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Every Symmetric 3 x 3 Global Game of Strategic Complementarities Is Noise Independent

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EVERY SYMMETRIC 3 × 3 GLOBAL GAME OF STRATEGIC COMPLEMENTARITIES IS NOISE INDEPENDENT

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ABSTRACT. We prove that the global game selection in all 3×3 payoff-symmetric supermodular games is independent of the noise structure. As far as we know, all other proofs of noise independence of such games rely on the existence of a so-called monotone potential (MP) maximiser. Our result is more general, since some 3×3 symmetric supermodular games do not admit an MP maximiser. Moreover, a corollary is that noise independence does not imply the existence of an MP maximiser.

Keywords: global game, noise independence, strategic complementarities, supermodular game
JEL codes: C72, D82.

1. Introduction

Global games are often used to select a unique equilibrium in models that would typically have multiple equilibria. There are many applications, particularly to the theory of financial crises (see Morris and Shin (2003) for an overview). In this note, we prove that in two-player, three-action, supermodular games with symmetric payoffs this *global game selection* is independent of the noise structure when the noise vanishes (see Frankel, Morris, and Pauzner (2003) (FMP) for the definition of global games used here). Games with this property are called noise independent.

Theorem. *Every* 3×3 *symmetric supermodular game is noise independent.*

The significance of this result is in its implication that 3×3 games clarify the connections between the noise independence of global games, robustness to incomplete information (Kajii and Morris, 1997), and the existence of a monotone potential (MP) maximiser (Morris and Ui, 2005). As far as we know, all proofs of the noise independence of (subclasses of) supermodular games so far rely on the existence of an MP maximiser. Existence of an MP maximiser guarantees

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existence of an equilibrium robust to incomplete information (Morris and Ui, 2005), and *a fortiori*, noise independence (Oury and Tercieux, 2007; Basteck, Daniëls, and Heinemann, 2010). The noise independence of 3×3 symmetric supermodular games with three Nash equilibria can be proved along these lines, see Oyama and Takahashi (2009). However, Honda (2010) has found a non-empty open set of 3×3 symmetric supermodular games, with two Nash equilibria, that have no MP maximiser. Oyama and Takahashi (2011) show that these games have no equilibrium robust to incomplete information either.

Our proof of their noise independence does not rely on the existence of an MP maximiser. Since it applies to all 3×3 games with symmetric payoffs, it is necessarily more general. In particular, combined with the results of Honda and Oyama and Takahashi, it shows that noise independence is not equivalent to the existence of an MP maximiser, nor to the existence of an equilibrium robust to incomplete information.

Carlsson and Van Damme (1993), who introduced global games, established the noise independence of all 2×2 games. FMP investigated the noise independence of 3×3 symmetric supermodular games. The cases of noise independence that FMP formally consider rely on the existence of an MP maximiser.[1] But they also give a heuristic argument for the noise independence of 3×3 games with symmetric payoffs when, in addition, the noise distributions of players' signals are symmetric in the mean. Unfortunately, in general it is not true[2] that if the global game selection is independent of the noise structure for all mean-symmetric noise distributions, the game is noise independent in general, as we show below per counter example.

2. Preliminaries

Consider a 3×3 game $\mathbf{g}$ with players $i \in I = \{1, 2\}$, both endowed with ordered action set $A = \{a, b, c\}$, $a < b < c$ and payoff function $g_i : A \times A \to \mathbf{R}$, where $g_i(a_i, a_{-i})$ is i's payoff if she chooses a_i and her opponent a_{-i}. We will typically denote an action profile $(a^*, a^*) \in A \times A$ also by a^*, economising slightly on notation.

Let $\Delta_{i x_i}^{y_i}(x_{-i}) := g_i(y_i, x_{-i}) - g_i(x_i, x_{-i})$, the payoff difference of playing y_i instead of x_i against an opposing action x_{-i} and recall that $\mathbf{g}$ is called (weakly) supermodular if[3]

$$(x_i < y_i \text{ and } x_{-i} < y_{-i}) \implies \Delta_{i x_i}^{y_i}(x_{-i}) \leq \Delta_{i x_i}^{y_i}(y_{-i}), \tag{1}$$

in other words, each $\Delta_{i x_i}^{y_i}(x_{-i})$ is a monotonic function for all $x_i < y_i$. A game $\mathbf{g}$ is called strictly supermodular if (1) still holds when the weak inequality is replaced by a strict one. The *dual* game of $\mathbf{g}$, denoted $\mathbf{g}^\partial$, is obtained by reversing the ordering on A. Note that $\mathbf{g}$ is supermodular if and only if $\mathbf{g}^\partial$ is supermodular.

To establish the noise independence of $\mathbf{g}$ while avoiding having to deal with sequences of global games with increasingly precise signals, we will define a new family of incomplete information games that allow us to establish the global game selection more directly. We will first give their definition and then discuss their intuition and the way we use them.

Let $f = (f_1, f_2)$ denote a pair of probability densities, whose supports are subsets of $[-\frac{1}{2}, \frac{1}{2}]$. For each f we define the game $\underline{e}(\mathbf{g}, f)$ as follows. A state parameter θ is drawn uniformly from

[1] More specifically, they rely on the existence of a Local Potential (LP) maximiser, which implies the existence of an MP maximiser in own-action concave games (Morris and Ui, 2005; Oyama and Takahashi, 2009).
[2] Nor, we should add, do FMP claim this is true.
[3] FMP use the terminology "game of strategic complementarities".

the interval $[-\frac{1}{2}, 8]$. Each player receives a signal $x_i = \theta + \eta_i$ about the true state, with η_i drawn according to the density f_i. The random variables θ, η_1, η_2 are independently distributed. Players' payoffs u_i, $i \in \{1, 2\}$, are given by

$$u_i(a_i, a_{-i}, x_i) := \begin{cases} \tilde{u}(a_i, a_{-i}) & \text{if } x_i < 0, \\ g_i(a_i, a_{-i}) & \text{if } x_i \geq 0, \end{cases}$$

where $\tilde{u}$ is an arbitrary payoff function such that the least action dominates all others, say for all a_{-i}, $\tilde{u}(a_i, a_{-i}) = 0$ if $a_i \in \{b, c\}$ and $\tilde{u}(a_i, a_{-i}) = 1$ if $a_i = a$.

A strategy profile s is a function that associates an action for each player with each pair of signals. We say a strategy profile s in $\underline{e}(\mathbf{g}, f)$ *attains* a^* $\in A \times A$ if $s(x) \geq a^*$ for some pair of signals x. An action profile $a^* \in A \times A$ is said to be *attained from below under* f if the greatest equilibrium strategy profile of $\underline{e}(\mathbf{g}, f)$ attains a^*. (As is usual, strategy profiles are ordered pointwise. The supermodularity of $\mathbf{g}$ guarantees the existence of a greatest equilibrium strategy profile). If s is a strategy profile, and $\beta(s)$ the greatest best reply to s, we can conduct upper-best reply iterations s, $\beta(s)$, $\beta(\beta(s))$, $\beta(\beta(\beta(s))), \ldots$ starting at some strategy profile s. If $\beta(s)$ is weakly greater than s, the resulting sequence of strategy profiles will be monotonically increasing, due to the supermodularity of $\mathbf{g}$. As the strategy space is bounded, this sequence converges pointwise to an equilibrium strategy profile, weakly smaller than the greatest equilibrium strategy profile.

Intuitively, the game $\underline{e}(\mathbf{g}, f)$ resembles a global game with arbitrarily precise signals drawn from a scaled down version of f: after receiving their signal, players know the true payoff function and use their signals to form beliefs about the actions that others may choose. FMP use an analogous construction in the proof of their theorem 4, and it is related to the "elaborations" of Kajii and Morris (1997). The connection with global games is further formalised in (Basteck, Daniëls, and Heinemann, 2010), where we use this construction to prove the following characterisation of the global game selection.

Let $G(v)$ be a global game, with noise structure f up to the usual scaling parameter v, such that its payoff structure equals that of $\mathbf{g}$ at some payoff parameter θ. As $v \to 0$, the action profile a^* is the greatest global game selection at θ if, and *only* if, a^* is the greatest action profile attained from below under f in $\underline{e}(\mathbf{g}, f)$. Dually, a^* is the least global game selection at θ if and only if it is the greatest action profile attained from below under f of the dual game $\mathbf{g}^\partial$. The greatest and least global game selection almost always coincide.

3. Proof of Noise Independence of 3 × 3 Symmetric Supermodular Games

Now suppose $\mathbf{g}$ is a symmetric game, so that we may write $g := g_1 = g_2$ and $\Delta := \Delta_1 = \Delta_2$. We will prove $\mathbf{g}$ is noise independent. We may assume without loss of generality that $\mathbf{g}$ is strictly supermodular, [4] and that no action strictly dominates another (that would imply that $\mathbf{g}$ can be

[4] If $\mathbf{g}$ is not strictly supermodular, we may embed it in a global game $G(v)$ where the payoff structure is symmetric and strictly supermodular almost everywhere. For instance, identify $a = -1, b = 0, c = 1$ and consider the global game where payoffs depend on a state variable θ as follows:

$$u_i(a_i, a_{-i}, \theta) := g(a_i, a_{-i}) + \theta a_i(2 + \operatorname{sgn}(\theta) a_{-i}).$$

One may verify that $g(a_i, a_{-i}) = u(a_i, a_{-i}, \theta)$ for $\theta = 0$ and that u satisfies the requirements of a global game as stated by FMP. By results in (Basteck, Daniëls, and Heinemann, 2010), the global game selection in $\mathbf{g}$ does not depend on the embedding chosen. Since the greatest (least) global game selection is continuous in θ from the right (left), and noise independent at almost all θ by our proof below, the global game selection for $\mathbf{g}$ is independent of the noise structure as well.

reduced to a 2×2 game known to be noise independent). Then, by supermodularity, both a and c must be equilibria.

Consider a strategy in the mixed extension of $\mathbf{g}$ that mixes over the actions a, b, c with probabilities ("weights") w_a, w_b, w_c. Define $S(w_c)$ to be the number w_a that solves the equation

$$(2) \quad w_a g(b,a) + (1 - w_a - w_c) g(b,b) + w_c g(b,c) = w_a g(c,a) + (1 - w_a - w_c) g(c,b) + w_c g(c,c).$$

Even though $S(w_c)$ is not necessarily in the interval $[0, 1]$, we can think of it intuitively as the weight that may be put on the least action, a, to make the opposing player indifferent between playing the middle action, b, and the greatest action, c, when the weight on c is w_c. Existence and uniqueness of the solution $S(w_c)$ are guaranteed by strict supermodularity. The function S has derivative

$$\varrho_S := \frac{\Delta_b^c(c) - \Delta_b^c(b)}{\Delta_b^c(b) - \Delta_b^c(a)} > 0,$$

thus is linear and (due to supermodularity) increasing. Analogously, define $N(w_a)$ to be the minimal weight that needs to be put on c to make the opposing player indifferent between playing a and b when the weight on a is w_a. That is, $N(w_a)$ is the solution w_c that solves

$$(3) \quad w_a g(a,a) + (1 - w_a - w_c) g(a,b) + w_c g(a,c) = w_a g(b,a) + (1 - w_a - w_c) g(b,b) + w_c g(b,c).$$

The function N has derivative

$$\varrho_N := \frac{\Delta_a^b(b) - \Delta_a^b(a)}{\Delta_a^b(c) - \Delta_a^b(b)} > 0.$$

Finally, let μ_a^c denote the mixed strategy that puts weight $w_a = w_c = \frac{1}{2}$ on a and c.

Lemma. *If either (I) c is a best reply to μ_a^c, or (II) b is a best reply to μ_a^c and in addition both $N(\frac{1}{2}) \leq S(\frac{1}{2})$ and $0 \leq S(\frac{1}{2})$ hold, then c is the global game selection, for any f.*

Proof. Fix f. We will show that there exists an increasing strategy profile s^* in $\underline{e}(\mathbf{g}, f)$ that attains c, and such that $s^* \leq \beta(s^*)$. In this case, c must be the global game selection.

First observe that any increasing strategy profile can be represented by the threshold signals at which players switch to higher actions. For a typical increasing strategy profile, let $\underline{z}_i$ denote the threshold where player i switches from a to b, and $\overline{z}_i$ is the threshold at which she switches from b to c. Note that we must have $-1 \leq \underline{z}_i \leq \overline{z}_i \leq 9$.

In $\underline{e}(\mathbf{g}, f)$, θ is drawn uniformly from $[-\frac{1}{2}, 8]$ and since the error terms η_i are within $[-\frac{1}{2}, \frac{1}{2}]$, the distribution over signal differences $x_i - x_{-i}$ conditional on the signal x_i received is the same for all $x_i \in [0, 7\frac{1}{2}]$. Let H be the cumulative distribution function of the signal difference $x_1 - x_2$ and, without loss of generality, assume $H(0) = \frac{1}{2}$. If all threshold signals are within $[0, 7\frac{1}{2}]$, we may deduce the following weights from H, which are straightforward to verify. If player 2 receives the signal $\overline{z}_2$, she assigns weight

$$w_c(\overline{z}_2|\overline{z}_1) := \mathbf{P}(x_1 \geq \overline{z}_1 | x_2 = \overline{z}_2) = \mathbf{P}(x_1 - x_2 \geq \overline{z}_1 - \overline{z}_2 | x_2 = \overline{z}_2) = 1 - H(\overline{z}_1 - \overline{z}_2)$$

to player 1 playing c. Player 1 at $x_1 = \overline{z}_1$ assigns weight $w_c(\overline{z}_1|\overline{z}_2) := H(\overline{z}_1 - \overline{z}_2) = 1 - w_c(\overline{z}_2|\overline{z}_1)$ to player 2 playing c. In a similar vein, at $\underline{z}_2$, Player 2 assigns weight $w_a(\underline{z}_2|\underline{z}_1) := H(\underline{z}_1 - \underline{z}_2)$ to player 1 playing a. At $\underline{z}_1$, player 1 assigns weight $w_a(\underline{z}_1|\underline{z}_2) := 1 - H(\underline{z}_1 - \underline{z}_2)$ to player 2 playing a. Also, we will make use of the fact that $w_a(\overline{z}_2|\underline{z}_1) := H(\underline{z}_1 - \overline{z}_2) =: w_c(\underline{z}_1|\overline{z}_2)$ and similarly $w_a(\overline{z}_1|\underline{z}_2) := 1 - H(\overline{z}_1 - \underline{z}_2) =: w_c(\underline{z}_2|\overline{z}_1)$.

If (I) holds, it is easy to verify the existence of the strategy profile s^*. Simply set $\underline{z}_1 = \underline{z}_2 = \overline{z}_1 = \overline{z}_2 = 0$.

If (II) holds, it suffices to show that c is the global game selection in games where one of the inequalities $(*)$ $N(\frac{1}{2}) \leq S(\frac{1}{2})$ or $(**)$ $0 \leq S(\frac{1}{2})$ is binding. Any game $\mathbf{g}$ where the inequalities are strict can be changed into a game $\mathbf{g}'$ where one of the inequalities becomes binding by lowering the payoffs from playing action c by some constant. Lowering the payoffs from c does not change the property that b is a best reply to μ_a^c, so (II) holds in both $\mathbf{g}$ and $\mathbf{g}'$. Then, if we can show that the greatest equilibrium strategy profile in the game $\underline{e}(\mathbf{g}', f)$ attains c, the same must be true in the original game.

Now consider the set M^* of all increasing strategy profiles satisfying: (i) at $\underline{z}_2$, b or c is a best reply for player 2; and (ii) at $\bar{z}_2$, c is a best reply for player 2; and (iii) $\bar{z}_1$ player 1 weakly prefers to play c over b (we do not make any assumptions concerning the expected payoff from playing a); and (iv) $\underline{z}_1 = 1$.

Note that due to supermodularity, if s satisfies (i)-(iii), decreasing $\underline{z}_2$ preserves (ii) and (iii); decreasing $\bar{z}_2$ preserves (i) and (iii), and decreasing $\bar{z}_1$ preserves (i) and (ii).

Claim 1: The set M^* is nonempty. To see this, first consider case $(**)$, $S(\frac{1}{2}) = 0$. Set $\underline{z}_2 = 2$, $\bar{z}_1 = \bar{z}_2 = 3$. At each $\bar{z}_i$, players face an opponent mixing equally over b and c and are therefore indifferent between b and c. At $\underline{z}_2$, player 2 faces an opponent who plays b with probability 1 and will strictly prefer b over a as $N(0) < N(\frac{1}{2}) \leq S(\frac{1}{2}) = 0$. Next, consider case $(*)$, $N(\frac{1}{2}) = S(\frac{1}{2})$. Set $\bar{z}_1 = \bar{z}_2 = 3$ and choose $\underline{z}_2 \in [2, 3]$ such that player 1 is indifferent between b and c at $\bar{z}_1$. To see that this is always possible, note that player 1 would prefer b at $\bar{z}_1$ if we would set $\underline{z}_2 = 3$, as b is a best reply against μ_a^c. Also, she would prefer c at $\bar{z}_1$ if we would set $\underline{z}_2 = 2$ as $S(\frac{1}{2}) \geq 0$. Thus we may satisfy (iii). Under the resulting profile s, player 2's best reply at $\bar{z}_2$ is c, since she faces an opponent that mixes over b and c with probability $\frac{1}{2}$, thus (ii) is satisfied. For player 1, we have $w_c(\bar{z}_1|\bar{z}_2) = \frac{1}{2}$ and since she is indifferent between b and c, we know that $w_a(\bar{z}_1|\underline{z}_2) = S(\frac{1}{2})$. As $\underline{z}_2 \geq 2$, we know that player 2 at the threshold $\underline{z}_2$ puts weight $w_a(\underline{z}_2|\underline{z}_1) = 0$ on her opponent playing a. In addition, $w_c(\underline{z}_2|\bar{z}_1) = w_a(\bar{z}_1|\underline{z}_2) = S(\frac{1}{2}) = N(\frac{1}{2}) > N(0)$ so that she strictly prefers b over a, and this means (i) is satisfied, and our claim is proved.

Claim 2: The set M^* has a maximal element. Let C be any chain, i.e. linearly ordered subset, of M^*. Construct a strategy profile s by setting each of the thresholds $\underline{z}_1, \underline{z}_2, \bar{z}_1, \bar{z}_2$ to the infimum of the corresponding thresholds of the strategy profiles in C. To see that $s \in M^*$, consider an increasing, countable subsequence $s^0 \leq s^1 \leq s^2 \ldots$ of C converging pointwise to s. The inequalities on expected payoff implied by (i)-(iii) are satisfied for each s^n, so are satisfied for the limit point s by the dominated convergence theorem. Clearly then, s is a least upper bound for C in M^*. Since C was arbitrary, M^* has a maximal element by Zorn's lemma. This proves our claim.

Choose s^* to be a maximal element of M^*. We will prove that player 1 weakly prefers b over a at the threshold $\underline{z}_1 = 1$ in s^*. Since s^* satisfies (i)-(iv), this means $s^* \leq \beta(s^*)$, and the proof of the lemma is complete.

We begin by examining the preferences at the other three thresholds. First, if player 1 strictly prefers c over b at $\bar{z}_1$, then it must be that $\underline{z}_1 = 1 = \bar{z}_1$—otherwise s^* would not be a maximal element. Yet if $\underline{z}_1 = 1 = \bar{z}_1$, we arrive at a contradiction as follows. Player 2 must be indifferent between a and b at $\underline{z}_2 \leq 1$, as b is a best reply to the opposing mixed strategy profile μ_a^c at signal $x_2 = 1$. Similarly, $\bar{z}_2 \geq 1$. But then either

$$(*) \qquad w_a(\bar{z}_1|\underline{z}_2)) < S(w_c(\bar{z}_1|\bar{z}_2)) \leq S(\tfrac{1}{2}) = N(\tfrac{1}{2}) \leq N(w_a(\underline{z}_2|\underline{z}_1)) = w_c(\underline{z}_2|\bar{z}_1)$$

or

$$(**)\qquad w_a(\bar z_1|\underline{z}_2)) < S(w_c(\bar z_1|\bar z_2)) \le S(\tfrac12) = 0 \le w_c(\underline{z}_2|\bar z_1),$$

contradicting the fact that $w_a(\bar z_1|\underline{z}_2) = w_c(\underline{z}_2|\bar z_1)$. Thus we conclude that player 1 is indifferent between b and c at $\bar z_1$.

Next, if player 2 would strictly prefer a over b at $\underline{z}_2$ then she would have to be indifferent between a and c at $\underline{z}_2 = \bar z_2 =: z_2$. Since b is a best reply for player 1 against the belief μ_a^c, and player 1 is indifferent at $\bar z_1$, we find that $z_2 \le \bar z_1$. Also, we have that $\underline{z}_1 \le z_2$, as otherwise at z_2 player 2 would face an opposing action distribution that is dominated by μ_a^c, so that her best reply would be weakly smaller than b. But then we arrive at the following contradiction:

$$w_c(z_2|\bar z_1) < N(w_a(z_2|\underline{z}_1)) \le N(\tfrac12) \le S(\tfrac12) \le S(w_c(\bar z_1|z_2)) = w_a(\bar z_1|z_2) = w_c(z_2|\bar z_1).$$

Finally, if player 2 would strictly prefer c over b at $\bar z_2$, maximality of s^* would imply that she is indifferent between a and c at $\underline{z}_2 = \bar z_2 = z_2$, and hence strictly prefers a over b at z_2, which we have just ruled out.

In sum, in the strategy profile s^*, each player i is indifferent between b and c at $\bar z_i$ and player 2 is indifferent between a and b at $\underline{z}_2$. Then, by definition, we have $S(w_c(\bar z_1|\bar z_2)) = w_a(\bar z_1|\underline{z}_2)$ and $N(w_a(\underline{z}_2|\underline{z}_1)) = w_c(\underline{z}_2|\bar z_1)$. In addition, it is always the case that $w_a(\bar z_1|\underline{z}_2) = w_c(\underline{z}_2|\bar z_1)$, so we conclude that $S(w_c(\bar z_1|\bar z_2)) = N(w_a(\underline{z}_2|\underline{z}_1))$. But then, as

$$\begin{aligned} S(w_c(\bar z_1|\bar z_2)) = S(\tfrac12) + (w_c(\bar z_1|\bar z_2) - \tfrac12)\varrho_S &= S(\tfrac12) + \tfrac12(w_c(\bar z_1|\bar z_2) - w_c(\bar z_2|\bar z_1))\varrho_S \\ &= N(\tfrac12) + \tfrac12(w_a(\underline{z}_2|\underline{z}_1) - w_a(\underline{z}_1|\underline{z}_2))\varrho_N = N(w_a(\underline{z}_2|\underline{z}_1)). \end{aligned}$$

and as $S(\tfrac12) \ge N(\tfrac12)$, we know that

$$(w_c(\bar z_1|\bar z_2) - w_c(\bar z_2|\bar z_1))\varrho_S \le (w_a(\underline{z}_2|\underline{z}_1) - w_a(\underline{z}_1|\underline{z}_2))\varrho_N.$$

Returning to the situation of player 1 at $\underline{z}_1$, we can now say that

$$\begin{aligned} N(w_a(\underline{z}_1|\underline{z}_2)) &= N(\tfrac12) - \tfrac12(w_a(\underline{z}_2|\underline{z}_1) - w_a(\underline{z}_1|\underline{z}_2))\varrho_N \\ &\le S(\tfrac12) - \tfrac12(w_c(\bar z_1|\bar z_2) - w_c(\bar z_2|\bar z_1))\varrho_S = S(w_c(\bar z_2|\bar z_1)). \end{aligned}$$

Since we know that player 2 is indifferent between b and c at $\bar z_2$ and that $w_a(\bar z_2|\underline{z}_1) = w_c(\underline{z}_1|\bar z_2)$, we conclude that

$$N(w_a(\underline{z}_1|\underline{z}_2)) \le S(w_c(\bar z_2|\bar z_1)) = w_a(\bar z_2|\underline{z}_1) = w_c(\underline{z}_1|\bar z_2).$$

But, by the definition of N, this simply says that player 1 weakly prefers b over a at $\underline{z}_1$, which is what we needed to show. □

Corollary. *If either (I) a is a best reply to μ_a^c, or (II) b is a best reply to μ_a^c and in addition both $S(\tfrac12) \le N(\tfrac12)$ and $0 \le N(\tfrac12)$ hold, then a is the global game selection, for any f.*

Proof. In the dual game of $\mathbf{g}$, the ordering on A is reversed. Define N^∂ and S^∂ for $\mathbf{g}^\partial$ analogous to N and S for $\mathbf{g}$, by replacing all the occurrences of a in expressions (2) and (3) by c, and all occurrences of c by a. We find that $N^\partial = S$ and similarly $S^\partial = N$. By our lemma, a is the noise independent selection in $\mathbf{g}^\partial$ (using, in case (II), that $N^\partial(\tfrac12) \le S^\partial(\tfrac12)$, and $0 \le S^\partial(\tfrac12)$). Since $\mathbf{g}$ and $\mathbf{g}^\partial$ differ only in their ordering, a is the noise independent selection in $\mathbf{g}$ as well. □

If $\mathbf{g}$ neither satisfies the conditions of the lemma nor of its corollary, b must be a Nash equilibrium, since it is a best reply to μ_a^c and both $N(\tfrac12) < 0$ and $S(\tfrac12) < 0$ hold. In this case, by results of

Oyama and Takahashi (2009), b is an MP maximiser,[5] and therefore the global game selection. This completes the proof that $\mathbf{g}$ is noise independent.

Remark. Oyama and Takahashi (2009) provide payoff conditions to find an MP maximiser in a game of three Nash equilibria. They coincide with our conditions to find the global game selection, even when applied to games with less than three Nash equilibria. The conditions given by FMP also coincide with ours, under the additional assumption that b is a best reply to μ_a^c.

4. Mean-symmetric noise independence versus noise independence

We say f is mean-symmetric if the induced distribution over signal differences $x_i - x_{-i}$, conditional on x_i, is symmetric in the mean.[6] We now ask whether, if a game has the property that the global game selection is independent of the noise structure for all mean-symmetric f, that game is noise independent. The answer is no. As symmetric supermodular 3×3 games are noise independent, we turn to an asymmetric 3×3 game to prove this.

		player 2		
		a	b	c
	a	$2,\ \ 1$	$0,\ \ 0$	$-3, -3$
player 1	b	$0, -1$	$0,\ \ 0$	$0,\ \ 0$
	c	$-3, -1$	$0,\ \ 0$	$2,\ \ 2$

Figure 1. Asymmetric two-player three-action game

Suppose $\mathbf{g}$ is given by the bimatrix in figure 1. Both players are indifferent between a and b when facing an opponent who plays (a, b, c) with probabilities $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ and indifferent between b and c when facing probabilities $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. We will see, that for a mean-symmetric f, in the games $\underline{e}(\mathbf{g}, f)$ and $\underline{e}(\mathbf{g}^\partial, f)$ we can find threshold values $\underline{z}_1, \bar{z}_1, \underline{z}_2, \bar{z}_2$ such that players hold the above mentioned beliefs over opponents play at their respective threshold and are thus indifferent between the two actions. Thus, a is attained from below under f in $\mathbf{g}^\partial$, and c is attained from below under f in $\mathbf{g}$, and the example is a knife-edge case where a is the least global game selection and c the greatest.

This is generally no longer possible if the noise distribution is not mean-symmetric and we will give an example of a distribution of signal differences where c is the global game selection in $\mathbf{g}$, as well as for slightly perturbed payoffs. Thus, c is the unique global game selection of $\mathbf{g}$.

By perturbing the payoff table slightly, we could create a game where a is the unique global game selection for any symmetric f, while under our example with non-mean-symmetric f, c would be the unique global game selection. However, in order to keep our example simple, we stick to the numbers in figure 1.

Mean-symmetric noise structures. Without loss of generality let us assume that the conditional densities over the opponents signal are symmetric in 0. Set $\underline{z}_1, \underline{z}_2 = 0$. Then, upon receiving a signal $x_i = 0$, both players expect their opponent to play a with probability $\frac{1}{2}$. Next, set $\bar{z}_{-i}$ such that on receiving a signal $x_i = 0$ an player expects his opponent to play b with probability $\frac{1}{6}$ and c with probability $\frac{1}{3}$. Due to symmetry, we find that $\bar{z}_1 = \bar{z}_2 = t$ for some t, so that an player at

[5] In the notation of their paper, the fact that b is a best reply to μ_a^c is expressed as $\Delta_{ba}^{ac} \geq 0$ and $\Delta_{bc}^{ac} \geq 0$. Also, $N(\frac{1}{2}) < 0 \Leftrightarrow \Delta_{ba}^{ba} > 0$ and $S(\frac{1}{2}) < 0 \Leftrightarrow \Delta_{bc}^{bc} > 0$. These conditions correspond to case (3b) in their Proposition 1.

[6] This is the case if $f_1 = f_2$, or if the individual f_i are symmetric in their mean.

$x_i = t$ holds belief $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$ over (a, b, c) being played. Thus, players will indeed be willing to switch to the higher action at each of the thresholds, so that c is attained from below under f, and hence the greatest global game selection. In the same way, one can show that a is attained from below under f in $\underline{e}(\mathbf{g}^{\partial}, f)$, making it part of the global game selection as well.

Non-mean-symmetric noise structures. To gain some intuition, assume a noise structure f such that players assign probability $\frac{1}{2}$ to the event that their opponent receives a signal smaller that their own. Set $\underline{z}_1 = 0$. Adjust $\underline{z}_2$ such that player 2 is indifferent between a and b at the threshold: this is the case for $\underline{z}_2 = \underline{z}_1 = 0$, irrespective of $\overline{z}_1, \overline{z}_2$. Next, set the $\overline{z}_i$ simultaneously to a level where both players are indifferent between b and c at their respective thresholds. In general, and in contrast to a situation with mean-symmetric f, we will find that $\overline{z}_1 \neq \overline{z}_2$, so the probability that player 2 assigns to her opponent playing c will be unequal to $\frac{1}{2}$. But this implies that the probability she assigns to player 1 playing a will be unequal to $\frac{1}{3}$. For player 1 with signal $\underline{z}_1 = 0$ this implies that, while she assigns probability $\frac{1}{2}$ to player 2 playing a, she assigns probability unequal to $\frac{1}{3}$ to player 2 playing c. Thus, she strictly prefers either a or b so that the global game selection is either a or c, uniquely.

We move on to a numerical example. Consider the following conditional probability density function of player 1 about player 2's signal:

$$\pi_1(x_2|x_1) := \begin{cases} 1 + x_2 - x_1 & \text{if } x_1 - 1 < x_2 < x_1, \\ x_2 - x_1 & \text{if } x_1 < x_2 < x_1 + 1 \end{cases}.$$

Player 2 holds a mirrored version, namely

$$\pi_2(x_1|x_2) := \begin{cases} x_2 - x_1 & \text{if } x_2 - 1 < x_1 < x_2, \\ 1 + x_2 - x_1 & \text{if } x_2 < x_1 < x_2 + 1 \end{cases}.$$

Set $\underline{z}_1 = 0$. The smallest value at which player 2 is willing to switch to b is then $\underline{z}_2 = 0$ at which she expects player 1 to play a with probability $\frac{1}{2}$. By numerical methods we establish that the smallest values $\overline{z}_i$ where players are willing to switch to c are $\overline{z}_1 \simeq 0.2214$ and $\overline{z}_2 \simeq 0.5224$. From the perspective of player 1 at $\underline{z}_1$, the probability that her opponent will play c is then approximately equal to $0.5 - 0.5(0.5224)^2 = 0.3635 > \frac{1}{3}$ while the probability for action a is $\frac{1}{2}$. Thus, player 1 *strictly* prefers to play b at her threshold, so that we have found a strategy profile that attains c, which must be the greatest global game selection. Since player 1 strictly prefers to switch at $\underline{z}_1$, we could slightly increase all other thresholds, so that both players strictly prefer to play a higher action at each of their thresholds. Slight perturbations of the payoff table would not alter these strict preferences, implying that c is the *unique* global game selection. This contrasts with the case of mean-symmetric f, so we have shown that $\mathbf{g}$ is not noise independent.

It may be hard to generate the π_i's using two independently distributed error terms with densities f_1, f_2. However, they can be approximated close enough for the numerical result to hold: assume that player 1 receives a very precise signal, while player 2's signal is distributed around θ just like x_2 is distributed around x_1 according to π_1.

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