The Use and Misuse of Orthogonal Regression in Linear Errors-in-Variables Models

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Orthogonal regression is one of the standard linear regression methods to correct for the effects of measurement error in predictors. We argue that orthogonal regression is often misused in errors-in-variables linear regression because of a failure to account for equation errors. The typical result is to overcorrect for measurement error, that is, overestimate the slope, because equation error is ignored. The use of orthogonal regression must include a careful assessment of equation error, and not merely the usual (often informal) estimation of the ratio of measurement error variances. There are rarer instances, for example, an example from geology discussed here, where the use of orthogonal regression without proper attention to modeling may lead to either overcorrection or undercorrection, depending on the relative sizes of the variances involved. Thus our main point, which does not seem to be widely appreciated, is that orthogonal regression, just like any measurement error analysis, requires careful modeling of error.

KEY WORDS: Functional regression; Linear regression; Measurement error models; Method of moments.

1. INTRODUCTION

One of the most widely known techniques for errors-in-variables (EIV) estimation in the simple linear regression model is orthogonal regression, also sometimes known as the functional maximum likelihood estimator under the constraint of known error variance ratio (see below). This is an old method; see Madansky (1959) and Kendall and Stuart (1979, chap. 29); Fuller (1987) is now the standard reference. A classical use of orthogonal regression occurs when two methods attempt to measure the same quantity, or when two variables are related by physical laws.

The thesis of this article is that orthogonal regression is a technique that is often misapplied, with the consequence usually being that the effects of measurement error are exaggerated. The main problem with using orthogonal regression is that doing so often ignores an important component of variability, the equation error. A rarer problem, illustrated in an example from geology in Section 5, occurs when the measurement error variance is overestimated be-

cause inhomogeneity in the true variates is confused with measurement error. This problem can lead to either over- or undercorrection for measurement error, depending upon the relative sizes of the variances involved. Stated more broadly, our thesis is that correction for measurement error requires careful modeling of all sources of variation. The danger in orthogonal regression is that it may focus attention away from modeling.

The article is organized as follows. In Section 2 we review orthogonal regression. In Section 3 we discuss equation error and method of moments estimation. Further sections give examples of use and misuse of orthogonal regression.

2. ORTHOGONAL REGRESSION

Orthogonal regression (OR) is derived from a "pure" measurement error perspective. It is assumed that there are theoretical constructs (constants) y_{true} and X that are linearly related through

$$y_{\text{true}} = \beta_0 + \beta_1 X. \tag{1}$$

Equation (1) states that if y_{true} and X could be measured, they would be exactly linearly related. Typical examples where this might be thought to be the case occur in the physical sciences when the variables are related by fundamental physical laws.

The notation in measurement error models is notorious for its lack of consistency. Table 1 gives a listing of the notation used in this article. The notation is consistent with that of Carroll, Ruppert, and Stefanski (1995).

In the classical orthogonal regression development, instead of observing (y_{true}, X), we observe them corrupted by measurement error; namely, we observe

$$Y = y_{\text{true}} + \varepsilon$$
$$W = X + U \tag{2}$$

where ε and U are independent mean zero random variables with variances σ_ε^2 and σ_u^2, respectively. Combining (1) and (2) we have a regression-like model

$$Y = \beta_0 + \beta_1 X + \varepsilon. \tag{3}$$

In the literature the case that the X's are fixed unknown constants is known as the functional case, while if the X's are random variables we are in the structural case.

For reasons that will be made clear later, the orthogonal regression estimator requires knowledge of the error variance ratio

$$\eta = \frac{\text{var}(Y|X)}{\text{var}(W|X)} = \frac{\sigma_\varepsilon^2}{\sigma_u^2}. \tag{4}$$

The orthogonal regression estimator is based on a sample of size n, $(Y_i, W_i)_{i=1}^n$ where, of course, the X's are unknown

Table 1. Notation for Measurement Error Models Used in this Article

y_{true}	"True" response, typically unobservable
X	"True" predictor, typically unobservable
Y	Observed response, $Y = y_{\text{true}} + \varepsilon$, $\text{var}(\varepsilon)$ $= \sigma_\varepsilon^2$
W	Observed predictor, $W = X + U$, $\text{var}(U)$ $= \sigma_u^2$
$y_{\text{true}} = \beta_0 + \beta_1 X$	Classical functional model, no equation error
$y_{\text{true}} = \beta_0 + \beta_1 X + q$	Classical functional model, but with added equation error q, $\text{var}(q) = \sigma_q^2$
$\eta = \sigma_\varepsilon^2/\sigma_u^2$	Variance ratio for classical orthogonal regression
$\eta_{EE} = (\sigma_\varepsilon^2 + \sigma_q^2)/\sigma_u^2$	Variance ratio for orthogonal regression taking equation error into account

and unobserved. The orthogonal regression estimator is obtained by minimizing

$$\sum_{i=1}^{n} \{(Y_i - \beta_0 - \beta_1 X_i)^2/\eta + (W_i - X_i)^2\} \qquad (5)$$

in the unknowns, namely $\beta_0, \beta_1, X_1, \ldots, X_n$. If $\eta = 1$, (5) is just the usual squared total Euclidean or orthogonal distance of $(Y_i, W_i)_{i=1}^{n}$ from the line $(\beta_0 + \beta_1 X_i, X_i)_{i=1}^{n}$. If $\eta \neq 1$, (5) is a weighted orthogonal distance.

Let s_y^2, s_w^2, and s_{wy} be the sample variance of the Y's, the sample variance of the W's, and the sample covariance between the Y's and the W's, respectively. The orthogonal regression estimate of slope is

$$\hat{\beta}_1(\text{OR}) = \frac{s_y^2 - \eta s_w^2 + \{(s_y^2 - \eta s_w^2)^2 + 4\eta s_{wy}^2\}^{1/2}}{2s_{wy}} . \qquad (6)$$

This well-known estimator is derived in many places; we find Fuller (1987, sec. 1.3) to be a useful source. The reader should note that Fuller does not use the term "orthogonal regression," and calls (6) a method of moments estimator, although it differs from the method of moments estimator we will introduce in the next subsection. The orthogonal regression estimator has the famous property that it lies between the slope of the regression of Y on W and the inverse of the slope of the regression of W on Y.

The estimator (6) can be justified in several ways. For example, it can be derived as the functional normal maximum likelihood estimator, "functional" meaning that the X's are considered nonrandom, although unknown. More precisely, if ε and U are normally distributed and η in (4) is known, then (6) is the maximum likelihood estimator when one treats $(\sigma_\varepsilon^2, \sigma_u^2, \beta_0, \beta_1, X_1, \ldots, X_n)$ as unknown constants. Fuller (1987, sec. 1.3.2) also derives (6) as a maximum likelihood estimator when the X's are normally distributed with mean μ_x and variance σ_x^2, and shows that (6) is consistent in this setting. Gleser (1983) shows the more general result that estimators which are consistent and asymptotically normally distributed (CAN) in a functional model are also CAN in a structural setting.

2.1 Why Assume Known Error Variance Ratio?

The restriction that the variance ratio (4) be known arises from the following considerations. In the structural case with (ε, U, X) independent and normally distributed,

there are six unknown parameters $(\mu_x, \sigma_x^2, \sigma_\varepsilon^2, \sigma_u^2, \beta_0, \beta_1)$, but only five sufficient statistics, these being the sample means, sample variances, and sample covariance of the Y's and the W's. This overparameterization means that β_1 is not identified (Fuller 1987, sec. 1.1.3). By assuming that η in (4) is known, we restrict to five unknown parameters, all of which can be estimated.

Of course, assuming that η is known is not the only possibility. Indeed, if σ_u^2 is known (or estimated from outside data), the classical method of moments estimator is

$$\hat{\beta}_1(\text{MM}) = \frac{s_w^2}{s_w^2 - \sigma_u^2} \, \hat{\beta}_1(\text{OLS}) \qquad (7)$$

where $\hat{\beta}_1(\text{OLS})$ is the ordinary least square slope estimate. Small sample improvements to this estimator are described in Fuller (1987, sec. 2.5). Often, $\hat{\beta}_1(\text{MM})$ is called the OLS estimator "corrected for attenuation."

3. EQUATION ERROR

The orthogonal regression (functional EIV) fitting method is an acceptable method as long as η in (4) is specified correctly. The problem in practice is that η is often incorrectly specified, in fact underestimated, leading to an overcorrection for attenuation.

The difficulty revolves around a missing component of variance from model (1). Model (1) states that in the absence of measurement error only, *the data would fall exactly on a straight line.* This can happen, of course, but in practice it is plausible only in rare circumstances. As Weisberg (1985, p. 6), in discussing the model (1), states:

"Real data almost never fall exactly on a straight line ... The random component of the errors can arise from several sources. Measurement errors, for now only in Y, not X, are almost always present ... The effects of variables not explicitly included in the model can contribute to the errors. For example, in Forbes' experiments (on pressure and boiling point), wind speed may have had small effects on the atmospheric pressure, contributing to the variability in the observed values. Also, random errors due to natural variability occur."

The point here is that data typically do not fall on a straight line in the absence of measurement error, so that (1) is implausible in most applications. Weisberg (1985) is talking about what Fuller (1987, p. 106) calls equation errors, namely that (1) should be replaced by

$$y_{\text{true}} = \beta_0 + \beta_1 X + q \qquad (8)$$

where q is a mean zero random variable with variance σ_q^2. Fuller's explication of equation errors is an important conceptual contribution to measurement error methodology.

Model (8) says that even in the absence of measurement error, y_{true} and X will not fall exactly on a straight line.

Combining (2) and (8) we have the expanded regression model

$$Y = \beta_0 + \beta_1 X + q + \varepsilon \qquad (9)$$

where q is the equation error and ε is the measurement error. To apply orthogonal regression one needs knowledge of

$$\eta_{EE} = \frac{\text{var}(Y|X)}{\text{var}(W|X)} = \frac{\sigma_q^2 + \sigma_\varepsilon^2}{\sigma_u^2} . \qquad (10)$$

In practice, what tends to be done is to run some additional studies to estimate the measurement error variances, so that for some observations, we observe replicates $(Y_{ij})_{j=1}^M$ and $(W_{ijk})_{k=1}^M$. These studies provide estimates of σ_ε^2 and σ_u^2, and then typically (4) is applied. Comparing (4) and (10) we see that this typical practice underestimates η_{EE}, sometimes severely. In the Appendix we show the following.

Theorem. In the presence of equation error, asymptotically the orthogonal regression estimator based on using (4) overcorrects; that is, in large samples it overestimates the slope β_1 in absolute value.

Fuller (1987, pp. 106–113) shows how one can estimate the equation error variance σ_q^2. Let $\hat{\beta}_1(\text{MM})$ be the method of moments estimated slope, and let $\hat{\sigma}_\varepsilon^2$ and $\hat{\sigma}_u^2$ be estimated error variances for Y and W, respectively. Then a consistent estimate of $\hat{\sigma}_q^2$ is

$$\hat{\sigma}_q^2 = s_v^2 - \hat{\sigma}_\varepsilon^2 - \hat{\beta}_1^2(\text{MM})\hat{\sigma}_u^2 \qquad (11)$$

where

$$s_v^2 = (n-2)^{-1}\sum_{i=1}^n \{Y_i - \bar{Y}. - \hat{\beta}_1(\text{MM})(W_i - \bar{W}.)\}^2.$$

There is no guarantee that $\hat{\sigma}_q^2 \geq 0$.

Under the assumption that $\nu_\varepsilon \hat{\sigma}_\varepsilon^2 / \sigma_\varepsilon^2$ and $\nu_u \hat{\sigma}_u^2 / \sigma_u^2$ are approximately chi-squared with ν_ε and ν_u degrees of freedom, respectively, and if (ε, q, U) are jointly normally distributed, an estimated variance for $\hat{\sigma}_q^2$ is

$$2\{(n-2)^{-1}s_v^4 + \hat{\sigma}_\varepsilon^4/\nu_\varepsilon + \hat{\beta}_1^4(\text{MM})\hat{\sigma}_u^4/\nu_u\}.$$

In practice, however, we use the bootstrap to assess the sampling distribution of $\hat{\sigma}_q^2$.

4. EXAMPLE: RELATING TWO MEASUREMENT METHODS

The following example is real, but we are not at liberty to give details of the experiment. A company was trying to relate two methods of measuring the same quantity. Because both devices were trying to measure the same quantity, it was thought a priori that the "no equation error" model (1) was appropriate, with any observed deviations from the line due purely to measurement error. The data are given in Table 2. In this example it was known from other experiments that $\sigma_u^2 \approx .0424$. Two replicates of the response were taken in order to ascertain the measurement error in Y. Suppose

Table 2. Example With Replicated Response

W_i	Y_{i1}	Y_{i2}
−1.8007	−.5558	−.9089
−.7717	.2076	.6499
−.4287	−1.7365	−1.8542
−.0857	−.9018	.2040
.2572	−.2312	−.3097
.6002	.2967	.5072
.9432	.5928	1.5381
1.2862	1.2420	1.2599

the observations are (Y_{ij}^*) for $i = 1, \ldots, n$ and $j = 1, 2$, and let Y_i be the mean response for unit i.

In the presence of replicates there are two important variants of model (2). In the first one assumes that the Y_{ij}^* are true replicates with the same mean. In this case an unbiased estimate of the error variance σ_ε^2 of the (Y_i) is

$$\hat{\sigma}_\varepsilon^2 = (4n)^{-1}\sum_{i=1}^n (Y_{i2}^* - Y_{i1}^*)^2.$$

A second model allows for "drift" in the replicates, so that the mean of the first replicate is systematically different from the mean of the second replicate. In this case an unbiased estimate of σ_ε^2 is the sample variance of the terms $(Y_{i2}^* - Y_{i1}^*)/2$; that is,

$$\hat{\sigma}_\varepsilon^2 = \{4(n-1)\}^{-1}\sum_{i=1}^n \{(Y_{i2}^* - Y_{i1}^*) - (\bar{Y}_{.2}^* - \bar{Y}_{.1}^*)\}^2 \quad (12)$$

where $\bar{Y}_{.j}$ is the sample mean of the jth replicate. In the data under consideration a paired t test for drift has a t statistic of 1.46 that, although not statistically significant, is large enough to suggest the choice of (12), which we used in our calculations.

We remind the reader here that the orthogonal regression is carried out using the average of the two responses; quite different fits are obtained if the two separate values are used.

We have rescaled the data so that $s_w^2 = 1.0$, where s_w^2 is the sample variance of the W's. From (7) this means that essentially no correction for measurement error is necessary, and in fact the method of moments estimated slope is $1.04 = 1.0/(1.0 - .0424)$ times the least squares slope.

The original analysis of these data was as follows. It is found that $\hat{\sigma}_\varepsilon^2 = .0683, \hat{\eta} = 1.6118$, and thus

$$\hat{\beta}_1(\text{OR}) = 1.35\hat{\beta}_1(\text{OLS});$$

contrast with

$$\hat{\beta}_1(\text{MM}) = 1.04\hat{\beta}_1(\text{OLS}).$$

There is thus quite a substantial difference between the methods of moments and orthogonal regression estimators, a difference we ascribe to equation error. In this example equation error is "unexpected" because Y and W are both trying to measure the same quantity. However, the presence of equation error is strongly suggested by Figure 1, where it is evident that the deviations of the observations about any line simply cannot be described by measurement error alone. In reading Figure 1, remember that there is hardly any measurement error in X because $\text{var}(W|X) = .0424$, but $\text{var}(W) = 1.0$.

Fuller's estimate of the equation error variance (11) is $\hat{\sigma}_q^2 = .582$. Using $\nu_\varepsilon = n - 1 = 7$ and $\nu_u = \infty$ (because σ_u^2 is assumed known), an estimated standard error for $\hat{\sigma}_q^2$ is .411, and hence the nominal one-sided t-significance level with six degrees of freedom is 10.3%, although with a sample of size $n = 8$, it may be stretching the point to use asymptotic theory. When we computed a one-sided confidence interval for σ_q^2 using the bootstrap, we found a significance level of
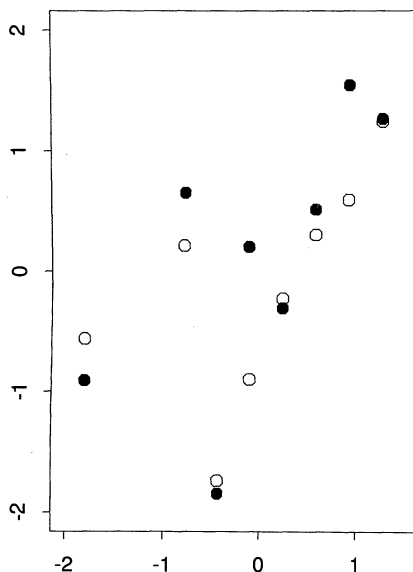
Figure 1. Plot of the Two Replicate Responses Against the Observed Predictor in the Example of Section 4.

12.5%. In this example the evidence points to the existence of substantial equation error. With $n = 8$ the evidence is not completely overwhelming, but that is not the point. Rather, the point is that orthogonal regression is based on an assumption that clearly cannot be supported by the data, and the orthogonal regression is sensitive to violation of this assumption.

5. EXAMPLE: REGRESSION IN GEOLOGY

Jones (1979) considered an example from geology, where $Y = \log$ permeability of indurated alluvial-deltaic sandstones in a petroleum reservoir of Eocene age, while $W = $ measured porosity and X is true porosity.

The sample mean and variance of W are 15.52 and 4.79, respectively, while the sample mean and variance of Y are 2.22 and .32, respectively. The sample covariance between Y and W is .75. The estimated least squares slope is $\hat{\beta}_1(\text{OLS}) = .15$.

Jones suggests that the data consist of measurements from a series of sand bodies. Treating the observations within each sand body as homogeneous and using them as replicates, he estimates that $\hat{\sigma}_u^2 = 3.87$ and $\hat{\sigma}_\varepsilon^2 = .25$, so that an estimate is $\hat{\eta} = .065$. With this value of $\hat{\eta}$ the orthogonal regression slope estimate is

$$\hat{\beta}_1(\text{OR}) = 1.71\hat{\beta}_1(\text{OLS}) = .26.$$

Note, however, that something is obviously amiss. This sample variance of W is $s_w^2 = 4.79$, while the estimated measurement error variance is $\hat{\sigma}_u^2 = 3.87$. From (7) this leads to a method of moments correction for attenuation of

$$\hat{\beta}_1(\text{MM}) = 5.20\hat{\beta}_1(\text{OLS}) = .80.$$

Note how the method of moments estimator is much larger than the orthogonal regression estimator, which should not happen if the only problem is failure to consider equation error. One possible source of the difficulty is the method of

obtaining the estimates $\hat{\sigma}_\varepsilon^2$ and $\hat{\sigma}_u^2$. Within each sand body, $\hat{\sigma}_\varepsilon^2$ and $\hat{\sigma}_u^2$ were obtained as the sample variances of Y and W. However, Jones points out that observations within a sand body are not truly homogeneous, so that $\hat{\sigma}_u^2$ is estimating a sum of two variance components: (1) σ_u^2, the measurement error variance; and (2) σ_a^2, the inhomogeneity of the true porosity X within a given sand body. A similar analysis applies to $\hat{\sigma}_\varepsilon^2$, with the missing component of variance being σ_b^2 and representing inhomogeneity of the true response, y, within a sand body. We have $\sigma_b^2 = \beta_1^2\sigma_a^2 + \sigma_q^2$ because inhomogeneity in y comes from two sources, inhomogeneity in X and equation error. To appreciate this, notice that if there were no inhomogeneity in X within a sand body and no equation error, then y would be constant within that sand body.

Therefore $\tilde{\eta}$ is an estimate not of $(\sigma_q^2 + \sigma_\varepsilon^2)/\sigma_u^2$, but instead of

$$(\sigma_q^2 + \sigma_\varepsilon^2 + \beta_1^2\sigma_a^2)/(\sigma_u^2 + \sigma_a^2).$$

Thus $\tilde{\eta}$ could be either too small or too big, depending on the unknown values of the variances involved.

Because we cannot obtain an unbiased estimate of σ_u^2 from the available data, the classic method of moments estimator will be biased and inconsistent. It is not clear what estimator should be used. The OLS estimator might be best— because most measurement error will be in Y, not W—since Jones states that "permeability is notoriously more difficult to measure than is porosity" (Jones 1979, p. 20). Since the orthogonal regression estimator is closer than the method of moments to OLS, this is one instance where we would prefer orthogonal regression over the method of moments if we had to choose between the two.

In examples like this where the ratio η_{EE} in (10) seems unfathomable, it is often suggested that one give up any hope of obtaining a point estimate of β_1. Instead, one might specify a range for η, say $a \le \eta \le b$, and compute the value of the orthogonal regression estimator for all values of η between a and b. There are three obvious objections to this criticism:

(1) To be useful, a narrow range $[a, b]$ is typically specified. In the presence of substantial equation error, all of the orthogonal regression estimators with η in a narrow range about $\sigma_\varepsilon^2/\sigma_u^2$ will overcorrect for measurement error.

(2) If the range is too wide, the results may not be of much use. For example, setting $a = 0, b = \infty$ means that we calculate all possible orthogonal regression estimates. In the geology data these range from .154 with $\eta = 0$ to .438 with $\eta = \infty$.

(3) Quite apart from its utility, which is often limited, our main objection to the "range of η's" rule is that it is another temptation to not think carefully about all the sources of variability in a problem, for example, equation error, inhomogeneity, and correlation.

6. EXAMPLE: CALIBRATING MEASURES OF GLUCOSE

The previous two examples show that, without a careful consideration of the components of variance, orthogonal

regression can lead one astray. This is not the fault of orthogonal regression, of course, but orthogonal regression is so widely misused because it does not lead one to consider the sources of variability.

Here is an example where there appears to be little equation error, if any. Leurgans (1980) considered two methods of measuring glucose, a test/new method and a reference/standard method. After some preliminary analysis, we used $Y = 100 \log(100 + \text{test})$ and $W = 100 \log(100 + \text{reference})$. The observation with the smallest value of Y (and W) was deleted.

Leurgans first evaluated the measurement error variances. For the test method there were three control studies done at low ($n = 41$), medium ($n = 37$), and high ($n = 36$) values, with sample standard deviations 1.41, 1.40, and 1.70, respectively. A weighted estimate is $\hat{\sigma}_\varepsilon^2 = 2.25$. There were also two control samples of the reference method with low ($n = 61$) and high ($n = 51$) levels, which had sample standard deviations 1.52 and 1.58, respectively. A weighted estimate of $\hat{\sigma}_u^2 = 2.41$, and thus $\hat{\eta} = .9345$. The sample variance of the Y's is $s_y^2 = 970.85$, and the sample variance of the W's is $s_w^2 = 1068.95$. The measurement error in W is clearly small, and indeed $\hat{\beta}_1(\text{MM}) = 1.0023\hat{\beta}_1(\text{OLS}) = .9529$, while the orthogonal regression estimator is $\hat{\beta}_1(\text{OR}) = .9530$. Also, $\hat{\sigma}_q^2 = .17$, which is tiny when one remembers that $\hat{\sigma}_\varepsilon^2 = 2.25$ and $s_y^2 = 970.85$. In this problem there is clearly little, if any, equation error or measurement error, and both orthogonal regression and the method of moments give essentially the same estimates as the ordinary least squares slope.

7. EXAMPLE: DIETARY ASSESSMENT

An important problem in dietary assessment is calibrating food frequency questionnaires (Y) to correct for their systematic bias in measuring true usual intake (X); see Freedman, Carroll, and Wax (1991). Instead of being able to measure true usual intake, one typically runs an experiment where some individuals fill out replicated 2–4 day food diaries, resulting in an error-prone version W of true usual intake. An important variable that we use here is "% Calories from Fat," which, as the name suggests, means the percentage of calories in a person's diet that comes from fat.

When we talk about systematic bias in questionnaires, we are especially concerned with the possibility that $\beta_1 \neq 1$ in (8). When this occurs, typically $\beta_1 < 1$, so the effect is that questionnaires, on average, underreport the usual intake of individuals with large amounts of fat in their diet. Understanding the bias in questionnaires is important in a number of contexts, ranging from designing clinical trials (Freedman, Schatzkin, and Wax 1990) to estimating the population distribution of usual intake.

Because food frequency questionnaires and diaries are trying to measure the same quantity, one might naively think that this is a good example for orthogonal regression. This is far from the case because equation error is a large and important component in food frequency questionnaires.

We illustrate the problem on data from the Helsinki Diet Pilot Study (Pietinen et al. 1988). For $n = 133$ individuals the investigators recorded first a questionnaire Y_{i1}^*, then a food diary W_{i1}^*, then a second food diary W_{i2}^*, and finally a second questionnaire Y_{i2}^*. There was a gap of over a month between each measurement, so that it is reasonable to assume that all of the random variables in our model (2) and (8) are uncorrelated; this was checked using the methods of Landin, Carroll, and Freedman (1995). The means within each individual yield W_i and Y_i. The W's have mean 38.85 and variance 29.54, while the Y's have mean 37.58 and variance 20.68.

The ordinary least squares slope is .40. In this analysis, because we have replicates, we can estimate the measurement error variances: $\hat{\sigma}_\varepsilon^2 = 6.36, \hat{\sigma}_u^2 = 7.15$, and hence $\hat{\eta} = .89$. This leads to

$$\hat{\beta}_1(\text{OR}) = 1.83\hat{\beta}_1(\text{OLS}) = .74,$$

while the method of moments slope is

$$\hat{\beta}_1(\text{MM}) = 1.32\hat{\beta}_1(\text{OLS}) = .53.$$

The fairly large differences between the method of moments and orthogonal regression are practically important. The method of moments strongly suggests that food frequency questionnaires systematically underestimate large values of true usual intake of % Calories from Fat (because the slope, equal to .53, is much less than 1.0, the bootstrap standard error is .10, and the upper end of the bootstrap 95% confidence interval is .73), while orthogonal regression does not so strongly suggest the bias (since the slope equals .74, the bootstrap standard error equals .15, and the upper end of a bootstrap 95% confidence interval is 1.04). The bias that we believe exists in food frequency questionnaires makes it difficult to use these instruments to estimate the distribution of usual intake in a population.

Again, the large difference between orthogonal regression and the method of moments suggests the presence of equation error. We obtained an estimated equation error variance $\hat{\sigma}_q^2 = 8.10$ with estimated standard error 2.06; both the asymptotics and the bootstrap give small significance levels. Instead of estimating η in (4) as .89, it appears that η_{EE} in (10) should be estimated by 2.02. With this value of η_{EE} the orthogonal regression and method of moments estimators both equal .53, and their bootstrap analyses are virtually identical as well.

8. CONCLUSIONS

When its assumptions hold, orthogonal regression is a perfectly justifiable method of estimation. As a method, however, it often lends itself to misuse by the unwary because orthogonal regression does not take equation error into account.

Our own favorite method of estimation in linear EIV problems is the method of moments. This requires that one obtain an estimate of the measurement error variance in the observed predictor W. No estimator is a panacea or should be used as a black box. However, by forcing us to supply an error variance estimate, the method of moments

requires careful consideration of error components. In the geology example, a rare case where orthogonal regression seems preferable to the method of moments, an analysis of sources of variation warns us of the problem with the method of moments.

We have alluded only briefly to the problem of correlation of measurement errors, and have checked for this in the nutrition example (Section 7). If the covariance between $q + \varepsilon$ and U is $\sigma_{q+\varepsilon,u}$, then the ordinary least squares estimated slope converges to $\{\beta_1 \sigma_x^2 + \sigma_{q+\varepsilon,u}\}/(\sigma_x^2 + \sigma_u^2)$. In this case both the standard orthogonal regression estimator (6) and the classic method of moments estimator are inconsistent. For example, the classic method of moments estimator converges to $\beta_1 + \sigma_{q+\varepsilon,u}/\sigma_x^2$. The existence of correlated measurement errors requires that one construct alternative estimators. Fuller (1987) should be consulted for details; see also Landin et al. (1995) for modifications appropriate for nutrition data. What do we recommend in practice? It seems to us that the routine use of the methods of moments will seldom lead one astray. The method requires no knowledge of the equation error variance or the variance of the measurement error in Y. Only the variance of the measurement error in X is needed. As we saw in the example of calibrating glucose measurements, when orthogonal regression can be properly applied, usually the method of moments can also be used, and the two methods will give similar estimates. The chief danger with the method of moments is that, as in the geology example, measurement error variance can be seriously missestimated by not using truly replicate measurements. However, practicing statisticians can usually identify such situations.

APPENDIX

Let $\eta = \sigma_\varepsilon^2/\sigma_u^2$ and $\xi = \sigma_r^2/\sigma_x^2$. Let $Y = \beta_0 + \beta_1 X + r + \varepsilon$ and $W = X + U$. By direct calculation, s_y^2, s_w^2, and s_{wy} converge in probability to $\beta_1^2 \sigma_x^2 + \sigma_r^2 + \sigma_\varepsilon^2, \sigma_x^2 + \sigma_u^2$, and $\beta_1 \sigma_x^2$, respectively. The orthogonal regression estimator $\hat{\beta}_1$ thus satisfies

$$
\hat{\beta}_1 \xrightarrow{p} \frac{\begin{array}{c}(\beta_1^2 - \eta)\sigma_x^2 + \sigma_r^2 \\ + [\{(\beta_1^2 - \eta)\sigma_x^2 + \sigma_r^2\}^2 + 4\eta\beta_1^2\sigma_x^4]^{1/2}\end{array}}{s\beta_1\sigma_x^2}
$$

$$
= \frac{\beta_1^2 - \eta + \xi + \{(\beta_1^2 - \eta + \xi)^2 + 4\eta\beta_1^2\}^{1/2}}{2\beta_1}.
$$

The claim is that this limiting value exceeds β_1 in absolute value. It suffices to take $\beta_1 > 0$, in which case we must show that

$$
\{(\beta_1^2 - \eta + \xi)^2 + 4\eta\beta_1^2\}^{1/2} \geq 2\beta_1^2 - (\beta_1^2 - \eta + \xi).
$$

If the right-hand side is negative, we are done, so we assume that it is nonnegative. Squaring, we find that we must show that

$$
(\beta_1^2 - \eta + \xi)^2 + 4\eta\beta_1^2 \geq 2\beta_1^4 - 4\beta_1^2(\beta_1^2 - \eta + \xi) \\ + (\beta_1^2 - \eta + \xi)^2,
$$

which is obvious.

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