamma(x) = 1 - \exp(-\gamma x)$ for a risk aversion $\gamma > 0$. The (random) expected utility of the agent conditional on the information $\mathcal{G}$ is therefore

$$U_\gamma(X) \triangleq E_P(1 - e^{-\gamma X} \mid \mathcal{G}) = 1 - E_P(e^{-\gamma X} \mid \mathcal{G}) \in L^0_{\mathcal{G}}.$$

Consider, as in the unconditional case, the acceptance set

$$\mathcal{A}_\gamma \triangleq \{X \in L^\infty \mid U_\gamma(X) \geq U_\gamma(0) = 0\} = \{X \in L^\infty \mid E_P(e^{-\gamma X} \mid \mathcal{G}) \leq 1\}.$$

It satisfies the conditions in Proposition 2.5 and thus leads to a conditional convex risk measure ρ_γ that has the explicit representation

$$\begin{aligned}\rho_\gamma(X) &\triangleq \text{ess.inf}\,\{Y \in L^\infty_{\mathcal{G}} \mid X + Y \in \mathcal{A}_\gamma\}\\ &= \text{ess.inf}\,\{Y \in L^\infty_{\mathcal{G}} \mid E_P(e^{-\gamma X} \mid \mathcal{G}) \leq e^{\gamma Y}\}\\ &= \frac{1}{\gamma}\log E_P(e^{-\gamma X} \mid \mathcal{G}).\end{aligned}$$

Definition 5.2 The conditional convex risk measure ρ_γ defined above is called the *conditional entropic risk measure* associated with the risk aversion $\gamma > 0$.

The name entropic derives, as in the unconditional case, from the form of the penalty function in the robust representation. Therefore, we extend the notion of relative entropy to the conditional setting.

Definition 5.3 For every $Q \in \mathcal{P}_{\mathcal{G}}$ the *conditional relative entropy* of Q w.r.t. P is

$$H_{\mathcal{G}}(Q|P) \triangleq E_P\left(\frac{dQ}{dP}\log\frac{dQ}{dP} \mid \mathcal{G}\right).$$

For $Q \in \mathcal{P}_{\mathcal{G}}$, we have the representation

$$H_{\mathcal{G}}(Q|P) = \frac{E_P\left(\frac{dQ}{dP}\log\frac{dQ}{dP} \mid \mathcal{G}\right)}{E_P(\frac{dQ}{dP} \mid \mathcal{G})} = E_Q\left(\log\frac{dQ}{dP} \mid \mathcal{G}\right),$$

because $E_P(\frac{dQ}{dP} \mid \mathcal{G}) = 1$ is the density of Q w.r.t. P on $\mathcal{G}$. Interpreting Q and P as regular conditional probabilities the conditional relative entropy can also be introduced pointwise as (unconditional) relative entropy. This approach coincides with definition 5.3 for $Q \in \mathcal{P}_{\mathcal{G}}$.

With this notion of conditional relative entropy, we can represent the minimal penalty function of entropic risk measures.

Proposition 5.4 *For any $\gamma > 0$, ρ_γ is representable and its minimal penalty function is*

$$\alpha^*(Q) = \frac{1}{\gamma}H_{\mathcal{G}}(Q|P), \quad Q \in \mathcal{P}_{\mathcal{G}}$$

Proof. Using monotone convergence it is straightforward to prove continuity from above. According to Theorem 3.2, ρ_γ is thus representable. The form of the minimal penalty function can be derived as

$$\begin{aligned}
\alpha^*(Q) &= \operatorname*{ess.sup}_{X \in L^\infty}\{-E_Q(X \mid \mathcal{G}) - \rho_\gamma(X)\}\\
&= \operatorname*{ess.sup}_{X \in L^\infty}\{-E_Q(X \mid \mathcal{G}) - \frac{1}{\gamma}\log E_P(e^{-\gamma X} \mid \mathcal{G})\}\\
&= \frac{1}{\gamma}\operatorname*{ess.sup}_{Z \in L^\infty}\{E_Q(Z \mid \mathcal{G}) - \log E_P(e^Z \mid \mathcal{G})\},\ Q \in \mathcal{P}_{\mathcal{G}}.
\end{aligned}$$

Finally, Lemma 5.5 proves the claimed representation of the minimal penalty function. $\square$

Lemma 5.5 *For any $Q \in \mathcal{P}_{\mathcal{G}}$ it holds*

$$\operatorname*{ess.sup}_{Z \in L^\infty}\{E_Q(Z \mid \mathcal{G}) - \log E_P(e^Z \mid \mathcal{G})\} = H_{\mathcal{G}}(Q \mid P).$$

Proof. "$\leq$". For any fixed $Z \in L^\infty$, the random variable

$$\varphi^Z \triangleq \frac{e^Z}{E_P(e^Z \mid \mathcal{G})}$$

is strictly positive, integrable and $E_P \varphi^Z = 1$, so that it is the density w.r.t. P of a probability measure $P^Z \sim P$. Hence, we have $Q \ll P^Z$ and

$$Z - \log E_P(e^Z \mid \mathcal{G}) = \log \frac{dP^Z}{dP} = \log \frac{dQ}{dP} - \log \frac{dQ}{dP^Z}$$

which yields

$$E_Q(Z \mid \mathcal{G}) - \log E_P(e^Z \mid \mathcal{G}) = E_Q(\log \frac{dQ}{dP} \mid \mathcal{G}) - E_Q(\log \frac{dQ}{dP^Z} \mid \mathcal{G})$$

Because of $P^Z \in \mathcal{P}_\mathcal{G}$ applying Jensen's inequality to the convex function g defined by $g(x) = x \log x$ for $x > 0$ and $g(0) = 0$ yields

$$E_Q(\log \frac{dQ}{dP^Z} \mid \mathcal{G}) = E_{P^Z}(g(\frac{dQ}{dP^Z}) \mid \mathcal{G}) \geq g(E_{P^Z}(\frac{dQ}{dP^Z} \mid \mathcal{G})) = g(1) = 0.$$

We then conclude

$$E_Q(Z \mid \mathcal{G}) - \log E_P(e^Z \mid \mathcal{G}) = H_\mathcal{G}(Q \mid P) - E_Q(\log \frac{dQ}{dP^Z} \mid \mathcal{G}) \leq H_\mathcal{G}(Q \mid P).$$

"$\geq$". Set $\varphi \triangleq dQ/dP$ and define the sequence of bounded random variables

$$Z_n \triangleq (-n) \vee \log \varphi \wedge n.$$

By considering the conditional expectation $E_P(e^{Z_n} \mid \mathcal{G})$ separately on the sets $\{\varphi \geq 1\}$ and $\{\varphi < 1\}$ we find that

$$E_P(e^{Z_n} \mid \mathcal{G}) \to E_P(e^{\log \varphi} \mid \mathcal{G}) = 1.$$

Moreover, Fatou's lemma yields

$$\liminf_{n\to\infty} E_Q(Z_n \mid \mathcal{G}) = \liminf_{n\to\infty} E_P(\varphi Z_n \mid \mathcal{G}) \geq E_P(\varphi \log \varphi \mid \mathcal{G}) = H_\mathcal{G}(Q \mid P).$$

It then follows

$$\begin{aligned}
\operatorname*{ess.sup}_{Z \in L^\infty} \{E_Q(Z \mid \mathcal{G}) - \log E_P(e^Z \mid \mathcal{G})\} \geq \\
&\geq \liminf_{n\to\infty} \{E_Q(Z_n \mid \mathcal{G}) - \log E_P(e^{Z_n} \mid \mathcal{G})\} \\
&= \liminf_{n\to\infty} \{E_Q(Z_n \mid \mathcal{G})\} \geq \\
&\geq H_\mathcal{G}(Q \mid P).
\end{aligned}$$

$\square$

Remark 5.6 It is possible to consider random risk aversion coefficients $\Gamma \in L^\infty_{\mathcal{G}}$, $\Gamma > 0$ and random utility functions $u_\Gamma(x, \omega) = 1 - \exp(-\Gamma(\omega)x)$. This generalization could be employed to model the preferences of an agent whose utility is exponential, but whose risk aversion depends on the additional information $\mathcal{G}$. It is straightforward to see that Proposition 5.4 holds true also in this case under the assumption $\Gamma^{-1} \in L^\infty_{\mathcal{G}}$. $\Diamond$

6 Dynamic convex risk measures

In this section we investigate conditional risk measures in a dynamic framework where successive measurements are performed. Consider a finite set of dates $0 = t_0 < t_1 < \ldots < t_N = T$ when the riskiness of a final payoff at time T is assessed. We introduce a filtration $(\mathcal{F}_n)_{n=0}^N$ where $\mathcal{F}_n$ models the information available at time t_n. Moreover, we assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

Definition 6.1 A *dynamic convex risk measure* is a family $(\rho_n)_{n=0}^N$ where every $\rho_n : L^\infty \to L^\infty_n \triangleq L^\infty(\mathcal{F}_n)$ is a conditional convex risk measure.

Remark 6.2 The risk measurement at time $t_N = T$ has been introduced for ease of notation only. Note that the only conditional convex risk measure at time t_N is $\rho_N(X) = -X$ for all X.

The concept of a dynamic risk measure has first been introduced by T.Wang [15] and then elaborated on by Riedel [11] and Weber [16]. Consistently with our previous discussion, we additionally require that every component is a conditional convex risk measure. Plainly, a dynamic convex risk measure maps a random variable $X \in L^\infty$ into the adapted process $(\rho_n(X))_{n=0}^N$ and can be seen as the result of a risk assessment of a final payoff through time.
This definition of a dynamic convex risk measure is quite general: in fact it can be completed by some form of internal time consistency. Consider the following three properties for a dynamic convex risk measure $(\rho_n)_{n=0}^N$:

- **Time consistency**. For any $X, Y \in L^\infty$ and $0 \le n \le N-1$ it holds

$$\rho_{n+1}(X) = \rho_{n+1}(Y) \Longrightarrow \rho_n(X) = \rho_n(Y)$$

- **Recursiveness**. For any $X \in L^\infty$ and $0 \le n \le N-1$ it holds

$$\rho_n(X) = \rho_n(-\rho_{n+1}(X)).$$

- **Supermartingale**. For any $X \in L^\infty$ and $0 \le n \le N-1$ it holds

$$\rho_n(X) \ge E_P(\rho_{n+1}(X) \mid \mathcal{F}_n).$$

The financial meaning of time consistency is based on a general intuition: if two payoffs will have tomorrow the same riskiness in every state of nature, then the same conclusion should be drawn today as well. The case for recursiveness, on

the contrary, strongly relies on the interpretation of a conditional convex risk measure as a capital requirement. In fact, it requires that the riskiness $\rho_n(X)$ of a final payoff X today equals the riskiness of the capital requirement $\rho_{n+1}(X)$ that has to be set aside tomorrow. The property of recursiveness has been introduced by Roorda et al. [12] and Riedel [11] and it is actually equivalent to time consistency. Indeed, assume that $(\rho_n)_{n=0}^N$ is time consistent and fix n; by conditional translation invariance of ρ_{n+1} we have $\rho_{n+1}(-\rho_{n+1}(X)) = \rho_{n+1}(X)$, so that $\rho_n(-\rho_{n+1}(X)) = \rho_n(X)$. The converse implication is trivial.
Finally, the supermartingale property can be interpreted as follows: as time evolves, the information about the payoff X increases. This should lower the perceived riskiness - not almost surely, but in the (conditional) mean.

Remark 6.3 Arztner et al. [2] show in a two period example (see the remark after Definition 5.5) that the dynamic convex risk measure whose general component is a conditional Average Value at Risk with a fixed parameter is not time consistent. $\diamondsuit$

An example for a dynamic convex risk measure that shares all the three above mentioned properties is given by the dynamic entropic risk measure whose general components are

$$\rho_n(X) \triangleq \frac{1}{\gamma} \log E_P(e^{-\gamma X} \mid \mathcal{F}_n), \quad X \in L^\infty$$

with risk aversion $\gamma > 0$.

Proposition 6.4 *Every dynamic entropic risk measure is time consistent and satisfies the supermartingale property.*

Proof. For any n and $X \in L^\infty$ we have

$$\begin{aligned}
\rho_n(X) &= \frac{1}{\gamma} \log E_P(e^{-\gamma X} \mid \mathcal{F}_n) \\
&= \frac{1}{\gamma} \log E_P(\exp\{-\gamma(-\frac{1}{\gamma} \log E_P(e^{-X} \mid \mathcal{F}_{n+1}))\} \mid \mathcal{F}_n) \\
&= \rho_n(-\rho_{n+1}(X)).
\end{aligned}$$

The process $(\rho_n(X))_{n=0}^N$ is a P-supermartingale since it is a concave function of the P-martingale $(E_P(e^{-\gamma X} \mid \mathcal{F}_n))_{n=0}^N$. $\square$

We now assume that a dynamic convex risk measure $(\rho_n)_{n=0}^N$ is representable, meaning that every component is representable. In this case, for every n it holds

$$\rho_n(X) = \operatorname*{ess.sup}_{Q \in \mathcal{P}_n} \{-E_Q(X \mid \mathcal{F}_n) - \alpha_n^*(Q)\}, \quad X \in L^\infty$$

where $\mathcal{P}_n \triangleq \{Q \in \mathcal{P} \mid Q \equiv P \text{ on } \mathcal{F}_n\}$ and

$$\alpha_n^*(Q) \triangleq \operatorname*{ess.sup}_{X \in L^\infty} \{-E_Q(X \mid \mathcal{F}_n) - \rho_n(X)\}, \quad Q \in \mathcal{P}_n.$$

Plainly, $\mathcal{P}_{n+1} \subset \mathcal{P}_n$ for all n and $\mathcal{P}_N = \{P\}$. Our aim is now to relate the dynamic properties of the family $(\rho_n)_{n=0}^N$ to some properties of the family of minimal penalty functions $(\alpha_n^*)_{n=0}^N$. We begin with a sufficient condition for the supermartingale property.

Proposition 6.5 *If $E_P(\alpha_n^*(Q) \mid \mathcal{F}_{n-1}) \geq \alpha_{n-1}^*(Q)$ for any $Q \in \mathcal{P}_n$, then $(\rho_n(X))_{n=0}^N$ is a P-supermartingale for any $X \in L^\infty$.*

Proof. We have

$$\begin{aligned} E_P(\rho_n(X) \mid \mathcal{F}_{n-1}) &= \operatorname*{ess.sup}_{Q\in\mathcal{P}_n} \{-E_Q(X \mid \mathcal{F}_{n-1}) - E_P(\alpha_n^*(Q) \mid \mathcal{F}_{n-1})\} \\ &\leq \operatorname*{ess.sup}_{Q\in\mathcal{P}_{n-1}} \{-E_Q(X \mid \mathcal{F}_{n-1}) - \alpha_{n-1}^*(Q)\} = \rho_{n-1}(X), \end{aligned}$$

where Lemma 3.4 has been applied in the first equality. □

From now on, we assume that for any probability measure P on $\mathcal{F}$ and any sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, a regular conditional probability of P given $\mathcal{G}$ exists. Remind that a regular conditional probability of P given $\mathcal{G}$ is a map $P_\mathcal{G} : \Omega\times\mathcal{F} \to [0,1]$ such that $P_\mathcal{G}(\omega, \cdot)$ is a probability measure for each ω and $P_\mathcal{G}(., A)$ is a version of $E_P(I_A|\mathcal{G})$ for every $A \in \mathcal{F}$. This assumption is satisfied if $(\Omega, \mathcal{F})$ is a Polish measurable space or, more generally, if $\mathcal{F}$ is countably generated (see [5] for other sufficient conditions). In what follows, P_n will denote the regular conditional probability of P given $\mathcal{F}_n$.

Definition 6.6 If $Q \ll P$ and $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-algebra, then the pasting of P and Q in $\mathcal{G}$ is the probability measure $PQ_\mathcal{G}$ defined by $PQ_\mathcal{G}(A) \triangleq E_P(Q_\mathcal{G}(\cdot, A)),\ A \in \mathcal{F}$.

It is not difficult to prove that

$$E_{PQ_\mathcal{G}}(X|\mathcal{H}) = E_P[E_Q(X|\mathcal{G})|\mathcal{H}]$$

for every sub-σ-algebra $\mathcal{H} \subseteq \mathcal{G}$.

Proposition 6.7 *The family $(\rho_n)_{n=0}^N$ is time consistent if and only if for every n the map*

$$\alpha_n(Q) \triangleq E_Q(\alpha_{n+1}^*(PQ_{n+1})|\mathcal{F}_n) + \operatorname{ess.inf}\{\alpha_n^*(R) \mid R \equiv Q \text{ on } \mathcal{F}_{n+1}\}, \quad Q \in \mathcal{P}_n$$

is a penalty function for ρ_n.

Proof. For every $X \in L^\infty$ Lemma 3.4 yields

$$\begin{aligned} &\rho_n(- \rho_{n+1}(X)) = \\ &\quad = \operatorname*{ess.sup}_{R\in\mathcal{P}_n} \{E_R(\operatorname*{ess.sup}_{S\in\mathcal{P}_{n+1}} \{-E_S(X \mid \mathcal{F}_{n+1}) - \alpha_{n+1}^*(S)\} \mid \mathcal{F}_n) - \alpha_n^*(R)\} \\ &\quad = \operatorname*{ess.sup}_{R\in\mathcal{P}_n,\, S\in\mathcal{P}_{n+1}} \{-E_R(E_S(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) - \alpha_n^*(R) - E_R(\alpha_{n+1}^*(S) \mid \mathcal{F}_n)\} \\ &\quad = \operatorname*{ess.sup}_{R\in\mathcal{P}_n,\, S\in\mathcal{P}_{n+1}} \{-E_{RS_{n+1}}(X \mid \mathcal{F}_n) - \beta_n(R, S)\}, \end{aligned}$$

where $\beta_n(R,S) \triangleq \alpha_n^*(R) + E_R(\alpha_{n+1}^*(S) \mid \mathcal{F}_n)$.
Note that $\{RS_{n+1} \mid R \in \mathcal{P}_n,\, S \in \mathcal{P}_{n+1}\} = \mathcal{P}_n$. The inclusion "$\subseteq$" is easy to see. For the converse, observe that a probability measure $Q \in \mathcal{P}_n$ can be written in the form $Q = RS_{n+1}$ if and only if $Q \equiv R$ on $\mathcal{F}_{n+1}$ and $Q_{n+1} \equiv S_{n+1}$ (P-a.s. as probability measures): these conditions are met by taking $R = Q$ and $S = PQ_{n+1}$. Hence

$$\rho_n(-\rho_{n+1}(X)) = \operatorname*{ess.sup}_{Q\in\mathcal{P}_n} \{-E_Q(X \mid \mathcal{F}_n) - \alpha_n(Q)\},$$

where

$$\begin{aligned}
\alpha_n(Q) &\triangleq \text{ess.inf}\,\{\beta_n(R,S) \mid R \in \mathcal{P}_n,\ S \in \mathcal{P}_{n+1},\ Q = RS_{n+1}\} \\
&= \text{ess.inf}\,\{\alpha_n^*(R) + E_R(\alpha_{n+1}^*(S) \mid \mathcal{F}_n) \mid R \equiv Q \text{ on } \mathcal{F}_{n+1}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad S \in \mathcal{P}_{n+1},\ S_{n+1} \equiv Q_{n+1}\} \\
&= \text{ess.inf}\,\{\alpha_n^*(R) + E_R(\alpha_{n+1}^*(PQ_{n+1}) \mid \mathcal{F}_n) \mid R \equiv Q \text{ on } \mathcal{F}_{n+1}\}.
\end{aligned}$$

Finally, observe that if $R \equiv Q$ on $\mathcal{F}_{n+1}$, then $E_R(Y|\mathcal{F}_n) = E_Q(Y|\mathcal{F}_n)$ for any $Y \in L^0_{n+1}$ for which the expectations exist; consequently

$$\alpha_n(Q) = E_Q(\alpha_{n+1}^*(PQ_{n+1}) \mid \mathcal{F}_n) + \text{ess.inf}\,\{\alpha_n^*(R) \mid R \equiv Q \text{ on } \mathcal{F}_{n+1}\}.$$

We easily conclude, observing that $\rho_n(-\rho_{n+1}(X)) = \rho_n(X)$ if and only if α_n defined above is a penalty function for ρ_n. □

A *dynamic coherent risk measure* is defined to be a dynamic convex risk measure whose components are coherent. If it is representable, then for every n we have

$$\rho_n(X) = \operatorname*{ess.sup}_{Q\in\mathcal{Q}_n^*} \{-E_Q(X \mid \mathcal{F}_n)\},$$

where $\mathcal{Q}_n^* \triangleq \{Q \in \mathcal{P}_n \mid E_Q(X \mid \mathcal{F}_n) \geq -\rho_n(X)\, \forall X \in L^\infty\}$ is the set where α_n^* vanishes. Therefore Proposition 6.5 and Proposition 6.7 can be stated in a simpler way.

Corollary 6.8 *Let $(\rho_n)_{n=0}^N$ be a representable dynamic coherent risk measure.*
1. It satisfies the supermartingale property provided $\mathcal{Q}_n^* \subseteq \mathcal{Q}_{n-1}^*$ for any n.
2. It is time consistent if and only if for any n, ρ_n can be represented in terms of

$$\mathcal{Q}_n \triangleq \{Q \in \mathcal{P}_n \mid PQ_{n+1} \in \mathcal{Q}_{n+1}^* \text{ and } \exists Q' \in \mathcal{Q}_n^* \text{ s.t. } Q' \equiv Q \text{ on } \mathcal{F}_{n+1}\}.$$

Conclusions

We characterize the class of conditional risk measures which can be interpreted as *good* conditional capital requirements. In particular, under a weak technical assumption which is essentially the same as in the unconditional setting, we provide a representation for these risk measures as worst conditional loss with respect to a set of probabilistic models and a penalty function. The main difference in comparison with the unconditional setting is provided by the random nature of these two objects. This is natural, since they describe, in some sense, the degree of *trustworthiness* towards different models, which depends on available information and thus may change in time. In the representation we propose, additional information is reflected both in the conditional nature of the expectations and in the penalty function. This issue is particularly important when successive risk measurements of the same payoff are performed or, in our terminology, when a dynamic risk measure has to be constructed. In this case, a *penalty process* has to be chosen, describing how the degree of trustworthiness of different models evolves through time. In the last section it is shown how this choice is constrained by some basic natural consistency properties. Notwithstanding, in our opinion the class of penalty processes is still too large from an economic viewpoint, so that other consistency properties have to be discussed even in connection with the theory of updating information. Finally, a complete economic interpretation of the penalty term still lacks, even in the classical setting. This interpretation could be related to some sort of preference structure in the dual space, that of probabilistic models. We leave this important issue to further investigation.

A Appendix

Let $(\Omega, \mathcal{F}, P)$ be a probability space and denote by $L^0(\overline{\mathbb{R}})$ the space of *extended random variables*, i.e. P-equivalence classes of $\mathcal{F}$-measurable maps from Ω to $\overline{\mathbb{R}} \triangleq [-\infty, +\infty]$, where the natural extension of the Borel σ-algebra is considered on $\overline{\mathbb{R}}$. The preorder initially defined on L^0 naturally extends to this larger space; we refer to Section II.4 in [14] for other natural conventions. For any subset $\mathcal{X} \subseteq L^0(\overline{\mathbb{R}})$, the family of dominating random variables $\mathcal{D}(\mathcal{X}) \triangleq \{Z \in L^0(\overline{\mathbb{R}}) \mid Z \geq X,\ \forall X \in \mathcal{X}\}$ is not empty, since it contains $+\infty$.

Theorem A.1 *For any $\mathcal{X} \subseteq L^0(\overline{\mathbb{R}})$ there exists a unique element $X^* \in \mathcal{D}(\mathcal{X})$ such that $X^* \leq Z$ for any $Z \in \mathcal{D}(\mathcal{X})$. If in addition $\mathcal{X}$ is upward directed, i.e. for any $X_1, X_2 \in \mathcal{X}$ there exists $X \in \mathcal{X}$ such that $X \geq \max(X_1, X_2)$, then there is an increasing sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $X_n \nearrow X^*$ P-a.s.*

Proof. See [8], Theorem A.18 $\square$

The random variable X^* characterized in the previous theorem is called the *essential supremum* of $\mathcal{X}$ and denoted by $\text{ess.sup}\,\mathcal{X}$. The *essential infimum* is defined by $\text{ess.inf}\,\mathcal{X} \triangleq -\text{ess.sup}\,(-\mathcal{X})$.
The definition of (conditional) expectation is naturally extended to $L^0(\overline{\mathbb{R}})$: see again [14] for details.

Lemma A.2 *If $\mathcal{X} \subseteq L^0(\overline{\mathbb{R}})$ is upward directed then it holds*

$$E_P(\text{ess.sup}\,\mathcal{X}) = \sup_{X\in\mathcal{X}} E_P X,$$

provided the expectations exist.

Proof. "$\geq$" This relation follows from $\text{ess.sup}\,\mathcal{X} \geq X$ for all $X \in \mathcal{X}$.
"$\leq$" According to Theorem A.1 there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $X_n \nearrow \text{ess.sup}\,\mathcal{X}$. Then

$$E_P(\text{ess.sup}\,\mathcal{X}) = E_P(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E_P(X_n) \leq \sup_{X\in\mathcal{X}} E_P(X),$$

thanks to the monotone convergence theorem. $\square$

The previous result holds as well if expectations are replaced by conditional expectations with respect to some sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and in the right hand side the essential supremum is considered.

Lemma A.3 *If $\mathcal{X}, \mathcal{Y} \subseteq L^0(\overline{\mathbb{R}})$ and $A \in \mathcal{F}$, then*

$$\text{ess.sup}\,(\mathcal{X}I_A + \mathcal{Y}I_{A^c}) = (\text{ess.sup}\,\mathcal{X})I_A + (\text{ess.sup}\,\mathcal{Y})I_{A^c},$$

where $\mathcal{X}I_A + \mathcal{Y}I_{A^c} \triangleq \{XI_A + YI_{A^c} \mid X \in \mathcal{X},\, Y \in \mathcal{Y}\}$.

Proof. "$\leq$" This relation is immediate.
"$\geq$" If $Z \in \mathcal{D}(\mathcal{X}I_A + \mathcal{Y}I_{A^c})$ then consider $Z' \triangleq ZI_A + (\text{ess.sup}\,\mathcal{X})I_{A^c}$. Since $Z' \geq X$ for all $X \in \mathcal{X}$, then $Z' \geq \text{ess.sup}\,\mathcal{X}$. Hence, we have $ZI_A = Z'I_A \geq (\text{ess.sup}\,\mathcal{X})I_A$ and the claim follows from a similar argument on A^c. $\square$

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