Goodness-of-fit Test for Specification of Semiparametric Copula Dependence Models

Shulin Zhang*
Ostap Okhrin**
Qian M. Zhou***
Peter X.-K. Song****

* Southwestern University of Finance and Economics, China
** Humboldt-Universität zu Berlin, Germany
*** Simon Fraser University, Canada
**** University of Michigan, USA

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Shulin Zhang[a], Ostap Okhrin[b,*], Qian M. Zhou[c], Peter X.-K. Song[d]

[a] Center of Statistical Research, School of Statistics, Southwestern University of Finance and Economics, China

[b] School of Business and Economics, C.A.S.E. – Center for Applied Statistics and Economics, Humboldt University of Berlin, Germany

[c] Department of Statistics and Actuarial Science, Simon Fraser University, Canada

[d] Department of Biostatistics, University of Michigan, USA

ABSTRACT

This paper concerns goodness-of-fit test for semiparametric copula models. Our contribution is two-fold: we first propose a new test constructed via the comparison between "in-sample" and "out-of-sample" pseudo-likelihoods, which avoids the use of any probability integral transformations. Under the null hypothesis that the copula model is correctly specified, we show that the proposed test statistic converges in probability to a constant equal to the dimension of the parameter space and establish the asymptotic normality for the test. Second, we introduce a hybrid mechanism to combine several test statistics, so that the resulting test will make a desirable test power among the involved tests. This hybrid method is particularly appealing when there exists no single dominant optimal test. We conduct comprehensive simulation experiments to compare the proposed new test and hybrid approach with the best "blank test" shown in Genest et al. (2009). For illustration, we apply the proposed tests to analyze three real datasets.

JEL classification: C12; C22; C32; C52; G15.

KEY WORDS: hybrid test; in-and-out-of sample likelihood; power; tail dependence.

1 Introduction

Assessing dependency among multiple variables is a primary task in business economics or financial applications. Copula is becoming increasingly popular in such fields due to its flexibility in seamlessly integrating sophisticated dependence structures and varying marginal distributions of multivariate random variables. For example, in Finance, copulas are widely applied to study dependency in asset pricing, asset allocation and risk management; see Klugman and Parsa (1999), Cherubini et al. (2004) and Cherubini et al. (2011), among others. More examples in other fields can be found in Frees and Valdez (1998), Wang and Wells (2000), Song (2007) and Danaher and Smith (2011), just to name a few.

Essentially, a parametric copula is a cumulative distribution function (CDF) specified by a certain known functional form up to some unknown dependence parameters. When a parametric copula is used in applications, misspecification on any of its parametric structure may cause misleading statistical estimation and inference. To check for the adequacy of a copula model, specification test has been extensively investigated in the literature. Wang and Wells (2000) proposed a rank based test in bivariate copulas. Malevergne and Sornette (2003) developed a test for the specification of Gaussian copula. Fermanian (2005) and Scaillet (2007) established goodness-of-fit tests through kernel techniques. Other types of specification tests include Panchenko's (2005) V-statistic type test, Prokhorov and Schmidt's (2009) conditional moment based test, Mesfioui et al.'s (2009) Spearman dependence based test, and Genest et al.'s (2011) Pickands dependence based test. Very recently, Huang and Prokhorov (2013) adopted White's test based on information matrix (White, 1982) to derive a test for copula models specification. With the utility of either Kendall's or Rosenblat's probability integral transformations, several other versions of specification tests have been proposed in the literature, including those proposed by Breymann et al. (2003), Dobrić and Schmid (2007) and Genest and Favre (2007), among others.

In a recent paper, Genest et al. (2009) made a thorough comparison for most of the existing "blank tests". A blank test refers to a test whose implementation does not require either an arbitrary categorization of data or any strategic choice of smoothing parameter, weight function, kernel or bandwidth. It is demonstrated by Genest et al. (2009) that none of these blank tests performs uniformly the best. It is interesting to note that , almost all of them had illustrated nearly no power in differentiating Gaussian copula and Student's t copula, both of which are very important

symmetric copulas with different tail dependence properties. Another challenge in the use of the tests considered in Genest et al. (2009) is that they rely on certain probability integral transformations, which may be difficult to derive analytically in many popular copula dependence models, e.g. Student's t copula and vine copulas (e.g. Kurowicka and Joe (2011)).

To overcome the difficulties above, we propose an alternative specification test for semiparametric copulas in this paper. The proposed test statistic takes a form of ratio constructed via two types of pseudo-likelihoods, one is "in-sample" pseudo-likelihood and the other is "out-of-sample" pseudo-likelihood. The idea behind the construction of the new test is rooted in the fact that, heuristically, a goodness-of-fit test is to examine how model fit the data. Thus, we vary data by the means of jackknife and quantify how sensitive the pseudo likelihood is to the varying data. Naturally, a comparison of pseudo likelihoods over different data sets are utilized to characterize how well the model fits the data. Inspired by Presnell and Boos's (2004) likelihood based in-and-out-of-sample test, we term our proposed test as the pseudo in-and-out-sample (PIOS) test. In comparison to the tests in Genest et al. (2009), which are all indeed rank-based tests, our PIOS test is a pseudo likelihood based test, which does not require any probability integral transformation. Thus, as demonstrated later in the paper, the PIOS test is computationally simple and numerically stable.

Under the null hypothesis of copula model being correctly specified, we show that under some mild regularity conditions, the PIOS test statistic converges in probability to a constant equal to the dimension of parameter space of the null copula model. Also, we establish the asymptotic normality for the PIOS test statistic. Compared to the fully parametric in-and-out-of-sample test proposed by Presnell and Boos (2004), our work makes the following new contributions. First, the PIOS test is applicable to a semiparametric copula model in which the marginal CDFs may be fully unspecified. Second, Presnell and Boos's (2004) test is based on a single point data in-and-out-of-sample procedure. As a useful extension, the PIOS test is based on a data block in-sample and out-of-sample procedure, where the size of block is allowed to increase with the sample size. Such flexibility is critical to extend the original idea to serially dependent time series data, as well as fast numerically calculation. Third, the development of asymptotic properties of the PIOS test is involved in the use of the theory of empirical processes with varying block size, and therefore such theoretical work is new and fundamentally different from that established in Presnell and

Boos (2004).

Another primary focus of the paper is the development of a mechanism to combine several test statistics and the resulting test is termed as the hybrid test in this paper. As demonstrated in Genest et al. (2009), there exists no single dominant asymptotically optimal test against general alternatives; see also Freedman (2009). The proposed hybrid test offers a compromise of several different tests, which is particularly appealing when there is no *a prior* knowledge about the top performer in the hypothesis test. We show that the proposed hybrid test can fully control type I error, as long as each of them does, and that it will be a consistent test as long as there exists one consistent test in the involved tests, regardless of the performance of the remaining tests. Our simulation studies clearly illustrate that, in general, the hybrid test enjoy the best finite sample performance.

This paper is organized as follows. Section 2 is devoted to the detail for the construction of the PIOS test. Section 3 discusses the hybrid test. Section 4 presents the large sample properties of the proposed PIOS test statistic. Section 5 concerns Monte Carlo simulation studies to evaluate finite sample performances of the proposed PIOS test and hybrid test. In Section 6, the proposed tests are applied to three real datasets. The final section provides some concluding remarks. All technical details are included in the appendix.

2 Pseudo in-and-out-of-sample test (PIOST)

Suppose that $X_1 = (X_{11}, \cdots, X_{1d})^T, \ldots, X_n = (X_{n1}, \ldots, X_{nd})^T$ is a random sample of size n drawn from a multivariate distribution $H(x) = H(x_1, x_2, \ldots, x_d)$ with continuous marginal CDF $F(x) \triangleq \{F_1(x_1), \ldots, F_d(x_d)\}$. According to Sklar's theorem (Sklar, 1959), we suppose that the joint distribution $H(\cdot)$ can be expressed by the following representation:

$$H(x_1, x_2, \cdots, x_d) \triangleq C_0\{F(x)\} = C_0\{F_1(x_1), \ldots, F_d(x_d)\},$$

where $C_0(\cdot)$ is the true copula function. The corresponding joint density function of $H(\cdot)$, denoted by $h(\cdot)$, takes the form of

$$h(x_1, x_2, \cdots, x_d) = c_0\{F_1(x_1), \ldots, F_d(x_d)\} \prod_{k=1}^{d} f_k(x_k),$$

where, $c_0(u)$, $u = (u_1, \ldots, u_d) \in (0,1)^d$ is the resulting copula density function of copula $C_0(\cdot)$ and $f_k(\cdot)$ are the corresponding marginal density functions of $F_k(\cdot)$, $k = 1, \ldots, d$. Throughout this paper, the marginal CDF $F(\cdot)$ is not specified by any parametric forms.

In practice, we often assume that the underlying true copula C_0 belongs to a parametric class , *say,*

$$\mathcal{C} \triangleq \{C(\cdot; \theta), \theta \in \Theta\},$$

where Θ is a p-dimensional parameter space. It is well known that misspecification on any of its parametric structure of $C(\cdot; \theta)$ may ruin likelihood based statistical estimation and inference. Hence, checking the model specification is an important task in model diagnosis. In the following, we are concern of with the development of a goodness-of-fit test on the hypotheses

$$\mathcal{H}_0 : C_0 \in \mathcal{C} = \{C(\cdot; \theta) : \theta \in \Theta\} \quad vs. \quad \mathcal{H}_1 : C_0 \notin \mathcal{C} = \{C(\cdot; \theta) : \theta \in \Theta\}.$$

To begin, we first apply the so-called two-step pseudo maximum likelihood (PMLE) method (e.g. Oakes (1994), Genest and Rivest (1995), Shih and Louis (1995) and Chen and Fan (2005)) to estimate the dependence parameter θ. In order to avoid the estimated copula function from blowing up at the boundary of 0 or 1, let $\tilde{F}(x) = \{\tilde{F}_1(x_1), \ldots, \tilde{F}_d(x_d)\}$ be the set of rescaled empirical marginal distributions, where the k-th component is given by

$$\tilde{F}_k(x_k) = \frac{1}{n+1} \sum_{t=1}^{n} I(X_{tk} \leq x_k), \tag{1}$$

where $I(\cdot)$ is the indicator function. The corresponding set of empirical marginal density functions is denoted as $\tilde{f}(x) = \{\tilde{f}_1(x_1), \ldots, \tilde{f}_d(x_d)\}$. Let $l\{\tilde{F}(X_t); \theta\} = \log c\{\tilde{F}_1(X_{t1}), \ldots, \tilde{F}_d(X_{td}); \theta\}$, and let $\hat{\theta}$ be the two-step PMLE of θ given by

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \sum_{t=1}^{n} l\{\tilde{F}(X_t); \theta\}. \tag{2}$$

Genest and Rivest (1995) investigate large sample properties of the above PMLE under the assumption of the copula function being correctly specified, and Chen and Fan (2005) established asymptotic properties of the PLME $\hat{\theta}$ under a misspecified models.

To present our new test, let us randomly divide the original data $\{X_1, \ldots, X_n\}$ into B blocks and denote the b-th block as $X^b = (X_1^b, \ldots, X_{n_b}^b)$, $b = 1, \ldots, B$. Without loss of generality, suppose $X_i^b = X_{n_1 + \cdots + n_{b-1} + i}$ and the k-th element of X_i^b is denoted by X_{ik}^b, $k = 1, \ldots, d$, $i = 1, \ldots, n_b$,

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and $n_1 + \cdots + n_B = n$. For the simplicity of exposition, we assume that all blocks have equal size, *say*, $n_b \equiv m$, and hence $mB = n$. With little technical effort, all arguments presented in the rest of this paper can be extended to the case of unequal block size. In a similar spirit to the "jackknife" resampling method (e.g. Efron (1982)), we can yield a set of delete-one-block PLMEs $\hat{\theta}_{-b}, 1 \leq b \leq B$, according to the following procedure:

$$\hat{\theta}_{-b} = \arg\max_{\theta \in \Theta} \sum_{b' \neq b}^{B} \sum_{i=1}^{m} l\{\tilde{F}(X_i^{b'}); \theta\}, \quad b = 1, \ldots, B. \tag{3}$$

Note that the delete-one-block pseudo likelihood (out-of-sample) $\prod_{i=1}^{m} c\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\} \prod_{k=1}^{d} \tilde{f}(X_{ik}^b)$ measures how well the hypothesized model predicts the b-th block of observations $X^b = (X_1^b, \ldots, X_m^b)$. Obviously, if the full pseudo likelihood (in- sample) $\prod_{i=1}^{m} c\{\tilde{F}(X_i^b); \hat{\theta}\} \prod_{k=1}^{d} \tilde{f}_k(X_{ik}^b)$ appears to be much larger than the out-of-sample counterpart, $\prod_{i=1}^{m} c\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\} \prod_{k=1}^{d} \tilde{f}(X_{ik}^b)$, then the fitted model is very sensitive to the b-th block of observations, implying that the hypothesized model may be inadequate to fit the data. Thus, we can establish a global measure for goodness-of-fit using a comparison between the "in-sample" pseudo-likelihood, $\prod_{b=1}^{B} \prod_{i=1}^{m} c\{\tilde{F}(X_i^b); \hat{\theta}\} \prod_{k=1}^{d} \tilde{f}_k(X_{ik}^b)$ and the "out-of-sample" pseudo-likelihood, $\prod_{b=1}^{B} \prod_{i=1}^{m} c\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\} \prod_{k=1}^{d} \tilde{f}(X_{ik}^b)$. Precisely, we propose a test statistic of the following form:

$$T_n(m) \triangleq \sum_{b=1}^{B} \sum_{i=1}^{m} \left[l\{\tilde{F}(X_i^b); \hat{\theta}\} - l\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\} \right]. \tag{4}$$

The resulting test is termed as the pseudo in-and-out-of-sample (PIOS) test. It is worth pointing out that, when the margins are known and the block size is fixed as $m \equiv 1$, $T_n(m)$ in (4) reduces to the IOS test statistic proposed by Presnell and Boos (2004).

Under the null hypothesis of correct model specification, statistic $T_n(m)$ in (4) is shown to converge in probability to p, the dimension of the parameter vector θ. Here, we present in a heuristic argument as to why its limiting value is p. First, we define two types of Fisher information matrices (Song (2007), Chapter 3), negative sensitivity matrix and variability matrix as follows:

$$S(\theta) \triangleq -\mathbb{E}_0 \left[l_{\theta\theta}\{F(X_1); \theta\} \right],$$
$$V(\theta) \triangleq \mathbb{E}_0 \left[l_{\theta}\{F(X_1); \theta\} l_{\theta}^T\{F(X_1); \theta\} \right],$$

where $l_{\theta}(u; \theta) = \frac{\partial}{\partial \theta} \log c(u; \theta)$, $l_{\theta\theta}(u; \theta) = \frac{\partial^2}{\partial \theta \partial \theta^T} \log c(u; \theta)$, and $\mathbb{E}_0(\cdot)$ represents the expectation under the true copula C_0. Throughout this paper, we assume there exists a θ^* such that $\hat{\theta} \to \theta^*$

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in probability under some regularity conditions. Refer to Chen and Fan (2005) for the regularity conditions required to establish such consistency under mispecified model. The point of interest is that, under suitable regularity conditions given in Theorem 2, we can show that

$$T_n(m) \xrightarrow{pr} \mathbb{E}_0 \left[l_\theta^T \{F(X_1); \theta^*\} S(\theta^*)^{-1} l_\theta \{F(X_1); \theta^*\} \right] = \text{tr} \left\{ S(\theta^*)^{-1} V(\theta^*) \right\}, \text{ as } n \to \infty,$$

where $\text{tr}(A)$ denotes the trace of a matrix A. As a result of the Bartlett's identity (White, 1982), a correct model specification implies $V(\theta^*) = S(\theta^*)$, so $\text{tr}\left\{S(\theta^*)^{-1}V(\theta^*)\right\} = p$, the trace of the p-dimensional identity matrix. Furthermore, $T_n(m) - p$, adjusted with a proper standard deviation, is asymptotically normalized distributed, which is the theoretical basis to define the rejection rule for the hypothesis test.

To implement the proposed test statistic $T_n(m)$ in practice we need to estimate dependence parameter $[n/m]$ (the largest integer less than n/m) times, which may be computationally demanding. Indeed, we can approximate $T_n(m)$ by the following test statistic, which is shown to be asymptotically equivalent to $T_n(m)$ in Theorem 2(ii):

$$R_n \overset{\Delta}{=} \frac{1}{n} \sum_{t=1}^n l_\theta^T \{\tilde{F}(X_t); \hat{\theta}\} \hat{S}^{-1}(\hat{\theta}) l_\theta \{\tilde{F}(X_t); \hat{\theta}\} = \text{tr} \left\{ \hat{S}^{-1}(\hat{\theta}) \hat{V}(\hat{\theta}) \right\}, \tag{5}$$

where $\hat{S}(\hat{\theta})$ and $\hat{V}(\hat{\theta})$ are the sample counterparts of the negative sensitivity matrix and variability matrix, respectively, defined by

$$\hat{S}(\hat{\theta}) = -\frac{1}{n} \sum_{t=1}^n l_{\theta\theta} \{\tilde{F}(X_t); \hat{\theta}\},$$

$$\hat{V}(\hat{\theta}) = \frac{1}{n} \sum_{t=1}^n l_\theta \{\tilde{F}(X_t); \hat{\theta}\} l_\theta^T \{\tilde{F}(X_t); \hat{\theta}\}.$$

The statistic R_n given in (5) is similar to the information ratio (IR) test statistic proposed by Zhou et al. (2012) for cross-sectional and longitudinal data in the framework of estimating equations, which was later extended to time series data using martingale estimating equations in Zhang et al. (2012).

3   Hybrid Test

In most of scenarios for goodness-of-fit test, including the one for copula models (e.g. Genest et al. (2009)) there exists no single dominate optimal test. It is often the case that at one occasion,

one test is more powerful, but at other occasions, other tests were more powerful. See also Freed-man (2009). The same phenomenon also occurs in our simulation studies. At some occasions, PIOS outperforms others, but at other settings, other tests (such as the one proposed by Genest et al. (2009)) perform better. Inspired by Zhou et al. (2013), here we propose the following hybrid test that enables us to combine several different tests to achieve certain compromise in the test power. The resulting test, although may not have the highest power, outperforms on average any of individual tests used in the combination. This strategy is particularly appealing when there is no *a prior* knowledge regarding the top performer at a given occasion.

Consider q test statistics, denoted by $T_n^{(1)}, T_n^{(2)}, \ldots, T_n^{(q)}$, where subscript n is the sample size. Suppose that all of them have type I error controlled at a given significance level α under a common null hypothesis. A hybrid test is constructed as follows: Let $p_n^{(i)}$ denote the corresponding p-value obtained from the test statistic $T_n^{(i)}$, $i = 1, \ldots, q$. A hybrid test, denoted by T_n^{hybrid}, will make decision according to a p-value, defined as

$$p_n^{hybrid} = q \times min\{p_n^{(1)}, \ldots, p_n^{(q)}\}.$$

Consequently, the rejection rule of the hybrid test is that, if $p_n^{hybrid} \leq \alpha$, the null hypothesis is rejected. This is equivalent to the situation where there is at least one test rejecting the null at the level of $\frac{\alpha}{q}$.

Under the null hypothesis H_0 and a significance level α, we have the type I error for the hybrid test:

$$\begin{aligned} p_r(p_n^{hybrid} \leq \alpha | H_0) &= p_r(p_n^{(1)} \leq \alpha/q \text{ or } \cdots \text{ or } p_n^{(q)} \leq \alpha/q | H_0) \\ &\leq \sum_{i=1}^{q} p_r(p_n^{(i)} \leq \alpha/q | H_0) \\ &\leq \alpha. \end{aligned}$$

The above inequality shows that, provided that all of the test $T_n^{(i)}, i = 1, \ldots, q$, have controlled type I errors, the hybrid test T_n^{hybrid} has its type I error controlled at α.

Let $\beta_n^{(i)}(\alpha)$ be the power function of test $T_n^{(i)}$ at a given significance level α and sample size n, $i = 1, \ldots, q$. That is, under the alternative hypothesis H_A, $\beta_n^{(i)}(\alpha) = p_r(p_n^{(i)} \leq \alpha | H_A)$. The power

function of the hybrid test T_n^{hybrid} has the following lower bound:

$$
\begin{aligned}
\beta_n^{hybrid}(\alpha) &= p_r\left(p_n^{hybrid} \leq \alpha | H_A\right) \\
&= p_r\left(p_n^{(1)} \leq \frac{\alpha}{q} \text{ or } \cdots \text{ or } p_n^{(q)} \leq \frac{\alpha}{q} | H_A\right) \\
&\geq \max\left\{\beta_n^{(1)}(\frac{\alpha}{q}), \ldots, \beta_n^{(q)}(\frac{\alpha}{q})\right\}.
\end{aligned}
$$

The above inequality implies that (i) the power of the hybrid test is better than the test given by the strategy of Bonferroni correction for multiplicity; and (ii) more importantly, if there is at least one test that is consistent (namely, the power tends to 1 as the sample size increases to ∞), then the hybrid test is consistent. Our simulation studies also show that the hybrid test behave more desirably than any of individual tests.

4 Asymptotic Properties of PIOS test

In this section, we establish several asymptotic properties of the proposed PIOS test as well as the relationship between $T_n(m)$ in (4) and R_n in (5). Throughout this paper, we denote $\|x\|$ as the usual Euclidean metric of any vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, namely, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$ and for any $d \times d$ matrix A, $\|A\| = \sqrt{\sum_{i,j=1}^d A_{ij}^2}$, where A_{ij} is the (i,j)-th element of A. Let $\mathcal{N}(\theta^*)$ denote an open neighborhood of θ^*.

Firstly, we establish the consistency of the test statistics R_n. To proceed, we need the following regularity conditions.

A1. The first-order and second-order derivatives, $l_\theta(u; \theta)$ and $l_{\theta\theta}(u; \theta)$, are continuous with respective to θ for any $u \in [0, 1]^d$; and there exist integrable functions $G_1(u)$ and $G_2(u)$ such that $\|l_\theta(u; \theta) l_\theta^T(u; \theta)\| \leq G_1(u)$ and $\|l_{\theta\theta}(u; \theta)\| \leq G_2(u)$ for all $\theta \in \mathcal{N}(\theta^*)$.

A2. Matrix $S(\theta^*) = -\mathbb{E}_0 l_{\theta\theta}\{F(X_1); \theta^*\}$ is finite and nonsingular.

Assumption (A1) is the so-called dominating condition, which is commonly imposed in order to establish the uniform law of large number theorem (e.g. Wooldridge (1994)). Assumption (A2) requires the sensitivity matrix $S(\theta^*)$ to be invertible, so that the test statistic R_n in (5) will be well-defined.

Theorem 1 *Under conditions (A1)-(A2), we have*

$$R_n \xrightarrow{pr} tr\left\{ S(\theta^*)^{-1} V(\theta^*) \right\}, \ as \ n \to \infty,$$

where θ^* is the limiting value of PMLE $\hat{\theta}$.

The following regularity conditions are used to establish the central limit theorem for both R_n and $T_n(m)$.

B1. Denote $J_i(u) = const \times \prod_{k=1}^{d} \{u_k(1 - u_k)\}^{-\xi_{ik}}$, where $\xi_{ik} \geq 0$, $i = 1, 2$, ξ_{ik} are some constants. Suppose that for all $\theta \in \mathcal{N}_{\theta^*}$, $\|l_\theta(u;\theta)l_\theta^T(u;\theta)\| \leq J_1(u)$, $\|l_{\theta\theta}(u;\theta)\| \leq J_2(u)$, and $\mathbb{E}_0 J_i^2\{F(X_1)\} < \infty$.

B2. Suppose that both $\frac{\partial}{\partial u_k} l_\theta(u;\theta)$ and $\frac{\partial}{\partial u_k} l_{\theta\theta}(u;\theta)$, $k = 1, 2, \ldots, d$ exist and are continuous. Denote $\tilde{J}_i^k(u) = const \times \{u_k(1 - u_k)\}^{-\tilde{\xi}_{ik}} \prod_{j=1, j \neq k}^{d} \{u_j(1 - u_j)\}^{-\tilde{\xi}_{ij}}$, where $\tilde{\xi}_{ij} > \xi_{ij}$ are some constants, such that for all $\theta \in \mathcal{N}(\theta^*)$, $\|\frac{\partial}{\partial u_k} l_\theta(u;\theta)\| \leq \tilde{J}_1^k(u)$ and $\|\frac{\partial}{\partial u_k} l_{\theta\theta}(u;\theta)\| \leq \tilde{J}_2^k(u)$, and furthermore, $\mathbb{E}_0 \tilde{J}_i\{F(X_1)\} < \infty$, $i = 1, 2$ and $k = 1, 2, \ldots, d$.

B3. Suppose $\frac{\partial l_{\theta\theta}(u;\theta)}{\partial \theta_k}$, $k = 1, 2, \ldots, p$ exist and are continuous with $\theta \in \mathcal{N}(\theta^*)$, and there exist an integrable function $G_3(u)$ such that $\|\frac{\partial l_{\theta\theta}(u;\theta)}{\partial \theta_k}\| \leq G_3(u)$ for all $\theta \in \mathcal{N}(\theta^*)$, $k = 1, \ldots, d$.

Assumptions (B1) and (B2) are similar to the conditions in Lemma 2 of Chen and Fan (2005). Obviously, Assumption (B1) implies assumptions (A1). Assumption (B3) is commonly required in the literature to establish the uniform law of large number theorem .

C1. The block size m is of order $o(n^a)$ with $a \leq \frac{1}{4}$.

This Assumption (C1) is needed to bound the difference between R_n and $T_n(m)$, so that these two statistics have the same limiting distribution.

Under the above regularity conditions, we have the following results.

Theorem 2 *(i) Under the null hypothesis, if (A2) and (B1)- (B3) hold, then we have*

$$\sqrt{n}\{R_n - p\} \xrightarrow{d} N(0, \sigma_R^2), \ as \ n \to \infty,$$

where σ_R^2 is the asymptotic variance given by equation (9) in the appendix, which can be consistently estimated by equation (10) in the appendix.

(ii) Under assumptions (A2),(B1)-B(3) and (C1), we have

$$R_n - T_n(m) = o_p(n^{-1/2}).$$

Remark 1 *To establish a general theory of consistency for the proposed goodness-of-fit test is challenging due to the fact that it is difficulty to characterize proper alternatives against which the test is consistent. However, given a restricted set of alternatives, it is possible to establish the test consistency. For example, for the alternatives under which the sensitivity matrix $S(\theta^*)$ differs from the variability matrix $V(\theta^*)$, following similar arguments as those given in the proof of Theorem 1, we can show that the PIOS test is consistent.*

Remark 2 *One issue in the use of the above PIOS test is how to select block-size m to achieve better performance. Our Monte Carlo simulations show that the choice of m depends on the underlying data generating process, and in most cases the PIOS test with block-size $m = 1$ behaves satisfactorily for independent cross-sectional data.*

5   Simulation study

5.1   Setup

In this section, we conduct extensive Monte Carlo simulation experiments to evaluate the finite performance of the proposed tests $T_n(m)$ in (4) and R_n in (5). We chose $m = 1$ and $m = 3$ that satisfy Condition (C1) for independent cross-sectional data. In the implementation of our tests, the asymptotic variance σ_R^2 needs to be estimated, what is not easy to be done directly. This leads us to the following semiparametric bootstrap procedure to numerically establish the null distribution of R_n. A similar bootstrap approach was considered in Genest et al. (2009). This bootstrap technique is also applied for the null distribution of test $T_n(m)$ as well.

Let $\hat{\theta}$ be the PMLE and let $\tilde{F}(x)$ be the rescaled empirical distribution in (1). The semiparametric bootstrap proceeds as follows:

Step 1. Generate a bootstrap sample $\left\{ X_1^{(k)}, \ldots, X_n^{(k)} \right\}$ from the estimated copula $C(u; \hat{\theta})$ under the null hypothesis and the marginal distribution $\tilde{F}(x)$;

Step 2. Estimate the dependence parameter θ of the copula in the null hypothesis by the two-step PMLE method and compute the test statistic R_n, denoted by R_n^k;

Step 3. Repeat Steps 1- 2 N times and obtain N statistics $R_n^k, k = 1, \ldots, N$;

Step 4. Compute empirical p-value as $p_e = \frac{1}{N} \sum_{k=1}^{N} I \left(|R_n^k| < |R_n| \right)$.

For the purpose of comparison, we include a test proposed by Genest et al. (2009), termed in short as GRB test. GRB test has been shown to have the best performance on average among all the existing "blank tests". In fact, GRB test statistic is a Cramér-von Mises statistic based on Rosenblatt's transform (Rosenblatt, 1952), defined by

$$
\begin{aligned}
S_n &= n \int_{[0,1]^d} \{D_n(u) - C_\perp(u)\}^2 \, du \\
&= n/3^d - 1/2^{d-1} \sum_{t=1}^{n} \prod_{k=1}^{d} \left(1 - E_{tk}^2\right) + 1/n \sum_{t=1}^{n} \sum_{s=1}^{n} \prod_{k=1}^{d} \{1 - \max(E_{tk}, E_{sk})\},
\end{aligned}
$$

where $E_t = (E_{t,1}, \cdots, E_{t,d})^T, t = 1, \ldots, n$, are pseudo observations derived from the following Rosenblatt's transform:

$$
E_{tk} = \frac{\partial^{k-1} C(U_{t,1}, \cdots, U_{t,k}, 1, \cdots, 1)/\partial U_{t,1} \cdots \partial U_{t,k-1}}{\partial^{k-1} C(U_{t,1}, \cdots, U_{t,k-1}, 1, \cdots, 1)/\partial U_{t,1} \cdots \partial U_{t,k-1}}, k = 1, 2, \cdots, d,
$$

and $D_n(u) = \frac{1}{n} \sum_{t=1}^{n} I(E_t \leq u)$ is the d-dimensional empirical distribution function based on the pseudo observations $E_1, \ldots, E_n$, and $C_\perp(u) = u_1 \times u_2 \times \cdots \times u_d$ is the d-dimensional independent copula.

Genest et al. (2009) compared the finite-sample performance among seven types of blanket tests, and concluded that none of these tests was uniformly superior. However, on average the above GRB test S_n performed the best. To be fair, in our comparison setting, GRB is also implemented via the bootstrap method suggested by Genest et al. (2009), with the equal number of bootstrap sample paths. We also consider two types of hybrid tests in the comparison. One is a hybrid of S_n and R_n, denoted as SR_n; and the other is a hybrid of S_n and $T_n(m)$, denoted as $ST_n(m)$. In test $ST_n(m)$ for the sake of brevity we took $m = 1$

We consider four most popular bivariate copula families, namely Gaussian, Student's t, Clayton and Gumbel. All of them have been investigated extensively in a vast literatures; see for example, Song (2000), Chen and Fan (2005), Cossin and Schellhorn (2007), Song et al. (2009) and Genest et al. (2009), just to name a few. The former two copulas are prominent examples of the elliptical families and the latter two are mostly used Archimedean copulas. For the sake of self-containedness, we present some brief descriptions of these four copula families in Appendix B.

In the implementation, whenever it is needed to estimate a parametric copula we use the PLME instead of the means of an inversion of the Kendall's τ proposed by Genest et al. (2009).

To investigate the impact of dependence strength on the finite performance of tests, we set three values of dependence parameters in terms of Kendall's tau, $\tau = 0.25, 0.50$ and 0.75, respectively. For every possible choice of copula and a fixed dependence parameter, we also set two values of sample size as $n = 100$ and 300, respectively. In each experiment, we conduct $M = 1000$ rounds of simulations, in which $N = 1000$ bootstrap sample paths are generated for each simulation case to yield the null distribution. The simulation study has been undertaken on 64 Intel Xeon CPU Cores mit 2.67 GHz of Windows 2003 Server, over a period of two months.

5.2 Results

Tables 1 and 2 report the simulation results at nominal level 5% for all four copulas being true under H_0 hypothesis. From both tables, we can see that the proposed tests R_n, $T_n(m)$ and two hybrid tests SR_n and $ST_n(m)$ perform well on type I error control. The empirical type I error rates are marked with bold font for all cases. Regardless of the choice of sample size, the choice of dependence strength or the choice of copula family, the type I error is satisfactorily controlled at the level close to the nominal level. In this aspect, our new test R_n or $T_n(m)$ is clearly comparable to the GRB test S_n.

In regard to the power comparison, we may draw following conclusions:

1. In general, there is no significant difference between the proposed test $T_n(m)$ and its approximation version R_n, as well as between both hybrid tests $ST_n(m)$ and SR_n, regardless of choice of the dependence strength, the choice of sample size and chosen copula family.

2. The $T_n(1)$ has overall better or equal performance to the $T_n(3)$, because in the case of independent cross-sectional data using $m = 3$ shrinks the effective sample size. Thus in the later discussion on the comparison with other methods we only focus on $T_n(1)$. This numerical evidence also is the basis for our use of $T_n(1)$ in three empirical studies

3. In the cases where Student's t copula is given under H_0, test R_n performs much better than $T_n(1)$ test in most of cases, except the case when $\tau = 0.75$ and the true copula is the Clayton. Similar is observed for SR_n and $ST_n(1)$ tests, where SR_n performs better than $ST_n(1)$ in the

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cases mentioned above.

4. The performance of the proposed R_n, $T_n(1)$, SR_n, $ST_n(1)$ tests as well as GRB test is relied on the strength of the dependence. When $\tau = 0.25$ and sample size $n = 100$, with no surprise, all of the tests have almost no power. Up to our knowledge, there exists no single test that has desirable performance in such a setting of low correlation. Similar results are reported by Genest et al. (2009). This is because when Kendall's tau approximates to 0, the simulated data are drawn from a copula close to the independent copula. In this case, it becomes very hard to differentiate one copula from others. Obviously, in this case making a choice of the copula function form does not really matter. It is interesting to note that even in this situation of weak dependence all the proposed tests has demonstrated in contrary to GRB test to have relatively high power of rejecting Gaussian copula when the underlying is the Student's t copula. In the situation, when Student's t copula is under H_0 and Gaussian copula is the true one, R_n and SR_n tests perform much better than GRB, $T_n(1)$ and $ST_n(1)$ tests.

5. It is interesting to observe that all the proposed tests are significantly superior to GRB test to differentiate between Student's t copula and Gaussian copula. When the sample size increases to $n = 300$, R_n, $T_n(1)$, SR_n and $ST_n(1)$ tests almost reaches 100% power as opposed to the power of GRB test lower than 70% .

6. When Kendall's tau is not too small ($\tau = 0.5$ or $\tau = 0.75$), and the sample size is large enough ($n = 300$), all the proposed tests behave very well. It is worth pointing out that in the following cases both R_n and $T_n(1)$ tests perform poorly and are inferior to GBR, SR_n and $ST_n(1)$ tests: (a) when Gaussian is true and Gumbel is under H_0; (b) when Gumbel is true and t under is H_0; (c) Gaussian is true and Clayton under H_0 for $\tau = 0.5$; and (d) Clayton is true and Student's t copula under H_0 for $\tau = 0.5$. However, in all the remaining cases, the proposed R_n and $T_n(1)$ tests together with hybrid ones SR_n and $ST_n(1)$ perform comparably or better than GRB test S_n.

7. As demonstrated clearly in the simulation studies both hybrid tests $ST_n(1)$ and SR_n showed clear advantages by getting individual tests (R_n and S_n or $T_n(1)$ and S_n) involved. The hybrid tests have shown superior performances in all the cases, regardless of the choice of copula family or choice of dependency strength, and hence they are recommended to be

applied in practice.

6   Applications

In this section we present three empirical analyses using the proposed tests in this paper. In the first example we examine changes of the dependence structure over time between stock returns; the second example investigates joint behavior of residuals between high-frequency time series; and the third example studies the dependence structure of insurance data on losses and expenses.

6.1   Detecting structural changes in the dependency

We use daily returns of Citigroup (C) and Bank of America (BAC), over the period 2004-2012 for this first empirical study. We take each pair belonging to the same industry in order to have a strong dependence. It is known from recent studies (c.g. Hafner and Manner (2012), Patton (2012), Härdle et al. (2013)) that during the global financial crisis over years 2008-2010 the dependency between various financial instruments has been changed, which provides a venue to examine which of the tests would stay most stable in the calm period and be able to capture changes the dependency in the turbulent periods. First, we remove some temporal dependencies by the means of an ARMA(1,1)-GARCH(1,1) process for each year separately, and yield residuals for the analysis at the next step. Following the discussion in the paper, we use empirical marginal distributions to avoid influence of the margin misspecification. To visualize potential dependency changes, in Figure 1 we display via the scatterplots of the residuals transformed to standard normal distribution for years 2004 (left) and 2009 (right). The left plot of residuals for year 2004 shows a usual Gaussian elliptical shape, whereas in the right plot for year 2009, an asymmetric shape like "water-drop" appears in the scatterplot. This makes us wonder if a Gumbel copula would fit the underlying dependence. The p-values of five tests, namely, $T_n(1)$, R_n, S_n and the two hybrid tests, for these two particular years are listed in Table 3, where the two hybrid tests confirm our visual inspection. In addition, we run all five tests over the period of 2004-2012 and select the copula dependency with the largest p-value for each year. These results are summarized in Table 4, which clearly indicate changes in the dependency structure before, during and after the crisis period of 2008-2009.

6.2 Copulas in high-frequency data

In the modern financial econometrics, multiplicative error models (MEM) are standard tools in modeling non-negative time series, e.g. Hautsch (2012), Cipollini and Gallo (2010) and Engle (2002). In this section of empirical study we consider the Vector MEM (VMEM) proposed by Cipollini and Gallo (2010), which is defined as follow,

$$x_t = \mu_t \odot \varepsilon_t = diag(\mu_t)\varepsilon_t,$$

where innovation vector ε_t is a d-dimensional conditionally i.i.d. stochastic process and "$\odot$" denotes the Hadamard (element-by-element) product. Its density is specified over R_+^d with a unit vector 1 as expectation and variance-covariance matrix Σ, namely

$$\varepsilon_t | \mathcal{F}_{t-1} \sim D(1, \Sigma),$$

where $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \cdots)$ and $D(1, \Sigma)$ is a multivariate distribution with mean vector 1 and covariance matrix Σ. It follows that

$$E(x_t | \mathcal{F}_{t-1}) = \mu_t;$$
$$Var(x_t | \mathcal{F}_{t-1}) = \mu_t \mu_t' \odot \Sigma.$$

In order to model short-run effects and exclude low frequency patterns of the marginal time series, the marginal mean vectors $\mu_{i,t}$ for $i = 1, \ldots, d$ have been specified through a fractional integrations of order $(1,1)$, e.g. Hautsch (2012)

$$\mu_{i,t} = [\omega_i + \{1 - \beta_i - (\alpha_i + \beta_i)(1 - L)^{\delta_i}\} x_{i,t}] + \beta_i \mu_{i,t-1}, \tag{6}$$

where L is the lag operator with $L^j x_t = x_{t-j}$; $(1 - L)^\delta = \sum_{j=0}^{\infty} \binom{\delta}{j}(-1)^j L^j$ is the fractional difference operator with different δ for each series; ω_i, α_i and β_i with $\alpha_i + \beta_i < 1$ for $i = 1, \ldots, d$ are the parameters of the model. This model may be further extended to a case when β and α are matrices for more precise multivariate modelling.

We apply the VMEM to analyze the high-frequency stock data for Apple as one of the most liquid stocks on NYSE. We extract the raw data of executed trades for the period of December 17th, 2009 to December 31st, 2009. The data has been cleaned using a similar method to that proposed in Hautsch (2012). The resulting cleaned data contain 10 min based numbers of trades (NT), volumes (Vol) and high-low differences (HL). Thus, in this example the process is three-dimensional,

namely NT, Vol and HL. The sample size is $n = 300$, consistent with that considered in the simulation study. Also we remove the U-shaped daily cyclic patterns via cubic splines. The issue of interest concerns the specification of multivariate distribution D for these three variables. A popular choice would be multivariate Gamma distribution, which does not necessarily describe the data well. Alternatively, some researchers suggest to separate margins and dependency in the VMEM (e.g. Cipollini and Gallo (2010)). To date, there is no work on specification test for copula model, and simply Gaussian copula has been used in the VMEM. As D is unknown, we first estimate the parameters of (6) through the PMLE under assumption of independent copula, implying diagonal covariance matrix Σ. Having residuals obtained from MEM (6) we analyze them upon the needed appropriate multivariate dependence D and after specification of the copula re-estimate fully specified model. A similar approach is used in SCOMDY models by Chen and Fan (2006) where one first estimates univariate GARCH, then find the copula based on the residuals and later re-estimate multivariate GARCH with correct copula specification. Figure 2 shows scatterplots of the residuals from univariate MEMs, which are transformed to the univariate normal distribution. These scatterplots suggest a hypothesis of Gumbel copula. Thus we perform all 5 tests for all pairs, namely (NT, Vol), (NT, HL) and (Vol, HL) as well as for the three-dimensional (NT, Vol, HL). The results are summarized in Tables 5. Here, we consider classical 3-dimensional Gaussian, Gumbel and Clayton copulas, as well as recently developed hierarchical Archimedean copulas (HAC) with Gumbel and Clayton generators (e.g. Okhrin et al. (2013a), Okhrin et al. (2013b)) and Vine copulas (e.g. Aas et al. (2009)). As seen from Table 5, for most of the pairs, the Gumbel copula is preferred for (NT, Vol), Gaussian copula for (HL, NT) and Clayton or Gumbel for (HL, Vol). For the 3-dimensional case, we are unable to derive GRB S_n test statistic for the vine copulas because of very complex forms of the copula density. So, no results on vine copulas have been reported for S_n nor for both hybrid tests (SR_n and $ST_n(1)$). In the empirical study, either HAC Gumbel or simple Gumbel is accepted by the hybrid tests, as oppose to the conflicting results draws by the S_n and $T_n(1)$ tests. This analysis illustrating an approach to determining an appropriate dependency of characteristics (e.g. Vol, HL or NT) of a stock.

6.3 Analysis of Losses and Allocated Loss Adjustment Expenses

Now, we apply the proposed tests to a well-known insurance dataset on losses and ALAEs, which are collected by the US Insurance Service Office. Such data has been previously analyzed by many authors, including, Frees and Valdez (1998), Genest et al. (1998), Klugman and Parsa (1999), Chen and Fan (2005) and Denuit et al. (2006), among others.

The dataset consists of 1500 general liability claims, among which 34 claims are censored due to late settlement lags. Each claim consists of an indemnity payment (i.e. the loss) and an allocated loss adjustment expense (ALAE). Here we determine a dependence model using the 1466 complete data. We run the proposed goodness-of-fit tests on four families of copulas, including Gaussian copula, Student's t copula, Gumbel copula and Clayton copula. For each copula, we estimate the dependence parameter by the PMLE approach described in Section 2.

Table 6 reports the results of PMLE, test statistics and p-values. The estimated freedom of degree of Student's t copula is 11.11. From Table 6, we find that Gumbel copula appears to be the most adequate and Gaussian copula is least suitable among the four copula models.

Our findings obtained by the hybrid tests are consistent with the model selection results reported by Frees and Valdez (1998), Genest et al. (1998), Chen and Fan (2005) and Denuit et al. (2006). Frees and Valdez (1998) and Denuit et al. (2006) point out a positive upper-tail dependence between loss and ALAE, implying that large losses tend to be associated to large ALAEs. This is because expensive claims usually take some time to be settled and induce extra costs for the insurance company. Thus, it is reasonable to observe a positive upper-tail dependence. On the other hand, no lower tail dependence is detected. Among the four copula models, Gumbel copula exhibits a strong upper-tail dependence, which properly reflects the relationship between loss and ALAE. The other copula models do not have similar features of upper tail dependence.

7 Conclusion

In this paper, we focus on goodness-of-fit test for specification of semiparametric copula dependence models. We propose a new method based on pseudo likelihood of cross-validation leading to the construction of a test statistics by comparing the "in-sample" pseudo-likelihood and "out-of-sample" pseudo-likelihood. As shown in theory and numerical examples, the pro-

posed comparison of pseudo likelihoods over different data sets has provide a highly competitive performance to indicate how well a model fits the data. To mitigate the computational burden of of the proposed $T_n(m)$ test, we introduce R_n test, which shown similar performance to $T_n(m)$ test. We establish the large sample properties for both $T_n(m)$ and R_n tests. In comparison to the blank tests considered in Genest et al. (2009), all of which are rank-based tests, the proposed test enable us to avoid using any probability integral transformations.

Another main contribution of this paper is that we propose a hybrid mechanism to combine several different tests. In term of average performance, the hybrid test is clearly superior to any of individual tests used in the combination. An important property is that if there is at least one consistent test in the combination, then the hybrid test is consistent. This hybrid strategy is particularly appealing when there is no *a prior* knowledge which test might be the top performer at a given occasion.

We conduct extensive simulation experiments to investigate and compare the finite-sample performances between our proposed tests and the GRB test. The results of Monte Carlo simulations show that the proposed tests perform satisfactorily in type I error control and that they are very comparable to the best performer (i.e. GRB test) given by Genest et al. (2009). In particular, when the data are generated from Student's *t* copula, the proposed tests are more powerful than GRB test. Also, the proposed hybrid tests have shown a superior performance in all the cases, regardless of the choice of copula or choice of dependency strength, and hence they are highly recommended as a desirable method to be applied in practice. We also applied these proposed tests to three real datasets.

It is interesting to extend the proposed tests with independent cross-sectional data to time series data. Also, it is worth exploring the effect of block size on the test power. Finally, in this paper, we focus only on the occasion of 2-dimensional copula families, and it is of great interest to evaluate theses tests to multi-dimensional copulas, such as vine copulas.

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APPENDIX A

This appendix is devoted to the proofs of the theorems given in Section 4.

Proof of Theorem 1:

Define the rescaled empirical copula of $X_1, \cdots, X_n$ by

$$\tilde{C}(u) = \frac{1}{n+1} \sum_{t=1}^{n} I\left\{\tilde{F}(X_t) \leq u\right\} = \frac{1}{n+1} \sum_{t=1}^{n} I\left\{\tilde{F}_1(X_{t1}) \leq u_1, \cdots, \tilde{F}_d(X_{td}) \leq u_d\right\}.$$

For any $\theta \in \Theta$, we can rewrite $S(\theta)$, $\hat{S}(\theta)$, $V(\theta)$ and $\hat{V}(\theta)$ as follows:

$$S(\theta) = -\int_{u \in [0,1]^d} l_{\theta\theta}(u;\theta)dC_0(u); \qquad \hat{S}(\theta) = -\frac{n+1}{n}\int_{u \in [0,1]^d} l_{\theta\theta}(u;\theta)d\tilde{C}(u),$$

and

$$V(\theta) = \int_{u \in [0,1]^d} l_{\theta}(u)l_{\theta}^T(u)dC_0(u); \qquad \hat{V}(\theta) = \frac{n+1}{n}\int_{u \in [0,1]^d} l_{\theta}(u)l_{\theta}^T(u)d\tilde{C}(u),$$

where $C_0(\cdot)$ and $\tilde{C}(\cdot)$ are the true copula and the rescaled empirical copula.

By condition (A1), applying Lemma 1(c) in Chen and Fan (2005)(see also Fermanian et al. (2004)), we have

$$\sup_{\theta \in \mathcal{N}(\theta^*)} \left\|\hat{S}(\theta) - S(\theta)\right\| = \sup_{\theta \in \mathcal{N}(\theta^*)} \left\|\int_{u \in [0,1]^d} l_{\theta\theta}(u;\theta)d\left\{\frac{n+1}{n}\tilde{C}(u) - C_0(u)\right\}\right\| \xrightarrow{pr} 0, \quad \text{as} \quad n \to \infty.$$

Hence, using the two facts $\left\|\hat{S}(\hat{\theta}) - S(\theta^*)\right\| \leq \left\|\hat{S}(\hat{\theta}) - S(\hat{\theta})\right\| + \left\|S(\hat{\theta}) - S(\theta^*)\right\|$, and $\hat{\theta} \xrightarrow{pr} \theta^*$, we obtained $\hat{S}(\hat{\theta}) \xrightarrow{pr} S(\theta^*)$.

Applying the same arguments above, we can show $\hat{V}(\hat{\theta}) \xrightarrow{pr} V(\theta^*)$.

Furthermore, by condition (A2) and Slutsky's theorem, we have

$$R_n = \text{tr}\left\{\hat{S}(\hat{\theta})^{-1}\hat{V}(\hat{\theta})\right\} \xrightarrow{pr} \text{tr}\left\{S(\theta^*)^{-1}V(\theta^*)\right\}.$$

Proof of Theorem 2(i):

First note that, $\hat{\theta}$ solves the equation $\sum_{t=1}^{n} l_{\theta}\{\tilde{F}(X_t);\hat{\theta}\} = 0$. Applying the mean-value theorem, we have

$$0 = \sum_{t=1}^{n} l_{\theta}\{\tilde{F}(X_t);\theta^*\} + \sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t);\tilde{\theta}\}(\hat{\theta} - \theta^*),$$

25

where $\tilde{\theta}$ lies between θ^* and $\hat{\theta}$. Thus

$$\hat{\theta} - \theta^* = -\left[\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \tilde{\theta}\}\right]^{-1} \frac{1}{n}\sum_{t=1}^{n} l_{\theta}\{\tilde{F}(X_t); \theta^*\}.$$

For any $1 \leq i, j \leq p$, expanding $l_{\theta\theta}\{\tilde{F}(X_t); \hat{\theta}\}_{ij}$ around θ^* leads to

$$
\begin{aligned}
\hat{S}(\hat{\theta})_{ij} &= \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \hat{\theta}\}_{ij} \\
&= \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \theta^*\}_{ij} + \frac{1}{n}\sum_{t=1}^{n} \frac{\partial l_{\theta\theta}\{\tilde{F}(X_t); \check{\theta}\}_{ij}}{\partial\theta^T}(\hat{\theta} - \theta^*) \\
&= \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \theta^*\}_{ij} - \frac{1}{n}\sum_{t=1}^{n} \frac{\partial l_{\theta\theta}\{\tilde{F}(X_t); \check{\theta}\}_{ij}}{\partial\theta^T}\left[\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \tilde{\theta}\}\right]^{-1}\frac{1}{n}\sum_{t=1}^{n} l_{\theta}\{\tilde{F}(X_t); \theta^*\},
\end{aligned}
$$

where $\check{\theta}$ lies between θ^* and $\hat{\theta}$.

By condition (B3), applying again Lemma 1(c) in Chen and Fan (2005), we obtain

$$\frac{1}{n}\sum_{t=1}^{n} \frac{\partial l_{\theta\theta}\{\tilde{F}(X_t); \check{\theta}\}_{ij}}{\partial\theta^T} \xrightarrow{pr} \mathbb{E}_0\left[\frac{\partial l_{\theta\theta}\{F(X_1); \theta^*\}_{ij}}{\partial\theta^T}\right].$$

Also, we know $\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t); \tilde{\theta}\} \xrightarrow{pr} S(\theta^*)$ as $n \to \infty$. Therefore

$$
\begin{aligned}
\hat{S}(\hat{\theta})_{ij} &= \frac{1}{n}\sum_{t=1}^{n}\left[l_{\theta\theta}\{\tilde{F}(X_t); \theta^*\}_{ij} + M_1^{ij} S^{-1}(\theta^*) l_{\theta}\{\tilde{F}(X_t); \theta^*\}\right] + o_p(1) \\
&\triangleq \frac{1}{n}\sum_{t=1}^{n} h_S\{\tilde{F}(X_t); \theta^*\}_{ij} + o_p(1),
\end{aligned}
\tag{7}
$$

where, $M_1^{ij} \triangleq \mathbb{E}_0\left[\frac{\partial l_{\theta\theta}\{F(X_1); \theta^*\}_{ij}}{\partial\theta^T}\right]$ is a $1 \times p$ vector, h_S is a $p \times p$ matrix with element $h_S\{\tilde{F}(X_t); \theta^*\}_{ij}$.

Employing the same arguments above, we have

$$
\begin{aligned}
\hat{V}(\hat{\theta})_{ij} &= \frac{1}{n}\sum_{t=1}^{n}\left[l_{\theta}\{\tilde{F}(X_t); \theta^*\}_i l_{\theta}\{\tilde{F}(X_t); \theta^*\}_j + M_2^{ij} S^{-1}(\theta^*) l_{\theta}\{\tilde{F}(X_t); \theta^*\}\right] + o_p(1) \quad (8) \\
&\triangleq \frac{1}{n}\sum_{t=1}^{n} h_V\{\tilde{F}(X_t); \theta^*\}_{ij} + o_p(1),
\end{aligned}
$$

where $M_2^{ij} = \mathbb{E}_0\left[\frac{\partial l_{\theta}\{F(X_1); \theta^*\}_i}{\partial\theta^T} l_{\theta}\{F(X_1); \theta^*\}_j + \frac{\partial l_{\theta}\{F(X_1); \theta^*\}_j}{\partial\theta^T} l_{\theta}\{F(X_1); \theta^*\}_i\right]$ and h_V is a $p \times p$ matrix with element $h_V\{\tilde{F}(X_t); \theta^*\}_{ij}$.

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Under the null hypothesis of the copula model being correctly specified, by Bartlett identity, we have $S(\theta^*) = V(\theta^*)$, moreover the test statistic R_n given in (5) can be represented as follows:

$$
\begin{aligned}
\sqrt{n}\,(R_n - p) &= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat{\theta})\hat{V}(\hat{\theta}) - I_p\right\} \\
&= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat{\theta})\hat{V}(\hat{\theta}) - S^{-1}(\theta^*)V(\theta^*)\right\} \\
&= \mathrm{tr}\left[S^{-1}(\theta^*)\sqrt{n}\left\{\hat{V}(\hat{\theta}) - V(\theta^*)\right\}\right] \\
&\quad + \mathrm{tr}\left[S^{-1}(\theta^*)\hat{V}(\hat{\theta})S^{-1}(\theta^*)\sqrt{n}\left\{S(\theta^*) - \hat{S}(\hat{\theta})\right\}\right] \\
&\quad + \mathrm{tr}\left[\hat{S}^{-1}(\hat{\theta})\hat{V}(\hat{\theta})S^{-2}(\theta^*)\sqrt{n}\left\{S(\theta^*) - \hat{S}(\hat{\theta})\right\}^2\right].
\end{aligned}
$$

Utilizing the asymptotic expansion in (7) and (8), we have

$$
\begin{aligned}
\sqrt{n}\left\{\hat{S}(\hat{\theta}) - S(\theta^*)\right\} &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left[h_S\{\tilde{F}(X_t); \theta^*\} - S(\theta^*)\right] + o_p(1) \\
&= \sqrt{n}\int_{u \in (0,1)^d} h_S(u; \theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u) - C_0(u)\right\} + o_p(1),
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{n}\left\{\hat{V}(\hat{\theta}) - V(\theta^*)\right\} &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left[h_V\{\tilde{F}(X_t); \theta^*\} - V(\theta^*)\right] + o_p(1) \\
&= \sqrt{n}\int_{u \in (0,1)^d} h_V(u; \theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u) - C_0(u)\right\} + o_p(1).
\end{aligned}
$$

By conditions (B1) and (B2), employing Lemma 2 in Chen and Fan (2005) (see also Ruymgaart et al. (1972), Ruymgaart (1974) or Genest and Rivest (1995)), we have $\|\hat{S}(\hat{\theta}) - S(\theta^*)\| = O_p(n^{-1/2})$ and $\|\hat{V}(\hat{\theta}) - V(\theta^*)\| = O_p(n^{-1/2})$. In addition, giving these facts: $\sqrt{n}\|\hat{S}(\hat{\theta}) - S(\theta^*)\|^2 = o_p(1)$, $\hat{S}(\hat{\theta}) \xrightarrow{pr} S(\theta^*)$ and $\hat{V}(\hat{\theta}) \xrightarrow{pr} V(\theta^*)$, we reach the following expression:

$$
\sqrt{n}\,(R_n - p) = \sqrt{n}\int_{u \in [0,1]^d} h_R(u; \theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u) - C_0(u)\right\} + o_p(1),
$$

where

$$
h_R(u; \theta^*) = \sum_{i,j=1}^{p} S^{-1}(\theta^*)_{ij}\left\{h_S(u; \theta^*)_{ji} + h_V(u; \theta^*)_{ji}\right\}.
$$

Again, applying Lemma 2 in Chen and Fan (2005), we have

$$\sqrt{n}\,(R_n - p) \xrightarrow{d} N(0, \sigma_R^2),$$

where

$$\sigma_R^2 = var_0\left[h_R(u; \theta^*) + D\{F(X_1); \theta^*\}\right], \tag{9}$$

and

$$D\{F(X_1); \theta^*\} = \sum_{j=1}^{d} \int_{u \in [0,1]^d} \frac{\partial h_R(u; \theta^*)}{\partial u_j} I\{F_j(X_{1j}) \le u_j\} dC_0(u).$$

Note that the additional term $D\{F(X_1); \theta^*\}$ comes from the uncertainty of the estimator for the marginal distribution function $F(X_1) = \{F_1(x_1), \cdots, F_d(x_d)\}$. It vanishes when $F(X_1)$ is known.

The asymptotic variance σ_R^2 may be consistently estimated by

$$\hat{\sigma}_R^2 = \frac{1}{n} \sum_{t=1}^{n} \left[h_R\{\tilde{F}(X_t); \hat{\theta}\} - \sum_{i,j=1}^{p} \hat{S}(\hat{\theta})_{ij}^{-1} \hat{V}(\hat{\theta})_{ji} + D[\tilde{F}(X_t); \hat{\theta}]\right]^2. \tag{10}$$

To prove Theorem 2(ii), we need the following lemma.

Lemma 1 : Under the conditions (A1) and (C1), we have

$$\sup_{1 \le b \le B} \|\hat{\theta} - \hat{\theta}_{-b}\| = o_p(n^{-\frac{3}{4}}). \tag{11}$$

Proof of Lemma 1: By equation (3), $\hat{\theta}_{-b}$ solves the following equation

$$0 = \sum_{b'=1, b' \ne b}^{B} \sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{b'}); \hat{\theta}_{-b}\}.$$

Expanding $l_\theta\{\tilde{F}(X_i^{b'}); \hat{\theta}_{-b}\}$ around $\hat{\theta}$ leads to

$$\begin{aligned}
0 &= \sum_{b'=1, b' \ne b}^{B} \sum_{i=1}^{m} l_\theta(\tilde{F}(X_i^{b'}); \hat{\theta}_{-b}) \\
&= -\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b); \hat{\theta}\} + \sum_{b'=1, b' \ne b}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^{b'}); \tilde{\theta}_{-b}\}(\hat{\theta}_{-b} - \hat{\theta}),
\end{aligned}$$

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where $\tilde{\theta}_{-b}$ lies between $\hat{\theta}$ and $\hat{\theta}_{-b}$. It follows that

$$\hat{\theta}_{-b} - \hat{\theta} = \left[\frac{1}{n} \sum_{b'=1,b'\neq b}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^{b'}); \tilde{\theta}_{-b}\}\right]^{-1} \frac{1}{n} \sum_{i=1}^{m} l_{\theta}\{\tilde{F}(X_i^b); \hat{\theta}\}.$$

By conditions (A1) and (C1),

$$\begin{aligned}
\sup_{1\leq b\leq B} \frac{1}{n} \left\|\sum_{i=1}^{m} l_{\theta}\{\tilde{F}(X_i^b); \hat{\theta}\}\right\| &\leq \frac{m}{n} \sup_{1\leq b\leq B} \sup_{\theta\in\mathcal{N}(\theta^*)} \left\|l_{\theta}\{\tilde{F}(X_i^b); \theta\}\right\| \\
&= o_p(n^{-\frac{3}{4}})O_p(1) \\
&= o_p(n^{-\frac{3}{4}}).
\end{aligned}$$

In addition, by condition (A1), using the similar arguments in the proof of Theorem 1, we can show

$$\frac{1}{n} \sum_{b'=1,b'\neq b}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^{b'}); \tilde{\theta}_{-b}\} \xrightarrow{pr} S(\theta^*).$$

Moreover,

$$\begin{aligned}
\sup_{1\leq b\leq B} \|\hat{\theta}_{-b} - \hat{\theta}\| &\leq \sup_{1\leq b\leq B} \left\|\left[\frac{1}{n} \sum_{b'=1,b'\neq b}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^{b'}); \tilde{\theta}_{-b}\}\right]^{-1}\right\| \\
&\quad \times \sup_{1\leq b\leq B} \left\|\frac{1}{n} \sum_{i=1}^{m} l_{\theta}\{\tilde{F}(X_i^b); \hat{\theta}\}\right\| \\
&= o_p(n^{-\frac{3}{4}}).
\end{aligned}$$

Proof of Theorem 2(ii): By definition

$$T_n(m) = \sum_{b=1}^{B} \sum_{i=1}^{m} l\{\tilde{F}(X_i^b); \hat{\theta}\} - \sum_{b=1}^{B} \sum_{i=1}^{m} l\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\},$$

expanding $l\{\tilde{F}(X_i^b); \hat{\theta}_{-b}\}$ around $\hat{\theta}$ leads to

$$
\begin{aligned}
T_n(m) &= -\sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}(\hat{\theta}-\hat{\theta}_{-b}) - \frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat{\theta}-\hat{\theta}_{-b})^T l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\}(\hat{\theta}-\hat{\theta}_{-b}) \\
&= -\sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}\left[\frac{1}{n}\sum_{b'=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^{b'});\hat{\theta}\} + e_{1b} + e_{2b}\right]^{-1}\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\} \\
&\quad - \frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat{\theta}-\hat{\theta}_{-b})^T l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\}(\hat{\theta}-\hat{\theta}_{-b}) \\
&\triangleq R_n - W_1 - W_2,
\end{aligned}
$$

where

$$
W_1 = \frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat{\theta}-\hat{\theta}_{-b})^T l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\}(\hat{\theta}-\hat{\theta}_{-b}),
$$

and

$$
\begin{aligned}
W_2 &= \sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}\left[\frac{1}{n}\sum_{b'=1,b'\neq b}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{b'});\tilde{\theta}_{-b}\}\right]^{-1}(e_{1b}+e_{2b}) \\
&\quad \times \left[\frac{1}{n}\sum_{b'=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{b'});\hat{\theta}\}\right]^{-1}\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\},
\end{aligned}
$$

with

$$
e_{1b} = \frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\},
$$

and

$$
e_{2b} = \frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\} - \frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b);\hat{\theta}\}.
$$

Lemma 1 implies that

$$
\begin{aligned}
\sup_{1\leq b\leq B}\|W_1\| &= \sup_{1\leq b\leq B}\|\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat{\theta}-\hat{\theta}_{-b})^T l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}(\hat{\theta}-\hat{\theta}_{-b})\| \\
&= o_p(n^{-\frac{1}{2}})\sup_{1\leq b\leq B}\sup_{\theta\in\Theta}\|l_\theta\{\tilde{F}(X_i^b);\theta\}\| \\
&= o_p(n^{-\frac{1}{2}})O_p(1) \\
&= o_p(n^{-\frac{1}{2}}).
\end{aligned}
$$

We now prove that $W_2 = o_p(n^{-\frac{1}{2}})$. By conditions (A1) and (C1),

$$\sup_{1 \le b \le B} \|e_{1b}\| \le \frac{m}{n} \sup_{1 \le b \le B} \sup_{\theta \in \mathcal{N}(\theta^*)} \left\| l_\theta\{\tilde{F}(X_i^b); \theta\} \right\| = o_p(n^{-\frac{3}{4}}) O_p(1) = o_p(n^{-\frac{3}{4}}).$$

Expanding $l_{\theta\theta}(\tilde{F}(X_i^b); \tilde{\theta}_{-b})$ around $\hat{\theta}$ leads to, under condition (B3),

$$
\begin{aligned}
\|e_{2b}\| &= \|\frac{1}{n} \sum_{b=1}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b); \tilde{\theta}_{-b}\} - \frac{1}{n} \sum_{b=1}^{B} \sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b); \hat{\theta}\}\| \\
&\le \|\frac{1}{n} \sum_{b=1}^{B} \sum_{i=1}^{m} \sum_{k,l=1}^{p} \frac{\partial}{\partial \theta} l_{\theta\theta}\{\tilde{F}(X_i^b); \tilde{\tilde{\theta}}_{-b}\}_{kl}\| \sup_{1 \le b \le B} \|\hat{\theta} - \tilde{\theta}_{-b}\| \\
&= o_p(n^{-\frac{3}{4}}) O_p(1) = o_p(n^{-\frac{3}{4}}).
\end{aligned}
$$

where $\tilde{\tilde{\theta}}_{-b}$ lies between $\hat{\theta}$ and $\hat{\theta}_{-b}$. Therefore,

$$
\begin{aligned}
\sup_{1 \le b \le B} \|W_2\| &\le n \left\| \frac{1}{n} \sum_{b=1}^{B} \sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b); \hat{\theta}\} \right\| \sup_{1 \le b \le B} \left\| \left[\frac{1}{n} \sum_{b'=1, b' \ne b}^{B} \sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{b'}); \tilde{\theta}_{-b}\} \right]^{-1} \right\| \\
&\quad \times \sup_{1 \le b \le B} \|(e_{1b} + e_{2b})\| \left\| \left[\frac{1}{n} \sum_{b'=1}^{B} \sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^q); \hat{\theta}\} \right]^{-1} \right\| \sup_{1 \le b \le B} \left\| \frac{1}{n} \sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b); \hat{\theta}\} \right\| \\
&= n \times O_p(1) O_p(1) o_p(n^{-\frac{3}{4}}) O_p(1) o_p(n^{-\frac{3}{4}}) \\
&= o_p(n^{-\frac{1}{2}}).
\end{aligned}
$$

In summary, we prove that

$$T_n(m) - R_n = o_p(n^{-\frac{1}{2}}).$$

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APPENDIX B

This appendix is devoted to the copula models used in the Simulation and Empirical Studies.

1. The bivariate Gaussian copula is defined as

$$C(u_1, u_2; \theta) = \Phi\{\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta\},$$

where $\Phi(\cdot)$ is the standard univariate Gaussian distribution, $\Phi(u_1, u_2; \theta)$ is the bivariate Gaussian distribution with zeros mean and unit variance, and $\theta \in [-1, 1]$ is the dependence coefficient corresponding to Kendall's $\tau = \frac{2}{\pi} \arcsin(\theta)$.

2. The Student's t copula is given by

$$C(u_1, u_2; \nu; \theta) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{\sqrt{2\pi(1-\theta^2)^{1/2}}} \left(1 + \frac{x^2 - 2\theta xy + y^2}{\nu(1-\theta^2)}\right)^{-\nu/2-1} dxdy,$$

where ν is the number of degrees of freedom, and θ is the dependence coefficient that correspond to Kendall's $\tau = \frac{2}{\pi} \arcsin(\theta)$. Student's t copula has received increasing attention in the modeling of multivariate financial time series. Many empirical studies have shown that the Student's *t* copula fit better than Gaussian copula since the former captures better the phenomenon of dependent extreme values, which is often observed in financial asset return data. See for example Breymann et al. (2003) and Demarta and McNeil (2005). Throughout the simulation study we fix the number of degrees of freedom at $\nu = 4$ as in Genest et al. (2009). Varying ν in the numerical analysis of the GRB test is numerically challenging and computationally unstable because of the Rosenblatt's transform.

3. Clayton copula (Clayton, 1978)

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta},$$

where the dependence coefficient $\theta \geq 0$, which corresponds to Kendall's $\tau = \theta/(\theta + 2)$. This copula represents lower tail dependency and is of particular interest in the modelling of Value-at-Risk.

4. Gumbel copula (Gumbel, 1960)

$$C(u_1, u_2; \theta) = \exp\left[-\left\{(-\log u_1)^\theta + (-\log u_2)^\theta\right\}^{1/\theta}\right],$$

with dependence coefficient $\theta \geq 1$ and the corresponding Kendall's $\tau = 1 - 1/\theta$. This family is the only extreme value Archimedean copula which also represents an upper tail dependency. It is often used in the modeling of gains as well as in the modeling of joint maxima.

APPENDIX C

This appendix is devoted to the results of the Simulation and Empirical Studies.

| | | $n = 100$ | | | | | | | | | | | | | | | | | |
| | | $\tau = 0.25$ | | | | | | $\tau = 0.50$ | | | | | | $\tau = 0.75$ | | | | | |
True	H_0	S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$	S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$	S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$
Ga.	Ga.	**4.5**	**6.1**	**6.0**	**5.1**	**5.6**	**5.9**	**5.7**	**5.8**	**5.1**	**5.1**	**5.7**	**6.2**	**4.7**	**4.1**	**4.0**	**4.3**	**4.4**	**4.5**
Ga.	t	2.9	28.9	5.3	1.5	18.1	2.5	2.9	26.3	10.8	2.4	16.3	3.8	2.1	21.5	9.1	2.3	13.7	3.8
Ga.	Cl.	14.7	5.4	4.3	4.8	10.3	10.0	27.3	13.5	15.9	10.6	21.9	22.4	35.9	36.4	36.7	26.6	39.3	39.5
Ga.	Gu.	4.3	4.4	2.2	3.3	3.7	2.6	10.0	5.0	3.1	2.0	6.6	6.5	18.1	5.1	4.5	5.2	12.8	13.0
t	Ga.	11.1	55.4	54.9	37.8	49.8	48.9	20.0	62.7	64.3	47.5	55.0	55.8	25.4	59.2	59.9	48.0	51.7	52.5
t	t	**5.2**	**4.1**	**5.2**	**3.7**	**3.9**	**4.5**	**4.7**	**4.6**	**5.1**	**3.9**	**3.5**	**4.1**	**6.3**	**4.3**	**3.8**	**5.5**	**5.6**	**5.4**
t	Cl.	22.2	30.4	30.4	20.5	29.4	27.3	38.1	44.8	46.4	33.9	45.7	46.2	42.9	63.8	63.9	47.4	57.2	57.1
t	Gu.	7.8	29.9	31.7	18.2	24.6	24.2	18.8	30.8	32.1	20.7	29.3	30.0	36.9	32.2	32.5	24.0	38.5	38.9
Cl.	Ga.	22.9	10.5	9.3	9.1	18.3	18.1	79.0	36.5	32.0	23.7	73.7	73.4	99.9	84.9	72.5	55.9	99.2	99.2
Cl.	t	7.4	12.6	1.5	1.2	10.8	3.8	38.0	4.5	4.3	3.5	25.0	24.2	87.9	19.1	58.2	29.7	80.4	81.9
Cl.	Cl.	**4.9**	**4.5**	**4.0**	**4.7**	**4.1**	**3.8**	**4.2**	**6.2**	**5.8**	**4.6**	**5.7**	**5.7**	**5.3**	**6.3**	**6.0**	**4.3**	**5.5**	**5.2**
Cl.	Gu.	34.1	16.9	15.1	10.7	31.7	29.7	92.5	44.8	48.0	27.4	90.6	90.6	100.0	97.2	97.7	81.2	100.0	100.0
Gu.	Ga.	6.4	15.0	14.2	12.3	12.9	13.0	7.6	27.8	27.5	23.4	22.5	22.1	10.1	45.4	42.4	31.7	35.7	35.0
Gu.	t	8.6	10.8	0.9	1.5	8.6	4.1	10.7	7.1	3.0	2.3	7.8	5.6	8.9	4.7	4.4	5.0	6.5	6.8
Gu.	Cl.	36.8	11.0	9.0	7.6	30.0	29.1	68.1	30.5	35.3	27.2	62.5	62.8	82.6	80.9	81.6	61.4	86.5	86.6
Gu.	Gu.	**4.7**	**5.8**	**5.4**	**4.6**	**4.5**	**4.7**	**4.3**	**5.3**	**4.7**	**4.8**	**3.2**	**3.4**	**5.9**	**5.3**	**5.5**	**4.9**	**5.5**	**5.4**

Table 1: Percentage of rejection of H_0 by various tests of size $n = 100$ from different copula models with different τ, $N = 1000$, $M = 1000$.

True	H_0	\multicolumn{6}{c}{$\tau = 0.25$}	\multicolumn{6}{c}{$\tau = 0.50$}	\multicolumn{6}{c}{$\tau = 0.75$}
		S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$	S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$	S_n	R_n	$T_n(1)$	$T_n(3)$	SR_n	$ST_n(1)$
Ga.	Ga.	**5.1**	**5.1**	**5.4**	**5.0**	**4.7**	**5.1**	**4.3**	**5.2**	**5.3**	**5.3**	**4.9**	**4.8**	**5.5**	**4.4**	**4.5**	**4.2**	**4.7**	**4.2**
Ga.	t	9.7	80.8	69.8	21.2	70.3	51.6	6.7	75.0	73.9	27.1	64.9	58.8	5.1	66.0	61.7	22.4	55.3	46.4
Ga.	Cl.	41.7	5.8	5.6	5.9	30.9	30.8	88.3	31.8	33.9	21.2	81.3	81.4	99.1	77.7	78.8	62.5	98.3	98.3
Ga.	Gu.	19.8	6.0	4.7	3.8	15.7	14.9	42.4	8.3	6.1	4.8	33.8	32.7	60.2	7.3	6.9	6.3	49.5	49.1
t	Ga.	21.5	90.9	92.9	77.3	87.8	89.9	45.8	93.9	94.5	84.7	91.2	92.6	65.7	95.6	96.3	88.1	92.9	93.7
t	t	**4.5**	**5.0**	**6.0**	**3.8**	**5.5**	**5.1**	**4.8**	**4.4**	**5.8**	**2.8**	**4.9**	**5.8**	**4.3**	**5.5**	**4.6**	**6.2**	**5.6**	**4.5**
t	Cl.	63.6	61.9	64.9	44.2	71.8	72.9	90.4	81.1	82.4	66.7	91.8	91.8	98.3	98.0	98.0	86.5	99.6	99.6
t	Gu.	22.2	64.5	67.0	40.9	61.3	63.4	57.9	61.8	63.6	37.9	69.0	70.1	88.3	71.4	72.6	52.7	88.3	88.3
Cl.	Ga.	61.5	24.7	25.7	16.1	54.6	55.1	99.9	78.8	72.4	55.4	99.9	99.9	100.0	100.0	99.8	97.2	100.0	100.0
Cl.	t	35.9	40.9	22.8	9.0	46.0	32.2	96.0	4.6	4.1	3.3	93.2	92.5	100.0	36.6	97.7	75.9	100.0	100.0
Cl.	Cl.	**4.5**	**5.8**	**6.2**	**4.2**	**4.9**	**5.3**	**5.1**	**4.1**	**4.4**	**3.7**	**3.8**	**3.9**	**5.0**	**6.6**	**6.5**	**5.0**	**5.5**	**5.5**
Cl.	Gu.	96.3	50.3	53.2	29.0	94.6	94.6	100.0	94.9	95.7	68.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Gu.	Ga.	10.3	32.5	33.7	20.2	30.5	30.2	16.1	63.8	63.0	45.2	57.8	57.6	26.1	87.8	84.1	69.4	83.1	80.0
Gu.	t	22.6	31.3	16.4	5.3	30.3	18.7	36.8	15.5	5.5	3.4	29.7	26.0	47.0	5.5	4.3	5.9	32.2	31.8
Gu.	Cl.	92.5	18.8	20.8	13.6	87.0	87.0	100.0	79.2	81.5	60.5	99.9	99.9	100.0	100.0	100.0	97.5	100.0	100.0
Gu.	Gu.	**6.7**	**5.3**	**5.5**	**4.1**	**5.3**	**5.5**	**3.6**	**6.3**	**5.7**	**6.1**	**4.6**	**4.7**	**4.5**	**5.2**	**5.2**	**5.2**	**4.4**	**4.3**

$n = 300$

Table 2: Percentage of rejection of H_0 by various tests of size $n = 300$ from different copula models with different τ, $N = 1000$, $M = 1000$.

(a) year 2004   (b) year 2009
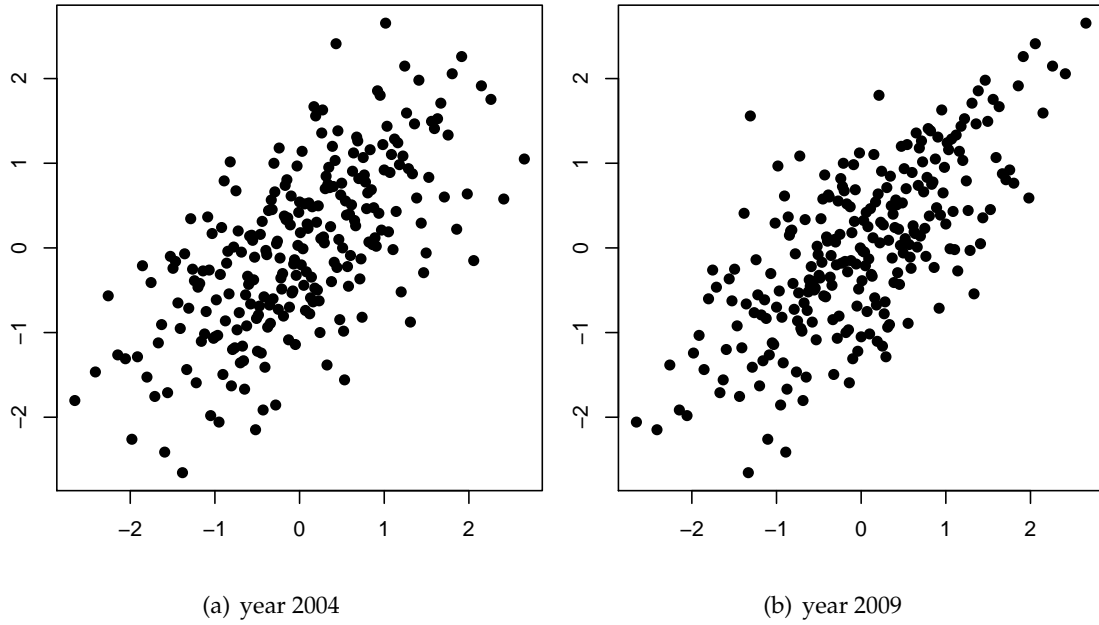
Figure 1: Scatterplots of residuals transformed to the standard normal for C/BAC for 2004 (left) and 2009 (right).

	2004					2009				
	$T_n(1)$	R_n	S_n	$ST_n(1)$	SR_n	$T_n(1)$	R_n	S_n	$ST_n(1)$	SR_n
Clayton	0.942	0.824	0.000	0.000	0.000	0.281	0.357	0.005	0.010	0.010
Gumbel	0.716	0.834	0.178	0.356	0.356	0.536	0.487	0.876	1.000	0.974
Gauss	0.235	0.423	0.660	0.470	0.846	0.101	0.229	0.788	0.202	0.458
t	0.019	0.003	0.100	0.038	0.006	0.118	0.377	0.652	0.236	0.754

Table 3: p-values for copula model specification for the residuals for the ARMA(1,1)+GARCH(1,1) on the log returns of C and BAC for 2004 and 2009.

		$T_n(1)$	R_n	S_n	$ST_n(1)$	SR_n
	2004	Clayton	Gumbel	Gauss	Gauss	Gauss
	2005	Gumbel	Gumbel	t	Gumbel	Gumbel
	2006	t	t	t	t	t
	2007	t	t	t	t	t
C/BAC	2008	Gumbel	Gumbel	t	t	Gauss
	2009	Gumbel	Gumbel	Gumbel	Gumbel	Gumbel
	2010	t	t	Gumbel	Gumbel	Gumbel
	2011	t	t	t	t	t
	2012	t	t	t	t	t

Table 4: Copulas that are preferred in each time period by each goodness-of-fit test.

NT, Vol	$T_n(1)$	R_n	S_n	$ST_n(1)$	SR_n	HL, Vol	$T_n(1)$	R_n	S_n	$ST_n(1)$	SR_n
Clayton	0.004	0.004	0.000	0.000	0.000	Clayton	0.070	0.088	0.006	0.012	0.012
Gumbel	0.139	0.114	0.946	0.278	0.228	Gumbel	0.596	0.520	0.467	0.934	0.934
Gauss	0.016	0.017	0.419	0.032	0.034	Gauss	0.000	0.001	0.589	0.000	0.002
t	0.039	0.027	0.121	0.078	0.054	t	0.077	0.026	0.430	0.154	0.052
HL, NT						HL, NT, Vol					
Clayton	0.196	0.262	0.000	0.000	0.000	Gauss	0.003	0.003	0.540	0.006	0.006
Gumbel	0.178	0.150	0.484	0.356	0.300	Clayton	0.000	0.000	0.000	0.000	0.000
Gauss	0.443	0.318	0.242	0.484	0.484	Gumbel	0.073	0.070	0.997	0.146	0.140
t	0.018	0.009	0.074	0.036	0.018	HAC Clayton	0.059	0.006	0.000	0.000	0.000
						HAC Gumbel	0.657	0.994	0.062	0.124	0.124
						Vine	0.000	0.000	NA	NA	NA

Table 5: p-values of the goodness-of-fit tests for different 2- and 3-dimensional copulas for Apple stock data.
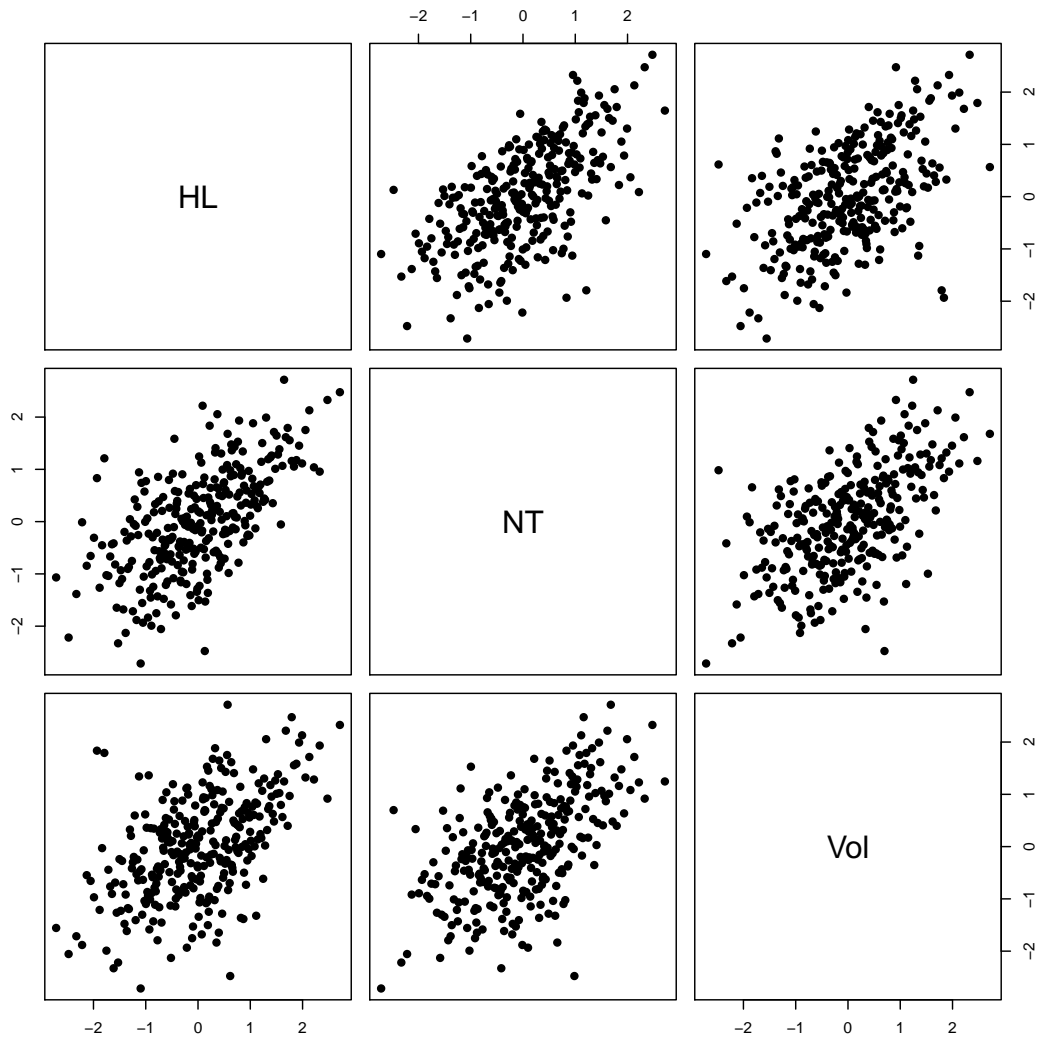
Figure 2: Scatterplots of residuals transformed to the standard normal for HL, NT and Vol, three characteristics of Apple.

Table 6: Summary of data analysis results obtained from the four copulas: Gaussian, Student's t, Clayton and Gumbel, including dependence parameter estimates, test statistics, and p-values.

copula	parameter	$T_n(1)$	p-val	R_n	p-val	S_n	p-val	$ST_n(1)$ (p-val)	SR_n (p-val)
Clayton	0.511	1.316	0.000	1.323	0.000	1.407	0.000	0.000	0.000
Gumbel	1.428	0.954	0.370	0.959	0.315	1.072	0.006	0.012	0.012
Gauss	0.456	1.223	0.000	1.274	0.000	1.118	0.000	0.000	0.000
t	0.466	0.998	1.000	1.654	1.000	1.163	0.000	0.000	0.000

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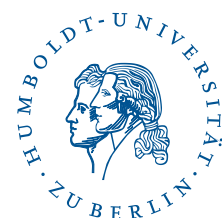
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